Chapter 1

Introduction

In the realm of fluid mechanics, the study of viscous fluids is revolved around the famous incompressible Navier-Stokes equations (NSEs) for more than a century now. The system comprises of partial differential equations with prescribed boundary values and in non-stationary case, with an initial value, and it represents fluid flows that are governed by the Newton's law of viscosity. The incompressible NSEs has been studied extensively both in pure and applied fields for its wide range of applications in engineering and scientific problems. However, a smooth solution of $3D$ NSEs still eludes us and is part of the Millennium prize problem [50].

There have been several works devoted to the solution of the incompressible Navier-Stokes system, starting with the work of Leray in 1934 [95, 96]. We want to bring to notice the celebrated works of Ladyzhenskaya [90], where she has proved the uniqueness for a large time without any restriction on the smallness of the given data or the domain in two-dimensional case. Also, in a three-dimensional case, a unique solution has been shown for all time under the condition that the given forces are derivable from potential and the Reynold number is less than one at the initial time. If conditions are not met, at least one solution exists, but it may not be unique. Uniqueness is proved only for smooth initial data for a specific time interval. For more details, see [90].

However, the unique solvability of the three-dimensional boundary value problem was still out of reach for a large time. And therefore, Ladyzhenskaya and her group resorted to the study of some new regularized models of the NSEs, based on the idea that the solvability of these models may lead to the solvability of the NSEs. Extending this idea, her pupils, namely, Oskolkov, Kotsiolis, Karzeeva, Sobolevskii, etc., have worked on a few generalized problems. For example, in [111], Oskolkov has considered a generalized problem where the solution approaches the solution of NSEs but for smallness condition on the data. Furthermore, at the same time, it has been recognized that these regularized models describe non-Newtonian fluids, fluids that are no longer governed only by Newton's law of viscosity. Again, in the work [110], Oskolkov has stated that the model considered there describes the laminar motion of aqueous solutions of polymers. The molecular interactions have been emphasized. And it has been realized that these new regularized equations are, in fact, results of some linearized rheological equations, equations that identify the underlying fluid. One such model that first appeared in the work of Oldroyd [106] and has been studied by Oskolkov *et al.* [110, 111] is a model that incorporates an integral term to NSEs, a memory term to take into account the elastic property of an otherwise viscous fluid, and is known as the Oldroyd model of order $L,\ L > 0$.

In this thesis, we will consider the model for $L = 1$ as the analysis for the general case does not differ by much. And we will call it as Oldroyd model of order one. This linear viscoelastic model represents a basic model for polymeric fluids such as molten plastic, engine oils, paints, gels, ointment, etc. and biological fluids such as egg white and blood. It is a non-Newtonian model which has been derived under the assumptions that the material can be regarded as a single stationary macroscopic element with small stress and strain rates, and finds applications in various industries, like, paints, DNA suspensions, biological fluids and some chemical industries.
We present a brief discussion about the model in our next section, that is Section 1.1.

1.1 Oldroyd Model of Order One

The incompressible fluid flow in a bounded domain Ω in $\mathbb{R}^n,\ n = 2, 3$ is represented by the following system of differential equations:

$$\frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot \boldsymbol{\sigma} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}(x, t), \quad \nabla \cdot \mathbf{u} = 0, \quad \text{for } t > 0 \tag{1.1}$$

with appropriate initial and boundary conditions. Here, $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ik})$ denotes the *deviator of the stress tensor* (also called the *extra-stress tensor*) with $tr\ \boldsymbol{\sigma} = 0$, $\mathbf{u} = (u_1, u_2)$ (or $\mathbf{u} = (u_1, u_2, u_3)$) represents the velocity vector, p is the pressure of the fluid and $\mathbf{f}$ is the external force. A *rheological equation* defined via $\boldsymbol{\sigma}$ and $\mathbf{D}$ generally describes the underlying fluid where $\mathbf{D}$ is the tensor of deformation velocities and is defined as

$$\mathbf{D} = (\mathbf{D}_{ik}) = \frac{1}{2}(\mathbf{u}_{ix_k} + \mathbf{u}_{kx_i}).$$

For example, if we consider the *rheological equation* $\boldsymbol{\sigma} = 2\nu\mathbf{D}$, then we obtain the well known Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}(x,t), \quad \nabla\cdot\mathbf{u} = 0.$$

Here $\nu > 0$ is the kinematic coefficient of viscosity. However, in the case of non-Newtonian fluids, this relation needs to take into consideration the molecular interactions. For example, in polymer fluids, the presence of a long chain of molecules changes the fluid dynamics. Keeping these things in mind, several new *rheological equations* have been defined, incorporating the history of the fluids and are known to describe linear viscoelastic fluids, or more precisely diluted polymeric solutions. One such relation, describing a class of non-Newtonian fluids is given by [47, 87, 145]

$$\left(1 + \sum_{l=1}^{L} \lambda_l \frac{\partial^l}{\partial t^l}\right)\boldsymbol{\sigma} = 2\nu\left(1 + \sum_{m=1}^{M} \kappa_m \nu^{-1} \frac{\partial^m}{\partial t^m}\right)\mathbf{D},$$

which represents linear viscoelastic fluids, namely, aqueous polymer solutions with discrete modes of relaxation time $\{\lambda_l\},\ l = 1, 2, \ldots, L$ and with retardation times (delay times) $\{\kappa_m\nu^{-1}\},\ m = 1, 2, \ldots, M$.

In its simplistic version, that is, in the case of single-mode, the above equation gives rise to three fluid models. For $L = 1, M = 0$, we recover the Maxwell fluid, in which case, the stress does not vanish immediately; it decreases like $e^{-\lambda_1^{-1}t}$ after the termination of the motion (λ_1 is called the relaxation time). Moreover, for $L = 0, M = 1$, we obtain the Kelvin-Voigt fluid. This type of fluid is characterized by an exponential condensation of delay deformation with the rate $e^{-\kappa_1^{-1}\nu t}$ after removal of the stress ($\kappa_1^{-1}\nu$ is called the retardation time). When $L = 1$ and $M = 1$, the resulting fluid is known as Oldroyd fluid which exhibits both the characteristics, that is, after instantaneous cessation of motion, the stresses in the fluid do not vanish immediately, but die out like $e^{-\lambda_1^{-1}t}$, and after instantaneous removal of stresses, the velocity of the fluid does not vanish immediately, but dies out like $e^{-\kappa_1^{-1}\nu t}$.

In general, when $M = L$ in the above *rheological equation*, we find a linear viscoelastic fluid model that is called the Oldroyd model of order L. For $L = 1$, it is called the Oldroyd model of order one. This model was first proposed and developed by Oldroyd (see [106]) in the mid-twentieth century. It is based on the following defining *rheological equation* $(M = L = 1)$

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\boldsymbol{\sigma} = 2\nu \left(1 + \kappa\nu^{-1}\frac{\partial}{\partial t}\right)\mathbf{D}, \tag{1.2}$$

where $\nu > 0$ is the kinematic coefficient of viscosity, $\lambda > 0$ is the relaxation time and $\kappa\nu^{-1} > 0$ is the retardation time with $\nu - \kappa\lambda^{-1} > 0$. Model was later developed to take into account a discretely distributed relaxation and retardation times and thus developed it for various order with *rheological equation*

$$\left(1 + \sum_{l=1}^{L} \lambda_l \frac{\partial^l}{\partial t^l}\right)\boldsymbol{\sigma} = 2\nu \left(1 + \sum_{l=1}^{L} \kappa_l \nu^{-1}\frac{\partial^l}{\partial t^l}\right)\mathbf{D},$$

where $L = 1, 2, 3, \ldots$. The analysis for the general case does not differ much to that of the $L = 1$ case, and hence, we propose to analyse the model for order one $(L = 1)$ only.

As mentioned earlier, in this dissertation, we consider the Oldroyd model of order one, that is, $L = 1$. From the *rheological equation* (1.2), one can find

$$\boldsymbol{\sigma} = 2\mu\mathbf{D} + 2\int_0^t \beta(t-s)\mathbf{D}(s)ds, \tag{1.3}$$

where $\mu = \kappa\lambda^{-1} > 0$, the kernel $\beta(t) = \gamma\exp(-\delta t)$, $\gamma = \lambda^{-1}(\nu - \kappa\lambda^{-1}) > 0$, $\delta = \lambda^{-1} > 0$, and $\nu > 0$ is the kinematic coefficient of viscosity, $\lambda > 0$ is the relaxation time and $\kappa > 0$ is the retardation time. Combining (1.1) and (1.3), we find the following equation of motion arising for the Oldroyd model of order one as:

$$\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \int_0^t \beta(t-s)\Delta\mathbf{u}(s)\,ds + \nabla p = \mathbf{f}, \qquad \text{in } \Omega,\ t > 0 \tag{1.4}$$

with incompressibility condition

$$\nabla\cdot\mathbf{u} = 0, \qquad \text{on } \Omega,\ t > 0, \tag{1.5}$$

and initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \ \text{ in } \Omega, \quad \mathbf{u} = \mathbf{0}, \ \text{ on } \partial\Omega,\ t \geq 0. \tag{1.6}$$

Here, Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Unknowns $\mathbf{u}$ and p represent the velocity and the pressure of the fluid, respectively. Further, the forcing term $\mathbf{f}$ and initial velocity $\mathbf{u}_0$ are given functions in their respective domains of definition. For more details on the model, we refer to [87, 106].

We would like to end this section by reminding ourselves that the Oldroyd model of order one was conceived and studied as a regularized model of NSEs and is an integral perturbation of NSEs where the integral term takes into account the memory effects of the viscoelastic fluid. When the parameter $\gamma = 0$, the system reduces to the well-known Navier-Stokes flows and as such numerical schemes and related results for both the models should be comparable. This motivates us to work on the Oldroyd model whenever satisfactory results are available for the NSEs and motivate us further if none exists for either of the model.

