Chapter 6

Nonconforming Finite Element Method

As mentioned earlier, the conforming finite element spaces, that have been used for our model, need to satisfy the discrete inf-sup condition for a stable solution, and this leads to the use of complex elements (conforming stable pairs like (P_1b, P_1), (P_2, P_1), etc.) of limited applicability. In [37], several combinations of simpler nonconforming finite elements which violate the inter-element continuity condition of the velocities have been analyzed for Stokes problem. In this chapter, we analyze one such nonconforming finite element to approximate the velocity space. We obtain optimal $\mathbf{L}^2$ and $\mathbf{H}^1$ velocity error bounds for nonsmooth initial data. For the time discretization, we employ a first-order backward method as well as Euler incremental pressure correction scheme and discuss about stability and error analysis for nonsmooth initial data.

6.1 Introduction

For semidiscrete formulation, we consider the triangulation $\mathcal{T}_h$. We now define two discrete spaces $\mathbf{H}_h$ and L_h that approximate the velocity space $\mathbf{H}_0^1$ and the pressure space L^2, respectively, as follows:

$$\mathbf{H}_h = \{\mathbf{v}_h \in \mathbf{L}^2 : \mathbf{v}_h|_K \text{ is linear } \forall K \in \mathcal{T}_h, \ \ \mathbf{v}_h \text{ is continuous at } C_b \text{ and } \mathbf{v}_h = \mathbf{0} \text{ at } C_{\partial\Omega}\},$$
$$L_h = \{q_h \in L^2 : q_h|_K = \text{ constant for all } K \in \mathcal{T}_h\},$$

where C_b is the set of mid-point of each inter-element boundary and $C_{\partial\Omega}$ is the set of mid-point of each edge along $\partial\Omega$.

It is clear that $\mathbf{H}_h$ is not a subspace of $\mathbf{H}_0^1$. We next define the discrete version of bilinear and trilinear forms on $\mathbf{H}_0^1 \oplus \mathbf{H}_h$ by

$$\begin{aligned} a_h(\mathbf{w}_h, \mathbf{v}_h) &= (\nabla_h \mathbf{w}_h, \nabla_h \mathbf{v}_h), \\ b_h(\mathbf{u}_h, \mathbf{w}_h, \mathbf{v}_h) &= \frac{1}{2}(\mathbf{u}_h \cdot \nabla_h \mathbf{w}_h, \mathbf{v}_h) - \frac{1}{2}(\mathbf{u}_h \cdot \nabla_h \mathbf{v}_h, \mathbf{w}_h). \end{aligned}$$

Here the discrete operator ∇_h on each $K \in \mathcal{T}_h$ is ∇, see next Section. Now the semidiscrete variational formulation reads as: For any $t > 0$, find a pair $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{H}_h \times L_h$ satisfying

$$\begin{aligned} (\mathbf{u}_{ht}, \mathbf{v}_h) + \mu a_h(\mathbf{u}_h, \mathbf{v}_h) + \int_0^t \beta(t-s) a_h(\mathbf{u}_h(s), \mathbf{v}_h) ds + b_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \\ -(p_h, \nabla_h \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{H}_h, \end{aligned} \tag{6.1}$$

with $(\nabla_h \cdot \mathbf{u}_h, \chi_h) = 0, \ \chi_h \in L_h$ and $\mathbf{u}_h(0) = \mathbf{u}_{0h} \in \mathbf{H}_h$ is an appropriate approximation of the initial velocity $\mathbf{u}_0 \in \mathbf{J}_1$.
Next we recall the discrete divergence free subspace $\mathbf{J}_h$ of $\mathbf{H}_h$:

$$\mathbf{J}_h := \{\mathbf{w}_h \in \mathbf{H}_h : (z_h, \nabla_h \cdot \mathbf{w}_h) = 0, \quad \forall z_h \in L_h\}.$$

It is noted that $\mathbf{J}_h \not\subset \mathbf{J}_1$. Now we consider an equivalent variational formulation. For $t > 0$ and for all $\mathbf{v}_h \in \mathbf{J}_h$, find $\mathbf{u}_h(t) \in \mathbf{J}_h$ with $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ satisfying

$$(\mathbf{u}_{ht}, \mathbf{v}_h) + \mu a_h(\mathbf{u}_h, \mathbf{v}_h) + \int_0^t \beta(t-s) a_h(\mathbf{u}_h(s), \mathbf{v}_h) ds = (\mathbf{f}, \mathbf{v}_h) - b_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h). \tag{6.2}$$

In [37], several combinations of simpler nonconforming finite elements which violate the inter-element continuity condition of the velocities have been analyzed for Stokes problem. The methods have been shown to be stable, and optimal velocity error estimates have been derived. Stable and optimal results have been shown even for constant pressures paired with nonconforming piecewise linear velocities. Later on, several works appeared, extending these to steady and unsteady NSEs; and with works on lower-order and equal order finite elements, for examples [88, 98, 100, 146, 155], to name a few.

To the best of our knowledge, there is no work available in nonconforming finite elements for the Oldroyd model of order one. Also the work on lower-order spaces is limited; for example, in [142], the lowest equal order conforming elements (P_1, P_1) triangle element and (Q_1, Q_1) quadrilateral element have been analyzed for the Oldroyd model of one with stabilization, based on two local Gauss integrations. And in [151], a characteristic scheme has been considered for (P_1, P_1). A stabilization term has also been added to the discrete weak formulation to get a stable solution. In the case of the lowest order nonconforming pair, i.e. (P_1^{NC}, P_0), since the discrete LBB condition is satisfied, a stable simulation can be performed without any stabilization. We have considered in this chapter the (P_1^{NC}, P_0) elements to approximate the Oldroyd model of order one.

We first apply the backward Euler (BE) method to discretize the temporal variable. Assuming $[0, T]$ to be the time interval, we proceed as follows: Let $k = \frac{T}{N} > 0$ be the time step with $t_n = nk,\ n \geq 0$ representing the n-th time step. Here N is a positive integer. We next define for a sequence $\{\boldsymbol{\phi}^n\}_{n\geq 0} \subset \mathbf{J}_h$, the backward difference quotient

$$\partial_t \boldsymbol{\phi}^n = \frac{1}{k}(\boldsymbol{\phi}^n - \boldsymbol{\phi}^{n-1}).$$

For any continuous function $\boldsymbol{\phi}(t)$ we set $\boldsymbol{\phi}^n = \boldsymbol{\phi}(t_n)$. We approximate the integral term in (6.2) by right rectangle rule, the BE method being of first-order, with the notation $\beta_{nj} = \beta(t_n - t_j)$:

$$q_r^n(\boldsymbol{\phi}) = k \sum_{j=1}^{n} \beta_{nj} \phi^j \approx \int_0^{t_n} \beta(t_n - s) \boldsymbol{\phi}(s)\ ds.$$

Now, the fully discrete formulation based on BE method applied to the semidiscrete Oldroyd problem (6.1) reads as follows: For $1 \leq n \leq N$, seek $\{\mathbf{U}^n\}_{1\leq n\leq N} \in \mathbf{H}_h$ and $\{P^n\}_{1\leq n\leq N} \in L_h$ satisfying

$$\begin{aligned}(\partial_t \mathbf{U}^n, \mathbf{v}_h) + \mu a_h(\mathbf{U}^n, \mathbf{v}_h) + a_h(q_r^n(\mathbf{U}), \mathbf{v}_h) - (P^n, \nabla \cdot \mathbf{v}_h) \\ = (\mathbf{f}^n, \mathbf{v}_h) - b_h(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h), \quad \forall\ \mathbf{v}_h \in \mathbf{H}_h,\end{aligned} \tag{6.3}$$

with $(\nabla_h \cdot \mathbf{U}^n, \chi_h) = 0,\ \forall\ \chi_h \in L_h,\ n \geq 0$. We choose $\mathbf{U}^0 = \mathbf{u}_{0h}$. When $\mathbf{v}_h \in \mathbf{J}_h$, then the equivalent formulation read as: seek $\{\mathbf{U}^n\}_{1\leq n\leq N} \in \mathbf{J}_h$ such that

$$(\partial_t \mathbf{U}^n, \mathbf{v}_h) + \mu a_h(\mathbf{U}^n, \mathbf{v}_h) + a_h(q_r^n(\mathbf{U}), \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h) - b_h(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{J}_h. \tag{6.4}$$

Here again, we choose $\mathbf{U}^0 = \mathbf{u}_{0h} \in \mathbf{J}_h$. Now using variant of the Brouwer fixed point theorem and standard uniqueness argument, we can show that well-posedness of the discrete problem (6.4). For a proof, we refer to [59].

Then, we also analyze an incremental pressure correction (EIPC) scheme with nonconforming setup for our model. It is a time discrete projection method. The continuous form of the EIPC scheme reads as:

Step 1: Find $\hat{\mathbf{U}}^n \in \mathbf{H}_0^1$ such that

$$\frac{\hat{\mathbf{U}}^n - \mathbf{U}^{n-1}}{k} - \mu\Delta\hat{\mathbf{U}}^n + (\mathbf{U}^{n-1}\cdot\nabla)\hat{\mathbf{U}}^n - q_r^n(\Delta\hat{\mathbf{U}}) + \nabla P^{n-1} = \mathbf{f}^n. \tag{6.5}$$

Step 2: Find $(\mathbf{U}^n, p^n) \in \mathbf{H}_0^1 \times L^2(\Omega)/\mathbb{R}$ such that

$$\frac{\mathbf{U}^n - \hat{\mathbf{U}}^n}{k} + \nabla(P^n - P^{n-1}) = 0, \tag{6.6}$$

$$\nabla\cdot\mathbf{U}^n = 0. \tag{6.7}$$

The variational formulation of the EIPC scheme reads as: With $p^0 = 0$ and $\mathbf{U}^0$ is the solution of $(\mathbf{U}^0, \boldsymbol{\phi}) = (\mathbf{U}_0, \boldsymbol{\phi})$ for all $\boldsymbol{\phi} \in \mathbf{H}_0^1$,

Step 1: Find $\hat{\mathbf{U}}^n \in \mathbf{H}_0^1$ such that

$$\begin{aligned}\left(\frac{\hat{\mathbf{U}}^n - \mathbf{U}^{n-1}}{k}, \boldsymbol{\phi}\right) + \mu a(\hat{\mathbf{U}}^n, \boldsymbol{\phi}) + b(\mathbf{U}^{n-1}, \hat{\mathbf{U}}^n, \boldsymbol{\phi}) + a(q_r^n(\hat{\mathbf{U}}), \boldsymbol{\phi})&\\ = (P^{n-1}, \nabla\cdot\boldsymbol{\phi}) + (\mathbf{f}^n, \boldsymbol{\phi}), \quad \forall\boldsymbol{\phi} \in \mathbf{H}_0^1.&\end{aligned} \tag{6.8}$$

Step 2: Find $(\mathbf{U}^n, p^n) \in \mathbf{H}_0^1 \times L^2(\Omega)/\mathbb{R}$ such that

$$\left(\frac{\mathbf{U}^n - \hat{\mathbf{U}}^n}{k}, \boldsymbol{\phi}\right) - (P^n - P^{n-1}, \nabla\cdot\boldsymbol{\phi}) = 0, \quad \forall\boldsymbol{\phi} \in \mathbf{H}_0^1 \tag{6.9}$$

$$(\nabla\cdot\mathbf{U}^n, \psi) = 0, \quad \forall\psi \in L^2. \tag{6.10}$$

The finite element variational formulation of the EIPC scheme reads as: With $P_h^0 = 0$ and $\mathbf{U}_h^0$ is the solution of $(\mathbf{U}_h^0, \boldsymbol{\phi}_h) = (\mathbf{U}_0, \boldsymbol{\phi}_h)$ for all $\boldsymbol{\phi}_h \in \mathbf{H}_h$,

Step 1: Find $\hat{\mathbf{U}}_h^n \in \mathbf{H}_h$ such that

$$\begin{aligned}\left(\frac{\hat{\mathbf{U}}_h^n - \mathbf{U}_h^{n-1}}{k}, \boldsymbol{\phi}_h\right) + \mu a_h(\hat{\mathbf{U}}_h^n, \boldsymbol{\phi}_h) + b_h(\mathbf{U}_h^{n-1}, \hat{\mathbf{U}}_h^n, \boldsymbol{\phi}_h) + a_h(q_r^n(\hat{\mathbf{U}}_h), \boldsymbol{\phi}_h)&\\ -(P_h^{n-1}, \nabla_h\cdot\boldsymbol{\phi}_h) = (\mathbf{f}^n, \boldsymbol{\phi}_h), \quad \forall\boldsymbol{\phi}_h \in \mathbf{H}_h.&\end{aligned} \tag{6.11}$$

Step 2: Find $(\mathbf{U}_h^n, P_h^n) \in \mathbf{H}_h \times L_h$ such that

$$\left(\frac{\mathbf{U}_h^n - \hat{\mathbf{U}}_h^n}{k}, \boldsymbol{\phi}_h\right) - (P_h^n - P_h^{n-1}, \nabla_h\cdot\boldsymbol{\phi}_h) = 0, \quad \forall\boldsymbol{\phi}_h \in \mathbf{H}_h, \tag{6.12}$$

$$(\nabla_h\cdot\mathbf{U}_h^n, \psi_h) = 0, \quad \forall\psi_h \in L_h. \tag{6.13}$$

It is a time discrete projection method. Projection methods first studied in the late 1960s by Chorin [34] and Temam [129] for the incompressible time-dependent NSEs.

We can classify this method in three classes: Consistent splitting scheme [65, 105, 126], velocity-correction [67] and pressure-correction [66, 124, 127, 144]. A second-order incremental pressure correction scheme for the NSEs has been developed by Van Kan in [134], while Shen *et al.* [68] provided a first-order incremental pressure correction approach.

6.2 Preliminaries

In this section, we present the approximation properties of the discrete spaces. And we talk about a few tools useful for our analysis.

