# Chapter 3
# Low-Density and High-Density Asymmetric CT-Burst Correcting Integer Codes

The contents of this chapter are based on the paper mentioned below:

- Pokhrel, N. K. and Das, P. K. Low-density and high-density asymmetric CT-burst correcting integer codes. *Advances in Mathematics of Communications*, 1-15, 2022, doi:10.3934/amc.2022030.

# Chapter 3

# Low-Density and High-Density Asymmetric CT-Burst Correcting Integer Codes

## 3.1 Overview

Continuing with the asymmetric CT-burst correcting integer codes discussed in Chapter 2, we study the codes applying the Hamming weight constraint. We make use of the observation made by Wyner (refer Section 1.3) that not all components may be affected by a burst. As discussed in Section 2.2 of Chapter 2, we categorise these bursts based on the intensity of the affected components, viz., low-density and high-density. The two classes of error-correcting codes are classified as follows:

- Integer codes capable of correcting low-density asymmetric CT-bursts of length $l$ with weight between 1 to $\lfloor \frac{l}{2} \rfloor$.
- Integer codes capable of correcting high-density asymmetric CT-bursts of length $l$ with weight between $\lceil \frac{l}{2} \rceil$ to $l$.

Both types of errors considered lie in an asymmetric CT-burst of length $l$ within a $b$-bit byte. Continuing with Definition 1.12 having the values, $m = 2^b - 1, M = 1$ and $N = k + 1$, two classes of integer codes correcting low-density and high-density asymmetric CT-bursts within a $b$-bit byte have been presented in Section 3.2. In Section 3.3, the proposed codes are compared with similar error-correcting codes in terms of various properties, viz. memory consumption and number of table look ups. The implementation of these codes in a quad-core processor is similar to the implementation discussed in Chapter 2. This chapter winds up with the idea of determining the probability of erroneous decoding and an approach for undetected error probability in Section 3.3.

## 3.2 Construction of codes

Based on Definition 1.7-1.8 of low-density and high-density asymmetric CT-bursts respectively, the definitions discussed below are for their integer values.

**Definition 3.1.** *The collection of $LACTB_{d/l}$ errors within a single b-bit byte is defined by*

- $\epsilon_{d/l,b} = \bigcup_{i=1}^{b-l+1} e^i_{d/l,b}$,
  *where* $e^i_{d/l,b} = \Big\{2^{i-1} + p_1 2^i + p_2 2^{i+1} + \ldots + p_{l-1} 2^{i+l-2} \,|\, p_1, p_2, \ldots, p_{l-1} \in \{0,1\}$ *are such that* $\sum_{j=1}^{l-1} p_j \leq \lfloor \frac{l}{2} \rfloor - 1 \Big\}$.

The integer codes capable of correcting all $LACTB_{d/l}$ errors occurring in $b$-bit bytes are termed as integer $LACTB_{(d/l,b)}C$ codes.

**Definition 3.2.** *The collection of $HACTB_{h/l}$ errors occurring within a single b-bit byte is defined by*

- $\varepsilon_{h/l,b} = \bigcup_{i=1}^{b-l+1} E^i_{h/l,b}$,
  *where* $E^i_{h/l,b} = \{2^{i-1} + p_1 2^i + p_2 2^{i+1} + \ldots + p_{l-1} 2^{i+l-2} \,|\, p_1, p_2, \ldots, p_{l-1} \in \{0,1\}$ *are such that* $\sum_{j=1}^{l-1} p_j \geq \lceil \frac{l}{2} \rceil - 1\}$.

The integer codes capable of correcting all $HACTB_{h/l}$ errors occurring in $b$-bit bytes are termed as integer $HACTB_{(h/l,b)}C$ codes.

### 3.2.1 Encoding procedure

Here, we describe the encoding and decoding procedures for both integer $LACTB_{(d/l,b)}C$ and integer $HACTB_{(h/l,b)}C$ codes. The encoding procedure, width representation of an encoded codeword, and process for choosing the coefficients are similar to the processes discussed in Section 2.2 of Chapter 2. The definitions below give us the syndrome sets used in the error-correcting procedure.

**Definition 3.3.** *The set of syndromes for an integer code correcting low-density asymmetric CT-bursts within a b-bit byte will be*

$$S_1 = \bigcup_{i=1}^{k+1} -[C_i \epsilon_{d/l,b}] \pmod{2^b - 1}. \tag{3.1}$$

**Definition 3.4.** *The set of syndromes for an integer code correcting high-density asymmetric CT-bursts within a b-bit byte will be*

$$S_2 = \bigcup_{i=1}^{k+1} -[C_i \varepsilon_{h/l,b}] \pmod{2^b - 1}. \tag{3.2}$$

In both of the cases above, the condition for choosing coefficient $C_i$'s will be same as Chapter 2. The Python programmes for finding the required coefficients for both of the codes are given in Appendices B and C.

Since every error in the set $\epsilon_{d/l,b}$ or $\varepsilon_{h/l,b}$ has a unique form in terms of its $b$-bit representation and also each $b$-bit form corresponds to a unique element in the ring $\mathbb{Z}_{2^b-1}$, thus the sets $\epsilon_{d/l,b}$ and $\varepsilon_{h/l,b}$ have no repetition, consequently the elements within the Syndrome set (3.1)–(3.2) will not repeat making way for an ambiguity free decoding. We take $2^{[m,r]}$ as the incomplete binomial expansion of $(1+1)^m$ taken up to $(r+1)^{th}$ term, viz. $2^{[m,r]} = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{r}$. Throughout the study, we shall consider $l \geq 2$ as $l = 1$ corresponds to single asymmetric error [80], theorems discussed below help us to construct the look up table for decoding purpose.

**Theorem 3.5.** *A necessary and sufficient condition of a* $((k+1)b, kb)$ *integer* $LACTB_{(d/l,b)}C$ *code is that there exist* $k$ *distinct coefficients from the set* $\mathbb{Z}_{2^b-1} \setminus \{0, 1\}$ *such that* $|S_1| = (k+1)(b-l+1)2^{[l-1, \lfloor \frac{l}{2} \rfloor - 1]}$.

*Proof.* Since the sets $-C_i \epsilon_{d/l,b} \pmod{2^b - 1}$, $1 \leq i \leq k+1$ are all mutually disjoint, so it is sufficient to show that $|\epsilon_{d/l,b}| = (b-l+1)2^{[l-1, \lfloor \frac{l}{2} \rfloor - 1]}$.

