Chapter 2

Backward Euler Method

In this chapter, we study a time discrete scheme based on the first-order implicit backward Euler (BE) method, applied to the semidiscrete approximation of the Oldroyd model of order one. We present uniform in time bounds and optimal error estimates for the velocity when the initial data is nonsmooth. We also show that the estimates are valid as $t \to \infty$ under the uniqueness condition. We conclude the chapter with some numerical examples verifying the theoretical findings. This work has been published in [13].

2.1 Introduction

We recall here the semi and fully discrete formulations for our problem. Find a pair $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{H}_h \times L_h$ that satisfy, for $t > 0$,

$$\begin{aligned}(\mathbf{u}_{ht}, \mathbf{v}_h) + \mu a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{v}_h) ds \\ - (p_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_h, \end{aligned} \tag{2.1}$$

and $(\nabla \cdot \mathbf{u}_h, \chi_h) = 0$, for all $\chi_h \in L_h$, with $\mathbf{u}_h(0) = \mathbf{u}_{0h} \in \mathbf{H}_h$ is an appropriate approximation of the initial velocity $\mathbf{u}_0$ in $\mathbf{J}_1$. Here, h is the mesh size and, $\mathbf{H}_h$ and L_h are the finite element spaces that approximate the velocity and the pressure spaces, respectively.

An equivalent formulation is: For $t > 0$ and for all $\mathbf{v}_h \in \mathbf{J}_h$, seek $\mathbf{u}_h(t)$ in $\mathbf{J}_h$ with $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ such that

$$(\mathbf{u}_{ht}, \mathbf{v}_h) + \mu a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{v}_h) ds = (\mathbf{f}, \mathbf{v}_h), \tag{2.2}$$

where $\mathbf{J}_h$ is the discrete divergence free space. And a fully discrete formulation reads as: For $1 \leq n \leq N$, find $\{\mathbf{U}^n\}_{1\leq n\leq N} \in \mathbf{H}_h$ and $\{P^n\}_{1\leq n\leq N} \in L_h$ satisfying the following system:

$$(\partial_t \mathbf{U}^n, \mathbf{v}_h) + \mu a(\mathbf{U}^n, \mathbf{v}_h) + a(q_r^n(\mathbf{U}), \mathbf{v}_h) + b\left(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h\right) - (P^n, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h), \tag{2.3}$$

for $\mathbf{v}_h \in \mathbf{H}_h$ with $(\nabla \cdot \mathbf{U}^n, \chi_h) = 0$, for all $\chi_h \in L_h,\ n \geq 0$. Here, k is the uniform time step, ∂_t is the backward difference operator and q_r^n is the approximation of the integral term by right rectangle rule and $\mathbf{U}^0 = \mathbf{u}_{0h}$. An equivalent formulation when $\mathbf{v}_h \in \mathbf{J}_h$ reads as: look for $\mathbf{U}^n$ in $\mathbf{J}_h$ with $1 \leq n \leq N$ satisfying

$$(\partial_t \mathbf{U}^n, \mathbf{v}_h) + \mu a(\mathbf{U}^n, \mathbf{v}_h) + a(q_r^n(\mathbf{U}), \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{J}_h. \tag{2.4}$$

Here again, we choose $\mathbf{U}^0 = \mathbf{u}_{0h} \in \mathbf{J}_h$.
The fully discrete formulation (2.4) has been studied for stability and error analysis on a couple of occasions; once for $\mathbf{f} = 0$ and in a linearized set up by Pani *et. al* [117], and on the other occasion, for non-zero $\mathbf{f}$ and for the full nonlinear problem by Wang *et. al* [137]. In both the cases, initial data has been considered in $\mathbf{H}_0^1 \cap \mathbf{H}^2$. We present below the results obtained in [137], which has been carried out under smallness condition on data and for smooth initial data:

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \leq C(h^2 + k), \quad (\tau^*)^{1/2}\|\mathbf{u}(t_n) - \mathbf{U}^n\|_1 \leq C(h + k),$$

where $\mathbf{u}(t_n)$ and $\mathbf{U}^n$ are the solution of (1.8) and (2.4) respectively, and $\tau^*(t_n) = \min\{1, t_n\}$.

In this chapter, we have shown similar global results as obtained in [137], but for nonsmooth initial data. Furthermore, we have established local optimal $\mathbf{L}^2$-velocity error estimate. We prove here the following estimate when $\mathbf{u}_0 \in \mathbf{H}_0^1$:

$$\|\mathbf{u}_h(t_n) - \mathbf{U}^n\| \leq K_n t_n^{-1/2} k \big(1 + \log \frac{1}{k}\big)^{1/2}, \quad 1 \leq n \leq N < +\infty,$$

with the error constant $K_n > 0$ which may depend on given data but not depend on h and k. It grows exponentially with time, that is, $K_n \sim O(e^{t_n})$ and therefore, the above error estimate is local (in time). For smallness condition on data (we term it as uniqueness condition), we have shown the error to be uniformly bounded as $t \to +\infty$.