For any $\mathbf{w}_h \in \mathbf{H}_h$, the discrete gradient operator is defined by ∇_h by

$$\nabla_h \mathbf{w}_h|_K = \nabla \mathbf{w}_h|_K, \quad \forall K \in \mathcal{T}_h.$$

It is clear that the elements of $\mathbf{H}_h$ are not continuous on the common side of two adjacent triangles (the continuity is required only in one point) the space $\mathbf{H}_h$ is not a subspace of $\mathbf{H}_0^1$. The operator ∇_h can be extended to $\mathbf{H}_0^1$ by element-wise calculation and it is equivalent to ∇ on each element.

Assume that the space $\mathbf{H}_h$ satisfy following patch-test [79]:

(**B1i**) Each $\mathbf{w}_h \in \mathbf{H}_h$ satisfies

$$\int_\Gamma \{\mathbf{w}_h|_K - \mathbf{w}_h|_{K'}\} d\tau = 0$$

on inter-element faces $\Gamma = K \cap K', \; K, K' \in \mathcal{T}_h$, and

$$\int_\Gamma \mathbf{w}_h|_K d\tau = 0$$

on boundary faces $\Gamma = \partial\Omega \cap K, \; K, \in \mathcal{T}_h$.

If the patch-test is satisfied then the bilinear operator a_h is elliptic on $\mathbf{H}_h \times \mathbf{H}_h$. We would like to state that the nonlinear operator $b_h(\cdot, \cdot, \cdot)$ retain the antisymmetric property of its continuous form, that is,

$$b_h(\mathbf{v}_h, \boldsymbol{\phi}_h, \boldsymbol{\phi}_h) = 0, \quad \forall \mathbf{v}_h, \boldsymbol{\phi}_h \in \mathbf{H}_0^1 \oplus \mathbf{H}_h. \tag{6.14}$$

The system (6.2) becomes a system of nonlinear integro differential equations and it has a unique solution for all time $t \geq 0$ [116]. If we set $\mathbf{v}_h = \mathbf{u}_h$ in (6.2), then with (6.14) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_h\|^2 + \mu\|\nabla_h \mathbf{u}_h\|^2 + \int_0^t \beta(t-s) a_h(\mathbf{u}_h(s), \mathbf{u}_h) ds \leq \|\mathbf{f}\|\|\mathbf{u}_h\|.$$

Once we find $\mathbf{u}_h$, then the discrete pressure p_h is recovered from (6.1). Uniqueness of the discrete pressure is derived in the quotient space L_h/N_h, where

$$N_h = \{q_h \in L_h : (z_h, \nabla_h \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{H}_h\}.$$

The associated norm on the quotient space L_h/N_h is defined as

$$\|z_h\|_{L^2/N_h} = \inf_{\chi_h \in N_h} \|z_h + \chi_h\|.$$

We make the following assumption on the discrete spaces [79]:

(**B2i**) For every pair $(\mathbf{w}, z) \in \mathbf{H}_0^1 \cap \mathbf{H}^2 \times H^1/\mathbb{R}$, there always exist approximations $i_h\mathbf{w} \in \mathbf{H}_h$ and $j_h z \in L_h$ such that

$$\|\nabla_h(\mathbf{w} - i_h\mathbf{w})\| \le Ch\|\mathbf{w}\|_2, \qquad \|z - j_h z\|_{L^2/\mathbb{R}} \le Ch\|z\|_{H^1/\mathbb{R}}.$$

Moreover, for $\mathbf{w}_h \in \mathbf{H}_h$ the following inverse hypothesis holds:

$$\|\nabla_h \mathbf{w}_h\| \le Ch^{-1}\|\mathbf{w}_h\|.$$

We also take the following approximation property:

(**B3i**) There is an approximation $r_h\mathbf{w} \in \mathbf{J_h}$ of $\mathbf{w} \in \mathbf{H}^2 \cap \mathbf{J}_1$ satisfying

$$\|\mathbf{w} - r_h\mathbf{w}\| + h\|\nabla_h(\mathbf{w} - r_h\mathbf{w})\| \le Ch^2\|\mathbf{w}\|_2.$$

We now consider that the L^2 projection $P_h : \mathbf{L}^2 \mapsto \mathbf{J}_h$ satisfy

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla_h P_h\boldsymbol{\phi}\| \le Ch\|\nabla_h\boldsymbol{\phi}\|, \tag{6.15}$$

for $\boldsymbol{\phi} \in \mathbf{J}_1 \oplus \mathbf{J}_h$, and

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla_h(\boldsymbol{\phi} - P_h\boldsymbol{\phi})\| + h^2\|\Delta_h P_h\boldsymbol{\phi}\| \le Ch^2\|\tilde{\Delta}\boldsymbol{\phi}\|, \tag{6.16}$$

for $\boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$. For a proof, see [79].

As earlier, the discrete Laplace operator $\Delta_h : \mathbf{H}_h \mapsto \mathbf{H}_h$ is defined by

$$a_h(\mathbf{w}_h, \mathbf{v}_h) = (-\Delta_h\mathbf{w}_h, \mathbf{v}_h), \quad \forall \mathbf{w}_h, \mathbf{v}_h \in \mathbf{H}_h,$$

and the discrete Stokes operator is defined via $\tilde{\Delta}_h = P_h\Delta_h$. The restriction of $\tilde{\Delta}_h$ to $\mathbf{J}_h$ is invertible and we denote the inverse by $\tilde{\Delta}_h^{-1}$. Note that $-\tilde{\Delta}_h$ is a positive definite and self-adjoint operator. We now define *discrete Sobolev norms* on $\mathbf{J}_h$ by

$$\|\mathbf{v}_h\|_r = \|(-\tilde{\Delta}_h)^{r/2}\mathbf{v}_h\|, \quad \mathbf{v}_h \in \mathbf{J}_h, \ r \in \mathbb{R}.$$

We note that in particular $\|\mathbf{v}_h\|_0 = \|\mathbf{v}_h\|$ and $\|\mathbf{v}_h\|_1 = \|\nabla_h\mathbf{v}_h\|$ for $\mathbf{v}_h \in \mathbf{J}_h$, and $\|\cdot\|_2$ and $\|\tilde{\Delta}_h \cdot\|$ are equivalent norms on $\mathbf{J}_h$. A detail discussion can be found in [79, 80].

Lemma 6.1 ([79])**.** *The map* $\tilde{\Delta}_h^{-1}P_h\tilde{\Delta} : \mathbf{J}_1 \cap \mathbf{H}^2 \to \mathbf{J}_h$ *satisfying*

$$\|\mathbf{v} - \tilde{\Delta}_h^{-1}P_h\tilde{\Delta}\mathbf{v}\| + h\|\nabla(\mathbf{v} - \tilde{\Delta}_h^{-1}P_h\tilde{\Delta}\mathbf{v})\| \le Ch^2\|\tilde{\Delta}\mathbf{v}\|, \tag{6.17}$$

and the map $\tilde{\Delta}^{-1}P_h\tilde{\Delta}_h^{-1} : \mathbf{J}_h \cap \mathbf{H}^2 \to \mathbf{J}_1$ *satisfying*

$$\|\mathbf{v} - \tilde{\Delta}^{-1}P_h\tilde{\Delta}_h^{-1}\mathbf{v}\| + h\|\nabla_h(\mathbf{v} - \tilde{\Delta}^{-1}P_h\tilde{\Delta}_h^{-1}\mathbf{v})\| \le Ch^2\|\tilde{\Delta}_h\mathbf{v}\|. \tag{6.18}$$

From the estimate (6.17), one can easily show that $\tilde{\Delta}_h^{-1}P_h\mathbf{g} - \tilde{\Delta}^{-1}P\mathbf{g} \le Ch^2\|\mathbf{g}\|$ for $\mathbf{g} \in \mathbf{L}^2$, that is, $\tilde{\Delta}_h^{-1}P_h$ converges to $\tilde{\Delta}^{-1}P$ as $h \to 0$. This implies $\lambda_h \to \lambda_1$ as $h \to 0$, where λ_h and λ_1 are the least eigenvalues of $-\tilde{\Delta}_h$ and$-\tilde{\Delta}$, respectively. In fact, from (6.17) and (6.18), one may directly show that $|\lambda_h - \lambda_1| \le Ch^2$. Finally, we note that

$$\|\mathbf{v}_h\|^2 \le \lambda_h^{-1}\|\nabla_h\mathbf{v}_h\|^2, \quad \forall \mathbf{v}_h \in \mathbf{J}_h. \tag{6.19}$$

Remark 6.1. *To avoid confusion as to whether* $\|\cdot\|_1$ *means standard or discrete Sobolev norm, we follow the convention that if* $\mathbf{v}$ *belongs to* $\mathbf{J}_h$ *then* $\|\mathbf{v}\|_1$ *represents* $\mathbf{v}$ *in discrete Sobolev norm, otherwise it is the standard Sobolev norm.*

Below we present some estimates of the nonlinear operator b for our subsequent use. The proofs of these estimates are well known and can be found in the literature based on NSEs (e.g., see [80, (3.7)]).

Lemma 6.2. *Suppose the conditions* (**A1**)*,* (**B1i**) *and* (**B2i**) *hold true. Then, the following holds for* $\mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_h$*:*

$$|b_h(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi})| \le C \begin{cases} \|\mathbf{v}\|^{\frac{1}{2}}\|\nabla_h\mathbf{v}\|^{\frac{1}{2}}\|\nabla_h\mathbf{w}\|^{\frac{1}{2}}\|\Delta_h\mathbf{w}\|^{\frac{1}{2}}\|\boldsymbol{\phi}\|, \\ \|\mathbf{v}\|^{\frac{1}{2}}\|\Delta_h\mathbf{v}\|^{\frac{1}{2}}\|\nabla_h\mathbf{w}\|\|\boldsymbol{\phi}\|, \\ \|\mathbf{v}\|^{\frac{1}{2}}\|\nabla_h\mathbf{v}\|^{\frac{1}{2}}\|\nabla_h\mathbf{w}\|\|\boldsymbol{\phi}\|^{\frac{1}{2}}\|\nabla_h\boldsymbol{\phi}\|^{\frac{1}{2}}, \\ \|\mathbf{v}\|\|\nabla_h\mathbf{w}\|\|\boldsymbol{\phi}\|^{\frac{1}{2}}\|\Delta_h\boldsymbol{\phi}\|^{\frac{1}{2}}, \\ \|\mathbf{v}\|\|\nabla_h\mathbf{w}\|^{\frac{1}{2}}\|\Delta_h\mathbf{w}\|^{\frac{1}{2}}\|\boldsymbol{\phi}\|^{\frac{1}{2}}\|\nabla_h\boldsymbol{\phi}\|^{\frac{1}{2}}. \end{cases}$$

The following Lemma deals with higher order estimates of $\mathbf{u}_h$, which will be useful in the error analysis of fully discrete case for nonsmooth data. We refrain from a proof since it would follow naturally from [59, 63].

Lemma 6.3. *Suppose* $0 < \alpha < \min\{\delta, \lambda_h\mu\}$*. Further, assume that* (**A1**)*,* (**A2**)*,* (**B1i**) *and* (**B2i**) *be satisfied. Moreover, let* $\mathbf{u}_h(0) \in \mathbf{J}_h$*. Then* $\mathbf{u}_h$*, the solutions of the* *semidiscrete Oldroyd problem (6.2), satisfies the following a priori estimates:*

$$\|\mathbf{u}_h(t)\|_r^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_{r+1}^2 ds \le K, \qquad r\in\{0,1\}$$

$$\tau^*\|\mathbf{u}_h\|_2^2 + (\tau^*)^{r+1}(t)\|\mathbf{u}_{ht}\|_r^2 \le K, \qquad r\in\{0,1\},$$

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau^*)^r(s)\|\mathbf{u}_{hs}\|_r^2\, ds \le K, \qquad r\in\{0,1,2\},$$

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau^*)^{r+1}(s)\|\mathbf{u}_{hss}\|_{r-1}^2\, ds \le K, \qquad r\in\{-1,0,1\},$$

where $\tau^*(t)=\min\{1,t\}$ and K depends on the given data, but not on time t.

6.3 Semidiscrete Error Analysis

This section deals with the error analysis due to the nonconforming finite element approximation for the velocity term. It is noted that $\mathbf{J}_h \not\subset \mathbf{J}_1$. Then, using integrating by parts, one can derive that the solution pair $\mathbf{u}(t)\in \mathbf{J}_1\cap\mathbf{H}^2$, $p(t)\in H^1/\mathbb{R}$ of the problem (1.4)-(1.6) satisfy

$$(\mathbf{u}_t,\mathbf{v}_h)+\mu a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{u},\mathbf{u},\mathbf{v}_h)+\int_0^t \beta(t-s)a_h(\mathbf{u}(s),\mathbf{v}_h)\, ds$$
$$= (p,\nabla_h\cdot\mathbf{v}_h)+(\mathbf{f},\mathbf{v}_h)+\Gamma_h(\mathbf{u},p,\mathbf{v}_h), \tag{6.20}$$

for all $\mathbf{v}_h\in\mathbf{H}_0^1\oplus\mathbf{H}_h$, where

$$\Gamma_h(\mathbf{u},p,\mathbf{v}_h)=\Gamma_h^1(\mathbf{u},\mathbf{v}_h)+\int_0^t\beta(t-s)\Gamma_h^1(\mathbf{u}(s),\mathbf{v}_h)ds+\Gamma_h^2(\mathbf{u},\mathbf{u},\mathbf{v}_h)+\Gamma_h^3(p,\mathbf{v}_h) \tag{6.21}$$

is a sum of boundary integral

$$\Gamma_h^1(\mathbf{w},\mathbf{v}_h)=\sum_{K\in\mathcal{T}_h}\int_{\partial K}\partial_n\mathbf{w}\cdot\mathbf{v}_h\, ds, \tag{6.22}$$

$$\Gamma_h^2(\mathbf{w},\boldsymbol{\phi},\mathbf{v}_h)=\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\mathbf{w}\cdot n)(\boldsymbol{\phi}\cdot\mathbf{v}_h)\, ds, \tag{6.23}$$

$$\Gamma_h^3(q,\mathbf{v}_h)=\sum_{K\in\mathcal{T}_h}\int_{\partial K}q(\mathbf{v}_h\cdot n)\, ds. \tag{6.24}$$