The bursts considered for low-density have weight at most $\lfloor \frac{l}{2} \rfloor$ with length $l$, so there are $(b-l+1)$ different beginning positions having same number of bursts as elaborated earlier. For bursts of weight at most $\lfloor \frac{l}{2} \rfloor$ beginning from first position, we have 1 ( or $\binom{l-1}{0}$) burst of weight 1 and length $l$, viz. $2^0$, $l-1$( or $\binom{l-1}{1}$) bursts of weight 2 and length $l$, viz. $2^0+2^1, 2^0+2^2, \ldots, 2^0+2^{l-1}$. Continuing this, we have $\binom{l-1}{\lfloor \frac{l}{2} \rfloor -1}$ bursts of weight $\lfloor \frac{l}{2} \rfloor$ and length $l$ beginning from the $1^{st}$ position. Thus the number of bursts up to weight $\lfloor \frac{l}{2} \rfloor$ of length $l$ beginning from the first position will be $\binom{l-1}{0}+\binom{l-1}{1}+\ldots+\binom{l-1}{\lfloor \frac{l}{2} \rfloor -1} = \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor -1}\binom{l-1}{j}$. We have $b-l+1$ beginning positions for the bursts discussed, viz. the bursts with $2^0, 2^1, \ldots, 2^{b-l}$ beginning values, thus the total number of error patterns in $\epsilon_{d/l,b}$ will be $(b-l+1)2^{[l-1, \lfloor \frac{l}{2} \rfloor -1]}$. Therefore, the cardinality of $S_1$ is $|S_1| = (k+1)(b-l+1)2^{[l-1, \lfloor \frac{l}{2} \rfloor -1]}$. □

**Remark 3.6.** *Since the last component of a CT-burst may be zero, so the length of a CT-burst may be increased provided it has sufficient number of components for it. For instance, CT-burst* (01011000) *may be considered of length* 4, 5, 6 *or* 7*, whereas the CT-burst* (00010110) *can only be considered of length* 4 *or* 5*. This approach will not be valid in case of a burst as the last component should be necessarily non-zero.*

**Theorem 3.7.** *A necessary and sufficient condition of a* $(kb+b, kb)$ *integer* $HACTB_{(h/l,b)}C$ *code is that there exist* $k$ *distinct coefficient* $C_i$*'s from the set* $\mathbb{Z}_{2^b-1} \setminus \{0, 1\}$ *such that* $|S_2| = (k+1)(b-l+1)(2^{l-1} - 2^{[l-1, \lceil \frac{l}{2} \rceil -2]})$.

*Proof.* Since $S_2 = (-C_1\varepsilon_{h/l,b}) \cup \ldots \cup (-C_k\varepsilon_{h/l,b}) \cup (\varepsilon_{h/l,b})$, and coefficient $C_i$'s are chosen such that the sets in the union above are all mutually disjoint, so it sufficient to show that $\varepsilon_{h/l,b} = (b-l+1)\Big(2^{l-1} - 2^{[l-1, \lceil \frac{l}{2} \rceil -2]}\Big)$. We determine the number of errors of this type as discussed in Theorem 3.5. For CT-bursts beginning from the $1^{st}$ position having weight $\lceil \frac{l}{2} \rceil$ and length $l$, there are $\binom{l-1}{\lceil \frac{l}{2} \rceil -1}$ possibilities, similarly there are $\binom{l-1}{\lceil \frac{l}{2} \rceil}$ CT-bursts of weight $\lceil \frac{l}{2} \rceil+1$ and length $l$, continuing this we have $\binom{l-1}{l-1}$ CT-bursts of length and weight $l$ beginning from the $1^{st}$ position. Since there are exactly $(b-l+1)$ beginning positions for the discussed type of CT-bursts, and the total number of CT-bursts of length $l$ for these beginning positions is $2^{l-1}$, hence $|\varepsilon_{h/l,b}| = (b-l+1)[2^{l-1} - \{\binom{l-1}{0}+\binom{l-1}{1}+\ldots+\binom{l-1}{\lceil \frac{l}{2} \rceil -2}\}] = (b-l+1)\Big(2^{l-1} - 2^{[l-1, \lceil \frac{l}{2} \rceil -2]}\Big)$. □

### 3.2.2 Decoding procedure

Again, the decoding procedure is similar to Chapter 2 with the $LUT_2$'s consuming $|S_1| \times (2b + \lceil log_2(k+1) \rceil)$ and $|S_2| \times (2b + \lceil log_2(k+1) \rceil)$ bits, respectively, for the low-density and high-density cases. Likewise, the steps for decoding are also analogous to the preceding chapter. We shall now explain the error-correcting procedures for both the codes with the help of suitable examples. Example 3.8 is for low-density and Example 3.10 is for high-density.

**Example 3.8.** *Let* $b = 8$*,* $l = 6$*, then* $\lfloor \frac{l}{2} \rfloor = 3$ *and* $C_1 = 2$*, Table 3.1 enumerates the syndrome elements for decoding a message. Suppose a message* 11010111 *is transmitted, then the corresponding check byte* $C_B = 11101011$*, so* 11010111 11101011 *will be the encoded message. For an asymmetric CT-burst of length* 6 *within an* 8*-bit byte having weight at most 3, we may have the following cases.*

***Case I*** *(Low-density asymmetric CT-burst in data byte): If the received message is* 1<u>000010</u>1 11101011*, then syndrome* $S = [C_{\bar{B}} - \bar{C}_B] \pmod{2^b - 1} = [2 \times 161 - 215] \pmod{255} = 107 = [-2 \times 74] \pmod{255}$*. Hence data byte* $B_1$ *has an error* $e = 2^1 + 2^3 + 2^6$*, so the corrected data byte will be* $B_1 = [\bar{B}_1 + e] \pmod{2^b - 1} = [161 + 74] \pmod{255} = 235 = 11010111$*.*

***Case II*** *(Low-density asymmetric CT-burst in check byte): If the received message is* 11010111 11<u>001000</u>*, then syndrome* $S = [C_{\bar{B}} - \bar{C}_B] \pmod{2^b - 1} = [215 - 19] \pmod{255} = 196$*. Hence check byte has an error* $e = 2^2 + 2^6 + 2^7$*, so the corrected check byte will be* $C_B = [\bar{C}_B + e] \pmod{2^b - 1} = [196 + 19] \pmod{255} = 215 = 11101011$*.*