There are practical motivations, the model being a linear viscoelastic fluid flow model. However as a system of differential equations, it comes with own set of problems. Problem (1.4)-(1.6) represents a system of nonlinear partial differential equations which are always difficult to solve. In general, nonlinearity is handled by means of successive approximation like Newton's iterative method which work very well under restrictive environment; however many a times, they are not suitable for complex problems, since they are time consuming and hence computationally inefficient. Also the problem (1.4)-(1.6) is a coupled one; there is coupling of the velocity and the pressure by the incompressibility condition $div\ u = 0$ along with the momentum equation. This forces us to employ the mixed finite element methods imposing restrictions on the finite element spaces. The finite dimensional spaces that approximate velocity and pressure, need to satisfy the discrete inf-sup (LBB) condition. In general, the equal order (like (P_1, P_1)) or lower order (like (P_1, P_0)) finite element spaces do not satisfy the discrete inf-sup condition. Hence we can not use these conforming spaces for stable numerical approximation although being lower order they are easy to implement and cost effective. Another difficulty arises when the co-efficient of viscosity, in our case, value of μ, becomes very small. The difficulty is due to the domination of the nonlinear convection term on the viscous term, which typically arises for small values of μ, and this plays a very important role in modelling turbulence.

Based on the difficulties identified above, in this thesis, we have analyzed our problem, the Oldroyd model of order one (1.4)-(1.6), with the help of various appropriate finite element methods, such as, two-grid method, penalty method, grad-div stabilization and nonconforming FEM. We have also performed numerical experiments to corroborate our theoretical findings.

Since for numerical computations, it is essential to study the time discretization schemes, we begin our study by analysing a first-order backward Euler method, an implicit but unconditionally stable scheme.

In our next section, we familiarize ourselves with the notations and preliminaries essential for our analysis.

1.2 Notations and Preliminaries

We begin this section by introducing the standard functional spaces and then the standard inequalities and a couple of versions of Gronwall's lemma, each of which appears very frequently in our analysis. We also briefly look at the variational/weak formulations of the continuous, semidiscrete and the fully discrete case and present the related notations, assumptions and results as we go along.

Let $\Omega \in \mathbb{R}^2$ be a bounded and convex polygonal domain with boundary $\partial\Omega$. For $1 \leq p < \infty$, $L^p(\Omega)$ denote the linear space of equivalence classes of measurable functions ϕ on Ω such that $\int_\Omega |\phi(x)|^p dx < \infty$ and associated norm define as

$$\|\phi\|_{L^p(\Omega)} = \left(\int_\Omega |\phi(x)|^p dx\right)^{1/p}.$$

For $p = \infty$, $L^\infty(\Omega)$ consists of measurable functions ϕ such that $\operatorname{ess\,sup}_{x\in\Omega} |\phi(x)| < \infty$ and associated norm define as

$$\|\phi\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |\phi(x)|.$$

Note that for $p = 2$, $L^2(\Omega)$ is a Hilbert space. Our analysis relies on this space and its closed subspaces and on the following quotient space (and its subspace)

$$L^2(\Omega)/\mathbb{R} = \{\phi \in L^2(\Omega) : \int_\Omega \phi(x)\ dx = 0\}.$$

For m, a non-negative integer and p such that $1 \leq p \leq \infty$, the Sobolev space of order (m, p) on Ω, denoted by $W^{m,p}(\Omega)$, is defined as a linear space of functions in $L^p(\Omega)$ whose distributional derivatives of order $\leq m$ are also in $L^p(\Omega)$, that is,

$$W^{m,p}(\Omega) := \big\{\phi \in L^p(\Omega) : D^\alpha\phi \in L^p(\Omega),\ 0 \leq |\alpha| \leq m\big\},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a 2-tuple with non-negative integer components and its order is defined by $|\alpha| = \alpha_1 + \alpha_2$. $D^\alpha \phi$ is the α-th derivative of $\phi(x)$ with $x = (x_1, x_2)$ defined by

$$D^\alpha \phi = \frac{D^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}.$$

The associated norm of the space $W^{m,p}(\Omega)$ is defined for $1 \leq p < \infty$ as

$$\|\phi\|_{m,p} = \left(\sum_{0\leq|\alpha|\leq m} \int_\Omega |D^\alpha \phi(x)|^p dx \right)^{1/p} = \left(\sum_{0\leq|\alpha|\leq m} \|D^\alpha \phi\|^p_{L^p(\Omega)} \right)^{1/p}.$$

When $p = \infty$, the norm on space $W^{m,\infty}(\Omega)$ is defined as

$$\|\phi\|_{m,\infty} = \max_{0\leq|\alpha|\leq m} \|D^\alpha \phi\|_{L^\infty(\Omega)}.$$

For $p = 2$, $W^{m,2}(\Omega)$ will be a Hilbert space and denoted by $H^m(\Omega)$ and its associated norm is denoted by $\|\cdot\|_{m,2}$ (for simplicity, we write $\|\cdot\|_m$). The natural inner product on the space $H^m(\Omega)$ is defined for all $\phi, \psi \in H^m(\Omega)$ by

$$(\phi, \psi) = \sum_{0\leq|\alpha|\leq m} \int_\Omega D^\alpha \phi(x) D^\alpha \psi(x) \; dx.$$

The closure of $C_c^\infty(\Omega)$, the space of infinitely differentiable functions with compact support, in $H^m(\Omega)$ is denoted by $H_0^m(\Omega)$, that is, $H_0^m(\Omega)$ is a subspace of $H^m(\Omega)$ with elements vanishing on boundary in the sense of trace [3]. The dual space of $H^m(\Omega)$ is defined as the completion of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\|\phi\|_{-m} := \sup \left\{ \frac{\langle \phi, \psi \rangle}{\|\psi\|_m} : \psi \in H^m(\Omega), \|\psi\|_m \neq 0 \right\},$$

and it is denoted by $H^{-m}(\Omega)$.

For our subsequent analysis we denote the $\mathbb{R}^2$-valued function spaces by bold face letters such as

$$\mathbf{H}^m = [H^m(\Omega)]^2, \quad \mathbf{H}_0^1 = [H_0^1(\Omega)]^2, \quad \text{and} \quad \mathbf{L}^2 = [L^2(\Omega)]^2.$$

The norm on the space $\mathbf{H}_0^1$ is defined as

$$\|\nabla \mathbf{w}\| = \left(\sum_{i,j=1}^{2} (\partial_j w_i, \partial_j w_i) \right)^{1/2} = \left(\sum_{i=1}^{2} (\nabla w_i, \nabla w_i) \right)^{1/2}.$$

Also for a given Banach space X with norm $\|\cdot\|_X$, let $L^p(0,T;X)$ be a space of all strongly measurable and p-th integrable X-valued functions $\psi : [0,T] \to X$ satisfying

$$\int_0^T \|\psi(t)\|_X^p dt < \infty, \quad 1 \leq p < \infty, \quad \text{and} \quad \operatorname*{ess\,sup}_{t\in[0,T]} \|\psi(t)\|_X < \infty, \quad p = \infty.$$

The norms for these spaces are defined as

$$\|\psi\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|\psi(t)\|_X^p dt \right)^{1/p} < \infty, & 1 \le p < \infty, \\ \text{ess sup}_{t\in[0,T]} \|\psi(t)\|_X < \infty, & p = \infty. \end{cases}$$

There are other spaces that are useful for our analysis as well. For example, we consider divergence free subspaces of usual solution spaces, since the fluid under consideration is incompressible, that is, the velocity vector is divergence free. We note here that the use of these spaces is limited to the analysis only and has not been considered for numerical computations. We now introduce below, the divergence free function spaces:

$$\mathbf{J} = \{\mathbf{v} \in \mathbf{L}^2 : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} \cdot \hat{\mathbf{n}}|_{\partial\Omega} = 0 \ \text{ holds weakly}\},$$
$$\mathbf{J}_1 = \{\mathbf{v} \in \mathbf{H}_0^1 : \nabla \cdot \mathbf{v} = 0\},$$

where $\hat{\mathbf{n}}$ is the outward normal vector to the boundary $\partial\Omega$ and $\mathbf{v} \cdot \hat{\mathbf{n}} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$ on the boundary $\partial\Omega$, see [130]. Also we use quotient spaces, since pressure is unique only up to a constant. Let $H^m/\mathbb{R}$ be the quotient space consisting of equivalence classes of elements of H^m differing by constants and the associated norm is defined by $\|\cdot\|_{m/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|\cdot + c\|_m$, see [81]. The underlying domain plays a crucial role, when we discuss about various spaces and their properties, namely, the trace inequalities. Therefore it is important to have some sort of smoothness of the domain. We make the following assumption on domain Ω: (see [79] for details)

(**A1**) Let $(\mathbf{v}, q) \in \mathbf{J}_1 \times L^2(\Omega)/\mathbb{R}$ be the unique solution of the steady state Stokes problem

$$-\Delta\mathbf{v} + \nabla q = \mathbf{g},\ \nabla \cdot \mathbf{v} = 0 \ in\ \Omega \ \text{ and } \ \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega,$$

Then, for $\mathbf{g} \in \mathbf{L}^2(\Omega)$, the following regularity result are valid

$$\|\mathbf{v}\|_2 + \|q\|_{1/\mathbb{R}} \le C\|\mathbf{g}\|.$$

Domains with C^2 boundary and convex polygon in two dimension are known to satisfy assumption (**A1**). This allows us the following norm inequalities:

$$\left.\begin{aligned} \|\mathbf{w}\|^2 \le&\ \ \lambda_1^{-1}\|\nabla\mathbf{w}\|^2, \quad \forall\ \mathbf{w} \in \mathbf{J}_1, \\ \|\nabla\mathbf{w}\|^2 \le&\ \ \lambda_1^{-1}\|\tilde{\Delta}\mathbf{w}\|^2, \quad \forall\ \mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2 \\ \|\mathbf{w}\|_2 \le&\ \ C\|\tilde{\Delta}\mathbf{w}\|, \quad \forall\ \mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2, \end{aligned}\right\} \tag{1.7}$$

where $\tilde{\Delta} = P\Delta : \mathbf{H}^2 \cap \mathbf{J}_1 \subset \mathbf{J} \to \mathbf{J}$ is known as the Stokes operator, P an orthogonal projection $P : \mathbf{L}^2(\Omega) \to \mathbf{J}$ and λ_1 is the least positive eigenvalue of the Stokes operator. We next list down a few standard inequalities:

(i) **Cauchy-Schwarz inequality:** The following inequality holds for all $a, b \geq 0$:

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}.$$

(ii) **Young inequality:** For all $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for all $a, b \geq 0, \epsilon > 0$, the following inequality holds:

$$ab \leq \frac{\epsilon a^p}{p} + \frac{b^q}{q\epsilon^{q/p}}.$$

(iii) **Hölder's inequality:** For all $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for $\phi \in L^p(\Omega)$ and $\psi \in L^q(\Omega)$, the following inequality holds:

$$\int_\Omega \phi\psi \; dx \leq \|\phi\|_p \|\psi\|_q.$$

Next in our list is the standard Gronwall's lemma. For a proof, we refer to [64].