The bound for above boundary integral as follow: (for a proof, see, [79, Lemma 4.1, Lemma 4.5])

Lemma 6.4. *Suppose the condition (**B1i**) is satisfied. Then for all $\mathbf{w}\in\mathbf{H}_0^1\cap\mathbf{H}^2$, $q\in H^1$ and $\boldsymbol{\phi},\mathbf{v}_h\in\mathbf{H}_0^1\oplus\mathbf{H}_h$, the following hold*

$$|\Gamma_h^1(\mathbf{w},\mathbf{v}_h)|\le Ch\|\nabla_h\mathbf{v}_h\|\|\Delta\mathbf{w}\|,\quad |\Gamma_h^3(q,\mathbf{v}_h)|\le Ch\|\nabla_h\mathbf{v}_h\|\|\nabla q\|,$$

$$|\Gamma_h^2(\mathbf{w},\boldsymbol{\phi},\mathbf{v}_h)| + |\Gamma_h^2(\mathbf{w},\mathbf{v}_h,\boldsymbol{\phi})| \le Ch\|\nabla\mathbf{w}\|^{\frac{1}{2}}\|\Delta\mathbf{w}\|^{\frac{1}{2}}\|\nabla_h\boldsymbol{\phi}\|\|\nabla_h\mathbf{v}_h\|.$$

We now define velocity error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ and subtract (6.2) from (6.20) to obtain

$$(\mathbf{e}_t,\mathbf{v}_h) + \mu a_h(\mathbf{e},\mathbf{v}_h) + \int_0^t \beta(t-s)a_h(\mathbf{e}(s),\mathbf{v}_h)\ ds = (p,\nabla_h\cdot\mathbf{v}_h)$$
$$+\Gamma_h(\mathbf{u},p,\mathbf{v}_h) + b_h(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) - b_h(\mathbf{u},\mathbf{u},\mathbf{v}_h), \quad \forall\mathbf{v}_h\in\mathbf{J}_h. \tag{6.25}$$

The equation (6.25) is a nonlinear problem and in order to find optimal error estimate in $L^\infty(\mathbf{L}^2)$-norm we divide it into two parts by considering an auxiliary function $\mathbf{w}_h \in \mathbf{J}_h$ satisfies

$$(\mathbf{w}_{ht},\mathbf{v}_h) + \mu a_h(\mathbf{w}_h,\mathbf{v}_h) + \int_0^t \beta(t-s)a_h(\mathbf{w}_h(s),\mathbf{v}_h)ds = (\mathbf{f},\mathbf{v}_h) - b_h(\mathbf{u},\mathbf{u},\mathbf{v}_h), \tag{6.26}$$

for all $\mathbf{v}_h \in \mathbf{J}_h$. We now write the error $\mathbf{e}$ as a summation of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, where $\boldsymbol{\xi} = \mathbf{u}-\mathbf{w}_h$ and $\boldsymbol{\eta} = \mathbf{w}_h - \mathbf{u}_h$. Clearly, $\boldsymbol{\xi}$ is the error occurred due to linear part and $\boldsymbol{\eta}$ is the error committed due to the nonlinear part. Now , the equation of $\boldsymbol{\xi}$ comes from subtracting (6.26) from (6.20) as

$$(\boldsymbol{\xi}_t,\mathbf{v}_h) + \mu a_h(\boldsymbol{\xi},\mathbf{v}_h) + \int_0^t \beta(t-s)a_h(\boldsymbol{\xi}(s),\mathbf{v}_h)\ ds$$
$$= (p,\nabla_h\cdot\mathbf{v}_h) + \Gamma_h(\mathbf{u},p,\mathbf{v}_h), \quad \forall\mathbf{v}_h\in\mathbf{J}_h. \tag{6.27}$$

Lemma 6.5. *Suppose the assumptions of Lemma 6.3 hold. Also, assume that $\mathbf{w}_h$ be the solution of (6.26) with $\mathbf{w}_h(0) = P_h\mathbf{u}_0$. Then, for any time $t>0$, $\boldsymbol{\xi}$ satisfies the following result:*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}\|^2 ds \le Ch^4,\ \ t>0.$$

Proof. Choose $\mathbf{v}_h = P_h\boldsymbol{\xi} = \boldsymbol{\xi} - (\mathbf{u} - P_h\mathbf{u})$ in (6.27) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\xi}\|^2 + \mu\|\nabla_h\boldsymbol{\xi}\|^2 + \int_0^t \beta(t-s)a_h(\boldsymbol{\xi}(s),\boldsymbol{\xi})\ ds = (\boldsymbol{\xi}_t,\mathbf{u}-P_h\mathbf{u}) + \mu a_h(\boldsymbol{\xi},\mathbf{u}-P_h\mathbf{u})$$
$$+\int_0^t \beta(t-s)a_h(\boldsymbol{\xi}(s),\mathbf{u}-P_h\mathbf{u})\ ds + (p,\nabla_h\cdot P_h\boldsymbol{\xi}) + \Gamma_h(\mathbf{u},p,P_h\boldsymbol{\xi}). \tag{6.28}$$

We use the properties of P_h to rewrite the following as

$$(\boldsymbol{\xi}_t,\mathbf{u}-P_h\mathbf{u}) = (\mathbf{u}_t - P_h\mathbf{u}_t + P_h\mathbf{u}_t - \mathbf{v}_{ht},\mathbf{u}-P_h\mathbf{u}) = \frac{1}{2}\frac{d}{dt}\|\mathbf{u}-P_h\mathbf{u}\|^2. \tag{6.29}$$

We use the "Young's inequality" and the "Cauchy-Schwarz inequality" with (6.16) to find

$$a_h(\boldsymbol{\xi},\mathbf{u}-P_h\mathbf{u}) \le \|\nabla_h\boldsymbol{\xi}\|\|\nabla_h(\mathbf{u}-P_h\mathbf{u})\| \le Ch\|\nabla_h\boldsymbol{\xi}\|\|\tilde{\Delta}\mathbf{u}\|. \tag{6.30}$$

We use the discrete incompressibility condition and approximation property with (6.16) to obtain

$$(p, \nabla_h \cdot P_h\boldsymbol{\xi}) \le (p - j_h p, \nabla_h \cdot P_h\boldsymbol{\xi}) \le Ch\|\nabla_h\boldsymbol{\xi}\|\|\nabla p\|. \tag{6.31}$$

From (6.21) with Lemma 6.4, we bound the following as

$$\begin{aligned}\Gamma_h(\mathbf{u}, p, P_h\boldsymbol{\xi}) &\le Ch(\|\tilde{\Delta}\mathbf{u}\| + \int_0^t \beta(t-s)\|\tilde{\Delta}\mathbf{u}(s)\|ds + \|\nabla p\| + \|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\|)\|\nabla_h\boldsymbol{\xi}\| \\ &\le Ch\mathcal{K}(t)\|\nabla_h\boldsymbol{\xi}\|, \end{aligned} \tag{6.32}$$

where $\mathcal{K}(t) = \|\tilde{\Delta}\mathbf{u}\| + \int_0^t \beta(t-s)\|\tilde{\Delta}\mathbf{u}(s)\|ds + \|\nabla p\| + \|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\|$.
Inserting (6.29)-(6.32) in (6.28). Multiplying both sides by $e^{2\alpha t}$ and using $\|\boldsymbol{\xi}\| \le \frac{1}{\sqrt{\lambda_h}}\|\nabla_h\boldsymbol{\xi}\| + Ch\|\tilde{\Delta}\mathbf{u}\|$ and the "Young's inequality", we reach at

$$\begin{aligned}&\frac{d}{dt}e^{2\alpha t}\|\boldsymbol{\xi}\|^2 + (\frac{3\mu}{2} - \frac{2\alpha}{\lambda_h})e^{2\alpha t}\|\nabla_h\boldsymbol{\xi}\|^2 + 2e^{2\alpha t}\int_0^t \beta(t-s)a_h(\boldsymbol{\xi}(s), \boldsymbol{\xi})\ ds \le \frac{d}{dt}e^{2\alpha t}\|\mathbf{u} - P_h\mathbf{u}\|^2 \\ &\quad - \alpha e^{2\alpha t}\|\mathbf{u} - P_h\mathbf{u}\|^2 + \frac{\mu(\delta-\alpha)^2}{2\gamma^2}e^{2\alpha t}\Big(\int_0^t \beta(t-s)\|\nabla_h\boldsymbol{\xi}(s)\|ds\Big)^2 + Ch^2e^{2\alpha t}\mathcal{K}^2(t).\end{aligned} \tag{6.33}$$

We now drop the second term on the right of inequality (6.33) and take time integration from 0 to t with $\boldsymbol{\xi}(0) = \mathbf{u}(0) - P_h\mathbf{u}(0)$ to obtain

$$\begin{aligned}&e^{2\alpha t}\|\boldsymbol{\xi}(t)\|^2 + (\frac{3\mu}{2} - \frac{2\alpha}{\lambda_h})\int_0^t e^{2\alpha s}\|\nabla_h\boldsymbol{\xi}(s)\|^2ds + 2\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)a_h(\boldsymbol{\xi}(\tau), \boldsymbol{\xi}(s))d\tau ds \\ &\quad \le e^{2\alpha t}\|\mathbf{u}(t) - P_h\mathbf{u}(t)\|^2 + \frac{\mu(\delta-\alpha)^2}{2\gamma^2}\int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\nabla_h\boldsymbol{\xi}(\tau)\|\ d\tau\Big)^2 ds \\ &\qquad + Ch^2\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds.\end{aligned} \tag{6.34}$$

Since the double integration term on the left of inequality (6.34) is positive, hence we drop it and the another double integration term can be handled similar to (2.15) as

$$\frac{\mu(\delta-\alpha)^2}{2\gamma^2}\int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\nabla_h\boldsymbol{\xi}(\tau)\|\ d\tau\Big)^2 ds \le \frac{\mu}{2}\int_0^t e^{2\alpha s}\|\nabla_h\boldsymbol{\xi}(s)\|^2ds. \tag{6.35}$$

Incorporating (6.34) in (6.35) and using the fact $\|\boldsymbol{\xi}(t)\| \ge \|\mathbf{u}(t) - P_h\mathbf{u}(t)\|$ (see, (3.20)), we find that

$$(\mu - \frac{2\alpha}{\lambda_h})\int_0^t e^{2\alpha s}\|\nabla_h\boldsymbol{\xi}(s)\|^2ds \le Ch^2\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds. \tag{6.36}$$

With $0 < \alpha < \min\{\delta, \mu\lambda_h/2\}$, we have $\mu - \frac{2\alpha}{\lambda_h} > 0$. Finally, we conclude

$$\int_0^t e^{2\alpha s}\|\nabla_h\boldsymbol{\xi}(s)\|^2ds \le Ch^2\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds. \tag{6.37}$$

We consider the following duality argument to find the L^2-error for the velocity: Suppose the solution pair $(\mathbf{w}^1(\tau), q^1(\tau)) \in \mathbf{J}_1 \times L^2/\mathbb{R}$ satisfies the backward Stokes-Volterra equation for a fixed t with $0 < \tau < t < T$ and $\mathbf{w}^1(t) = 0$

$$\mathbf{w}^1_\tau + \mu\Delta\mathbf{w}^1 + \int_\tau^t \beta(s-\tau)\Delta\mathbf{w}^1(s)ds - \nabla q^1 = e^{2\alpha t}\boldsymbol{\xi}, \tag{6.38}$$

With a change of variable $t \to t-\tau$, one can show that the following *a priori* estimate holds true (see [116, (5.9), page 766])

$$\int_0^t e^{-2\alpha\tau}\big(\|\Delta\mathbf{w}^1\|^2 + \|\mathbf{w}^1_\tau\|^2 + \|\nabla q^1\|^2\big)d\tau \le C\int_0^t e^{2\alpha\tau}\|\boldsymbol{\xi}(\tau)\|^2 d\tau. \tag{6.39}$$

Multiply (6.35) by $\boldsymbol{\xi}$ and take integration over Ω to find

$$\begin{aligned} e^{2\alpha\tau}\|\boldsymbol{\xi}\|^2 =& (\boldsymbol{\xi}, \mathbf{w}^1_\tau) - \mu a_h(\boldsymbol{\xi}, \mathbf{w}^1) - \int_\tau^t \beta(s-\tau)a_h(\boldsymbol{\xi}, \mathbf{w}^1(s))ds \\ &+ (q^1, \nabla\cdot\boldsymbol{\xi}) + \Sigma_1(\mathbf{w}^1, q^1, \boldsymbol{\xi}), \end{aligned} \tag{6.40}$$

where

$$\Sigma_1(\mathbf{v}, z, \boldsymbol{\phi}_h) = \Gamma^1_h(\mathbf{v}, \boldsymbol{\phi}_h) + \int_\tau^t \beta(s-\tau)\Gamma^1_h(\mathbf{v}(s), \boldsymbol{\phi}_h)ds - \Gamma^3_h(z, \boldsymbol{\phi}_h) \tag{6.41}$$

Using (6.27) with $\mathbf{v}_h = P_h\mathbf{w}^1$ and $t = \tau$ in (6.40), we obtain

$$\begin{aligned} e^{2\alpha\tau}\|\boldsymbol{\xi}\|^2 =& (\boldsymbol{\xi}, \mathbf{w}^1_\tau) + (\boldsymbol{\xi}_\tau, P_h\mathbf{w}^1) - \mu a_h(\boldsymbol{\xi}, \mathbf{w}^1 - P_h\mathbf{w}^1) - \int_\tau^t \beta(s-\tau)a_h(\boldsymbol{\xi}, \mathbf{w}^1(s))ds \\ &+ \int_0^\tau \beta(\tau-s)a_h(\boldsymbol{\xi}(s), P_h\mathbf{w}^1)ds + (q^1, \nabla\cdot\boldsymbol{\xi}) - (p, \nabla_h\cdot P_h\mathbf{w}^1) \\ &+ \Sigma_1(\mathbf{w}^1, q^1, \boldsymbol{\xi}) - \Gamma_h(\mathbf{u}, p, P_h\mathbf{w}^1). \end{aligned} \tag{6.42}$$