***Case III*** *(Error pattern beyond specification): If the received message is* 11<u>000000</u> 11101011*, then syndrome* $S = [C_{\bar{B}} - \bar{C}_B] \pmod{2^b - 1} = [6 - 215] \pmod{255} = 46 \neq 0$*. Since the syndrome value is not in the* $LUT_2$*, the decoder will declare an uncorrectable error.*

**Remark 3.9.** *It is interesting to note that for* $l = 6$ *and* $b = 8$*, we do not obtain* $C_i$*'s for asymmetric CT-bursts (refer Chapter 2), correspondingly we cannot transmit messages. But by choosing low-density asymmetric CT-bursts of weight up to*

3 *inside an asymmetric CT-burst of length* 6*, we can construct codes conveniently. Case I in Example 3.8 reflects the beauty of this method, where a low-density asymmetric CT-burst up to weight* 3 *and length* 6 *is easily detected as well as corrected. This would not have been possible otherwise.*

Table 3.1: $\boldsymbol{LUT_2}$ **for** $\boldsymbol{(16, 8)}$ **integer** $\boldsymbol{LACTB_{(3/6,8)}C}$ **code**

| **Sl. No.** | **Syndrome ($S_1$)** | **Error Loc. ($i$)** | **Error ($e$)** | **Sl. No.** | **Syndrome ($S_1$)** | **Error Loc. ($i$)** | **Error ($e$)** |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 49 | 107 | 1 | 74 |
| 2 | 2 | 2 | 2 | 50 | 115 | 1 | 70 |
| 3 | 3 | 2 | 3 | 51 | 118 | 1 | 196 |
| 4 | 4 | 2 | 4 | 52 | 119 | 1 | 68 |
| 5 | 5 | 2 | 5 | 53 | 123 | 1 | 66 |
| 6 | 6 | 2 | 6 | 54 | 132 | 2 | 132 |
| 7 | 7 | 2 | 7 | 55 | 140 | 2 | 140 |
| 8 | 9 | 2 | 9 | 56 | 148 | 2 | 148 |
| 9 | 10 | 2 | 10 | 57 | 151 | 1 | 52 |
| 10 | 11 | 2 | 11 | 58 | 155 | 1 | 50 |
| 11 | 12 | 2 | 12 | 59 | 157 | 1 | 49 |
| 12 | 13 | 2 | 13 | 60 | 164 | 2 | 164 |
| 13 | 14 | 2 | 14 | 61 | 167 | 1 | 44 |
| 14 | 17 | 2 | 17 | 62 | 171 | 1 | 42 |
| 15 | 18 | 2 | 18 | 63 | 173 | 1 | 41 |
| 16 | 19 | 2 | 19 | 64 | 179 | 1 | 38 |
| 17 | 20 | 2 | 20 | 65 | 181 | 1 | 37 |
| 18 | 21 | 2 | 21 | 66 | 182 | 1 | 164 |
| 19 | 22 | 2 | 22 | 67 | 183 | 1 | 36 |
| 20 | 25 | 2 | 25 | 68 | 185 | 1 | 35 |
| 21 | 26 | 2 | 26 | 69 | 187 | 1 | 34 |
| 22 | 28 | 2 | 28 | 70 | 189 | 1 | 33 |

Contd...

| **Sl. No.** | **Syndrome ($S_1$)** | **Error Loc. ($i$)** | **Error ($e$)** | **Sl. No.** | **Syndrome ($S_1$)** | **Error Loc. ($i$)** | **Error ($e$)** |
|---|---|---|---|---|---|---|---|
| 23 | 33 | 2 | 33 | 71 | 196 | 2 | 196 |
| 24 | 34 | 2 | 34 | 72 | 199 | 1 | 28 |
| 25 | 35 | 2 | 35 | 73 | 203 | 1 | 26 |
| 26 | 36 | 2 | 36 | 74 | 205 | 1 | 25 |
| 27 | 37 | 2 | 37 | 75 | 211 | 1 | 22 |
| 28 | 38 | 2 | 38 | 76 | 213 | 1 | 21 |
| 29 | 41 | 2 | 41 | 77 | 214 | 1 | 148 |
| 30 | 42 | 2 | 42 | 78 | 215 | 1 | 20 |
| 31 | 44 | 2 | 44 | 79 | 217 | 1 | 19 |
| 32 | 49 | 2 | 49 | 80 | 219 | 1 | 18 |
| 33 | 50 | 2 | 50 | 81 | 221 | 1 | 17 |
| 34 | 52 | 2 | 52 | 82 | 227 | 1 | 14 |
| 35 | 55 | 1 | 100 | 83 | 229 | 1 | 13 |
| 36 | 59 | 1 | 98 | 84 | 230 | 1 | 140 |
| 37 | 66 | 2 | 66 | 85 | 231 | 1 | 12 |
| 38 | 68 | 2 | 68 | 86 | 233 | 1 | 11 |
| 39 | 70 | 2 | 70 | 87 | 235 | 1 | 10 |
| 40 | 74 | 2 | 74 | 88 | 237 | 1 | 9 |
| 41 | 76 | 2 | 76 | 89 | 241 | 1 | 7 |
| 42 | 82 | 2 | 82 | 90 | 243 | 1 | 6 |
| 43 | 84 | 2 | 84 | 91 | 245 | 1 | 5 |
| 44 | 87 | 1 | 84 | 92 | 246 | 1 | 132 |
| 45 | 91 | 1 | 82 | 93 | 247 | 1 | 4 |
| 46 | 98 | 2 | 98 | 94 | 249 | 1 | 3 |
| 47 | 100 | 2 | 100 | 95 | 251 | 1 | 2 |
| 48 | 103 | 1 | 76 | 96 | 253 | 1 | 1 |

Example 3.10 illustrates an integer $HACTB_{(2/4,8)}C$ code with the help of Table 3.2 prepared using (3.2).

