Chapter 7

Conclusion and Future Work

In this chapter, we summarize the results derived in this thesis. Then we discuss possible extensions or new works in this direction.

7.1 Concluding Remarks

The main objective of this thesis has been to study and analyse a few finite element methods for the Oldroyd model of order one, and then to substantiate our findings by means of numerical computations. Since our model is a time-dependent problem, each finite element scheme further needs to be discretized in time for numerical simulation. We have therefore analysed the backward Euler method, a first-order time-discrete scheme at the very outset and then have gone on work on various finite element methods.

In Chapter 2, we have investigated the backward Euler method applying to the semidiscrete Galerkin approximation of the Oldroyd model of order one with non-smooth initial data, that is, $\mathbf{u}_0 \in \mathbf{H}_0^1$. We have employed the right rectangle rule to approximate the integral term and have preserved the positivity property of the memory term. The consistent scheme has been shown to be stable by obtaining new uniform a priori bounds for the fully discrete solution $\mathbf{U}^n,\ 1 \leq n \leq N$.

$$\Big(\|\mathbf{U}^n\|^2 + \mu e^{-2\alpha t_n} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\nabla \mathbf{U}^i\|^2\Big) + \|\nabla \mathbf{U}^n\|^2 + \tau^*(t_n)\|\tilde{\Delta}_h \mathbf{U}^n\|^2 \leq C,$$

where k is the time step and C is a positive constant depending on the given data but not on time; that is, bound is uniform in time.

We then have derived the following optimal error estimates for the velocity

$$\|\mathbf{u}_h(t_n) - \mathbf{U}^n\| \leq Ce^{Ct_n} t_n^{-1/2} k, \quad 1 \leq n \leq N < +\infty,$$

where $\mathbf{u}_h$ is the semidiscrete solution and $\mathbf{U}^n$ is the fully discrete solution. This has improved the existing results in the following sense: (i) this is for less regular initial data (ii) improved order of convergence, by an order of $\left(\log \frac{1}{k}\right)^{3/4}$. Under the uniqueness condition, we have shown this error is uniform in time. The analysis has been done for the nonsmooth initial data, and the proofs are more involved than the smooth case. Our numerical results conform to our theoretical results, especially the rate of convergence in both space and temporal directions with smooth and nonsmooth initial data. We have also verified the unconditional stability of the scheme and uniform in time bounds numerically.

A three-step two-grid finite element method is applied to the Oldroyd model of order one in Chapter 3. We solve a nonlinear problem in a coarse grid in the first step. Then, using this coarse grid solution, we linearize the problem and solve it in a fine grid by using one Newton iteration in the second step. In the third step, we correct the solution on the fine grid. Optimal error estimates for the velocity and the pressure have been obtained in the final step as

$$\begin{aligned}
\|(\mathbf{u} - \mathbf{u}_h)(t)\| &\leq K(t)(h^2 t^{-1/2} + H^{4-\ell} t^{-1}), \\
\|\nabla(\mathbf{u} - \mathbf{u}_h)(t)\| &\leq K(t)(h t^{-1/2} + H^{3-\ell} t^{-1}), \\
\|(p - p_h)\|_{L^2/N_h} &\leq K(t)(h t^{-1/2} + H^{3-\ell} t^{-1}),
\end{aligned}$$

where $\ell > 0$ is arbitrary small and $K(t) = Ce^{Ct}$. $(\mathbf{u}, p)$ and $(\mathbf{u}_h, p_h)$ are the continuous solution and the semidiscrete solution in step 3, respectively. We have proved that the largest scaling between the coarse mesh size H and fine mesh size h is $h = \mathcal{O}(H^{2-\ell})$ for optimal error estimates for velocity in $L^\infty(\mathbf{L}^2)$-norm and it is $h = \mathcal{O}(H^{3-\ell})$ for velocity in $L^\infty(\mathbf{H}^1)$-norm and for pressure in $L^\infty(\mathbf{L}^2)$-norm, for arbitrary small $\ell > 0$.

In the second part of Chapter 3, we have used backward Euler method for temporal discretization to the three step two-grid finite element approximations. We have obtained *a priori* estimates for the fully discrete solutions and have derived the following error estimates for the velocity and the pressure

$$\|\mathbf{u}_h(t_n) - \mathbf{U}_h^n\| \leq K_n k (1 + \log \frac{1}{k})^{\frac{1}{2}} t_n^{-1/2},$$

$$\|\nabla(\mathbf{u}_h(t_n) - \mathbf{U}_h^n)\| + \|p_h(t_n) - P_h^n)\| \leq K_n k^{\frac{1}{2}} (1 + \log \frac{1}{k})^{\frac{1}{2}} t_n^{-1/2},$$

where $K_n = Ce^{Ct_n}$ and $(\mathbf{u}_h, p_h)$ and $(\mathbf{U}_h^n, P_h^n)$ are the solutions of the semidiscrete and fully discrete cases, respectively. Moreover, under the uniqueness condition, the above estimates are uniform in time, that is, $K_n = C$. We have provided some numerical examples that validate our theoretical results. We have compared the CPU time for a standard Galerkin solution and two-grid solution, which indicates that the two-grid solution reduce the computation time by around 50%. Hence, it is a time-efficient and effective method.