We rewrite the following as

$$(\boldsymbol{\xi}, \mathbf{w}^1_\tau) + (\boldsymbol{\xi}_\tau, P_h\mathbf{w}^1) = \frac{d}{dt}(\boldsymbol{\xi}, P_h\mathbf{w}^1) + (\mathbf{u} - P_h\mathbf{u}, \mathbf{w}^1_\tau). \tag{6.43}$$

A use of approximation property **(B2i)** with definition of P_h gives

$$\begin{aligned} (q^1, \nabla\cdot\boldsymbol{\xi}) - (p, \nabla_h\cdot P_h\mathbf{w}^1) &= (q^1 - j_h q^1, \nabla\cdot\boldsymbol{\xi}) + (p - j_h p, \nabla_h\cdot(\mathbf{w}^1 - P_h\mathbf{w}^1)) \\ &\le Ch\|\nabla q^1\|\|\nabla_h\boldsymbol{\xi}\| + Ch^2\|\nabla p\|\|\tilde{\Delta}\mathbf{w}^1\|. \end{aligned} \tag{6.44}$$

From (6.41) and Lemma 6.4, we observe that

$$\Sigma_1(\mathbf{w}^1, q^1, \boldsymbol{\xi}) \le Ch(\|\tilde{\Delta}\mathbf{w}^1\| + \int_0^t \beta(t-s)\|\tilde{\Delta}\mathbf{w}^1(s)\|ds + \|\nabla q^1\|)\|\nabla_h\boldsymbol{\xi}\|. \tag{6.45}$$

Observing that $\Gamma_h(\mathbf{u},p,P_h\mathbf{w}^1)=0$, then from (6.32) we find that

$$
\begin{aligned}
|\Gamma_h(\mathbf{u},p,P_h\mathbf{w}^1)| &= |\Gamma_h(\mathbf{u},p,P_h\mathbf{w}^1-\mathbf{w}^1)|\\
&\le Ch\mathcal{K}(t)\|\nabla_h(P_h\mathbf{w}^1-\mathbf{w}^1)\|\\
&\le Ch^2\mathcal{K}(t)\|\tilde{\Delta}\mathbf{w}^1\|. \qquad (6.46)
\end{aligned}
$$

We rewrite the integral terms on the right of inequality (6.42) as

$$
\begin{aligned}
-\int_\tau^t \beta(s-\tau)a_h(\boldsymbol{\xi},\mathbf{w}^1(s)-P_h\mathbf{w}^1(s))ds+\int_0^\tau \beta(\tau-s)a_h(\boldsymbol{\xi}(s),P_h\mathbf{w}^1)ds\\
-\int_\tau^t \beta(s-\tau)a_h(\boldsymbol{\xi},P_h\mathbf{w}^1(s))ds. \qquad (6.47)
\end{aligned}
$$

Similar to [116], after integration the last two terms of (6.47) are cancelled out. Now, we incorporate (6.43)-(6.47) in (6.42) with (6.32) and take time integration on the resulting equation to obtain

$$
\begin{aligned}
\int_0^t e^{2\alpha\tau}\|\boldsymbol{\xi}\|^2d\tau \le& (\boldsymbol{\xi}(t),P_h\mathbf{w}^1(t))-(\boldsymbol{\xi}(0),P_h\mathbf{w}^1(0))+Ch^2\int_0^t e^{2\alpha\tau}\|\nabla_h\boldsymbol{\xi}\|^2d\tau\\
&+Ch^4\int_0^t e^{2\alpha\tau}\mathcal{K}^2(\tau)d\tau+\epsilon\int_0^t e^{-2\alpha\tau}\big(\|\Delta\mathbf{w}^1\|^2+\|\mathbf{w}^1_\tau\|^2+\|\nabla q^1\|^2\big)d\tau.
\end{aligned}
$$

Finally, we use (6.39) with $C\epsilon=\frac{1}{2}$ to conclude the proof. □

For optimal estimate of $\boldsymbol{\xi}$ in $L^\infty(\mathbf{L}^2)$, we consider a projection which is similar to Stokes-Volterra projection, see [63, 116]. Let $V_h:[0,T_0]\to\mathbf{H}_h$, for some $T_0>0$ satisfy

$$
\mu a_h(\mathbf{u}-V_h\mathbf{u},\mathbf{v}_h)+\int_0^t \beta(t-s)a_h((\mathbf{u}-V_h\mathbf{u})(s),\mathbf{v}_h)\,ds=(p,\nabla_h\cdot\mathbf{v}_h)+\Gamma_h(\mathbf{u},p,\mathbf{v}_h), \quad (6.48)
$$

for all $\mathbf{v}_h\in\mathbf{J}_h$. We note that the above system, similar to the Stokes-Volterra, has a positive definite operator, which in this case is $\tilde{\Delta}_h$. Therefore, we can establish the well-posedness of the system (6.48) as in the case of the Stokes-Volterra projection. For details, we refer to Chapter 4.

We now write

$$
\boldsymbol{\xi}=(\mathbf{u}-V_h\mathbf{u})+(V_h\mathbf{u}-\mathbf{w}_h^1)=:\boldsymbol{\zeta}+\boldsymbol{\theta}.
$$

Now we would like to estimate $\|\mathbf{u}-V_h\mathbf{u}\|$, $\|\nabla(\mathbf{u}-V_h\mathbf{u})\|$, as this is the first step towards obtaining the optimal estimate of $\boldsymbol{\xi}$. With the notation

$$
\boldsymbol{\zeta}=\mathbf{u}-V_h\mathbf{u},
$$

we present the following Lemma.

Lemma 6.6. *Suppose the hypothesis of Lemma 6.5 be satisfied. Then, for any $t > 0$, the following results hold:*

$$\|\boldsymbol{\zeta}(t)\|^2 + h^2\|\nabla_h\boldsymbol{\zeta}(t)\|^2 \le Ch^4\Big(\mathcal{K}^2(t) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds\Big).$$

Moreover, the following result holds:

$$\|\boldsymbol{\zeta}_t(t)\|^2 + h^2\|\nabla_h\boldsymbol{\zeta}_t(t)\|^2 \le Ch^4\Big(\mathcal{K}^2(t) + \mathcal{K}_t^2(t) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds\Big).$$

Proof. With $\boldsymbol{\zeta} = \mathbf{u} - V_h\mathbf{u}$, from (6.48), one can deduce that

$$\mu a_h(\boldsymbol{\zeta}, \mathbf{v}_h) + \int_0^t \beta(t-s)a_h(\boldsymbol{\zeta}(s), \mathbf{v}_h)\, ds = (p, \nabla_h\cdot\mathbf{v}_h) + \Gamma_h(\mathbf{u}, p, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{J}_h, \tag{6.49}$$

Setting $\mathbf{v}_h = P_h\boldsymbol{\zeta} = \boldsymbol{\zeta} - (\mathbf{u} - P_h\mathbf{u})$ in (6.49), we obtain

$$\begin{aligned}\mu\|\nabla_h\boldsymbol{\zeta}\|^2 + \int_0^t \beta(t-s)a_h(\boldsymbol{\zeta}(s), \boldsymbol{\zeta})\, ds &= \mu a_h(\boldsymbol{\zeta}, \mathbf{u} - P_h\mathbf{u}) \\ + \int_0^t \beta(t-s)a_h(\boldsymbol{\zeta}(s), \mathbf{u} - P_h\mathbf{u})\, ds &+ (p, \nabla_h \cdot P_h\boldsymbol{\zeta}) + \Gamma_h(\mathbf{u}, p, P_h\boldsymbol{\zeta}).\end{aligned} \tag{6.50}$$

Arguing with similar set of estimates of (6.30), (6.31) and (6.45) in (6.50), we reach at

$$\begin{aligned}\mu\|\nabla_h\boldsymbol{\zeta}\|^2 &+ \int_0^t \beta(t-s)a_h(\boldsymbol{\zeta}(s), \boldsymbol{\zeta})\, ds \\ &\le Ch^2\mathcal{K}_1^2(t) + \frac{\mu(\delta-\alpha)^2}{2\gamma^2}\Big(\int_0^t \beta(t-s)\|\nabla_h\boldsymbol{\zeta}(s)\|ds\Big)^2.\end{aligned} \tag{6.51}$$

After multiplying both sides by $e^{2\alpha t}$ and taking time integration, we drop the resulting double integration term from the left of inequality. The resulting double integration term on the right of inequality can be estimates similar to (6.35). Then, we finally deduce that

$$\int_0^t e^{2\alpha s}\|\nabla_h\boldsymbol{\zeta}(s)\|^2 ds \le Ch^2\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds. \tag{6.52}$$

Now, from (6.50), we easily find that

$$\|\nabla_h\boldsymbol{\zeta}(t)\|^2 \le Ch^2\Big(\mathcal{K}^2(t) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds\Big). \tag{6.53}$$

To estimate L^2-error, we considerthe following Stokes problem: For a fixed h, the solution pair $(\mathbf{w}^1, q^1) \in \mathbf{J}_1 \times L^2/\mathbb{R}$ satisfy

$$-\mu\Delta\mathbf{w}^1 + \nabla q^1 = \boldsymbol{\zeta}, \quad \text{in } \Omega. \tag{6.54}$$

By assumption (**A1**), we find that

$$\|\Delta \mathbf{w}^1\|^2 + \|\nabla q^1\|^2 \leq C\|\boldsymbol{\zeta}\|^2. \tag{6.55}$$

Multiply both sides of (6.54) by $\boldsymbol{\zeta}$ and take integration over Ω to find

$$\|\boldsymbol{\zeta}\|^2 = \mu a_h(\boldsymbol{\zeta}, \mathbf{w}^1) - (q^1, \nabla\cdot\boldsymbol{\zeta}) + \Gamma_h^1(\mathbf{w}^1, \boldsymbol{\zeta}) - \Gamma_h^3(q^1, \boldsymbol{\zeta}). \tag{6.56}$$

Using (6.49) with $\mathbf{v}_h = P_h\mathbf{w}^1$ in (6.56), we obtain

$$\begin{aligned}\|\boldsymbol{\zeta}\|^2 =& \mu a_h(\boldsymbol{\zeta}, \mathbf{w}^1 - P_h\mathbf{w}^1) - (q^1, \nabla\cdot\boldsymbol{\zeta}) + (p, \nabla\cdot P_h\mathbf{w}^1) + \Gamma_h^1(\mathbf{w}^1, \boldsymbol{\zeta}) - \Gamma_h^3(q^1, \boldsymbol{\zeta}) \\ &+ \Gamma_h(\mathbf{u}, p, P_h\mathbf{w}^1) - \int_0^t \beta(t-s) a_h(\boldsymbol{\zeta}(s), P_h\mathbf{w}^1)\, ds. \end{aligned} \tag{6.57}$$

All the terms on the right of inequality can be estimated similar to (6.30),(6.31) and (6.45) and using Lemma 6.4 except the last one. And the last one can be rewrite using (6.54) and the fact $\|\Delta_h P_h \mathbf{v}\| \leq C\|\tilde{\Delta}\mathbf{v}\|$ (see, (6.16)) as

$$\begin{aligned} -\int_0^t \beta(t-s) a_h(\boldsymbol{\zeta}(s), P_h\mathbf{w}^1)\, ds &= \int_0^t \beta(t-s)\Big[(\boldsymbol{\zeta}(s), \Delta_h P_h\mathbf{w}^1) + \Gamma_h^1(\mathbf{w}^1, \boldsymbol{\zeta})\Big] ds \\ &\leq \frac{(\delta-\alpha)^2}{2\gamma^2}\Big(\int_0^t \beta(t-s)\|\boldsymbol{\zeta}(s)\| ds\Big)^2 + C\|\tilde{\Delta}\mathbf{w}^1\|^2 + Ch^2\|\nabla_h\boldsymbol{\zeta}\|^2. \end{aligned} \tag{6.58}$$

Incorporate (6.58) in (6.57) and multiply both sides by $e^{2\alpha t}$ and take time integration to obtain

$$\begin{aligned}\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2 ds \leq& Ch^2 \int_0^t e^{2\alpha s}\|\nabla_h\boldsymbol{\zeta}(s)\|^2 ds + Ch^4 \int_0^t e^{2\alpha s}\mathcal{K}_1^2(s) ds \\ &+ \frac{\mu(\delta-\alpha)^2}{2\gamma^2}\int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\nabla_h\boldsymbol{\zeta}(\tau)\| d\tau\Big)^2 ds. \end{aligned} \tag{6.59}$$

The double integration term can be handled similar to (6.35), then we finally arrive at

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2 ds \leq Ch^4 \int_0^t e^{2\alpha s}\mathcal{K}^2(s) ds. \tag{6.60}$$

Now, from (6.57), we easily deduce that

$$\|\boldsymbol{\zeta}(t)\|^2 ds \leq Ch^4\Big(\mathcal{K}^2(t) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\mathcal{K}^2(s) ds\Big), \tag{6.61}$$

which concludes the first proof. For the second one, we differentiate (6.49) with respect to time and do similar as above. □

Armed with the estimates of $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}_t$, we now pursue the estimates of $\boldsymbol{\theta}$ to find the optimal error estimates of $\boldsymbol{\xi}$. From (6.26) and (6.48), we have

$$(\boldsymbol{\theta}_t, \mathbf{v}_h) + \mu a_h(\boldsymbol{\theta}, \mathbf{v}_h) + \int_0^t \beta(t-s) a_h(\boldsymbol{\theta}(s), \mathbf{v}_h)\, ds = -(\boldsymbol{\zeta}_t, \mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{J}_h. \quad (6.62)$$

Now, taking $\mathbf{v}_h = \sigma(t)\boldsymbol{\theta}$ in (6.62), we arrive at

$$\begin{aligned}\frac{d}{dt}\big(\sigma(t)\|\boldsymbol{\theta}\|^2\big) + 2\mu\sigma(t)\|\nabla_h\boldsymbol{\theta}\|^2 =& -2\sigma(t)(\boldsymbol{\zeta}_t, \boldsymbol{\theta}) + \sigma_t(t)\|\boldsymbol{\theta}\|^2 \\ & - 2\sigma(t)\int_0^t \beta(t-\tau) a_h(\boldsymbol{\theta}(\tau), \boldsymbol{\theta})\, d\tau. \end{aligned} \quad (6.63)$$