Table 3.2: $\boldsymbol{LUT_2}$ **for** $\mathbf{(16,8)}$ **integer** $\boldsymbol{HACTB_{(2/4,8)}C}$ **code**

| **Sl. No.** | **Syndrome ($S_2$)** | **Error Loc. ($i$)** | **Error ($e$)** | **Sl. No.** | **Syndrome ($S_2$)** | **Error Loc. ($i$)** | **Error ($e$)** |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 2 | 3 | 36 | 76 | 1 | 14 |
| 2 | 5 | 2 | 5 | 37 | 77 | 1 | 88 |
| 3 | 6 | 2 | 6 | 38 | 80 | 2 | 80 |
| 4 | 7 | 2 | 7 | 39 | 83 | 1 | 22 |
| 5 | 9 | 2 | 9 | 40 | 88 | 2 | 88 |
| 6 | 10 | 2 | 10 | 41 | 90 | 1 | 30 |
| 7 | 11 | 2 | 11 | 42 | 91 | 1 | 104 |
| 8 | 12 | 2 | 12 | 43 | 98 | 1 | 112 |
| 9 | 13 | 2 | 13 | 44 | 100 | 1 | 5 |
| 10 | 14 | 2 | 14 | 45 | 104 | 2 | 104 |
| 11 | 15 | 2 | 15 | 46 | 105 | 1 | 120 |
| 12 | 18 | 2 | 18 | 47 | 107 | 1 | 13 |
| 13 | 20 | 2 | 20 | 48 | 112 | 2 | 112 |
| 14 | 21 | 1 | 24 | 49 | 120 | 2 | 120 |
| 15 | 22 | 2 | 22 | 50 | 126 | 1 | 144 |
| 16 | 24 | 2 | 24 | 51 | 138 | 1 | 12 |
| 17 | 26 | 2 | 26 | 52 | 144 | 2 | 144 |
| 18 | 28 | 2 | 28 | 53 | 145 | 1 | 20 |
| 19 | 30 | 2 | 30 | 54 | 152 | 1 | 28 |
| 20 | 35 | 1 | 40 | 55 | 154 | 1 | 176 |
| 21 | 36 | 2 | 36 | 56 | 159 | 1 | 36 |
| 22 | 38 | 1 | 7 | 57 | 162 | 1 | 3 |
| 23 | 40 | 2 | 40 | 58 | 166 | 1 | 44 |
| 24 | 42 | 1 | 48 | 59 | 169 | 1 | 11 |
| 25 | 44 | 2 | 44 | 60 | 173 | 1 | 52 |
| 26 | 45 | 1 | 15 | 61 | 176 | 2 | 176 |

Contd...

| Sl. No. | Syndrome ($S_2$) | Error Loc. ($i$) | Error ($e$) | Sl. No. | Syndrome ($S_2$) | Error Loc. ($i$) | Error ($e$) |
|---|---|---|---|---|---|---|---|
| 27 | 48 | 2 | 48 | 62 | 180 | 1 | 60 |
| 28 | 49 | 1 | 56 | 63 | 182 | 1 | 208 |
| 29 | 52 | 2 | 52 | 64 | 200 | 1 | 10 |
| 30 | 56 | 2 | 56 | 65 | 207 | 1 | 18 |
| 31 | 60 | 2 | 60 | 66 | 208 | 2 | 208 |
| 32 | 63 | 1 | 72 | 67 | 210 | 1 | 240 |
| 33 | 69 | 1 | 6 | 68 | 214 | 1 | 26 |
| 34 | 70 | 1 | 80 | 69 | 231 | 1 | 9 |
| 35 | 72 | 2 | 72 | 70 | 240 | 2 | 240 |

**Example 3.10.** *Let* $b = 8$, $l = 4$, *then* $\lceil \frac{l}{2} \rceil = 2$ *and* $C_1 = 31$, *so the syndrome set will have* 70 *elements as listed in Table 3.2. Suppose we want to transmit message* 11111100, *then the encoded message will be* 11111100 00010101. *For an asymmetric CT-burst of length* 4 *within an* 8*-bit byte having weight at least* 2, *we consider the following cases.*

***Case I*** *(Data byte having error): Suppose the transmitted message* 11111100 01011101 *is received as* 11000000 01011101, *then syndrome value* $S = [C_{\bar{B}} - \bar{C}_B] \pmod{2^b - 1} = [3 \times 31 - 168] \pmod{255} = 180 = -31 \times 60$, *thus* $1^{st}$ *data byte has error* $e = 60 = (2^2 + 2^3 + 2^4 + 2^5) \pmod{255}$, *hence corrected data byte* $B_1 = [\bar{B}_1 + e] \pmod{2^b - 1} = [3 + 60] \pmod{255} = 63 = 11111100$.

***Case II*** *(Check byte having error): Suppose the message is received as* 11111100 00000001, *then syndrome* $S = [C_{\bar{B}} - \bar{C}_B] \pmod{2^b - 1} = [168 - 128] \pmod{255} = 40$, *thus the check byte has error* $e = 40 = (2^3 + 2^5) \pmod{255}$, *hence the corrected check byte will be* $C_B = [\bar{C}_B + e] \pmod{2^b - 1} = [128 + 40] \pmod{255} = 168 = 00010101$.

***Case III*** *(Error pattern beyond specification): Suppose the message is received as* 11111100 00010100, *then syndrome* $S = [C_{\bar{B}} - \bar{C}_B] \pmod{2^b - 1} = [168 - 40] \pmod{255} = 128 \neq 0$. *Since the syndrome value is not in the* $LUT_2$, *the decoder will declare an uncorrectable error.*

**Remark 3.11.** *In case of asymmetric CT-bursts of length* 4 *with* $b = 8$*, we do not find coefficients for the construction (refer Chapter 2). But by considering weight greater than or equal to* 2 *within length* 4*, we get the desired code as discussed above.*

## 3.3 Evaluation and comparison

In this section, we will analyse the implementation strategy, the probability of erroneous decoding, Bit Error Rate as well as undetected errors for the proposed codes.

### 3.3.1 Implementation and comparison

For implementation of these codes in a quad-core processor, we refer to Chapter 2.