Lemma 1.1 (Classical Gronwall's lemma)**.** *Let g, h, y be three locally integrable non-negative functions on the time interval $[t_0, \infty)$ such that for all $t \geq t_0$ the following holds*

$$\frac{dy}{dt} \leq gy + h,$$

where $\frac{dy}{dt}$ is locally integrable. Then,

$$y(t) \leq y(t_0) \exp\Big(\int_{t_0}^t g(\tau) d\tau\Big) + \Big(\int_{t_0}^t h(s) \exp\Big(\int_s^t g(\tau) d\tau\Big) \; ds\Big).$$

However in our analysis, we often resort to a modified version which we present below. For a proof, we refer to [59].

Lemma 1.2 (Gronwall's Lemma)**.** *Let g, h, y be three locally integrable non-negative functions on the time interval $[t_0, \infty)$ such that for all $t \geq t_0$*

$$y(t) + G(t) \leq C + \int_{t_0}^t h(s) \; ds + \int_{t_0}^t g(s) y(s) \; ds,$$

where $G(t)$ is a non-negative function on $[t_0, \infty)$ and $C \geq 0$ is a constant. Then,

$$y(t) + G(t) \leq \Big(C + \int_{t_0}^t h(s) \exp\Big(-\int_{t_0}^s g(\tau) \; d\tau\Big) \; ds\Big) \exp\Big(\int_{t_0}^t g(s) \; ds\Big).$$

Armed with the requisite spaces, we now briefly look at the continuous case. It is customary to study it in a weaker form and hence we first present the weak or variational formulation of the system (1.4)-(1.6): Find $\mathbf{u}(t) \in \mathbf{H}_0^1$ and $p(t) \in L^2/\mathbb{R}$ such that for $t > 0$

$$(\mathbf{u}_t, \boldsymbol{\phi}) + \mu a(\mathbf{u}, \boldsymbol{\phi}) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}) + \int_0^t \beta(t-s) a(\mathbf{u}(s), \boldsymbol{\phi}) ds - (p, \nabla \cdot \boldsymbol{\phi}) = (\mathbf{f}, \boldsymbol{\phi}), \quad (1.8)$$

for all $\boldsymbol{\phi} \in \mathbf{H}_0^1$ and $(\nabla \cdot \mathbf{u}, \chi) = 0, \ \forall \chi \in L^2$. If $\mathbf{u} \in \mathbf{J}_1$, the equivalent formulation is

$$(\mathbf{u}_t, \boldsymbol{\phi}) + \mu a(\mathbf{u}, \boldsymbol{\phi}) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}) + \int_0^t \beta(t-s) a(\mathbf{u}(s), \boldsymbol{\phi}) ds = (\mathbf{f}, \boldsymbol{\phi}), \quad \forall \boldsymbol{\phi} \in \mathbf{J}_1. \quad (1.9)$$

We have used above the bilinear form $a(\cdot, \cdot)$ and trilinear form $b(\cdot, \cdot, \cdot)$ which are defined as follows: For all $\mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1$,

$$a(\mathbf{v}, \boldsymbol{\phi}) = (\nabla \mathbf{v}, \nabla \boldsymbol{\phi}), \quad \text{and} \quad b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = \frac{1}{2}((\mathbf{v} \cdot \nabla)\mathbf{w}, \boldsymbol{\phi}) - \frac{1}{2}((\mathbf{v} \cdot \nabla)\boldsymbol{\phi}, \mathbf{w}). \quad (1.10)$$

From the very definition of the trilinear form, we easily conclude that,

$$b(\mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\phi}) = 0 \quad \text{and} \quad b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = -b(\mathbf{v}, \boldsymbol{\phi}, \mathbf{w}), \quad \forall \mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1. \quad (1.11)$$

And it is standard to bound the nonlinear term based on the following well-known Sobolev inequalities, see [128, 131].

Lemma 1.3. *For any open set $\Omega \in \mathbb{R}^2$ and for $\mathbf{v} \in \mathbf{H}_0^1$*

$$\|\mathbf{v}\|_{\mathbf{L}^4(\Omega)} \leq 2^{1/4} \|\mathbf{v}\|^{1/2} \|\nabla \mathbf{v}\|^{1/2}.$$

Moreover, when Ω is bounded, then following estimates hold

$$\|\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)} \leq C \begin{cases} \|\mathbf{v}\|^{1/2} \|\Delta \mathbf{v}\|^{1/2}, & \mathbf{v} \in \mathbf{H}^2 \\ \|\mathbf{v}\|^{1/2} \|\tilde{\Delta} \mathbf{v}\|^{1/2}, & \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2. \end{cases}$$

With the help of Lemma 1.3, one can easily obtain the estimates of the nonlinear operator $b(\cdot, \cdot, \cdot)$, see [80].

Lemma 1.4. *For any open and bounded set $\Omega \subset \mathbb{R}^2$, there exists a constant $C > 0$ such that for $\mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$, the followings hold:*

$$|(\mathbf{v} \cdot \nabla \mathbf{w}, \boldsymbol{\phi})| \leq C \begin{cases} \|\mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{w}\|^{\frac{1}{2}} \|\tilde{\Delta} \mathbf{w}\|^{\frac{1}{2}} \|\boldsymbol{\phi}\|, \\ \|\mathbf{v}\|^{\frac{1}{2}} \|\tilde{\Delta} \mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{w}\| \|\boldsymbol{\phi}\|, \\ \|\mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{w}\| \|\boldsymbol{\phi}\|^{\frac{1}{2}} \|\nabla \boldsymbol{\phi}\|^{\frac{1}{2}}, \\ \|\mathbf{v}\| \|\nabla \mathbf{w}\| \|\boldsymbol{\phi}\|^{\frac{1}{2}} \|\tilde{\Delta} \boldsymbol{\phi}\|^{\frac{1}{2}}, \\ \|\mathbf{v}\| \|\nabla \mathbf{w}\|^{\frac{1}{2}} \|\tilde{\Delta} \mathbf{w}\|^{\frac{1}{2}} \|\boldsymbol{\phi}\|^{\frac{1}{2}} \|\nabla \boldsymbol{\phi}\|^{\frac{1}{2}}. \end{cases}$$

In our integro-parabolic system, the kernel β enjoys a positivity property which is crucial for the well-poseness of the problem as well as in the error analysis. We present below the positivity property, for a proof of which, we refer to [102].

Lemma 1.5. *For any $\alpha > 0$ and $\psi \in L^2(0,t)$, the following positive definite property holds for any $t > 0$*

$$\int_0^t \Big(\int_0^s e^{-\alpha(s-\tau)} \psi(\tau) d\tau \Big) \psi(s) ds \geq 0.$$

We next consider the assumption on given data $\mathbf{u}_0$ and $\mathbf{f}$:

(**A2**). For a constant $M_0 > 0$, the initial velocity $\mathbf{u}_0$ and the external force $\mathbf{f}$ satisfy,

$$\mathbf{u}_0 \in \mathbf{J}_1 \ \text{ with } \ \|\mathbf{u}_0\|_1 \leq M_0, \ \text{ and } \ \mathbf{f}, \mathbf{f}_t \in L^\infty([0,\infty]; \mathbf{L}^2) \ \text{ with } \ \sup_{t>0} \big\{ \|\mathbf{f}\|, \|\mathbf{f}_t\| \big\} \leq M_0.$$

Under the assumptions of (**A1**) and (**A2**), we can show the well-posedness of the both weak and regular solution. For a proof, we refer to [59, 63].

The problem (1.4)-(1.6) is posed in an infinite dimensional function space and in finite element methods, we attempt a finite dimensional problem. This is achieved by discretizing the domain Ω into finitely many elements and then considering finite dimensional finite element spaces, where the problem is solved. When only space is discretized, time remaining continuous, we call it the semidiscrete case.

Let $\mathcal{T}_h = \{K\}$ be a finite decomposition of mesh size h with $0 < h < 1$, of the polygonal domain $\bar{\Omega}$ into closed subsets K, triangles or quadrilaterals. The decomposition $\mathcal{T}_h$ is assumed to be "face to face" and to satisfy a "uniform size" condition: *"Any two elements of $\mathcal{T}_h$ meet only in entire common sides or in vertices. Each element of $\mathcal{T}_h$ contains a circle of radius $\kappa_1 h$ and it is contained in a circle of radius $\kappa_2 h$, these constant κ_1, κ_2 being independent of h."*

We now define finite element spaces $\mathbf{H}_h$ and L_h that approximate the velocity space $\mathbf{H}_0^1$ and the pressure space L^2, respectively, as follows:

$$\mathbf{H}_h = \{\mathbf{v}_h \in (C^0(\bar{\Omega}))^2 \cap \mathbf{H}_0^1 : \mathbf{v}_h|_K \in P(K), \text{ for every } K \in \mathcal{T}_h\},$$
$$L_h = \{q_h \in C^0(\bar{\Omega}) : q_h|_K \in Q(K), \text{ for every } K \in \mathcal{T}_h\},$$

where $P(K)$ and $Q(K)$ are the polynomial spaces. It is noted that $\mathbf{H}_h$ and L_h are the subspaces of $\mathbf{H}_0^1$ and L^2, respectively.