An application of the "Young's inequality" and the "Cauchy-Schwarz inequality" with $\sigma_t(t) \le Ce^{2\alpha t}$ and $\frac{(\sigma(t))^2}{\sigma_t(t)} \le C\sigma_1(t)$ (where $\sigma_1(t) = (\tau^*(t))^2 e^{2\alpha t}$) yields

$$|2\sigma(t)(\boldsymbol{\zeta}_t, \boldsymbol{\theta})| \le \frac{(\sigma(t))^2}{\sigma_t(t)}\|\boldsymbol{\zeta}_t\|^2 + \sigma_t(t)\|\boldsymbol{\theta}\|^2 \le C\sigma_1(t)\|\boldsymbol{\zeta}_t\|^2 + Ce^{2\alpha t}\|\boldsymbol{\theta}\|^2.$$

Incorporate this in (6.63) and take time integration over $[0, t]$ to obtain

$$\begin{aligned}\sigma(t)\|\boldsymbol{\theta}(t)\|^2 + 2\mu\int_0^t \sigma(s)\|\nabla_h\boldsymbol{\theta}(s)\|^2 ds \le& C\Big(\int_0^t \sigma_1(s)\|\boldsymbol{\zeta}_s(s)\|^2 ds + \int_0^t e^{2\alpha s}\|\boldsymbol{\theta}(s)\|^2 ds\Big) \\ & -2\int_0^t \sigma(s)\int_0^s \beta(s-\tau) a_h(\boldsymbol{\theta}(\tau), \boldsymbol{\theta}(s)) d\tau ds.\end{aligned}$$

The double integration term no longer positive. Similar to (3.62), we rewrite this as

$$\begin{aligned}\int_0^t \sigma(s)\int_0^s \beta(s-\tau) a_h(\boldsymbol{\theta}(\tau), \boldsymbol{\theta}(s)) d\tau ds \le& C\int_0^t e^{2\alpha s}\|\nabla_h\widetilde{\boldsymbol{\theta}}(s)\|^2 ds \\ & + \frac{\mu}{2}\int_0^t \sigma(s)\|\nabla_h\boldsymbol{\theta}(s)\|^2, \end{aligned} \quad (6.64)$$

where $\widetilde{\boldsymbol{\theta}} = \int_0^t \boldsymbol{\theta}(s)\, ds$. Combine above two equations and write $\|\boldsymbol{\theta}\| \le \|\boldsymbol{\xi}\| + \|\boldsymbol{\zeta}\|$ to find

$$\begin{aligned}\sigma(t)\|\boldsymbol{\theta}(t)\|^2 + \mu\int_0^t \sigma(s)\|\nabla_h\boldsymbol{\theta}(s)\|^2 ds \le& C\int_0^t \sigma_1(s)\|\boldsymbol{\zeta}_s\|^2 ds. \\ & + C\int_0^t e^{2\alpha s}(\|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\zeta}\|^2) ds + C\int_0^t e^{2\alpha s}\|\nabla_h\widetilde{\boldsymbol{\theta}}(s)\|^2 ds. \end{aligned} \quad (6.65)$$

To find the bounds for last term on the right of inequality (6.65), we take time integration on (6.62) and write the double integral term as in (3.64) to obtain

$$\begin{aligned}(\widetilde{\boldsymbol{\theta}}_t, \mathbf{v}_h) + \mu a_h(\widetilde{\boldsymbol{\theta}}, \mathbf{v}_h) + & \int_0^t\int_0^s \beta(s-\tau) a_h(\boldsymbol{\theta}(\tau), \mathbf{v}_h)\, d\tau\, ds \\ &= -(\boldsymbol{\zeta}, \mathbf{v}_h) + (\mathbf{u}_0 - P_h\mathbf{u}_0, \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{J}_h. \end{aligned} \quad (6.66)$$

The last term vanishes due to P_h. We treat the double integral term as in (3.64) from Chapter 3 as

$$\int_0^t \int_0^s \beta(s-\tau) a_h(\boldsymbol{\theta}(\tau), \mathbf{v}_h)\ d\tau\ ds = \int_0^t \beta(t-\tau) a_h(\widetilde{\boldsymbol{\theta}}(\tau), \mathbf{v}_h)\ d\tau. \tag{6.67}$$

Thus from (6.66), we obtain

$$(\widetilde{\boldsymbol{\theta}}_t, \mathbf{v}_h) + \mu a_h(\widetilde{\boldsymbol{\theta}}, \mathbf{v}_h) + \int_0^t \beta(t-\tau) a_h(\widetilde{\boldsymbol{\theta}}(\tau), \mathbf{v}_h)\ d\tau\ ds = -(\boldsymbol{\zeta}, \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{J}_h. \tag{6.68}$$

Choose $\mathbf{v}_h = e^{2\alpha t}\widetilde{\boldsymbol{\theta}}$ in (6.68) and integrate the resulting equation. Drop the double integral term, as it is non-negative. Using Lemma 6.6, we reach at

$$e^{2\alpha t}\|\widetilde{\boldsymbol{\theta}}(t)\|^2 + (\mu - \frac{2\alpha}{\lambda_h}) \int_0^t e^{2\alpha s}\|\nabla_h \widetilde{\boldsymbol{\theta}}(s)\|^2\ ds \le Ch^4 \int_0^t e^{2\alpha s}\mathcal{K}^2(s) ds. \tag{6.69}$$

Incorporate (6.69) in (6.65) and use the Lemmas 6.5 and 6.6 to conclude

$$\tau^*(t)\|\boldsymbol{\theta}(t)\|^2 + e^{-2\alpha t}\int_0^t \sigma(s)\|A_{\varepsilon h}^{\frac{1}{2}}\boldsymbol{\theta}(s)\|^2\, ds \ \le\ Ch^4. \tag{6.70}$$

Note that the estimates of $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}_t$ involve the estimates of $\mathbf{u}$ and p (see Lemma 6.6). We have used the Lemma 1.8 to estimate $\mathbf{u}$ and p. Now with a use of triangle inequality and the inverse hypothesis with (6.70) and Lemma 6.6, we conclude the following results:

Lemma 6.7. *Suppose the hypothesis of Lemma 6.3 be satisfied. Then, the following result holds for any time $t > 0$,*

$$\|\boldsymbol{\xi}(t)\| + h\|\nabla\boldsymbol{\xi}(s)\| \ \le\ Ch^2 t^{-\frac{1}{2}}.$$

With the desired estimate of $\boldsymbol{\xi}$, we aim to achieve the estimates of $\mathbf{e}$ by means of $\boldsymbol{\eta}$. Note that $\mathbf{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$.

Lemma 6.8. *Suppose the assumptions of Lemma 6.3 hold and $\mathbf{u}_h(t)$ be a solution of (6.2) with $\mathbf{u}_h(0) = P_h\mathbf{u}_0$. Then, for $0 < t \le T_0$, the following error bound holds:*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{e}(s)\|^2\ ds \le K(t)h^4,$$

where $K(t) = Ce^{Ct}$ and $C > 0$ is a error constant.

Proof. As mentioned above, it suffices to estimate $\boldsymbol{\eta}$. From (6.2) and (6.26), one can find the equation of $\boldsymbol{\eta}$ as

$$(\boldsymbol{\eta}_t, \mathbf{v}_h) + \mu a_h(\boldsymbol{\eta}, \mathbf{v}_h) + \int_0^t \beta(t-s) a_h(\boldsymbol{\eta}(s), \mathbf{v}_h)\, ds = \Lambda_h(\mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{J}_h, \tag{6.71}$$

where

$$\Lambda_h(\mathbf{v}_h) = b_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - b_h(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) = -b_h(\mathbf{e}, \mathbf{u}_h, \mathbf{v}_h) - b_h(\mathbf{u}, \mathbf{e}, \mathbf{v}_h). \tag{6.72}$$

Choose $\mathbf{v}_h = e^{2\alpha t}(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}$ and use the "Poincaré inequality" to obtain

$$\frac{1}{2}\frac{d}{dt}(e^{2\alpha t}\|\boldsymbol{\eta}\|_{-1}^2) + \Big(\mu - \frac{\alpha}{\lambda_h}\Big)e^{2\alpha t}\|\boldsymbol{\eta}\|^2 + e^{2\alpha t}\int_0^t \beta(t-s)(\boldsymbol{\eta}(s), \boldsymbol{\eta})\, ds \\ \leq e^{2\alpha t}\Lambda_h((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}). \tag{6.73}$$

By writing $\mathbf{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$ and using Lemma 6.2, we estimate Λ_h as

$$|\Lambda_h((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta})| \leq \frac{\mu}{2}\|\boldsymbol{\eta}\|^2 + C(\mu)\Big(\|\mathbf{u}_h\|\|\nabla_h\mathbf{u}_h\| + \|\mathbf{u}\|\|\nabla_h\mathbf{u}\|\Big)\|\boldsymbol{\xi}\|^2 \\ + C(\mu)\|\boldsymbol{\eta}\|_{-1}^2\Big(\|\mathbf{u}_h\|^2\|\nabla_h\mathbf{u}_h\|^2 + \|\mathbf{u}\|^2\|\nabla_h\mathbf{u}\|^2\Big).$$

We now take time integration on (6.73) and remove the resulting double integration term due to positivity property to obtain

$$e^{2\alpha t}\|\boldsymbol{\eta}\|_{-1}^2 + \Big(\mu - \frac{2\alpha}{\lambda_h}\Big)\int_0^t e^{2\alpha s}\|\boldsymbol{\eta}\|^2 ds \leq C\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}\|^2 ds + C\int_0^t e^{2\alpha s}\|\boldsymbol{\eta}\|_{-1}^2 ds.$$

An application of the "Gronwall's Lemma" with Lemma 6.5 concludes the proof. □

We now present the main result of this section, that is, the finite element Galerkin approximation error estimate for the system.

Theorem 6.1. *Suppose the conditions* (**A1**)-(**A2**) *and* (**B1i**)-(**B2i**) *be satisfied. Further, assume that the discrete initial velocity* $\mathbf{u}_h(0) \in \mathbf{J}_h$ *with* $\mathbf{u}_h(0) = P_h\mathbf{u}_0$, *where* $\mathbf{u}_0 \in \mathbf{H}_0^1$. *Then, for* $0 < t \leq T_0$ *the following error estimates hold*

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| + h\|\nabla_h(\mathbf{u} - \mathbf{u}_h)(t)\| \leq K(t)h^2 t^{-\frac{1}{2}}.$$

where $K(t) = Ce^{Ct}$ *and* $C > 0$ *is arbitrary constant.*

Proof. Since $\mathbf{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$ and we have already obtained the estimate of $\boldsymbol{\xi}$, then it is enough to estimate $\boldsymbol{\eta}$. Choose $\mathbf{v}_h = \sigma(t)\boldsymbol{\eta}$ in (6.71) to deduce

$$\frac{1}{2}\frac{d}{dt}(\sigma(t)\|\boldsymbol{\eta}\|^2) + \mu\sigma(t)\|\nabla_h\boldsymbol{\eta}\|^2 = \frac{1}{2}\sigma_t(t)\|\boldsymbol{\eta}\|^2 - \sigma(t)\int_0^t \beta(t-s)a_h(\boldsymbol{\eta}(s), \boldsymbol{\eta})\, ds + \sigma(t)\Lambda_h(\boldsymbol{\eta}).$$

Use Lemma 6.2 to bound the nonlinear terms as

$$\Lambda_h(\boldsymbol{\eta}) = -b_h(\mathbf{e}, \mathbf{w}_h, \boldsymbol{\eta}) - b_h(\mathbf{u}, \mathbf{e}, \boldsymbol{\eta})$$

$$\le C(\|\nabla \mathbf{u}\|^{\frac{1}{2}}\|\tilde{\Delta}\mathbf{u}\|^{\frac{1}{2}} + \|\nabla_h \mathbf{w}_h\|^{\frac{1}{2}}\|\tilde{\Delta}_h \mathbf{w}_h\|^{\frac{1}{2}})\|\mathbf{e}\|\|\nabla_h \boldsymbol{\eta}\|.$$

A use of the inverse hypothesis and the approximation property yield

$$\begin{aligned}\|\tilde{\Delta}_h \mathbf{w}_h\| &\le \|\tilde{\Delta}_h \mathbf{w}_h - \tilde{\Delta}_h P_h \mathbf{u}\| + \|\tilde{\Delta}_h P_h \mathbf{u}\| \le Ch^{-2}\|\mathbf{w}_h - P_h\mathbf{u}\| + C\|\tilde{\Delta}\mathbf{u}\| \\ &\le Ch^{-2}(\|\boldsymbol{\xi}\| + \|\mathbf{u} - P_h\mathbf{u}\|) + C\|\tilde{\Delta}\mathbf{u}\| \le C\|\tilde{\Delta}\mathbf{u}\|. \qquad (6.74)\end{aligned}$$

Combining above three equations and integrating the resulting equation, we find that

$$\begin{aligned}\sigma(t)\|\boldsymbol{\eta}\|^2 + \mu\int_0^t \sigma(s)\|\nabla_h\boldsymbol{\eta}(s)\|^2 &\le 2(1+\alpha)\int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(s)\|^2 ds + C\int_0^t e^{2\alpha s}\|\nabla_h\widetilde{\boldsymbol{\eta}}(s)\|^2 ds \\ &+C\int_0^t \tau^*(s)(\|\nabla\mathbf{u}(s)\|\|\tilde{\Delta}\mathbf{u}(s)\|)e^{2\alpha s}\|\mathbf{e}(s)\|^2 ds. \qquad (6.75)\end{aligned}$$

Note that the resulting double integration term is estimated similar to (6) with $\widetilde{\boldsymbol{\eta}}(t) = \int_0^t \boldsymbol{\eta}(s)\ ds$. To find the bounds for the second term on the right of inequality (6.75), we integrate (6.71) and similar to (6.68) to obtain

$$(\boldsymbol{\eta}, \mathbf{v}_h) + \mu a_h(\widetilde{\boldsymbol{\eta}}, \mathbf{v}_h) + \int_0^t \beta(t-s)a_h(\widetilde{\boldsymbol{\eta}}(s), \mathbf{v}_h)\ ds = \int_0^t \Lambda_h(\mathbf{v}_h)\ ds. \qquad (6.76)$$