In Chapter 4, a penalized Oldroyd model of order one has been analyzed for non-smooth initial data, that is, $\mathbf{u}_{\varepsilon 0} \in \mathbf{H}_0^1$. Based on penalized Stokes operator and appropriate application of weighted time estimates with the positivity of the memory term, new regularity results have been established for the penalized problem, which are valid uniformly in time as $t \to \infty$ and in penalty parameter ε as it tends to zero. It is then followed by the semidiscrete analysis of the model based on the conforming finite element method. With the help of discrete penalty Stokes operator and uniform Gronwall's Lemma, uniform in time bound for the discrete velocity in the Dirichlet norm is derived. Subsequently, the following optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$-norm as well as $L^\infty(\mathbf{H}^1)$-norm and for the pressure in $L^\infty(L^2)$-norm have been established

$$\|(\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h})(t)\| + h\|\nabla(\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h})(t)\| + h\|(p_\varepsilon - p_{\varepsilon h})(t)\| \leq K(t) h^2 t^{-\frac{1}{2}},$$

where $(\mathbf{u}_\varepsilon, p_\varepsilon)$ and $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h})$ are the solution of penalized system and semidiscrte penalized system, respectively. These estimates are valid uniformly with respect to the penalty parameter as it goes to zero and in time under the smallness conditions on given data. Our analysis relies on the application of the inverse penalized Stokes operator with its discrete version, the penalized Stokes-Volterra projection, weighted time estimates, and positivity of the memory term. Then, we have applied the backward Euler method to the semidiscrete penalized system and have derived *a priori* bounds for the fully discrete penalized solution in $\mathbf{L}^2$ and Dirichlet norm. The following optimal error estimates have been established for the velocity and pressure:

$$\|\mathbf{u}_{\varepsilon h}(t_n) - \mathbf{U}_\varepsilon^n\| \leq K_n k t_n^{-1/2},$$

$$\|\nabla(\mathbf{u}_{\varepsilon h}(t_n) - \mathbf{U}_\varepsilon^n)\| + \|p_{\varepsilon h}(t_n) - P_\varepsilon^n\| \leq K_n k t_n^{-1},$$

where $(\mathbf{U}_\varepsilon^n, P_\varepsilon^n)$ is the solution of the fully discrete penalized system. Under the uniqueness condition, these bounds are shown uniform in time. We have presented some numerical simulations to validate our theoretical findings, mainly the convergence rate for nonsmooth initial data. Moreover, several numerical experiments are conducted on benchmark problems and for various small values of μ and γ.

In Chapter 5, we have considered an inf-sup stable mixed finite element method for the Oldroyd model of order one with grad-div stabilization. We have first investigated the stability of the discrete solution, observing that the scheme is valid only for a finite time when the discrete solution does not grow with high Reynold's number. We have obtained the error estimates for the velocity and the pressure in the semidiscrete case as well as in the fully discrete case with the error bounds independent of the inverse power of μ. We have carried out our analysis for both sufficiently smooth initial data ($\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^m,\ m > 2$) and smooth initial data ($\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^2$). The following error bounds have been obtained

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^{n} e^{2\alpha t_i} \|p(t_n) - P^n\|^2 \leq K_n(h^{2m} + k^2),$$

where m is the degree of the polynomial approximate the velocity space. We have verified our theoretical results by performing some numerical simulations. The numerical experiments have been presented for very small values of μ which confirm that the error bounds constant independent of the inverse power of μ. We also have conducted a few numerical simulations to find a suitable choice of grad-div parameter for the Oldroyd model of order one.

We have applied the nonconforming finite element method to the Oldroyd model of order one in Chapter 6. To discretize the velocity space, we have used $P1$ nonconforming finite element space, and for the pressure space, we have used piecewise constant space. The following optimal error estimates have been derived for nonsmooth initial data:

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| + h\|\nabla_h(\mathbf{u} - \mathbf{u}_h)(t)\| \leq K(t) h^2 t^{-\frac{1}{2}},$$

where ∇_h is the discrete gradient operator. Under uniqueness condition, these estimates are shown uniformly in time.