Assume the following approximation properties for the discrete spaces $\mathbf{H}_h$ and L_h:

(**B1**) For each $\boldsymbol{\phi} \in \mathbf{H}_0^1 \cap \mathbf{H}^2$ and $\psi \in H^1/\mathbb{R}$ there exist approximations $i_h\boldsymbol{\phi} \in \mathbf{H}_h$ and

$j_h\psi \in L_h$ such that

$$\|\boldsymbol{\phi} - i_h\boldsymbol{\phi}\| + h\|\nabla(\boldsymbol{\phi} - i_h\boldsymbol{\phi})\| \le Ch^2\|\boldsymbol{\phi}\|_2, \quad \|\psi - j_h\psi\|_{L^2/\mathbb{R}} \le Ch\|\psi\|_{1/\mathbb{R}}.$$

Further, we will assume that the following inverse hypothesis holds for $\mathbf{v}_h \in \mathbf{H}_h$, see [35, Theorem 3.2.6]

$$\|\mathbf{v}_h\|_{W^{m,p}(K)^d} \le Ch^{n-m-d(\frac{1}{q}-\frac{1}{p})}\|\mathbf{v}_h\|_{W^{n,q}(K)^d}, \tag{1.12}$$

where $0 \le n \le m < \infty$, $1 \le q \le p \le \infty$, h be the diameter of the mesh cell $K \in \mathcal{T}_h$ and $\|\cdot\|_{W^{m,p}(K)^d}$ is the norm in Sobolev space $W^{m,p}(K)^d$. In particular, for $m = 1, n = 0, p = q = 2$ the above inequality reads as for $\mathbf{v}_h \in \mathbf{H}_h$:

$$\|\nabla\mathbf{v}_h\| \le Ch^{-1}\|\mathbf{v}_h\|.$$

We now consider the discrete variational formulation of the problem (1.4)-(1.6) as follows: Find $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{H}_h \times L_h$ such that for $t > 0$

$$\begin{aligned}(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \mu a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s)a(\mathbf{u}_h(s), \boldsymbol{\phi}_h)ds \\ - (p_h, \nabla\cdot\boldsymbol{\phi}_h) = (\mathbf{f}, \boldsymbol{\phi}_h),\end{aligned} \tag{1.13}$$

for all $\boldsymbol{\phi}_h \in \mathbf{H}_h$ and $(\nabla\cdot\mathbf{u}_h, \chi_h) = 0,\ \forall\chi \in L_h$, with $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ where $\mathbf{u}_{0h} \in \mathbf{H}_h$ is a suitable approximation of $\mathbf{u}_0 \in \mathbf{J}_1$.

In order to consider a discrete space analogous to $\mathbf{J}_1$, we define a discrete divergence free space $\mathbf{J}_h \subset \mathbf{H}_h$ as

$$\mathbf{J}_h := \{\mathbf{w}_h \in \mathbf{H}_h : (\chi_h, \nabla\cdot\mathbf{w}_h) = 0, \quad \forall\chi_h \in L_h\}.$$

Note that $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$. We now introduce an equivalent Galerkin formulation in the space $\mathbf{J}_h$ as: Find $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that for $t > 0$

$$(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \mu a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s)a(\mathbf{u}_h(s), \boldsymbol{\phi}_h)ds = (\mathbf{f}, \boldsymbol{\phi}_h), \tag{1.14}$$

for all $\boldsymbol{\phi}_h \in \mathbf{J}_h$ with $\mathbf{u}_h(0) = \mathbf{u}_{0h}$. Since $\mathbf{J}_h$ is finite dimensional, the problem (1.14) leads to a system of nonlinear integro-differential equations. Using Picard's theorem with the continuity argument, we ensure the global existence and uniqueness of the discrete velocity $\mathbf{u}_h$ of (1.14), (see [116]). The discrete pressure $p_h \in L_h$ is unique in the quotient space L_h/N_h where

$$N_h = \{q_h \in L_h : (q_h, \nabla\cdot\boldsymbol{\phi}_h) = 0, \quad \forall\boldsymbol{\phi}_h \in \mathbf{H}_h\}.$$

The norm on L_h/N_h is given by

$$\|q_h\|_{L^2/N_h} = \inf_{\chi_h \in N_h} \|q_h + \chi_h\|.$$

Since the discrete pressure $p_h(t) \in L_h/N_h$ depends on the discrete velocity $\mathbf{u}_h(t) \in \mathbf{J}_h$, so we assume the following discrete inf-sup (LBB) condition for the discrete spaces $\mathbf{H}_h$ and L_h:

(**B2′**) For each $q_h \in L_h$, there is a $\boldsymbol{\phi}_h \in \mathbf{H}_h$ such that the following holds

$$|(q_h, \nabla \cdot \boldsymbol{\phi}_h)| \geq C\|\nabla \boldsymbol{\phi}_h\| \|q_h\|_{L^2/N_h}.$$

We also assume the following approximation property:

(**B2**) For every $\boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$, there exists an approximation $r_h\boldsymbol{\phi} \in \mathbf{J_h}$ such that

$$\|\boldsymbol{\phi} - r_h\boldsymbol{\phi}\| + h\|\nabla(\boldsymbol{\phi} - r_h\boldsymbol{\phi})\| \leq Ch^2\|\boldsymbol{\phi}\|_2.$$

As stated in [79]: This is a less restrictive condition than (**B2′**) and it has been used to derive the following properties of the L^2 projection $P_h : \mathbf{L}^2 \mapsto \mathbf{J}_h$.

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla(\boldsymbol{\phi} - P_h\boldsymbol{\phi})\| \leq Ch^2\|\tilde{\Delta}\boldsymbol{\phi}\|, \quad \text{for } \boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2. \tag{1.15}$$

Apart from these standard approximation properties, we also consider the interpolation properties for the Lagrange interpolant $I_h\boldsymbol{\phi} \in \mathbf{H}_h$ satisfying the following bounds (see [19, Theorem 4.4.4]), for $\boldsymbol{\phi} \in W^{n,p}(K)$,

$$\|\boldsymbol{\phi} - I_h\boldsymbol{\phi}\|_{W^{m,p}(K)} \leq Ch^{n-m}\|\boldsymbol{\phi}\|_{W^{n,p}(K)}, \quad 0 \leq m \leq n,$$

where $n > \frac{2}{p}$ when $1 < p \leq \infty$ and $n \geq 2$ when $p = 1$.

Examples of subspaces $\mathbf{H}_h$ and $\mathbf{L}_h$ satisfying assumptions (**B1**), (**B2**) and (**B2′**) are abundant in the literature. We mention a few below. The first one is the Taylor-Hood element [15]:

$$\mathbf{H}_h = \{\mathbf{v}_h \in (C^0(\bar{\Omega}))^2 \cap \mathbf{H}_0^1 : \mathbf{v}_h|_K \in (P_{k+1}(K))^2, \text{ for every } K \in \mathcal{T}_h\},$$
$$L_h = \{q_h \in C^0(\bar{\Omega}) : q_h|_K \in P_k(K), \text{ for every } K \in \mathcal{T}_h\},$$

where $k \geq 1$ and P_r represents the space of polynomials of degree less than or equal to r over the element K. And the second one is the MINI element [6]:

$$\mathbf{H}_h = \{\mathbf{v}_h \in (C^0(\bar{\Omega}))^2 \cap \mathbf{H}_0^1 : \mathbf{v}_h|_K \in (P_1 b(K))^2, \text{ for every } K \in \mathcal{T}_h\},$$

$$L_h = \{q_h \in C^0(\bar{\Omega}) : q_h|_K \in P_1(K), \text{ for every } K \in \mathcal{T}_h\},$$

where $P_1 b = P_1 \oplus b$, b stands for cubic bubble functions. Third one is due to Bercovier-Pironneau [9]:

$$\mathbf{H}_h = \{\mathbf{v}_h \in (C^0(\bar{\Omega}))^2 \cap \mathbf{H}_0^1 : \mathbf{v}_h|_K \in (P_1(K))^2, \text{ for every } K \in \mathcal{T}_{h/2}\},$$
$$L_h = \{q_h \in C^0(\bar{\Omega}) : q_h|_K \in P_1(K), \text{ for every } K \in \mathcal{T}_h\},$$

where $\mathcal{T}_{h/2}$ is obtained by dividing each triangle of $\mathcal{T}_h$ into four triangles. Since we carry out our (semi/fully)-discrete analysis in $\mathbf{J}_h$ whenever possible, it is fruitful to talk about the discrete Stokes operator, based on the discrete version of the Laplace operator $\Delta_h : \mathbf{H}_h \mapsto \mathbf{H}_h$ which is the bilinear form

$$a(\mathbf{w}_h, \boldsymbol{\phi}_h) = (-\Delta_h \mathbf{w}_h, \boldsymbol{\phi}), \quad \forall \mathbf{w}_h, \boldsymbol{\phi}_h \in \mathbf{H}_h.$$

Analogous to the Stokes operator $\tilde{\Delta} = P\Delta$ where P is the $\mathbf{L}^2$ projection onto J, the discrete version is defined as $\tilde{\Delta}_h = P_h \Delta_h$ where P_h is the $\mathbf{L}^2$ projection onto J_h. The restriction of $\tilde{\Delta}_h$ to $\mathbf{J}_h$ is invertible and we denote the inverse by $\tilde{\Delta}_h^{-1}$. Since $-\tilde{\Delta}_h$ is self-adjoint and positive definite, we define *discrete Sobolev norms* on $\mathbf{J}_h$ as follows:

$$\|\mathbf{v}_h\|_r = \|(-\tilde{\Delta}_h)^{r/2} \mathbf{v}_h\|, \quad \mathbf{v}_h \in \mathbf{J}_h, \ r \in \mathbb{R}.$$

We note that in particular $\|\mathbf{v}_h\|_0 = \|\mathbf{v}_h\|$ and $\|\mathbf{v}_h\|_1 = \|\nabla \mathbf{v}_h\|$ for $\mathbf{v}_h \in \mathbf{J}_h$, and $\|\cdot\|_2$ and $\|\tilde{\Delta}_h \cdot\|$ are equivalent norms on $\mathbf{J}_h$. For further detail, we refer to [79, 80].