Put $\mathbf{v}_h = e^{2\alpha t}\widetilde{\boldsymbol{\eta}}$ in (6.76) and take time integration. Then, drop the double integral term from the left of inequality due to positivity to find

$$e^{2\alpha t}\|\widetilde{\boldsymbol{\eta}}\|^2 + 2(\mu - \frac{\alpha}{\lambda_h})\int_0^t e^{2\alpha s}\|\nabla_h\widetilde{\boldsymbol{\eta}}(s)\|^2\ ds \le 2\int_0^t e^{2\alpha s}|\int_0^s \Lambda_h(\widetilde{\boldsymbol{\eta}}(s))\ d\tau|\ ds. \quad (6.77)$$

We bound the nonlinear terms using Lemma 6.2 as

$$2\int_0^t e^{2\alpha s}\int_0^s |\Lambda_h(\widetilde{\boldsymbol{\eta}}(s))| d\tau ds \le Ch^4 t^{\frac{1}{2}} e^{4\alpha t} + \mu\int_0^t e^{2\alpha s}\|\nabla_h\widetilde{\boldsymbol{\eta}}(s)\|^2 ds. \qquad (6.78)$$

Incorporate (6.78) in (6.77) with $(\mu - \frac{2\alpha}{\lambda_h}) > 0$ to obtain

$$\|\widetilde{\boldsymbol{\eta}}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla_h\widetilde{\boldsymbol{\eta}}(s)\|^2\ ds \le K(t)h^4 t^{\frac{1}{2}}. \qquad (6.79)$$

Now, insert (6.79) in (6.75) and apply the Lemmas 1.8, 6.8 and 6.3. Then, multiply by $e^{-2\alpha t}$ to yield

$$\tau^*(t)\|\boldsymbol{\eta}(t)\|^2 + e^{-2\alpha t}\mu\int_0^t \sigma(s)\|\nabla_h\boldsymbol{\eta}\|^2\ ds \le K(t)h^4.$$

Since $\boldsymbol{\eta} \in \mathbf{H}_h$, a use of the inverse hypothesis helps to find the bounds for $\|\nabla_h\boldsymbol{\eta}\|$. Now, we use the triangle inequality with Lemma 6.6 to complete the proof. □

6.3.1 Uniform in time bounds

The results derived in Theorem 6.1 are not uniform in time due to the exponential in time behaviour of the error bounds. But under the uniqueness condition (6.80), we find the following uniform (in time) estimates.

Theorem 6.2. *Suppose the hypothesis of Theorem 6.1 be satisfied. Then, under the uniqueness condition*

$$\mu - 2N\nu^{-1}\|\mathbf{f}\|_\infty > 0 \quad \text{and} \quad N = \sup_{\mathbf{u}_h,\mathbf{w}_h,\mathbf{v}_h\in\mathbf{H}_h} \frac{b_h(\mathbf{u}_h,\mathbf{w}_h,\mathbf{v}_h)}{\|\nabla_h\mathbf{u}_h\|\|\nabla_h\mathbf{w}_h\|\|\nabla_h\mathbf{v}_h\|}, \tag{6.80}$$

and for any $t>0$, the following result holds:

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\| + h\|\nabla_h(\mathbf{u}-\mathbf{u}_h)(t)\| \le Ch^2 t^{-\frac{1}{2}}.$$

Proof. Recall $\mathbf{e}=\boldsymbol{\xi}+\boldsymbol{\eta}$ and we have obtained that the bounds for $\boldsymbol{\xi}$ are uniformly in time (see, Lemma 6.5), but the estimates of $\boldsymbol{\eta}$ are not uniform (see, Lemma 6.8) due to use of the "Gronwall's lemma". Hence, it is enough to make the estimates of $\boldsymbol{\eta}$ are uniform in time. The idea is to estimate nonlinear term in a different manner using uniqueness condition (6.80) such that we can avoid the use of "Gronwall's lemma". For this, we choose $\mathbf{v}_h = e^{2\alpha t}\boldsymbol{\eta}$ in (6.71) to obtain

$$\frac{1}{2}\frac{d}{dt}(e^{2\alpha t}\|\boldsymbol{\eta}\|^2)+\mu e^{2\alpha t}\|\nabla_h\boldsymbol{\eta}\|^2+e^{2\alpha t}\int_0^t \beta(t-s)a_h(\boldsymbol{\eta}(s),\boldsymbol{\eta})\,ds = e^{2\alpha t}(\alpha\|\boldsymbol{\eta}\|^2+\Lambda_h(\boldsymbol{\eta})). \tag{6.81}$$

From (6.2) and (6.14), we rewrite the nonlinear terms as

$$\Lambda_h(\boldsymbol{\eta}) = -b_h(\mathbf{e},\mathbf{u}_h,\boldsymbol{\eta}) - b_h(\mathbf{u},\mathbf{e},\boldsymbol{\eta}) = b_h(\boldsymbol{\xi},\mathbf{v}_h,\boldsymbol{\eta}) - b_h(\boldsymbol{\eta},\mathbf{u}_h,\boldsymbol{\eta}) - b_h(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\eta}). \tag{6.82}$$

We use (6.80) to bound the second nonlinear term as

$$|b_h(\boldsymbol{\eta},\mathbf{u}_h,\boldsymbol{\eta})| \le N\|\nabla_h\mathbf{u}_h\|\|\nabla_h\boldsymbol{\eta}\|^2.$$

A use of Lemma 6.2 with (6.74) and the "Cauchy-Schwarz inequality" yields

$$\begin{aligned}|b_h(\boldsymbol{\xi},\mathbf{v}_h,\boldsymbol{\eta}) - b_h(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\eta})| &\le C(\|\nabla\mathbf{u}\|^{\frac{1}{2}}\|\tilde{\Delta}\mathbf{u}\|^{\frac{1}{2}} + \|\nabla_h\mathbf{v}_h\|^{\frac{1}{2}}\|\tilde{\Delta}_h\mathbf{v}_h\|^{\frac{1}{2}})\|\boldsymbol{\xi}\|\|\nabla_h\boldsymbol{\eta}\| \\ &\le C(\|\nabla\mathbf{u}\|^2+\|\tilde{\Delta}\mathbf{u}\|^2)\|\boldsymbol{\xi}\|^2 + \frac{\mu}{2}\|\nabla_h\boldsymbol{\eta}\|^2.\end{aligned}$$

Substitute the above two in (6.81) and take time integration to find

$$\begin{aligned}e^{2\alpha t}\|\boldsymbol{\eta}(t)\|^2 + 2\int_0^t e^{2\alpha s}(\frac{\mu}{2} - N\|\nabla_h\mathbf{u}_h\|)\|\nabla_h\boldsymbol{\eta}\|^2 ds + 2\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)a_h(\boldsymbol{\eta}(\tau),\boldsymbol{\eta}(s))d\tau ds \\ \le \|\boldsymbol{\eta}(0)\|^2 + 2\alpha\int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(s)\|^2 ds + C\int_0^t e^{2\alpha s}(\|\nabla\mathbf{u}(s)\|^2+\|\tilde{\Delta}\mathbf{u}(s)\|^2)\|\boldsymbol{\xi}(s)\|^2 ds.\end{aligned} \tag{6.83}$$

The last term on the right of inequality can be written as

$$\begin{aligned}\int_0^t & e^{2\alpha s}(\|\nabla \mathbf{u}(s)\|^2 + \|\tilde{\Delta}\mathbf{u}(s)\|^2)\|\boldsymbol{\xi}(s)\|^2 ds \\ &\leq \|\boldsymbol{\xi}(t)\|^2_{L^\infty(\mathbf{L}^2)} \int_0^t e^{2\alpha s}(\|\nabla \mathbf{u}(s)\|^2 + \|\tilde{\Delta}\mathbf{u}(s)\|^2) ds \\ &\leq Ce^{2\alpha t}\|\boldsymbol{\xi}(t)\|^2_{L^\infty(\mathbf{L}^2)}. \qquad (6.84)\end{aligned}$$

Use (6.84) with Lemma 1.8 in (6.83) and multiply both sides by $e^{-2\alpha t}$ to find

$$\begin{aligned}\|\boldsymbol{\eta}(t)\|^2 &+ 2e^{-2\alpha t}\int_0^t e^{2\alpha s}(\frac{\mu}{2} - N\|\nabla_h \mathbf{u}_h\|)\|\nabla_h \boldsymbol{\eta}\|^2 ds \\ &+ 2e^{-2\alpha t}\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)a(\boldsymbol{\eta}(\tau),\boldsymbol{\eta}(s))\ d\tau\ ds \\ &\leq e^{-2\alpha t}\|\boldsymbol{\eta}(0)\|^2 + 2\alpha e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(s)\|^2 ds + C\|\boldsymbol{\xi}(t)\|^2_{L^\infty(\mathbf{L}^2)}.\end{aligned}$$

Now, take limit supremum as $t \to \infty$ and L'Hospital rule with the followings from [63]

$$\begin{aligned}\lim_{t\to\infty}\sup e^{-2\alpha t}\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)a(\boldsymbol{\eta}(\tau),\boldsymbol{\eta}(s))\ d\tau\ ds &= \frac{\gamma}{2\alpha\delta}\lim_{t\to\infty}\sup\|\nabla_h \boldsymbol{\eta}\|^2, \\ \lim_{t\to\infty}\sup\|\nabla_h \mathbf{u}_h\| &\leq \nu^{-1}\|\mathbf{f}_\infty\|_{-1},\end{aligned}$$

to conclude

$$[\frac{\mu}{2} - N\nu^{-1}\|\mathbf{f}_\infty\|_{-1} + \frac{\gamma}{\delta}]\lim_{t\to\infty}\sup\|\nabla_h \boldsymbol{\eta}\|^2 \leq C\lim_{t\to\infty}\sup\|\boldsymbol{\xi}(t)\|^2_{L^\infty(\mathbf{L}^2)}.$$

With $1 - N\nu^{-2}\|\mathbf{f}_\infty\|_{-1} > 0$, we have $[\frac{\mu}{2} - N\nu^{-1}\|\mathbf{f}_\infty\|_{-1} + \frac{\gamma}{\delta}] = \frac{1}{\nu}[1 - N\nu^{-2}\|\mathbf{f}_\infty\|_{-1}] > 0$ and we obtain the following

$$\lim_{t\to\infty}\sup\|\boldsymbol{\eta}\| \leq \lim_{t\to\infty}\sup\|\nabla\boldsymbol{\eta}\| \leq C\lim_{t\to\infty}\sup\|\boldsymbol{\xi}(t)\|_{L^\infty(\mathbf{L}^2)}.$$

Combine with the estimates of $\boldsymbol{\xi}$, we conclude the rest of the proof. □

6.4 Fully Discrete Error Analysis

Here, we discretize the time variable where as the space is discretized based on non-conforming finite element. First we discuss about the Backward Euler (BE) method applied to our problem in nonconforming setups, then we apply Euler incremental pressure correction (EIPC) scheme and analysze it for our problem.

6.4.1 Backward Euler Method

All the analysis here goes similar as previous chapters. First we present *a priori* and regularity estimates of the fully discrete solution (We skip the proof, since, it is already given in Chapter 2, the only difference is that we replace ∇ is replaced by ∇_h due to the nonconforming space).

Lemma 6.9. *Suppose the conditions* $(\mathbf{A1})$ *and* $(\mathbf{A2})$ *be satisfied. Then, the following results hold:*

$$\|\mathbf{U}^n\|^2+\|\nabla_h\mathbf{U}^n\|^2+e^{-2\alpha t_n}k\sum_{i=1}^{n}e^{2\alpha t_i}(\|\nabla_h\mathbf{U}^i\|^2+\|\tilde{\Delta}_h\mathbf{U}^i\|^2)\le C,$$

$$\tau(t_n)\|\tilde{\Delta}_h\mathbf{U}^n\|^2+\tau(t_n)\|\partial_t\mathbf{U}^n\|^2+e^{-2\alpha t_n}k\sum_{i=1}^{n}e^{2\alpha t_i}\|\partial_t\mathbf{U}^i\|^2\le C.$$

where $\|\mathbf{f}\|_\infty=\|\mathbf{f}\|_{L^\infty(\mathbb{R}_+;\mathbf{L}^2(\Omega))}$, $\tau(t_n)=\min\{1,t_n\}$ *and* α *is a parameter of our choice satisfying* $0<\alpha<\min\{\delta,\frac{\mu\lambda_h}{2}\}$*and*

$$1+\big(\frac{\mu\lambda_h}{2}\big)k\ge e^{\alpha k}. \tag{6.85}$$

For the error analysis, we set, for fixed $n\in\mathbb{N},\ 1\le n\le N,\ \mathbf{e}_n=\mathbf{U}^n-\mathbf{u}_h(t_n)=\mathbf{U}^n-\mathbf{u}_h^n$. We now rewrite (6.2) at $t=t_n$ and subtract from (6.4) to obtain

$$(\partial_t\mathbf{e}_n,\mathbf{v}_h)+\mu a(\mathbf{e}_n,\mathbf{v}_h)+a(q_r^n(\mathbf{e}),\mathbf{v}_h)=R_h^n(\mathbf{v}_h)+E_h^n(\mathbf{v}_h)+\Lambda_h^n(\mathbf{v}_h),$$

where,

$$\begin{aligned}
R_h^n(\mathbf{v}_h)&=(\mathbf{u}_{ht}^n,\mathbf{v}_h)-(\partial_t\mathbf{u}_h^n,\mathbf{v}_h)\\
E_h^n(\mathbf{v}_h)&=\int_0^t\beta(t-s)a(\mathbf{u}_h(s),\mathbf{v}_h)ds-a(q_r^n(\mathbf{u}_h),\mathbf{v}_h),\\
\Lambda_h^n(\mathbf{v}_h)&=b(\mathbf{u}_h^n,\mathbf{u}_h^n,\mathbf{v}_h)-b(\mathbf{U}^n,\mathbf{U}^n,\mathbf{v}_h)
\end{aligned}$$

It is noted that the error $\mathbf{e}\in\mathbf{H}_h$, so there is no effect of nonconformity on the error. Hence all the analysis go similar to conforming case, which is already discussed in the previous chapters. So we only borrow the results here for the sake of completeness.