Then, an incremental pressure correction scheme has been applied to the Oldroyd model of order one to discretize in the temporal direction. For the space discretization, we have considered above mentioned nonconforming finite element spaces. Stability analysis of the scheme has also been performed. We have derived optimal error bounds for fully discrete approximation with nonsmooth initial data. We have considered a few numerical examples for the numerical validation and have presented some results that support our theoretical findings.

7.2 Future Plan

In future, we would continue working on the Oldroyd model of order one and then would like to move on to coupled problems and to other related problems.

For higher-order finite element approximations, it is desirable to use higher-order time schemes. We therefore plan to employ a couple of time discretizations of second-order to the Oldroyd model order one. Also combinations of explicit and implicit time schemes are often more effective than the individual scheme itself and we plan to study a few of them for our model. We will carry out stability and convergence analysis for these time schemes for nonsmooth data. Stability of various first-order scheme has been carried out in [150] and the Crank-Nicolson/Adams-Bashforth, a second order time discrete scheme, has been applied for our model in [69]. However the first work does not address the second order schemes and the second work is limited to a specific scheme. Also these work do not take into account less regular data.

The combination of incompressible flow and porous media flow is getting popular in the present day [32, 43, 93]. Such complicated phenomena have practical applications in different areas: Geosciences (modeling river-groundwater interactions, simulating the effect of flooding in dry areas), health sciences (filtration of blood via vessel walls, modeling blood flow and organs), industry (petroleum extraction, air or oil filters), and so on. Predicting how pollution dumped into rivers, lakes, and streams makes its way into the water supply is one of these problems. The coupling is also crucial in filtration-related technical applications. In order to simulate such physical phenomena, various systems of partial differential equations must be considered in each subregion of the domain of interest. The Navier-Stokes equations (or Stokes equations) in the fluid region are typically coupled across an interface with the Darcy equations for the filtration velocity in the porous media to provide a model that is most accessible to large-scale computations. As a result, the coupled system of equations of various orders in a different region causes mathematical complications.

For literature involving the coupled Stokes/Darcy model, we refer to [52, 58, 93, 103, 118, 119, 123] and for the steady-state coupled Navier-Stokes/Darcy equations, we refer to [23, 32, 33, 44–46, 57], and references therein. The unsteady case, being of interest, we note that it has been studied in the following context: existence and uniqueness of time-dependent problem [29], finite element methods [27], discontinuous Galerkin finite element methods [28, 30], partitioned time-stepping method [84], characteristic finite element methods [26, 85], mortar finite element methods [31], modular grad-div method [135], incremental pressure correction method [143], to name a few. There are several other schemes such as two-grid or multi-grid method, projection method, penalty method, stabilized method, etc. that can be analysed for the time-dependent Navier-Stokes/Darcy coupled model. In [23, 83, 154] a two-grid or multi-grid method have been applied for the steady Navier-Stokes/Darcy model, but there are no results for the unsteady Navier-Stokes/Darcy problem. Since there are a few basic works on unsteady problem using discontinuous Galerkin method [28, 30] and modular grad-div method [135], hence, there is a huge scope for analyzing this model in this direction. Recently, a second-order incremental pressure correction method for solving the Navier-Stokes/Darcy equation can be seen in [143]. But there are none on velocity correction or splitting scheme or non-incremental pressure correction scheme.

Apart from Navier-Stokes/Darcy, one can find Cahn-Hilliard/Navier-Stokes model, Cahn-Hilliard/ Navier-Stokes/Darcy model, Navier-Stokes/Forchheimer model which are very popular in these present days and we plan to look at them as well.

Furthermore, we would like to look into stabilized schemes, namely, pressure robust mixed finite element methods which are of interest in the research community these days. In case of standard mixed finite element method, the velocity error depends on the continuous pressure with coefficient $1/\nu$ (ν be the coefficient of viscosity) and the divergence constraints are not robust against large irrotational forces in the momentum balance. But, in pressure robust method, it is possible to find the velocity error which is pressure independent. In recent years, there are several works have been found in the literature based on pressure robust scheme for the time-dependent incompressible Navier-Stokes equations like $H(div)$-conforming discontinuous Galerkin method [72, 149], divergence-free reconstruction for Taylor-Hood elements and high order hybridizable discontinuous Galerkin methods [94], inf-sup stable FEM [122], divergence-free FEM [121] and so on. This motivates us to analyse this robust scheme for a possible study in combination with two-grid/multi-grid method, different stabilized methods like grad-div stabilization, pressure stabilized scheme, equal order stabilized scheme, (local) discontinuous Galerkin method and so on, for NSE and Oldroyd model.