Remark 1.1. *To avoid confusion as to whether* $\|\cdot\|_1$ *means standard or discrete Sobolev norm, we follow the convention that if* $\mathbf{v}$ *belongs to* $\mathbf{J}_h$ *then* $\|\mathbf{v}\|_1$ *represents* $\mathbf{v}$ *in discrete Sobolev norm, otherwise it is the standard Sobolev norm.*

The semidiscrete formulation(s) mentioned above are still continuous in time and in a fully discrete scheme, we further discretize (it) in the temporal direction. For time discretization, we consider the first-order implicit backward Euler method. Assuming $[0, T]$ to be the time interval, we proceed as follows: Let $k = \frac{T}{N} > 0$ be the time step with $t_n = nk, \ n \geq 0$ representing the n-th time step. Here N is a positive integer. We next define for a sequence $\{\boldsymbol{\phi}^n\}_{n \geq 0}$ the backward difference quotient

$$\partial_t \boldsymbol{\phi}^n = \frac{1}{k}(\boldsymbol{\phi}^n - \boldsymbol{\phi}^{n-1}), \quad n > 0.$$

We write any continuous function $\boldsymbol{\phi}(t_n)$ as $\boldsymbol{\phi}^n$. And we approximate the integral term in (1.14) by the right rectangle rule (since backward Euler method is of first-order in time) with the notation $\beta_{nj} = \beta(t_n - t_j)$:

$$q_r^n(\boldsymbol{\phi}) = k \sum_{j=1}^{n} \beta_{nj} \phi^j \approx \int_0^{t_n} \beta(t_n - s)\boldsymbol{\phi}(s)\, ds.$$

Now, the fully discrete scheme based on backward Euler method for the semidiscrete Oldroyd problem (1.13) reads as follows: Find $\{\mathbf{U}^n\}_{1\le n\le N} \in \mathbf{H}_h$ and $\{P^n\}_{1\le n\le N} \in L_h$ as solutions of the recursive nonlinear algebraic equations ($1 \le n \le N$):

$$\left.\begin{aligned} &(\partial_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \mu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{U}), \boldsymbol{\phi}_h) = (P^n, \nabla\cdot\boldsymbol{\phi}_h) \\ &\qquad\qquad\qquad\qquad + (\mathbf{f}^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h), \quad \forall\, \boldsymbol{\phi}_h \in \mathbf{H}_h, \\ &(\nabla\cdot\mathbf{U}^n, \chi_h) = 0, \quad \forall\, \chi_h \in L_h, \quad n \ge 0. \end{aligned}\right\} \tag{1.16}$$

We choose $\mathbf{U}^0 = \mathbf{u}_h(0)$. Equivalently, for $\boldsymbol{\phi}_h \in \mathbf{J}_h$ we seek $\{\mathbf{U}^n\}_{1\le n\le N} \in \mathbf{J}_h$ such that

$$(\partial_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \mu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{U}), \boldsymbol{\phi}_h) = (\mathbf{f}^n, \boldsymbol{\phi}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{1.17}$$

Here again, we choose $\mathbf{U}^0 = \mathbf{u}_h(0) \in \mathbf{J}_h$. Now using variant of the Brouwer fixed point theorem and standard uniqueness argument, one can show that the discrete problem (1.17) is well-posed. For a proof, we refer to [59].

And in the fully discrete case, as mentioned earlier, we have used right rectangle rule in (1.16) to approximate the integral term, and it is positive in the following sense:

$$k \sum_{i=1}^{n} q_r^i(\phi)\phi^i = k^2 \sum_{i=1}^{n}\sum_{j=1}^{i} \beta_{ij}\phi^j\phi^i \ge 0, \qquad \phi = (\phi^0, \cdots, \phi^N)^T, \tag{1.18}$$

where $\beta_{nj} = \beta(t_i - t_j) = e^{-\delta(t_i - t_j)}$. For details, see [102].

The error incurred due to right rectangle rule in approximating the integral term is given by

$$\begin{aligned} \varepsilon_r^n(\phi) &= \int_0^{t_n} \beta(t_n - \tau)\boldsymbol{\phi}(\tau)\, d\tau - k\sum_{j=1}^{n} \beta_{nj}\boldsymbol{\phi}^j \\ &= -\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (\tau - t_{j-1}) \frac{\partial}{\partial\tau}(\beta(t_n - \tau)\boldsymbol{\phi}(\tau))\, d\tau \\ &\le k \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big|\frac{\partial}{\partial\tau}(\beta(t_n - \tau)\boldsymbol{\phi}(\tau))\Big|\, d\tau. \end{aligned} \tag{1.19}$$

Due to the presence of the summation term, we often need to resort to summation by part formula, a discrete version of *integration by parts.* For sequences $\{a_i\}$ and $\{b_i\}$ of real numbers, the following summation by parts holds

$$k\sum_{j=1}^{i} a_j b_j = a_i \widehat{b}_i - k\sum_{j=1}^{i-1}(\partial_t a_{j+1})\widehat{b}_j, \quad \text{where } \widehat{b}_i = k\sum_{j=1}^{i} b_j. \tag{1.20}$$

Here, we observe that for sequences $\{a_i\}$ and $\{b_i\}$ of real numbers,

$$\sigma_i(a_i, \partial_t b_i) = \partial_t(\sigma_i(a_i, b_i)) - \sigma_i(\partial_t a_i, b_i) - (\partial_t \sigma_i)(a_i, b_i), \tag{1.21}$$

where, $\sigma_n = e^{2\alpha t_n}\tau^*(t_n)$ and $\tau^*(t_n) = \min\{1, t_n\}$.

We next present below the discrete version of Schwarz's inequality, which will be used in our later analysis.

Cauchy inequality: For a finite pair of positive real numbers $\{\phi_j, \psi_j\}_{j=1,2,\ldots,n}$, the following holds

$$\sum_{j=1}^{n} \phi_j \psi_j \leq \Big(\sum_{j=1}^{n} \phi_j^2\Big)^{1/2} \Big(\sum_{j=1}^{n} \psi_j^2\Big)^{1/2}.$$

We consider the following version of discrete Gronwall's Lemma. The proof can be found in [75, 113].

Lemma 1.6. *Let $\{a_n\}$ and $\{d_n\}$ be finite sequences of nonnegative real numbers and $\{b_n\}$ be a nondecreasing real finite sequence satisfying*

$$a_n \leq b_n + \sum_{i=0}^{n-1} d_i a_i, \quad \forall n \geq 0,$$

Then,

$$a_n \leq b_n \exp\Big(\sum_{i=0}^{n-1} d_i\Big), \quad \forall n \geq 0.$$

But for our subsequent analysis, we use more general version of the discrete Gronwall's Lemma, which is simply a reproduction of Lemma 5.1 from [80].

Lemma 1.7. *Let k, B and $\{a_i, b_i, c_i, d_i\}_{i\in\mathbb{N}}$ be non-negative numbers such that*

$$a_n + k\sum_{i=1}^{n} b_i \leq B + k\sum_{i=1}^{n} c_i + k\sum_{i=1}^{m} d_i a_i, \;\; n \geq 1, \tag{1.22}$$

for $m = n$ or $n-1$. Then,

$$a_n + k\sum_{i=1}^{n} b_i \leq \Big\{B + k\sum_{i=1}^{n} c_i\Big\} exp\Big(k\sum_{i=1}^{m} \gamma_i d_i\Big), \tag{1.23}$$

where $\gamma_i = 1$ when $m = n-1$ and $\gamma_i = (1 - kd_i)^{-1}, \; kd_i < 1$ when $m = n$.

Remark 1.2. *We note here that for $m = n$ case, (1.23) is valid only when $kd_i < 1$, for all i, and that put sever restriction on the time step k. However for $m = n-1$ case, no such restriction is applicable, and hence we would use this version of the discrete Gronwall's lemma wherever possible.*

We next state the discrete analogue of the L'Hospital rule, which will be useful in analysing the discrete case in Chapter 4. For a proof, see [104, pp. 85-87].

Theorem 1.1. *(Stolz-Cesaro Theorem) Let $\{\phi^n\}_{n=0}^{\infty}$ be a sequence of real numbers. Further, let $\{\psi^n\}_{n=0}^{\infty}$ be a strictly monotone and divergent sequence. If*

$$\lim_{n\to\infty}\left(\frac{\phi^n - \phi^{n-1}}{\psi^n - \psi^{n-1}}\right) = l,$$

then the following holds:

$$\lim_{n\to\infty}\left(\frac{\phi^n}{\psi^n}\right) = l.$$

Some Useful Results

Since the Oldroyd model of order one has been studied in details both in continuous and semidiscrete cases, several useful results are available pertaining to those. We recollect a few of them, which are recorded in [59, 63], and which will be used in our subsequent chapters. We begin by presenting *a priori* and regularity estimates of the continuous solution $(\mathbf{u}, p)$.

Lemma 1.8. *Let the assumptions* $(\mathbf{A1})$ *and* $(\mathbf{A2})$ *hold. Then, for any time T with $0 \le T < \infty$ and for some $\alpha > 0$ satisfying $0 < \alpha < \min\{\delta, \mu\lambda_1\}$, there exists a positive constant C such that the solution $(\mathbf{u}, p)$ of (1.8) satisfies,*

$$\begin{aligned}
&\|\mathbf{u}(t)\|_r^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}(s)\|_{r+1}^2 + \|p(s)\|_{1/\mathbb{R}}^2)ds \le C, \qquad r \in \{0,1\}\\
&(\tau^*)^{1/2}(t)\{\|\mathbf{u}(t)\|_2 + \|\mathbf{u}_t(t)\| + \|p(t)\|_{1/\mathbb{R}}\} \le C,\\
&(\tau^*(t))^{r+1}\|\mathbf{u}_t(t)\|_r^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau^*(s))^r\|\mathbf{u}_s(s)\|_r^2 ds \le C, \qquad r \in \{0,1,2\}\\
&e^{-2\alpha t}\int_0^t e^{2\alpha s}((\tau^*(s))^{r+1}\|\mathbf{u}_{ss}(s)\|_{r-1}^2 + (\tau^*(s))^2\|p_s(s)\|_{1/\mathbb{R}}^2)ds \le C, \; r \in \{-1,0,1\}.
\end{aligned}$$

where, $\tau^*(t) = \min\{1, t\}$ and C depends only on given data and not on time.

Next on the list is the *a priori* and regularity estimates of the semidiscrete solution.