Theorem 6.3. *Assume that* $(\mathbf{A1})$-$(\mathbf{A2})$ *and* $(\mathbf{B1i})$-$(\mathbf{B3i})$ *hold true. Then, the following error estimates hold:*

$$\|\mathbf{e}_n\|\le K_nt_n^{-1/2}k,\quad \|\nabla_h\mathbf{e}_n\|\le K_nt_n^{-1}k,$$

where $K_n = Ce^{Ct_n}$. The estimates are uniform in time under the smallness condition (6.80), that is, $K_n = C$.

Combining this above results with Theorem 6.1, we finally conclude that

Theorem 6.4. *Suppose the hypothesis of Theorem 6.1 and 6.3 be satisfied. Then, the following results hold:*

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \leq K_n(h^2 + k)t_n^{-1/2}, \quad \|\nabla_h(\mathbf{u}(t_n) - \mathbf{U}^n)\| \leq K_n(ht^{-1/2} + kt_n^{-1}).$$

where $K_n = Ce^{Ct_n}$. Under the uniqueness condition (6.80), the above estimates are valid uniformly in time, that is, $K_n = C$.

6.4.2 Euler Incremental Pressure Correction Method

Here, we analyze a first-order time discrete projection scheme, namely, Euler incremental pressure correction (EIPC) scheme to the Oldroyd model of order one. We start with the stability analysis of the scheme.

Lemma 6.10. *Suppose the conditions* (**A1**),(**A2**), (**B1i**) *and* (**B2i**) *hold. Then, for $0 < n < N$, the discrete solution satisfy the following stability result:*

$$\|\mathbf{U}_h^n\|^2 + \sum_{i=1}^{n} \|\mathbf{U}_h^i - \mathbf{U}_h^{i-1}\|^2 + \mu k \sum_{i=1}^{n} \|\nabla_h \hat{\mathbf{U}}_h^i\|^2 + k^2 \|\nabla_h P_h^n\|^2 \leq Ct_n.$$

Proof. Choose $\mathbf{v}_h = \hat{\mathbf{U}}_h^i$ in (6.11) with $n = i$ and using the fact $(a - b, a) = \frac{1}{2}(\|a\|^2 - \|b\|^2 + \|a - b\|^2)$ to arrive at

$$\frac{1}{2k}(\|\hat{\mathbf{U}}_h^i\|^2 + \|\hat{\mathbf{U}}_h^i - \mathbf{U}_h^{i-1}\|^2 - \|\mathbf{U}_h^{i-1}\|^2) + \mu\|\nabla_h \hat{\mathbf{U}}_h^i\|^2 + a_h(q_r^i(\hat{\mathbf{U}}_h), \hat{\mathbf{U}}_h^i)$$
$$= (P_h^{i-1}, \nabla_h \cdot \hat{\mathbf{U}}_h^i) + (\mathbf{f}^i, \hat{\mathbf{U}}_h^i). \tag{6.86}$$

Note that the nonlinear term $b_h(\mathbf{U}_h^{i-1}, \hat{\mathbf{U}}_h^i, \hat{\mathbf{U}}_h^i) = 0$ due to (6.14). A use of the "Poincaré inequality" and the "Cauchy-Schwarz inequality" helps to bound the last term on the right of inequality (6.86) as

$$(\mathbf{f}^i, \hat{\mathbf{U}}_h^i) \leq \|\mathbf{f}^i\|\|\hat{\mathbf{U}}_h^i\| \leq \frac{1}{\sqrt{\lambda_h}}\|\mathbf{f}^i\|\|\nabla_h \hat{\mathbf{U}}_h^i\| \leq \frac{1}{2\mu\lambda_h}\|\mathbf{f}^i\|^2 + \frac{\mu}{2}\|\nabla_h \hat{\mathbf{U}}_h^i\|^2. \tag{6.87}$$

For the first term, we choose $\mathbf{v}_h = \nabla_h P_h^{i-1}$ in (6.12) and use the fact $(a - b, b) = \frac{1}{2}(\|a\|^2 - \|b\|^2 - \|a - b\|^2)$ and (6.13) to obtain

$$(P_h^{i-1}, \nabla_h \cdot \hat{\mathbf{U}}_h^i) = (P_h^{i-1}, \nabla_h \cdot \mathbf{U}_h^i) - k(\nabla_h P_h^i - \nabla_h P_h^{i-1}, \nabla_h P_h^{i-1})$$

$$= k\|\nabla_h P_h^{i-1}\|^2 - k\|\nabla_h P_h^i\|^2 + k\|\nabla_h(P_h^i - P_h^{i-1})\|^2. \qquad (6.88)$$

To estimate the last term of (6.88), we choose $\mathbf{v}_h = \nabla_h(P_h^i - P_h^{i-1})$ in (6.12) to deduce

$$\|\nabla_h(P_h^i - P_h^{i-1})\|^2 = \frac{1}{k}(\hat{\mathbf{U}}_h^i - \mathbf{U}_h^i, \nabla_h(P_h^i - P_h^{i-1})) \leq \frac{1}{k}\|\hat{\mathbf{U}}_h^i - \mathbf{U}_h^i\|\|\nabla_h(P_h^i - P_h^{i-1})\|.$$

Cancelling one $\|\nabla_h(P_h^i - P_h^{i-1})\|$ from both sides, we arrive at

$$\|\nabla_h(P_h^i - P_h^{i-1})\| \leq \frac{1}{k}\|\hat{\mathbf{U}}_h^i - \mathbf{U}_h^i\|\|. \qquad (6.89)$$

Incorporating (6.87)-(6.89) in(6.86), we obtain

$$\|\hat{\mathbf{U}}_h^i\|^2 + \|\hat{\mathbf{U}}_h^i - \mathbf{U}_h^{i-1}\|^2 - \|\mathbf{U}_h^{i-1}\|^2 + \mu k\|\nabla_h\hat{\mathbf{U}}_h^i\|^2 + 2ka_h(q_r^i(\hat{\mathbf{U}}_h), \hat{\mathbf{U}}_h^i)$$
$$\leq \frac{k}{\mu\lambda_h}\|\mathbf{f}^i\|^2 + k^2\|\nabla_h P_h^{i-1}\|^2 - k^2\|\nabla_h P_h^i\|^2 + \|\hat{\mathbf{U}}_h^i - \mathbf{U}_h^i\|^2. \qquad (6.90)$$

Taking $\mathbf{v}_h = \mathbf{U}_h^i$ in (6.12) and using (6.13), we easily find that

$$\|\mathbf{U}_h^i\|^2 + \|\hat{\mathbf{U}}_h^i - \mathbf{U}_h^i\|^2 - \|\hat{\mathbf{U}}_h^i\|^2 = 0. \qquad (6.91)$$

A use of (6.91) in (6.90) yields

$$\|\mathbf{U}_h^i\|^2 - \|\mathbf{U}_h^{i-1}\|^2 + \|\mathbf{U}_h^i - \mathbf{U}_h^{i-1}\|^2 + \mu k\|\nabla_h\hat{\mathbf{U}}_h^i\|^2 + 2ka_h(q_r^i(\hat{\mathbf{U}}_h), \hat{\mathbf{U}}_h^i)$$
$$+k^2\|\nabla_h P_h^i\|^2 - k^2\|\nabla_h P_h^{i-1}\|^2 \leq \frac{k}{\mu\lambda_h}\|\mathbf{f}^i\|^2. \qquad (6.92)$$

Take sum from $i = 1$ to n to achieve

$$\|\mathbf{U}_h^n\|^2 + \sum_{i=1}^n \|\mathbf{U}_h^i - \mathbf{U}_h^{i-1}\|^2 + \mu k\sum_{i=1}^n \|\nabla_h\hat{\mathbf{U}}_h^i\|^2 + 2k\sum_{i=1}^n a_h(q_r^i(\hat{\mathbf{U}}_h), \hat{\mathbf{U}}_h^i) + k^2\|\nabla_h P_h^n\|^2$$
$$\leq \|\mathbf{U}_h^0\|^2 + \frac{1}{\mu\lambda_h}k\sum_{i=1}^n \|\mathbf{f}^i\|^2. \qquad (6.93)$$

Now forth term of the left of inequality (6.93) is positive due to (1.18), so we drop it, which conclude the rest of the proof. □

This proves that the scheme is unconditionally stable.

Now, we set $\mathbf{e}_h^n = \mathbf{U}_h^n - \mathbf{U}^n$, $\hat{\mathbf{e}}_h^n = \hat{\mathbf{U}}_h^n - \mathbf{U}^n$ and $\mathbf{e}_p^{n-1} = P_h^{n-1} - P^n$. Now, subtract (6.3) from (6.11) with $n = i$ to obtain

$$\Big(\frac{\hat{\mathbf{e}}_h^i - \mathbf{e}_h^{i-1}}{k}, \mathbf{v}_h\Big) + \mu a_h(\hat{\mathbf{e}}_h^i, \mathbf{v}_h) + a_h(q_r^i(\hat{\mathbf{e}}_h), \mathbf{v}_h) - (\mathbf{e}_p^{i-1}, \nabla_h \cdot \mathbf{v}_h) = \Lambda_h^i(\mathbf{v}_h), \qquad (6.94)$$

where $\Lambda_h^i(\mathbf{v}_h) = b_h(\mathbf{U}_h^{i-1}, \hat{\mathbf{U}}_h^i, \mathbf{v}_h) - b_h(\mathbf{U}^i, \mathbf{U}^i, \mathbf{v}_h)$.

We also rewrite (6.12) with $\mathbf{e}_q^n = P_h^n - P^n$ and $n = i$ as

$$\left(\frac{\mathbf{e}_h^i - \hat{\mathbf{e}}_h^i}{k}, \mathbf{v}_h\right) - (\mathbf{e}_q^i - \mathbf{e}_p^{i-1}, \nabla_h \cdot \mathbf{v}_h) = 0. \tag{6.95}$$

We now find the optimal error velocity estimates.

Lemma 6.11. *Suppose the conditions of Lemma 6.10 holds true. Then, for* $1 \leq n \leq N$*, the velocity and pressure errors satisfy the following results:*

$$\|\mathbf{e}_h^n\|^2 + \sum_{i=1}^{n} \|\mathbf{e}_h^i - \mathbf{e}_h^{i-1}\|^2 + \mu k \sum_{i=1}^{n} \|\nabla_h \hat{\mathbf{e}}_h^i\|^2 + k^2 \|\nabla_h \mathbf{e}_p^{n-1}\|^2 \leq C e^{Ct_n} k^2 t_n^{-1}.$$

Proof. Choose $\mathbf{v}_h = \hat{\mathbf{e}}_h^i$ in (6.94) to obtain

$$\frac{1}{2k}(\|\hat{\mathbf{e}}_h^i\|^2 + \|\hat{\mathbf{e}}_h^i - \mathbf{e}_h^{i-1}\|^2 - \|\mathbf{e}_h^{i-1}\|^2) + \mu\|\nabla_h \hat{\mathbf{e}}_h^i\|^2 + a_h(q_r^i(\hat{\mathbf{e}}_h), \hat{\mathbf{e}}_h^i)$$
$$-(\mathbf{e}_p^{i-1}, \nabla_h \cdot \hat{\mathbf{e}}_h^i) = \Lambda_h^i(\hat{\mathbf{e}}_h^i). \tag{6.96}$$

A use of Lemma 6.2 with the “Poincaré inequality” and the “Cauchy-Schwarz inequality” helps to bound the nonlinear terms as

$$\begin{aligned}\Lambda_h^i(\hat{\mathbf{e}}_h^i) &= -b_h(\mathbf{U}_h^{i-1}, \hat{\mathbf{e}}_h^i, \hat{\mathbf{e}}_h^i) - b_h(\mathbf{e}_h^{i-1}, \mathbf{U}^i, \hat{\mathbf{e}}_h^i) - b_h(\mathbf{U}^i - \mathbf{U}^{i-1}, \mathbf{U}^i, \hat{\mathbf{e}}_h^i) \\ &\leq \frac{1}{\mu}(\|\mathbf{e}_h^{i-1}\|^2 + \|\mathbf{U}^i - \mathbf{U}^{i-1}\|^2)\|\tilde{\Delta}_h \mathbf{U}^i\|^2 + \frac{\mu}{4}\|\nabla_h \hat{\mathbf{e}}_h^i\|^2.\end{aligned} \tag{6.97}$$

For the last term on the left of inequality (6.96), we choose $\mathbf{v}_h = \nabla_h \mathbf{e}_p^{i-1}$ in (6.95) and use (6.13) to obtain

$$\begin{aligned}(\mathbf{e}_p^{i-1}, \nabla_h \cdot \hat{\mathbf{e}}_h^i) &= (\mathbf{e}_p^{i-1}, \nabla_h \cdot \mathbf{e}_h^i) - k(\nabla_h \mathbf{e}_q^i - \nabla_h \mathbf{e}_p^{i-1}, \nabla_h \mathbf{e}_p^{i-1}) \\ &= k\|\nabla_h \mathbf{e}_p^{i-1}\|^2 - k\|\nabla_h \mathbf{e}_q^i\|^2 + k\|\nabla_h(\mathbf{e}_q^i - \mathbf{e}_p^{i-1})\|^2.\end{aligned} \tag{6.98}$$

To bound the last term on the right of inequality (6.98), we choose $\mathbf{v}_h = \nabla_h(\mathbf{e}_q^i - \mathbf{e}_p^{i-1})$ in (6.95) to deduce

$$\|\nabla_h(\mathbf{e}_q^i - \mathbf{e}_p^{i-1})\|^2 = \frac{1}{k}(\hat{\mathbf{e}}_h^i - \mathbf{e}_h^i, \nabla_h(\mathbf{e}_q^i - \mathbf{e}_p^{i-1})) \leq \frac{1}{k}\|\hat{\mathbf{e}}_h^i - \mathbf{e}_h^i\|\|\nabla_h(\mathbf{e}_q^i - \mathbf{e}_p^{i-1})\|.$$