Table 3.3: **First 32 possible coefficients for integer $LACTB_{(d/l,b)}C$ codes**

| $b$ | $l$ | $\lfloor \frac{l}{2} \rfloor$ | **Coefficients** |
|---|---|---|---|
| 7 | 5 | 2 | 2, 33, 47, 100 |
| 8 | 4 | 2 | 2 |
| 8 | 6 | 3 | 2 |
| 10 | 4 | 2 | 2, 7, 13, 15, 23, 37, 41, 47, 49, 83 |
| 10 | 6 | 3 | 2, 135 |
| 10 | 7 | 3 | 2 |
| 16 | 4 | 2 | 2, 7, 11, 13, 15, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 81, 83, 89, 91, 97, 101, 105, 107, 109 |
| 16 | 5 | 2 | 2, 7, 11, 13, 15, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 67, 71, 73, 77, 79, 81, 83, 89, 97, 101, 105, 107, 109, 121, 125, 127 |
| 16 | 6 | 3 | 2, 15, 23, 29, 31, 43, 47, 53, 59, 67, 71, 73, 77, 79, 83, 89, 97, 101, 107, 117, 131, 137, 139, 149, 157, 163, 167, 181, 199, 227, 233, 251 |
| 16 | 8 | 4 | 2, 31, 61, 207, 776, 7769 |

Contd...

| $b$ | $l$ | $\lfloor \frac{l}{2} \rfloor$ | **Coefficients** |
|---|---|---|---|
| 16 | 9 | 4 | 2, 31, 413, 1536, 16904 |
| 32 | 6 | 3 | 2, 15, 23, 29, 31, 43, 47, 53, 59, 61, 67, 71, 73, 77, 79, 81, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163 |
| 32 | 7 | 3 | 2, 15, 29, 31, 43, 47, 53, 59, 61, 71, 77, 79, 83, 89, 101, 103, 107, 109, 113, 117, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181 |
| 32 | 8 | 4 | 2, 31, 61, 63, 79, 95, 103, 107, 121, 127, 151, 157, 167, 173, 179, 181, 191, 199, 211, 221, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277 |

Table 3.5-3.6 represent the look up table sizes and the corresponding number of look ups required, codes in these tables are constructed upon the existence of coefficients as depicted in Table 3.3-3.4.

Table 3.4: **First 32 possible coefficients for integer $HACTB_{(h/l,b)}C$ codes**

| $b$ | $l$ | $\lceil \frac{l}{2} \rceil$ | **Coefficients** |
|---|---|---|---|
| 8 | 3 | 2 | 2, 3, 29, 37 |
| 8 | 4 | 2 | 31 |
| 8 | 5 | 3 | 239 |
| 10 | 4 | 2 | 2, 13, 41 |
| 10 | 6 | 3 | 991 |
| 16 | 5 | 3 | 2, 5, 11, 17, 35, 37, 39, 43, 47, 53, 59, 61, 67, 71, 73, 77, 79, 83, 97, 101, 107, 113, 119, 127, 131, 137, 149, 151, 157, 163, 169, 173 |
| 16 | 6 | 3 | 2, 11, 67, 71, 73, 79, 95, 103, 129, 137, 179, 193, 217, 267, 293, 311, 327, 373, 389, 393, 449, 461, 517, 725, 761, 1001, 2501, 2527, 2999, 3481, 3643, 4517 |
| 16 | 7 | 4 | 2, 9, 43, 131, 139, 163, 183, 197, 199, 209, 251, 491, 2477, 4727 |

Contd...

| $b$ | $l$ | $\lceil \frac{l}{2} \rceil$ | Coefficients |
|---|---|---|---|
| 16 | 8 | 4 | 7, 61, 22447 |
| 16 | 9 | 5 | 2389, 21769, 65279 |
| 32 | 6 | 3 | 2, 11, 17, 65, 67, 69, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 121, 127, 131, 133, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 187 |
| 32 | 8 | 4 | 2, 19, 87, 97, 131, 137, 161, 193, 257, 263, 265, 269, 271, 277, 281, 283, 289, 293, 307, 311, 313, 317, 331, 337, 341, 347, 349, 353, 359, 361, 367, 373 |
| 32 | 9 | 5 | 2, 17, 47, 77, 129, 131, 139, 193, 197, 257, 263, 265, 269, 277, 281, 289, 293, 321, 337, 353, 389, 401, 449, 521, 523, 529, 531, 533, 541, 547, 551, 557 |

Table 3.5: **Lookup sizes for integer $LACTB_{(d/l,b)}C$ codes**

| Codes | $b$ | $l$ | $\lfloor \frac{l}{2} \rfloor$ | $LUT_1$ size | $LUT_2$ size | No of table look ups |
|---|---|---|---|---|---|---|
| (144,128) | 16 | 4 | 2 | 4× 16B | 2.11 KB | $1 \leq \eta_{TL} \leq 10$ |
| (528,512) | 16 | 5 | 2 | 4× 64B | 9.41 KB | $1 \leq \eta_{TL} \leq 12$ |
| (512,480) | 32 | 6 | 3 | 4× 60B | 58.75 KB | $1 \leq \eta_{TL} \leq 14$ |
| (1056,1024) | 32 | 8 | 4 | 4× 128B | 0.46 MB | $1 \leq \eta_{TL} \leq 17$ |

Table 3.6: **Lookup sizes for integer $HACTB_{(h/l,b)}C$ codes**

| Codes | $b$ | $l$ | $\lceil \frac{l}{2} \rceil$ | $LUT_1$ size | $LUT_2$ size | No of table look ups |
|---|---|---|---|---|---|---|
| $(240, 224)$ | 16 | 7 | 4 | $4 \times 28$ B | 28.35 KB | $1 \leq \eta_{TL} \leq 14$ |
| $(512, 496)$ | 16 | 6 | 3 | $4 \times 62$ B | 42.33 KB | $1 \leq \eta_{TL} \leq 15$ |
| $(544, 512)$ | 32 | 6 | 3 | $4 \times 64$ B | 0.1 MB | $1 \leq \eta_{TL} \leq 15$ |
| $(1056, 1024)$ | 32 | 8 | 4 | $4 \times 128$ B | 0.71 MB | $1 \leq \eta_{TL} \leq 18$ |