Lemma 1.9. *Let the assumptions of Lemma 1.8 hold. Then, there exists a positive constant C such that the semidiscrete solution $(\mathbf{u}_h, p_h)$ of (1.13) satisfies,*

$$\|\mathbf{u}_h(t)\|_r^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_h(s)\|_{r+1}^2 + \|p_h(s)\|_{H^1/N_h}^2)ds \le C, \qquad r\in\{0,1\},$$
$$(\tau^*)^{1/2}(t)\{\|\mathbf{u}_h(t)\|_2 + \|\mathbf{u}_{ht}(t)\| + \|p_h(t)\|_{1/N_h}\} \le C,$$
$$(\tau^*(t))^{r+1}\|\mathbf{u}_{ht}(t)\|_r^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau^*(s))^r\|\mathbf{u}_{hs}(s)\|_r^2 ds \le C, \qquad r\in\{0,1,2\},$$

where, $\tau^*(t) = \min\{1,t\}$ and C depends only on given data and not on time.

And finally the optimal error estimates due to the space discretization.

Theorem 1.2. *Let Ω be a convex polygon and let the conditions (**A1**)-(**A2**) and (**B1**)-(**B2**) be satisfied. Further, let the discrete initial velocity $\mathbf{u}_{0h}\in\mathbf{J}_h$ with $\mathbf{u}_{0h} = P_h\mathbf{u}_0$, where $\mathbf{u}_0\in\mathbf{J}_1$. Then, there exists a positive constant K such that for $0<T<+\infty$ with $t\in(0,T]$*

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\| + h\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\| + h\|(p-p_h)(t)\| \le Ke^{Kt}h^2t^{-1/2}.$$

Moreover under the assumption of the uniqueness condition, that is,

$$\frac{N}{\nu^2}\|\mathbf{f}\|_\infty < 1 \quad and \quad N = \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{H}_0^1(\Omega)}\frac{b(\mathbf{u},\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|}, \tag{1.24}$$

where $\nu = \mu + \frac{\gamma}{\delta}$ and $\|\mathbf{f}\|_\infty := \|\mathbf{f}\|_{L^\infty(\mathbb{R}_+;\mathbf{L}^2(\Omega))}$, we have the following uniform estimate:

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\| + h\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\| + h\|(p-p_h)(t)\| \le Kh^2t^{-1/2}.$$

1.3 A Brief Literature Review

This section presents a brief survey of the literature concerning the equation of motion arising in Oldroyd model of order one and a few finite element methods, which will be analyzed in the later chapters of this thesis.

The Oldroyd model of order one has been studied for more than three decades now; early work on the Oldroyd model can be traced back to Oskolkov, Kot-siolis, Karzeeva, Sobolevski (for details, see [59, 63] and references therein) who studied well-posedness of the problem, asymptotic analysis and dynamical system (or long time solution behavior) following the works of Ladyzhenskaya [90] on NSEs. After a decade, Pani *et. al.* [116], have obtained a few new *a priori* estimates and have analyzed the long time behavior of the exact solution for realistically assumed data, that is, the initial data $\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^2$ (we call it as smooth initial data) and for forcing term $\mathbf{f} = 0$. This work has been extended in [59, 63] for non-zero-forcing term $\mathbf{f}$ independent of time or in L^∞, and for realistically assumed less regular given data, that is, the initial data $\mathbf{u}_0 \in \mathbf{H}_0^1$ and not in $\mathbf{H}^2$ (we call it as nonsmooth initial data).

The semidiscrete finite element approximation has first been studied by Canon *et. al.* [25] in the context of a modified nonlinear Galerkin method. However, He *et al.* [76] have studied the finite element formulation and have obtained optimal error estimate for the velocity in $\mathbf{H}^1$-norm and the pressure in L^2-norm. The work has been continued by Pani *et al.* [116], obtaining optimal error bounds for the velocity in $\mathbf{L}^2$ as well as $\mathbf{H}^1$-norm and the pressure in L^2-norm, for the zero-forcing term and smooth initial data. The estimates obtained there are valid uniformly in time $t > 0$ under the uniqueness condition. In further continuation to this, in [63], optimal semidiscrete error estimates have been derived for the non-zero forcing term and nonsmooth initial data.

In [63], the model has been thoroughly investigated in both continuous and semidiscrete setups. A step-by-step proof of the energy norm estimate, which is crucial for the existence of a weak solution, has been established. New regularity results for the solution have been obtained, which conclude the behaviour of the solution as $t \to 0$ and as $t \to \infty$. Optimal error estimates for the velocity in $\mathbf{L}^2$ and $\mathbf{H}^1$-norms and the pressure in L^2-norm have been proved and uniform in time bounds have been shown under uniqueness conditions.

In the phd dissertation [59], a complete analysis for the Oldroyd model of order one in the continuous and semidiscrete cases can be found for the forcing term $\mathbf{f} \in L^\infty(\mathbf{L}^2)$ and for nonsmooth initial data. Apart from incorporating the work of [63], the author has made an attempt to study the fully discrete schemes using the first-order backward Euler method and second order Crank-Nicolson scheme in the temporal direction. The analysis has been carried out for both smooth and nonsmooth initial data in the case of the backward Euler method and only for smooth initial data in the case of the Crank-Nicolson scheme. Finally, the penalty method has been applied to the model. It has been shown there the penalty solution goes to the original solution as the penalty parameter goes to zero. A semidiscrete finite element approximation of the penalized model has been analyzed, and error estimates have been derived. However, the error bounds were dependent on the inverse power of the penalty parameter; a long standing problem which was not resolved.

Our work here can be viewed as a continuation of these earlier works. We discuss below the literature of finite element methods that we have analysed in this thesis. Since each of these methods is validated by means of numerical computations, we do consider fully discrete versions by employing a first-order time discretization scheme for each case. Therefore, it is imperative that we first analyse a fully discrete scheme for the standard Galerkin finite element.

1.3.1 Backward Euler Method

As mention earlier, in fully discrete scheme, we discretize both space and time variables. We employ a first-order finite difference scheme for temporal discretization to approximate the time derivative and an appropriate quadrature rule to approximate the integral term. Literature for the fully discrete approximation to the Oldroyd model of order one (1.4)-(1.6) is limited. In [5], Akhmatov and Oskolkov have discussed stable and convergent finite difference schemes for the problem (1.4)-(1.6). On the other hand, Pani *et al.*, in [117], have considered a linearized backward Euler method to discretize in the temporal direction only, keeping spatial direction continuous and have used semi-group theoretic approach to establish *a priori* error estimates. The following time discrete error bounds are proved in [117] for $0 < \alpha < \min\{\delta, \lambda_1\}$,

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \le Ce^{-\alpha t_n} k, \quad \|\mathbf{u}(t_n) - \mathbf{U}^n\|_1 \le Ce^{-\alpha t_n} k(t_n^{-1/2} + \log \frac{1}{k}),$$

for smooth initial data, i.e., $\mathbf{u}_0 \in \mathbf{H}^2 \cap \mathbf{H}_0^1$ and for zero forcing term ($\mathbf{f} \equiv 0$). Here k is the uniform time-step size, $t_n = nk$ is the n-th time level and $t_N = Nk$ is the final time, $\mathbf{U}^n$ is the approximation of semidiscrete solution $\mathbf{u}_h$ at $t = t_n$ and fully discrete approximation of $\mathbf{u}$ at $t = t_n$ and λ_1 is the smallest positive eigenvalue of the Stokes operator.

In [137], Wang *et al.* have extended this work for non-zero forcing function. They have used energy arguments, along with uniqueness condition to obtain for fully discrete solution $\mathbf{U}^n$, the following uniform error estimates

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \le C(h^2 + k), \quad (\tau^*)^{1/2}\|\mathbf{u}(t_n) - \mathbf{U}^n\|_1 \le C(h + k),$$

where $\tau^*(t_n) = \min\{1, t_n\}$ and again for smooth data.

There are few related works in the literature based on the time-discrete scheme for the Oldroyd model of order one. For example, in [141], long-time numerical stability has been studied, where the Euler semi-implicit scheme is used. Global H^2-stability of the discrete solution and discrete asymptotic behavior has been discussed. In [69, 70], Guo *et al.* have worked with a second-order time discretization scheme based on Crank-Nicolson/Adams-Bashforth as part of the fully discrete analysis and have derived optimal error estimates under smooth initial data. In the works of [71, 150], we see first and second-order time-discrete schemes (implicit, semi-implicit, implicit/explicit, and explicit schemes) and stability analysis of fully discrete solutions. A fully discrete fractional step method has been applied in [153] where the stability analysis of the fully discrete solutions and optimal error estimates for the velocity and the pressure is derived. In [99], a fully discrete finite element approximation has been analyzed where the space is discretized based on the conforming finite element method and the time is discretized based on Euler incremental projection scheme. Unconditional stability and error estimates for velocity and pressure have been derived for smooth initial data.

We only examine the backward Euler method for less regularity on the initial velocity in the present work and reserve second-order and explicit/implicit schemes for future. Similar works can be seen in [59, 60]. We observe in [59, Chapter 4] that the author has applied backward Euler method and has obtained optimal $\mathbf{L}^2$ error for the velocity for nonsmooth initial data

$$\|\mathbf{u}_h(t_n) - \mathbf{U}^n\| \leq Ce^{Ct_n} t_n^{-1/2} k\big(1 + \log \frac{1}{k}\big)^{3/4}, \quad 1 \leq n \leq N < +\infty,$$

which is local in nature (that is, the estimates are valid only for a finite time). An improved result and uniform in time estimate have been seen in an unpublished work [60]. However, there are a few crucial technical mistakes in some of the proofs which render the results invalid. Also additional assumptions on time step (for example, Lemma 4.1, 4.2, 4.3, 5.1 and so on) make the results very restrictive and less usable. Hence we have revisited the backward Euler method very carefully and prove the following (see, Remark 2.11), when $\mathbf{u}_0 \in \mathbf{H}_0^1$:

$$\|\mathbf{u}_h(t_n) - \mathbf{U}^n\| \leq Ce^{Ct_n} k t_n^{-1/2}, \quad 1 \leq n \leq N < +\infty,$$

where the error bound constant depends only on the given data and, in particular, is independent of both h and k. However, it grows exponentially with time and therefore, the above error estimate is local (in time). Under uniqueness condition, we have shown the error to be uniformly bounded as $t \to +\infty$, see Chapter 2. Unlike [59, 60], we have given considerable importance to the numerical experiments and validation. Firstly, we have verified the rates of convergence in both space and time variables with smooth as well as nonsmooth initial data. Then we have shown the uniform in time bounds by taking few numerical examples. This work has been published [13].

