ABSTRACT

This thesis analyzes a few finite element methods for the equations of motion arising in the two-dimensional Oldroyd model of order one; a model that represents linear viscoelastic fluid flows and that can be viewed as an integral perturbation of Navier-Stokes equations. Each method is analyzed in both semi and fully discrete set-ups, and therefore it is imperative for us to analyse a time-discrete scheme at the very outset.

We study a first-order time-discrete scheme, namely, the backward Euler method, in conjunction with the Galerkin finite element method when the forcing term is independent of time or is in L^∞ in time. We obtain the fully discrete solution in the Dirichlet norm, bounded uniformly in time. An optimal error estimate in $\mathbf{L}^2$-norm for the velocity is derived. Under the assumption of uniqueness condition, this estimate is shown to be uniform in time. We present some numerical experiments to validate our theoretical findings, mainly the convergence rate and uniform stability with respect to time.

We next employ a three-step two-grid method for the Oldroyd model of order one. This method involves a three-step numerical scheme. In the first step, the problem is solved on a coarse grid, and we use this coarse grid solution to linearize the problem and solve it in the second step, on a finer grid. The third step is a correcting step done on the finer grid. Optimal error estimate for the velocity in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}^1)$-norms and for the pressure in $L^\infty(L^2)$-norm in the semidiscrete case are established. Estimates are shown to be uniform under the uniqueness assumption. Then, based on the backward Euler method, a completely discrete scheme is analyzed, and optimal *a priori* error estimates are derived. Finally, we present some numerical results to validate our theoretical results; showing that the three-step two-grid method is efficient than solving a nonlinear problem directly, as is expected.

Then, a penalty formulation is proposed and analyzed in both continuous and discrete setups for the two-dimensional Oldroyd model of order one. New regularity results, which are valid uniformly in time as $t \to \infty$ and the penalty parameter ε as $\varepsilon \to 0$, are derived for the solution of the penalized problem. Next, a semidiscrete penalized problem is analyzed based on the conforming finite element method and keeping the temporal variable continuous. We obtain a uniform in time *a priori* bound of the discrete velocity in Dirichlet norm with the help of penalized discrete Stokes operator and a modified uniform Gronwall's Lemma. Further, optimal error estimates for the penalized velocity in $L^\infty(\mathbf{L}^2)$ as well in $L^\infty(\mathbf{H}^1)$-norms and for the penalized pressure in $L^\infty(L^2)$-norm have been established for the semidiscrete problem.

These error estimates are uniform in time under the uniqueness assumption and uniform in the penalty parameter as it goes to zero. Then, a completely discrete scheme is analyzed where the time is discretized based on the backward Euler method. *A priori* bounds for the fully discrete solution and optimal error estimates for the velocity and the pressure are derived. Our analysis relies on the appropriate use of inverse the penalized Stokes operator, penalized Stokes-Volterra projection, and judicious application of weighted time estimates with positivity property of the memory term. A few numerical experiments are conducted on benchmark problems which confirm our theoretical findings.

Next, an inf-sup stable Galerkin mixed finite element method, for the Oldroyd model of order one, with a grad-div stabilization, is discussed, and an error analysis is carried out. Optimal error estimates for the velocity and the pressure are established in the semidiscrete case. Then, based on the backward Euler method, a completely discrete scheme is analyzed, and optimal error estimates are derived. This grad-div stabilization scheme, which adds a stabilization term to the momentum equation, is known to be suitable for high Reynolds number. In conformation with this, the error estimates, in both semi and fully discrete cases, are obtained with constants independent of the inverse power of viscosity. We present some numerical results to substantiate our theoretical results. Furthermore, we give a few numerical examples to find a suitable choice of grad-div parameter for the Oldroyd model of order one.

Finally, we turn our attention to the nonconforming finite element method for the Oldroyd model of order one. We approximate the velocity space by a simple $P1$ nonconforming finite element and the pressure space by a piecewise constant polynomial. Optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$ as well as $L^\infty(\mathbf{H}^1)$-norms and the pressure in $L^\infty(L^2)$-norm are established. We then consider an incremental pressure correction scheme to discretize the time variable. With nonconforming setup, we obtain optimal error estimates for the velocity and the pressure. We give some numerical examples to validate our theoretical findings.

All these methods, except the grad-div, are analysed for nonsmooth initial data, that is, the initial velocity $\mathbf{u}_0$ in $\mathbf{H}_0^1$ which we feel is more realistic and technically challenging. In case of grad-div stabilization, nonsmooth initial data leads to suboptimal results that are independent of inverse power of viscosity. Hence smooth ($\mathbf{u}_0$ in $\mathbf{H}_0^1 \cap \mathbf{H}^2$) and sufficiently smooth ($\mathbf{u}_0$ in $\mathbf{H}_0^1 \cap \mathbf{H}^m,\ m > 2$) initial data are considered. We conclude the thesis with a brief summary of our results and a brief discussion of our future problems.