To the best of our knowledge, no error-correcting and detecting code has been developed for low-density and high-density asymmetric CT-burst errors, therefore we shall consider some similar codes in this regard and try to compare them by matching the parameters on same lines. In [31], linear codes correcting low-density CT-bursts are discussed with fixed burst length, upon existence consider a $((k+1)b, kb)$ linear code with $q=2$ capable of correcting CT-bursts up to weight $d$ of length $l$ satisfying $d \leq l < \frac{b}{2}$. This linear code will have $k+1$ blocks of length $b$ each, thus the number of error patterns will be equal to $(k+1)(b-l+1)2^{[l-1,d-1]}$. By considering same parameters in the proposed code, we observe same number of error patterns. Since the syndrome table in [31] consists of error patterns and the corresponding syndromes, so the code requires storage of $(k+1)(b-l+1)2^{[l-1,d-1]}(kb+2b)$ bits for decoding purpose. Whereas the bit requirement for the proposed low-density integer codes is $(k+1)(b-l+1)2^{[l-1,d-1]}(\lceil log_2(k+1)\rceil + 2b)$, which clearly justifies less memory requirement for the proposed codes. For example in [31], consider code length $=20$, redundancy $=10$, burst length $=4$, weight (inverted bits) $\leq 2$, number of blocks $=2$ and $t=10$. Now using the necessary and sufficient conditions from [31] (refer Result 1.46-1.47), we can construct a $(20,10)$ linear code with the parameters specified above. Similarly by using Table 3.3, we can construct a $(20,10)$ integer $LACTB_{(2/4,10)}C$ code, where each codeword is of same bit-width similar to the linear counterpart. Thus both codes will have 56 error patterns of length 4 with weight up to 2, hence the bit requirement for syndrome table will be 1680 and 1176 bits respectively for linear and integer codes. It may also be noted that the existence of error-correcting capability of the linear codes is restricted for $d \leq l < \frac{b}{2}$, whereas for the proposed code we do not have such restriction, in fact from Table 3.3 we have many codes surpassing this constraint.

Similarly, the bit requirement for high-density asymmetric CT-bursts can be shown lesser in number compared to the linear codes discussed in [32]. This can be done by just considering parameters having same number of error patterns and replicating the steps followed above for low-density case. From Table 3.4 and Result 1.48-1.49, it is evident that $(16,8)$ codes exist for both cases by considering burst length 4, weight $\geq 2$. Memory consumed by linear code with these parameters is $2 \times 35 \times (16+8) = 1680$ bits whereas by proposed integer code it will be $2 \times 35 \times (16+1) = 1190$ bits. Also, the entries for a linear code can not be arranged in an increasing order, so the number of table look ups will be $\eta_{TL}$, where $1 \leq \eta_{TL} \leq |X|$

(see [63]), where $|X|$ denotes the number of error patterns for both [31] and [32]. Thus the number of table look ups in linear codes is significantly higher than the proposed codes. Memory consumed by $LUT_2$ of integer codes correcting different types of errors along with their range of table look ups are portrayed in Table 3.7. The integer codes discussed in the table are capable of correcting different types of errors in different ways, so it will facilitate in choosing a suitable code as per the error-correcting requirement and its cost effectiveness before its implementation.

Table 3.7: **Comparison of some integer codes with 32 information bytes**

| **Codes** | **Error patterns** | $\boldsymbol{b}$ | $\boldsymbol{l}$ | $\boldsymbol{LUT_2}$ **size** | **No of table look ups** |
|---|---|---|---|---|---|
| $LACTB_{(d/l,b)}C$ | Proposed low-density | 32 | 8 | 0.46 MB | $1 \leq \eta_{TL} \leq 17$ |
| $HACTB_{(h/l,b)}C$ | Proposed high-density | 32 | 8 | 0.71 MB | $1 \leq \eta_{TL} \leq 18$ |
| $(CT_lB)_b$ ( Chapter 2) | Asymmetric CT-bursts | 32 | 8 | 0.96 MB | $1 \leq \eta_{TL} \leq 18$ |
| Definition 1.20 | Symmetric bursts | 32 | 8 | 3.84 MB | $1 \leq \eta_{TL} \leq 20$ |
| Result 1.39 | Bursts and random asymmetric | 32 | 8 | 2.32 MB | $1 \leq \eta_{TL} \leq 20$ |
| Result 1.36 | Asymmetric bursts and double random asymmetric | 32 | 8 | 8.91 MB | $1 \leq \eta_{TL} \leq 21$ |

### 3.3.2 The probability of erroneous decoding

Since the codes are studied over a $Z$-channel, so consider probability of the occurrence of the pattern $1 \to 0$ per bit be $\epsilon$, thus the probability of the occurrence of $1 \to 1$ and $0 \to 1$ will become $1 - \epsilon$ and 0 respectively. Probability of erroneous decoding for both the codes are discussed in the theorems below.

**Theorem 3.12.** *For transition probability $\epsilon$ pertaining to $1 \to 0$, the erroneous decoding probability $P_d(LD)$ for a $(kb+b, kb)$ integer $LACTB_{(d/l,b)}C$ code will be* $(k+1)(b-l+1) \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor - 1} \binom{l-1}{i} \epsilon^{i+1} (1-\epsilon)^{(k+1)b-i-1}$.