1.3.2 Two-grid Method

The two-grid method is a highly efficient, accurate, and well-established method for solving nonlinear problems. The idea of the two-grid/multi-grid method was initially introduced by Fedorenko [48, 49] for constructing a fast iterative solver for solving an elliptic problem. This work has been extended for a general elliptic problem with variable coefficient by Bakhvalov [8], and in [17], Brandt has shown the computational ability of the method mentioned above. These results lead to mass acceptance of the method and a vast amount of followed. However recent works on two-grid/multi-grid method have been motivated by Xu for the linear and nonlinear elliptic problems [see, J.Xu, Two grid finite element discretizations for linear and nonlinear elliptic equations, Tech. Report, AM105, Dept. of Mathematics, Pennsylvania State University, University Park, July, 1992] that involves two grids of different sizes for solving the problems. It is observed that even for a very coarse mesh, one Newton iteration on a fine mesh with coarse mesh solution as an initial guess is rather an optimal approximation. This idea is subsequently extended to semi-linear/nonlinear elliptic equations [147, 148] and steady-state NSE by Layton *et al.* [91, 92].

The main idea of the two-grid method is to solve a nonlinear problem over a coarse mesh and then to use the coarse mesh solution to solve an associated linear problem over a fine mesh, thereby making the scheme computationally efficient. The method and the efficiency may vary with the linearized problem that we solve on the fine mesh. For example, Layton *et al.* [91, 92] have considered different algorithms for linearizing the NSE; linearizing based on the discrete Stokes problem or discrete steady Oseen problem or one step Newton method. The key feature of this method is that it can reduce the complexity of the original problem and save computational time.

To our knowledge, two-grid method has been applied to our problem only in [60]. The analysis has been carried out for less regular initial data, and the optimal $\mathbf{H}^1$ error for velocity and L^2 error for the pressure of order $\mathcal{O}(H^2t^{-1})$ has been obtained. However, the $\mathbf{L}^2$ error estimate is sub-optimal, which is a drawback of the method applied there. In the framework of NSE also, this has been observed, see [40, Remark 2].

Literature of two-grid/multi-grid method is abundant in the case of NSEs. For example, Girault *et al.* [55] in their work on steady-state NSEs have obtained the error estimate, and these works have been extended for transient NSE in [56]. In [1], Abboud *et al.* have further extended this analysis to the fully discrete case, and in [2] the second-order Hood-Taylor finite element has been used for the spatial discretization.

In [40], a two-grid method for the transient Navier-Stokes equations has been studied, employing three mixed-finite elements of first, second, and third-order, namely, the mini-element, the quadratic, and cubic Hood-Taylor elements , and two time discretization schemes, namely, the first-order backward Euler method and the second-order backward difference method. The rate of convergence in $\mathbf{H}^1$-norm is recovered by taking $h = H^2$ which is an improvement over the result with $h = H^{3/2}$ obtained in [55]. In addition, they have considered "the lack of regularity of the solution" at the initial time and have assumed $\mathbf{u}_0$ to be in $\mathbf{H}_0^1 \cap \mathbf{H}^2$. We note here that demanding further regularity requires the data to satisfy nonlocal compatibility conditions unlikely to be fulfilled in practical situations [79]. The regularity has been further reduced in an article by Goswami *et al.* [62] where a two-grid method for Navier-Stokes equations has been considered but only for linear approximation. There $\mathbf{u}_0 \in \mathbf{H}_0^1$ has been considered and no more, that is, $\|\mathbf{u}\|_2 \approx \mathcal{O}(t^{-1/2})$.

As mentioned earlier, the only work in Oldroyd model [60] has a sub-optimal $\mathbf{L}^2$ error estimate for velocity. Our main objective of Chapter 3 is to remedy this. Also, unlike [60], here we consider the fully discrete case using the backward Euler method for time discretization. In this work, we are going to extend the algorithm which was proposed earlier by Xu [147] for a nonlinear elliptic problem and adapted by Dai *et. al.* [38] for steady-state NSE and Pani *et. al.* [7] for transient NSE. We want to note here that our work here is quite close to the work in [7], for NSE. There, the authors have obtained optimal error estimate in $\mathbf{H}^1$-norm in velocity and L^2-norm in pressure with choice $h = H^{4-\ell}$. Also they have obtained the $\mathbf{L}^2$ error estimate for velocity with choice of $h = H^{2-\ell}$ for arbitrary small $\ell > 0$. All these analyses have been done by taking the initial velocity $\mathbf{u}_0$ in $\mathbf{H}_0^1 \cap \mathbf{H}^2$. In Chapter 3 of this thesis, we have obtained optimal error bounds for the semidiscrete as well as fully discrete approximations for nonsmooth initial data, that is, the initial velocity $\mathbf{u}_0$ in $\mathbf{H}_0^1$ not in $\mathbf{H}^2$. However, our work differs in several instances. First of all, this is the first time that this method has been applied to our model. The presence of the memory term along with the nonlinear term demands new technique and more sophistication. Also, we consider nonsmooth initial data, and this loss of regularity presents technical challenges, more notably in the fully discrete case. Finally, we give some numerical examples to verify the rates of convergence and time efficiency of our scheme. This work has been published [11].

1.3.3 Penalty Method

It is observed for a long time that the velocity $\mathbf{u}$ and the pressure p are coupled together by the incompressibility condition $div\ u = 0$ in the problem (1.4)-(1.6). This makes it difficult to solve the system numerically. A common way to handle this difficulty is to address the incompressibility condition, in other words, to relax this condition appropriately. The methods that come to our mind are the penalty method, the artificial compressibility method, the pressure stabilized method, and the projection method (see, for instance, Shen[125] and references, therein).

In Chapter 4 of this thesis, we will discuss the penalty method, which is the most straightforward and most effective finite element implementation to handle the incompressibility. This method is often employed in order to decouple the pressure equation from the system of nonlinear algebraic equations in velocity, which is obtained from finite element (or finite difference) discretizations.

The penalty approach was introduced by Courant [36] in the context of the calculus of variations. This idea has been widely used in the different areas of computational fluid dynamics (for instances, see [73, 74] and references therein). Besides the applications to constrained variational problems and variational inequalities, the penalty method is now a useful tool for numerical computations in continuum fluid and solid mechanics. Note that Temam, in the late 1960's [128], has initiated its application to Navier-Stokes equations. From that period, many works were devoted to studying the penalty method for the steady Stokes and Navier-Stokes equations and unsteady Navier-Stokes equations. However, error estimates are not optimal for the penalized (unsteady) Navier-Stokes equation for a long time. In 1995, Shen [125] has obtained optimal error estimates for the penalized system and its time discretizations for the unsteady Navier-Stokes, mainly of order $O(\varepsilon)$ (where ε is the penalty parameter) for both $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}^1)$. Backward Euler method is also employed for the time discretization of the penalized system, and optimal error estimates are obtained. In 2005, He *et al.* [73] extended Shen's analysis to the finite element approximations to the Navier-Stokes equations and the following error estimate has been derived for the conforming fully discrete finite element method for all $t_n \in [0,T], T > 0$

$$\tau(t_n)\|\mathbf{u}(t_n) - \mathbf{u}^n_{\varepsilon h}\|_1 + \Big(k\sum_{m=0}^{n}\tau^2(t_m)\|p(t_m) - p^n_{\varepsilon h}\|^2\Big)^{\frac{1}{2}} \le C(\varepsilon + h + k), \tag{1.25}$$

where $(\mathbf{u}(t_n), p(t_n))$ and $(\mathbf{u}^n_{\varepsilon h}, p^n_{\varepsilon h})$ are the solutions of the Navier-Stokes equation and its fully discrete penalized system, respectively, C is a positive constant, h is the mesh size, k is the time step, $t_n = nk, 0 \le n \le N = T/k, \tau(t_n) = \min\{t_n, 1\}$. We would like to note here that both the above mentioned works have been carried out for smooth initial data, that is, $\mathbf{u}_{\varepsilon 0} \in \mathbf{H}^2 \cap \mathbf{H}^1_0$.

For our problem (1.4)-(1.6), the literature is relatively limited. Only in the early '90s, Kotsiolis and Oskolkov [89] and later Oskolkov [112] have studied the penalty method for the Oldroyd model of order one and also of higher orders. In [89], the solvability of the initial boundary value problem for the equations of slightly compressible Jeffrey-Oldroyd model and penalized equations of Jeffrey-Oldroyd model for smooth boundary and smooth initial data has been studied with the forcing term in $L^\infty(\mathbf{L}^2)$. We would like to observe here that the Jeffrey-Oldroyd model of order one turns out to be our model (1.4)-(1.6). In [112], the authors have discussed the penalty method for three different equations of viscoelastic media, namely, the Maxwell equations, the Jeffrey-Oldroyd equations, and the Kelvin-Voight equations. Unlike in [89], here, the constraint $div\ u = 0$ is penalized in a different manner, that is, adding the integral term to the penalization. ε-dependent global classical solvability has been obtained for all three models, with the forcing term in $L^\infty(\mathbf{L}^2)$. Also, sub-optimal error estimates for the time discretization have been observed for the penalized systems.

Wang *et. al.* [138] have investigated the relations between the penalty parameter and the time step for the linearized Oldroyd model of order one. In fact, they have obtained optimal error estimates for the penalized system and the time discretized (backward Euler) penalized system. In Wang and He [136], similar results are observed as in [138], except for the fact that the problem is now nonlinear and the estimates are uniform, derived under the smallness assumption on given data. Subsequently, Wang *et al.* have extended the analysis in [139, 140] to the finite element approximations of the problem (1.4)-(1.6) and have derived optimal error estimate in $\mathbf{H}^1$-norm similar to the estimate in (1.25) for smooth initial data.

However there has been no published work in the literature for the optimal semidiscrete penalty error estimate in $L^\infty(\mathbf{L}^2)$-norm not only for our model but also for Navier-Stokes equations. Also, there is hardly any result on optimal error estimate in $L^\infty(\mathbf{L}^2)$-norm for nonsmooth initial data for the time discretization. In [59, Chapter 6], an attempt has been made to find an optimal $L^\infty(\mathbf{L}^2)$ error estimates for the semidiscrete penalized velocity for nonsmooth initial data. But these bounds are not ε-uniform (the bounds depend on $1/\varepsilon$) along standing problem till recently.