Cancelling one $\|\nabla_h(\mathbf{e}_q^i - \mathbf{e}_p^{i-1})\|$ from both sides, we arrive at

$$\|\nabla_h(\mathbf{e}_q^i - \mathbf{e}_p^{i-1})\| \leq \frac{1}{k}\|\hat{\mathbf{e}}_h^i - \mathbf{e}_h^i\|\|. \tag{6.99}$$

Incorporating (6.97)-(6.99) in(6.96), we obtain

$$\|\hat{\mathbf{e}}_h^i\|^2 + \|\hat{\mathbf{e}}_h^i - \mathbf{e}_h^{i-1}\|^2 - \|\mathbf{e}_h^{i-1}\|^2 + \mu k\|\nabla_h \hat{\mathbf{e}}_h^i\|^2 + 2k a_h(q_r^i(\hat{\mathbf{e}}_h), \hat{\mathbf{e}}_h^i) - k^2\|\nabla_h \mathbf{e}_p^{i-1}\|^2$$

$$+ k^2\|\nabla_h \mathbf{e}_q^i\|^2 + \|\hat{\mathbf{e}}_h^i - \mathbf{e}_h^i\|^2 \le \frac{1}{\mu}(\|\mathbf{e}_h^{i-1}\|^2 + \|\mathbf{U}^i - \mathbf{U}^{i-1}\|^2)\|\tilde{\Delta}_h \mathbf{U}^i\|^2. \quad (6.100)$$

Taking $\mathbf{v}_h = \mathbf{e}_h^i$ in (6.95) and using (6.13), we easily find that

$$\|\mathbf{e}_h^i\|^2 + \|\hat{\mathbf{e}}_h^i - \mathbf{e}_h^i\|^2 - \|\hat{\mathbf{e}}_h^i\|^2 = 0. \quad (6.101)$$

A use of (6.101) in (6.100) yields

$$\|\mathbf{e}_h^i\|^2 - \|\mathbf{e}_h^{i-1}\|^2 + \|\mathbf{e}_h^i - \mathbf{e}_h^{i-1}\|^2 + \mu k\|\nabla_h \hat{\mathbf{e}}_h^i\|^2 + 2k a_h(q_r^i(\hat{\mathbf{e}}_h), \hat{\mathbf{e}}_h^i) + k^2\|\nabla_h \mathbf{e}_q^i\|^2$$
$$- k^2\|\nabla_h \mathbf{e}_p^{i-1}\|^2 \le \frac{k}{\mu}(\|\mathbf{e}_h^{i-1}\|^2 + \|\mathbf{U}^i - \mathbf{U}^{i-1}\|^2)\|\tilde{\Delta}_h \mathbf{U}^i\|^2. \quad (6.102)$$

Take sum from $i = 1$ to n to achieve

$$\|\mathbf{e}_h^n\|^2 + \sum_{i=1}^n \|\mathbf{e}_h^i - \mathbf{e}_h^{i-1}\|^2 + \mu k \sum_{i=1}^n \|\nabla_h \hat{\mathbf{e}}_h^i\|^2 + 2k \sum_{i=1}^n a_h(q_r^i(\hat{\mathbf{e}}_h), \hat{\mathbf{e}}_h^i) + k^2\|\nabla_h \mathbf{e}_p^{n-1}\|^2$$
$$\le \frac{1}{\mu} k \sum_{i=1}^{n-1} \|\tilde{\Delta}_h \mathbf{U}^{i+1}\|^2 \|\mathbf{e}_h^i\|^2 + \frac{1}{\mu} k^3 \sum_{i=1}^n \|\partial_t \mathbf{U}^i\|^2 \|\tilde{\Delta}_h \mathbf{U}^i\|^2. \quad (6.103)$$

Now forth term of the left of inequality (6.93) is positive due to (1.18), so we drop it. And the last term on the right of inequality can be bound using Lemma 6.9 as

$$\frac{1}{\mu} k^3 \sum_{i=1}^n \|\partial_t \mathbf{U}^i\|^2 \|\tilde{\Delta}_h \mathbf{U}^i\|^2 \le C k^2 t_n^{-1}.$$

Finally, we use the "discrete Gronwall's lemma" to conclude proof. □

Similarly, one can obtain the following bounds by choosing $\mathbf{v}_h = \tilde{\Delta}_h \hat{\mathbf{e}}_h^i$ in (6.94):

Lemma 6.12. *Suppose the assumption of Lemma 6.10 holds true. Then, for* $1 \le n \le N$*, the velocity and pressure errors satisfy the following results:*

$$\|\nabla_h \mathbf{e}_h^n\|^2 + \sum_{i=1}^n \|\nabla_h(\mathbf{e}_h^i - \mathbf{e}_h^{i-1})\|^2 + \mu k \sum_{i=1}^n \|\tilde{\Delta}_h \hat{\mathbf{e}}_h^i\|^2 + k^2\|\tilde{\Delta}_h \mathbf{e}_p^{n-1}\|^2 \le C e^{C t_n} k^2 t_n^{-2}.$$

Combining the Lemma 6.11 and 6.12 and Theorem 6.4, we finally derive our main result of this chapter.

Theorem 6.5. *Suppose the assumptions of Theorem 6.1 and 6.3 hold true. Then, for* $1 \le n \le N$*, the velocity errors satisfy the following results:*

$$\|\mathbf{u}(t_n) - \mathbf{U}_h^n\| \le K_n(h^2 + k)t_n^{-1/2}, \quad \|\nabla_h(\mathbf{u}(t_n) - \mathbf{U}_h^n)\| \le K_n(ht^{-1/2} + kt_n^{-1}).$$

where $K_n = Ce^{Ct_n}$.

6.5 Numerical Experiments

Here, we validate our theoretical findings by taking numerical examples, mainly the order of convergence of the errors. For simplicity, we will use examples with known solution.

We consider the Oldroyd model of order one subject to homogeneous Dirichlet boundary conditions. We approximate the equation using (P_1^{NC}, P_0) element over a regular triangulation of Ω. We take the domain $\Omega = [0,1] \times [0,1]$, which is partitioned into triangles with size $h = 2^{-i},\ i = 2, 3, \ldots, 6$. We discretize the time based on first-order schemes that is, the backward Euler method (BE) and Euler incremental pressure correction scheme (EIPC) with uniform time step k. To verify the theoretical result, we consider the Example 2.1 from Chapter 2 and perform the following numerical simulations.

Table 6.1: Numerical results for the BE method for $\mu = 1$

h	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P^n - p(t_n)\|_{L^2}$	C.R.
1/4	0.02101079	-	0.26681521	-	0.15528319	-
1/8	0.00600509	1.8069	0.14546382	0.8751	0.05462960	1.5071
1/16	0.00157418	1.9316	0.07493866	0.9569	0.01803537	1.5988
1/32	0.00039982	1.9772	0.03785948	0.9850	0.00589277	1.6138
1/64	0.00010088	1.9866	0.01899798	0.9948	0.00207394	1.5066

Table 6.2: Numerical results for the BE method for $\mu = 0.1$

h	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P^n - p(t_n)\|_{L^2}$	C.R.
1/4	0.19748704	-	2.52966400	-	0.17796012	-
1/8	0.05799317	1.7678	1.42219263	0.8308	0.06119446	1.5401
1/16	0.01535055	1.9176	0.73969247	0.9431	0.01913284	1.6773
1/32	0.00391219	1.9722	0.37467570	0.9813	0.00561911	1.7676
1/64	0.00098432	1.9908	0.18814111	0.9938	0.00159110	1.8203

Table 6.3: Numerical results for the BE method for $\mu = 0.01$

h	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P^n - p(t_n)\|_{L^2}$	C.R.
1/4	43.7280103	-	665.561000	-	360.273464	-
1/8	0.50447807	6.4376	12.8890702	5.6303	0.16034756	11.1336
1/16	0.14241880	1.8246	7.17361742	0.8454	0.04075008	1.9763
1/32	0.03714366	1.9390	3.71649406	0.9488	0.01113022	1.8723
1/64	0.00940476	1.9816	1.87740122	0.9852	0.00294092	1.9201

Table 6.4: Numerical results for EIPC method for $\mu = 1$

h	$\|\mathbf{U}_h^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}_h^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P_h^n - p(t_n)\|_{L^2}$	C.R.
1/4	0.14112123	-	1.78240100	-	1.65368748	-
1/8	0.07746573	0.8653	1.83341188	-0.0407	0.94276186	0.8107
1/16	0.02461565	1.6540	1.14241055	0.6824	0.34826242	1.4367
1/32	0.00696360	1.8217	0.64763701	0.8188	0.11621285	1.5834
1/64	0.00181868	1.9369	0.33998376	0.9297	0.03670366	1.6628

Table 6.5: Numerical results for EIPC method for $\mu = 0.1$

h	$\|\mathbf{U}_h^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}_h^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P_h^n - p(t_n)\|_{L^2}$	C.R.
1/4	0.50546203	-	6.46751795	-	0.56599962	-
1/8	0.15197188	1.7338	3.68881222	0.8100	0.21713558	1.3822
1/16	0.04354579	1.8032	2.07716343	0.8285	0.07276996	1.5771
1/32	0.01134962	1.9399	1.07936986	0.9444	0.02372306	1.6170
1/64	0.00287758	1.9787	0.54769765	0.9787	0.00829561	1.5158

Table 6.6: Numerical results for EIPC method for $\mu = 0.01$

h	$\|\mathbf{U}_h^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}_h^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P_h^n - p(t_n)\|_{L^2}$	C.R.
1/4	7.86482873	-	106.197266	-	1.96913018	-
1/8	0.62338984	3.6572	16.1745429	2.7150	0.18877357	3.3828
1/16	0.17164906	1.8607	8.70646794	0.8935	0.05205958	1.8584
1/32	0.04501354	1.9310	4.53027279	0.9425	0.01448709	1.8453
1/64	0.01141575	1.9793	2.29226523	0.9828	0.00401723	1.8505

Table 6.7: Comparison of CPU time of BE and EIPC method

h	BE (in second)	EIPC (in second)
1/4	0.895	0.194
1/8	1.847	0.569
1/16	11.4968	3.887
1/32	102.578	34.077
1/64	712.128	356.833

In Tables 6.1-6.3, we present the velocity and pressure errors and rates of convergence derived on successive meshes using (P_1^{NC}, P_0) element for BE scheme applied to the system (1.4)-(1.6) with $\mu = 1, 0.1, 0.01$ and $\gamma = 0.1, \delta = 0.1$ and time $t = [0, 1]$. Also, in Tables 6.4-6.6, we give the numerical errors and convergence rates for EIPC method applied to the system (1.4)-(1.6) with similar choice of parameters as above. The numerical results show that the rate of convergence for the velocity are 2 and 1 in $\mathbf{L}^2$ and $\mathbf{H}^1$-norms, respectively. And the rate of convergence is 1 for the pressure. For experiments, we choose the time step $k = \mathcal{O}(h^2)$ and time $T = 1$. All these numerical results support the theoretical findings. In Table 6.7, we give a comparison of the cpu time for the BE method and the EIPC method and we observe that EIPC method takes almost half time than BE method. The error graphs are presented in Figures 6.1 and 6.2.

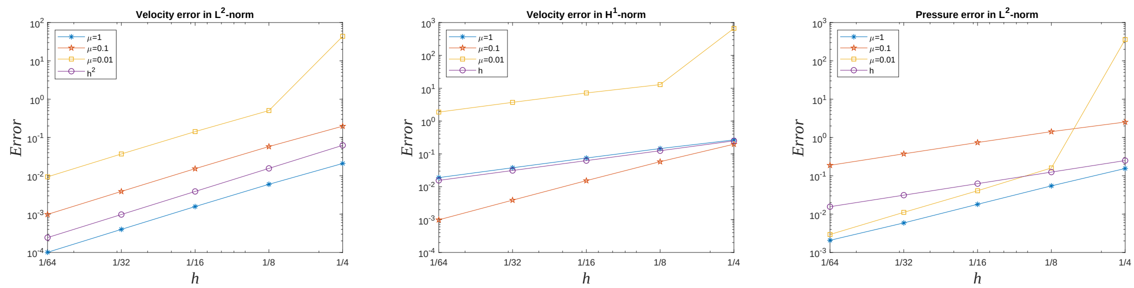

Figure 6.1: Velocity and pressure errors for BE method

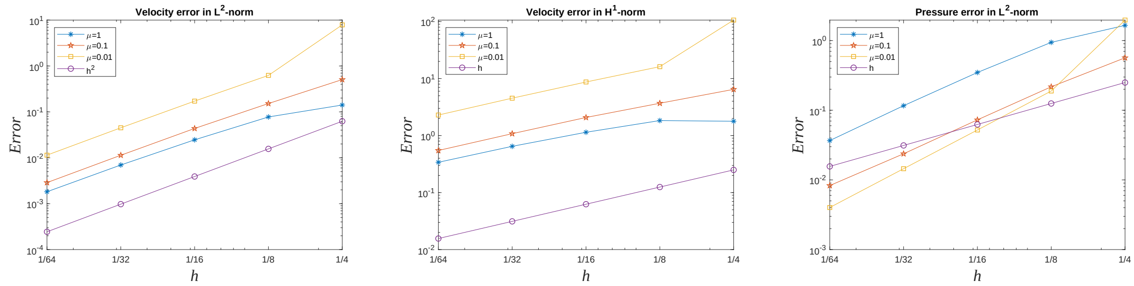

Figure 6.2: Velocity and pressure errors for EIPC method

6.6 Conclusion

In this chapter, we have analyzed the lowest order nonconforming finite element spaces for the velocity component whereas the space is discretized based on piecewise constant polynomial. We have established optimal velocity errors and have shown to be uniform in time under uniqueness condition. Then, based on backward Euler scheme and Euler incremental pressure correction scheme, we have analyzed the fully discrete error with nonsmooth initial data. Some numerical experiments have been performed to validate our theoretical findings.