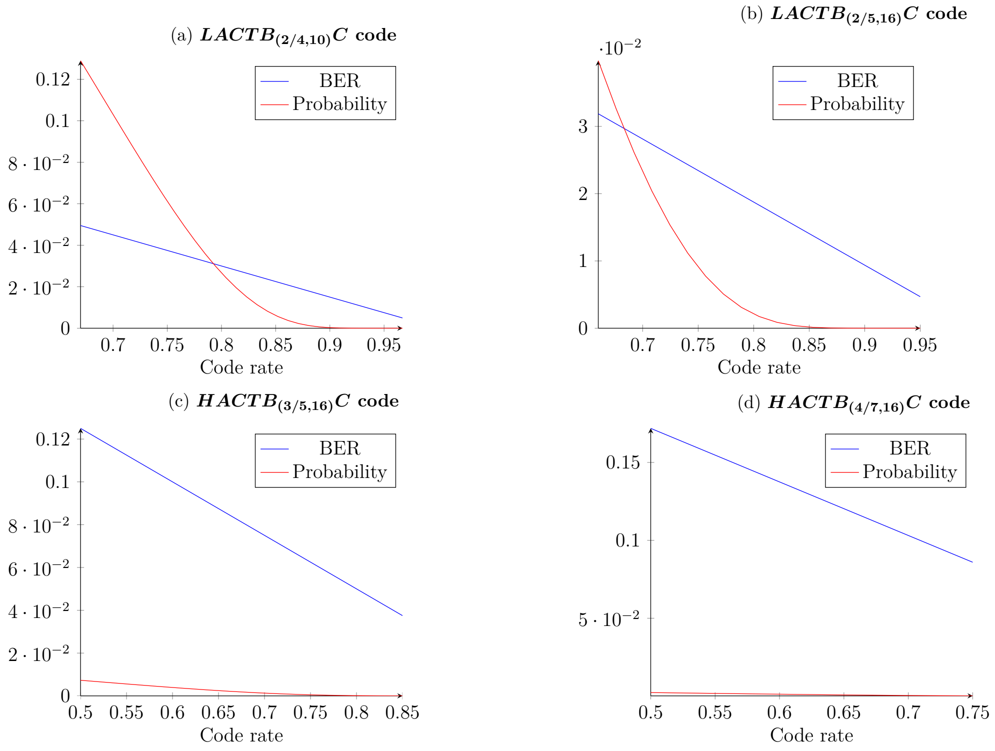
*Proof.* A transmitted codeword here is of $(k+1)b$ bits divided into $(k+1)$ equal $b$-bit bytes and the code is capable of correcting one asymmetric CT-burst having length $l$ and weight at most $\lfloor \frac{l}{2} \rfloor$. So a sent message affected by an asymmetric CT-burst of weight 1 in the first $b$-bit byte will have $(k+1)b-1$ non-corrupted bits, hence the probability of erroneous decoding in this case will be $\binom{l-1}{0}\epsilon^1(1-\epsilon)^{(k+1)b-1}$. Similarly the probability of erroneous decoding for an asymmetric CT-burst of weight 2 and length $l$ in the first $b$-bit byte will be $\binom{l-1}{1}\epsilon^2(1-\epsilon)^{(k+1)b-2}$, continuing this, the probability of erroneous decoding for an asymmetric CT-burst of length $l$ having weight $\lfloor \frac{l}{2} \rfloor$ will be $\binom{l-1}{\lfloor \frac{l}{2} \rfloor-1}\epsilon^{\lfloor \frac{l}{2} \rfloor}(1-\epsilon)^{(k+1)b-\lfloor \frac{l}{2} \rfloor}$. Since there are $(b-l+1)$ beginning positions for asymmetric CT-bursts in the first $b$-bit byte of length $l$ having weight up to $\lfloor \frac{l}{2} \rfloor$, so the probability in this case will be $(b-l+1)\sum_{i=0}^{\lfloor \frac{l}{2} \rfloor-1}\binom{l-1}{i}\epsilon^{i+1}(1-\epsilon)^{(k+1)b-i-1}$. As there are $k+1$ $b$-bit bytes in a transmitted codeword, so the total probability of erroneous decoding for the integer $LACTB_{(d/l,b)}C$ code will be $P_d(LD) = (k+1)(b-l+1)\sum_{i=0}^{\lfloor \frac{l}{2} \rfloor-1}\binom{l-1}{i}\epsilon^{i+1}(1-\epsilon)^{(k+1)b-i-1}$. □

**Theorem 3.13.** *For transition probability $\epsilon$ pertaining to $1 \to 0$, the erroneous decoding probability $P_d(HD)$ for a $(kb+b, kb)$ integer $HACTB_{(h/l,b)}C$ code will be $(k+1)(b-l+1)\sum_{i=\lceil \frac{l}{2} \rceil-1}^{l-1}\binom{l-1}{i}\epsilon^{i+1}(1-\epsilon)^{(k+1)b-i-1}$.*

*Proof.* Structure of a transmitted codeword here will be similar to Theorem 3.12, however the occurrence of CT-bursts will be different. Here an asymmetric CT-burst in the first $b$-bit byte of length $l$ having weight $\lceil \frac{l}{2} \rceil$ will have $(k+1)b-\lceil \frac{l}{2} \rceil$ non-corrupted bits, so the corresponding probability of erroneous decoding will be $\binom{l-1}{\lceil \frac{l}{2} \rceil-1}\epsilon^{i+1}(1-\epsilon)^{(k+1)b-i-1}$. Analogous to the discussion done in Theorem 3.12, the probability of erroneous decoding for asymmetric CT-bursts of length $l$ having weight at least $\lceil \frac{l}{2} \rceil$ will be $(b-l+1)\sum_{i=\lceil \frac{l}{2} \rceil-1}^{l-1}\binom{l-1}{i}\epsilon^{i+1}(1-\epsilon)^{(k+1)b-i-1}$. Again as there are $(k+1)$ $b$-bit bytes in a transmitted message, our claim gets proved. □

Bit Error Rate (BER) is the number of bits affected by error divided by the total number of bits received during a transmission. In the case of the proposed low-density and high-density codes, as the number of bits affected lie between 0 to $\lfloor \frac{l}{2} \rfloor$ and $\lceil \frac{l}{2} \rceil$ to $l$ respectively, so we consider the average of the number of er-

Figure 3.1: **Code rate vs BER and probability**




roneous bits for calculating BER. Thus BER for the proposed low-density will be $\frac{1}{(k+1)b}\frac{1+2+\ldots+\lfloor\frac{l}{2}\rfloor}{\lfloor\frac{l}{2}\rfloor} = \frac{\lfloor\frac{l}{2}\rfloor+1}{2b(k+1)}$. Similarly the BER for proposed high-density will be $\frac{(\lceil\frac{l}{2}\rceil)+(\lceil\frac{l}{2}\rceil+1)+\ldots+(\lceil\frac{l}{2}\rceil+l-\lceil\frac{l}{2}\rceil)}{b(k+1)(l-\lceil\frac{l}{2}\rceil+1)} = \frac{(l-\lceil\frac{l}{2}\rceil+1)\lceil\frac{l}{2}\rceil+1+2+\ldots+l-\lceil\frac{l}{2}\rceil}{b(k+1)(l-\lceil\frac{l}{2}\rceil+1)} = \frac{2(l-\lceil\frac{l}{2}\rceil+1)\lceil\frac{l}{2}\rceil+(l-\lceil\frac{l}{2}\rceil)(l-\lceil\frac{l}{2}\rceil+1)}{2b(k+1)(l-\lceil\frac{l}{2}\rceil+1)}$ $= \frac{\lceil\frac{l}{2}\rceil+l}{2b(k+1)}$. A few graphs are plotted in Fig 3.1 where BER and probability of erroneous decoding vs code rate is considered for both proposed codes, here transition probability $\epsilon = 0.1$ is considered. We observe that as the code rate increases, BER and probability of erroneous decoding decrease.