Therefore, in Chapter 4 of our thesis, we make an attempt to establish the ε-uniform error estimates in $L^\infty(\mathbf{L}^2)$-norm for both spatial and time-discretization schemes. Also, in [139], the results have been obtained for the smooth initial data, that is, when the initial data $\mathbf{u}_{\varepsilon 0}$ belongs to $\mathbf{H}_0^1 \cap \mathbf{H}^2$, but we aim to discuss error analysis for the nonsmooth initial data, that is, the initial data $\mathbf{u}_{\varepsilon 0}$ in $\mathbf{H}_0^1$ not in $\mathbf{H}^2$. This work has been published [14].

The result above can easily be carried over to the Navier-Stokes problem and similar ε-uniform error estimates in $L^\infty(\mathbf{L}^2)$-norm for both spatial and time-discretization schemes can be derived. In this thesis, we restrict ourselves only to the final results, omitting the analysis for the penalized Navier-Stokes equations. This work has been submitted for publication [12].

1.3.4 Grad-div Stabilization

As has been mentioned earlier, the Olrdoyd model of the order one suffers from the coupling of the momentum and continuity equations. Although Galerkin mixed finite element for the model has been successfully analysed on a few occasions [63, 76] with optimal error estimates, the coupling of the velocity and the pressure through the divergence-free term, is not desirable. Methods for decoupling by various means, like the penalty method, the artificial compressibility method, the pressure correction method, the projection method, etc., attempt to overcome this difficulty by introducing artificial conditions. Work in these directions for the Oldroyd model can be found in [14, 99, 136, 139, 152]. Unfortunately, these methods do not address the instability due to the high Reynolds number. This is due to the domination of the advection term on the viscous term, which typically arises for small values of viscosity. It is handled via methods based on stabilization techniques, like streamline upwind/Petrov-Galerkin(SUPG) method, residual-free bubbles enrichment method, local projection stabilization, and interior-penalty methods, see, [16, 20–22]. In particular, in the SUPG method, a grad-div stabilization is included, which allows achieving stability and accuracy for small values of viscosity.

In Chapter 5, we analyze the effect of grad-div stabilization for the Oldroyd model of order one when the Reynolds number is very high. The main idea is to add a stabilization term with respect to the continuity equation to the momentum equation. Franca and Hughes [51] first proposed it to improve the conservation of mass in the finite element method. However, the method comes with several other benefits. For example, the use of grad-div stabilization results in (i) improved convergence of preconditioned iteration when the stabilization parameter is too small [107], (ii) the well-posedness of the continuous solution, as well as the accuracy and convergence of the numerical approximation for small values of viscosity [108], (iii) the local mass balance of the system in numerical experiments [39]. Moreover, it has been observed that using grad-div stabilization in the simulation of turbulent flows is sufficient for performing a stable simulation, see [86, fig. 3] and [120, fig. 7].

These observations lead us to the work of Chapter 5: to derive the error bounds that do not depend on inverse powers of viscosity, for the Galerkin mixed finite element method with grad-div stabilization applied to the Oldroyd model of order one. This is not the first time where similar results have been achieved. In fact, in [41, 42], de Frutos *et al.* have obtained error bounds with constants independent of inverse powers of viscosity for evolutionary Oseen equations and Navier-Stokes equations, respectively. There are a few more works in this direction for incompressible flow problems, but there is no work available for the Oldroyd model of order one to the best of our knowledge.

In Chapter 5, we extend the analysis of [42] to the Oldroyd model of order one. As in [42], we have carried out our analysis for sufficiently smooth initial data, that is, $\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^k$ $(k > 2)$, as well as for smooth initial data, $\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^2$. However, our proofs are shorter and technically less involved than the ones from [42], especially when $\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^2$.

We note here that the assumption of sufficiently smooth data comes at the cost of non-local compatibility conditions of various order, for the given data, at time $t = 0$. Without these conditions, which do not arise naturally, the solutions of the Oldroyd model of order one can not be assumed to have more than second-order derivatives bounded in $L^2(\Omega)$ at $t = 0$ (see [63]). The analysis for smooth initial data takes into account this lack of regularity at $t = 0$.

We would also like to point out that the analysis in both these cases does not differ by much. However, the analysis suggests that less regularity of the initial velocity restricts the order of finite element approximation when keeping estimates independent of the inverse of viscosity. For example, with only smooth initial data, we may get a maximum of second-order convergence rate in case of velocity, even if we employ higher-order approximations, see Remark 5.5.

Another important aspect of our study is the appropriate choice of the stabilization parameter. It is well known that the suitable choice of stabilization parameter for any stabilized scheme is vital for accuracy in numerical simulations. In the case of grad-div stabilization, a suitable choice of grad-div parameter ρ is shown to be $\mathcal{O}(1)$ for the Navier-Stokes equations and inf-sup stable finite element pairs, in [107, 109]. And in [101], it is shown that error can be minimized for $\rho \approx 10^{-1}$. However, larger values of ρ may be needed in special cases, see [53]. A detailed investigation of the choice of grad-div stabilization parameter for steady Stokes problem has been discussed in [82]. They have observed that the choice of grad-div parameter depends on the used norm, the mesh size, the type of mesh, the viscosity, the finite element spaces, and the solution. A similar analysis and numerical simulations have been seen in [4] for the steady-state Oseen problem and Navier-Stokes equations. The end of this chapter, we briefly look into this aspect. Based on the error estimate from Theorem 5.1, we have observed that $\rho = \mathcal{O}(1)$ is a suitable choice for stable mixed finite element spaces. And for MINI element, the choice of ρ can be in the range of h^2 to 1, see Remark 5.4. Further, some numerical experiments are carried out. First, we have shown numerically that the grad-div parameter depends on the mesh size, the viscosity and the finite element spaces. Next we have obtained values of grad-div parameter ρ that minimize the $\mathbf{L}^2$ and $\mathbf{H}^1$ errors for the velocity and L^2 error for the pressure, for a known solution. This work has been submitted for publication [10].

1.3.5 Nonconforming Finite Element Method

The conforming finite element spaces, that have been used for our model, need to satisfy the discrete inf-sup condition for a stable solution, and this leads to the use of complex elements (conforming stable pairs like (P_1b, P_1), (P_2, P_1), etc.) of limited applicability. In [37], several combinations of simpler nonconforming finite elements which violate the inter-element continuity condition of the velocities have been analyzed for Stokes problem. The methods have been shown to be stable, and optimal error estimates have been derived in the energy norm and the L^2-norm. Stable and optimal results have been shown even for constant pressures paired with nonconforming piecewise linear velocities. Later on, several works appeared, extending these to steady and unsteady NSEs; and with works on lower-order and equal order finite elements, for examples [88, 98, 100, 146, 155], to name a few.

To the best of our knowledge, there is no work available in nonconforming finite elements for the Oldroyd model of order one. Also the work on the lower-order spaces is limited; for example, in [142], the lowest equal order conforming elements (P_1, P_1) triangle element and (Q_1, Q_1) quadrilateral element have been analyzed for the Oldroyd model of one with stabilization, based on two local Gauss integrations. And in [151], a characteristic scheme has been considered for (P_1, P_1). A stabilization term has also been added to the discrete weak formulation to get a stable solution. In the case of the lowest order nonconforming pair, i.e. (P_1^{NC}, P_0), since the discrete LBB condition is satisfied, a stable simulation can be performed without any stabilization. We have considered in our thesis the (P_1^{NC}, P_0) elements to approximate the Oldroyd model of order one.

We then consider the Euler incremental pressure correction method for time discretization. It is a time discrete projection method. Projection methods were first studied in the late 1960s by Chorin [34] and Temam [129] for the incompressible time-dependent Navier-Stokes equations. We can classify this method in three classes: Velocity-correction [67], pressure-correction [66, 124, 127, 144], and consistent splitting scheme [65, 105, 126]. A second-order incremental pressure correction scheme for the Navier-Stokes equations has been developed by Van Kan in [134], while Shen *et al.* [68] provided a first-order incremental pressure correction approach.

There is no work on this scheme for Oldroyd model of order one except in [99] where the Euler incremental pressure correction scheme is analyzed for conforming finite element and for smooth initial data. In our work, we analyze this scheme for nonconforming setups with nonsmooth initial data. Stability analysis of the scheme and optimal error analysis for the fully discrete velocity have been discussed. This work will be communicated soon for publication.

1.4 Chapter-wise Outline of the Thesis

The thesis comprises of seven chapters which have been organized as follows.

In Chapter 2, we employ a fully discrete finite element method based on the first-order backward difference scheme for time discretization. *A priori* estimates and regularity estimates of the discrete solution and optimal error estimate for the velocity are shown there. Also, under the uniqueness condition, the error is shown to be uniform in time. We carry out our analysis for nonsmooth initial data. Numerical experiments are given in support of the theoretical results. Chapter 3 deals with the difficulties due to the nonlinearity of the problem. We apply a three-step two-grid method to the Oldroyd model of order one. Optimal error estimates are presented for the semidiscrete as well as for the fully discrete scheme with nonsmooth initial data. We present numerical simulations to substantiate our theoretical findings and establish the time efficiency of this method.

In Chapter 4, we consider a penalty finite element method for the Oldroyd model of order one. We obtain new *a priori* and regularity estimates for the penalized solution, semidiscrete penalized solution as well as for the fully discrete penalized solution. Optimal error estimates for the semidiscrete and the fully discrete penalized problems have been derived for nonsmooth initial data. Numerical experiments are presented to defend the theoretical results. In Chapter 5, we analyse an inf-sup stable finite element Galerkin method with grad-div stabilization. Optimal error estimates are obtained when the value of the viscosity parameter is minimal. Also, we show the effect of the grad-div stabilization parameter for small values of μ by doing few numerical experiments.

In Chapter 6, we consider a lower order nonconforming finite element space and obtain optimal error estimates for the semidiscrete solution. Then, a fully discrete scheme is anayzed where time is discretized based on Euler incremental pressure correction method. All the analysis are carried out for nonsmooth initial data. Finally, in our last chapter, Chapter 7, we critically analyze our findings and our plan for future.