So far, study of determining the probability of erroneous decoding has been done. But sometimes an error may go undetected, for instance, due to an error $e' \notin \epsilon_{d/l,b}$ or $e' \notin \varepsilon_{h/l,b}$, sometimes the resultant syndrome $S = -C_i \times e' \pmod{2^b - 1} = 0$, $1 \leq i \leq k+1$. In such situation we call the error as undetected, however this is not possible for the errors occurred as per our specification. For example in the integer $HACTB_{(2/4,8)}C$ code, for coefficient $C_2 = 3$, we have $-3\times 85 \pmod{255} = -3\times 170 \pmod{255} = 0$, both the errors $85 = (10101010)$ and $170 = (01010101)$ are of length 7 and weight 4. Since the syndrome values of the proposed codes depend upon the coefficient $C_i$'s, which are obtained by a computer search result and do not follow any particular pattern, so to determine the exact probability of undetected error becomes an open problem. For an error $e'$ to remain undetected, $-C_i \times e' \pmod{2^b-1} = 0$, as $C_i$'s are fixed during transmission, so to have this undetected error, the value of $e'$ can not be a unit, so the probability of undetected errors will lie in the interval $[0, \frac{2^b-2-\varphi(2^b-1)-|\epsilon_{d/l,b}|}{2^b-2}]$ and $[0, \frac{2^b-2-\varphi(2^b-1)-|\varepsilon_{h/l,b}|}{2^b-2}]$, where $\varphi(2^b-1)$ is the Euler's phi function [23]. Now we give below an approach to determine the maximum probability of undetected errors with respect to asymmetric CT-burst whose length is longer than the specified value for the proposed codes.

**Theorem 3.14.** *The probability of asymmetric CT-bursts with length $r$ $(> l)$ having weight at most $\lfloor \frac{l}{2} \rfloor$ that can be undetected within a $b$-bit byte by a $(kb+b, kb)$ integer $LACTB_{(d/l,b)}C$ code with parity check matrix $H = (C_1\, C_2\, \dots\, C_k\, -1)$ is at most $\frac{2^{p_i+r-b+1}}{2^{[r-1,\lfloor \frac{l}{2} \rfloor -1]}}$, where $p_i$ is the highest power of 2 in the binary representation of $C_i$.*

*Proof.* Consider an asymmetric CT-burst of length $r$ with weight at most $\lfloor \frac{l}{2} \rfloor$ starting at $j^{th}$ position and ending at $(j+r-1)^{th}$ position in any $b$-bit byte. An asymmetric CT-burst $(E_r)$ of length $r$ having weight at most $\lfloor \frac{l}{2} \rfloor$ with beginning position as specified above will go undetected by the integer $LACTB_{(d/l,b)}C$ code if

$$C_i E_r \pmod{2^b-1} = 0, \quad i.e., \quad 2^b-1 \text{ divides } C_i E_r. \tag{3.3}$$

The binary representation of $2^b-1$ is $1+2+2^2+\cdots+2^{b-1}$, and let $C_i(2)$ and $E_r(2)$ be the binary representations corresponding to $C_i$ and $E_r$ respectively. As $E_r(2)$ represents an asymmetric CT-burst of length $r$ with weight $\lfloor \frac{l}{2} \rfloor$ or less, so $E_r(2) = 2^i E'_r(2)$, $0 \le i \le b-r$ where highest power of 2 in the binary representation of $E'_r(2) = r-1$ and the number of non-zero components in $E'_r(2) \le \lfloor \frac{l}{2} \rfloor$. So for a particular beginning position,

$$\#\ of\ E_r(2) = \#\ of\ E'_r(2) = 2^{[r-1,\lfloor \frac{l}{2} \rfloor -1]}. \tag{3.4}$$

From (3.3), $1+2+2^2+\cdots+2^{b-1}$ divides $C_i(2)E_r(2)$, i.e., $C_i(2)2^iE'_r(2)$, and since $1+2+2^2+\cdots+2^{b-1}$ and $2^i$ are relatively prime, so

$$(1+2+2^2+\cdots+2^{b-1}) | C_i(2)E'_r(2).$$

This means

$C_i(2)E'_r(2) = m_i(2)(1 + 2 + 2^2 + \cdots + 2^{b-1})$ for some binary representation $m_i(2)$.

If the highest power of 2 in the binary representation of $C_i(2) = p_i$, then the highest possible power of 2 for $m_i(2)$ will be $p_i + r - 1 - b + 1 = p_i + r - b$. Thus the maximum possible number of $m_i(2)$ here will be

$$2^{p_i+r-b+1}. \tag{3.5}$$

Dividing (3.5) by (3.4) proves the theorem. □

Similarly, an analogous result for high-density asymmetric CT-burst is derived below.

**Theorem 3.15.** *The probability of an asymmetric CT-bursts with length $r$ $(> l)$ having weight at least $\lceil \frac{l}{2} \rceil$ that can be undetected within a $b$-bit byte by a $(kb + b, kb)$ integer $HACTB_{(h/l,b)}C$ code with parity check matrix $H = (C_1\, C_2\, \ldots\, C_k\, -1)$ is at most $\dfrac{2^{p_i+r-b+1}}{2^{r-1} - 2^{[r-1,\lceil \frac{l}{2} \rceil - 1]}}$, where $p_i$ is the highest possible power in the binary representation of $C_i$.*

*Proof.* Consider $E_r$ be a high-density asymmetric CT-burst of length $r$ having weight between $\lceil \frac{l}{2} \rceil$ and $l$ occurring within a $b$-bit byte. If such burst begins at $j^{th}$ position, then it ends at $(j + r - 1)^{th}$ position in any $b$-bit byte. As discussed in Theorem 3.14, the burst $E_r$ will go undetected in $i^{th}$ $b$-bit byte if $C_iE_r \pmod{2^b - 1} = 0$. Since there are $(2^{r-1} - 2^{[r-1,\lceil \frac{l}{2} \rceil - 2]})$ possibilities for $E_r$ in any beginning position, so by following the approach similar to Theorem 3.14, we have the desired probability at most $\dfrac{2^{p_i+r-b+1}}{2^{r-1} - 2^{[r-1,\lceil \frac{l}{2} \rceil - 1]}}$. □

## 3.4 Conclusion

In this chapter, we have presented two classes of integer codes capable of correcting low-density and high-density asymmetric CT-bursts within a $b$-bit byte using weight constraint, also a probabilistic approach has been developed for such errors. Compared to similar codes, the proposed ones require less memory, use a smaller number of table look ups, and have a better code rate. This work can be further carried out for such CT-bursts which are spread over adjacent $b$-bit bytes. We can also look for similar works on moderate-density CT-bursts.