As mentioned earlier, we carry out our analysis for nonsmooth initial data, that is, for less regular data, that leads to realistic regularity of the exact solution of the problem (1.4)-(1.6). This forces singular behaviour in higher order Sobolev norms of the solution as $t \to 0$. For example, Lemma 2.1 says that $\|\mathbf{u}_h(t)\|_2$ and $\|\mathbf{u}_{ht}\|$ are of $O(t^{-1/2})$. As in [63], this breakdown at $t = 0$ is a major bottle-neck in our error analysis. To illustrate our point, consider $\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^2$ (smooth initial data). Then, the error $\mathbf{e}_n = \mathbf{U}^n - \mathbf{u}_h(t_n)$ satisfies the following estimate (see, [137, Lemma 4.2]):

$$\|\mathbf{e}_n\| \sim O(k), \quad 1 \leq n \leq N,$$

Following similar argument but now with $\mathbf{u}_0 \in \mathbf{H}_0^1$, we would only obtain (see, (2.101))

$$\|\mathbf{e}_n\| \sim O(k^{1/2}\big(1 + \log \frac{1}{k}\big)^{1/2}), \quad 1 \leq n \leq N.$$

The loss in the order of k, in fact, is due to the singularity of the higher-order norms of the solution at $t = 0$. The standard technique, in such cases, is to multiply by a weight $t^r,\ r \in \mathbb{N}$ to compensate for this singularity, thereby, recovering full order of convergence. But in our case, a direct application of this technique fails due to the presence of the memory term. It is noted that the kernel β present in the equation (1.4) has a certain positivity property (see Lemma 1.5) and we choose our quadrature rule to conform with it, see (1.18). This is crucial to our analysis. But when we opt for weighted Sobolev norm with a weight $t^r,\ r \in \mathbb{N}$, it nullifies the positivity property of the quadrature rule. Hence the main effort, when dealing with nonsmooth data, is to overcome this difficulty and to recover optimal fully discrete error estimate for the velocity, while working with weighted Sobolev norms. This requires borrowing certain tools from the realm of linear parabolic integro-differential equations that works for less regular data (see; [114, 115, 132]), like, the summation technique (we call it here "hat operator", see (2.79)), which adds to the technicality. Since singularity at $t = 0$ is more prominent for higher order Sobolev norm (see Lemma 2.1), analysis gets more technically involved in case of higher order time discretization scheme.

We now summarize our main results of this chapter as follows:

(i) Uniform bound in time for the fully discrete solution in the Dirichlet norm depicting long term stability (Lemma 2.6).

(ii) New uniform estimates for the error associated with fully discrete linearized problem (Lemma 2.12).

(iii) Local optimal $\mathbf{L}^2$ velocity error estimate (Theorem 2.1).

(iv) Optimal global fully discrete error estimates under the uniqueness assumption (Theorem 2.2).

(v) Numerical experiments to validate the theoretical findings (Section 2.5).

We would like to point out here that our main focus in this chapter is to find the optimal $\mathbf{L}^2$ velocity error estimate and hence we have avoided presenting here the optimal L^2 pressure and $\mathbf{H}^1$ velocity error estimates. These are presented in Chapter 3 and have been carried out in a slightly different fashion (see Remark 2.10).

2.2 *A Priori* and Regularity Estimates

We begin this section by presenting two Lemmas that deal with *a priori* and regularity bounds as well as negative norm estimate of the semidiscrete solution $\mathbf{u}_h$. These will be of use susbsequently, when we do error analysis for the nonsmooth data. Next Lemmas will be on the *a priori* estimates of the fully discrete solution $\mathbf{U}^n$.

Lemma 2.1. *Suppose* (**A1**)-(**A2**), (**B1**)-(**B2**) *hold. Moreover, let* $\mathbf{u}_h(0) \in \mathbf{J}_h$. *Then, for* $0 < \alpha < \min\{\delta, \mu\lambda_1\}$, *the semidiscrete solution* $\mathbf{u}_h$ *of (2.2) satisfies the following bounds:*

$$\|\mathbf{u}_h(t)\|_r^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_{r+1}^2 ds \le C, \qquad r \in \{0,1\} \tag{2.5}$$

$$\tau^*\|\mathbf{u}_h\|_2^2 + (\tau^*)^{r+1}(t)\|\mathbf{u}_{ht}\|_r^2 \le C, \qquad r \in \{0,1\}, \tag{2.6}$$

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau^*)^r(s)\|\mathbf{u}_{hs}\|_r^2\, ds \le C, \qquad r \in \{0,1,2\}, \tag{2.7}$$

where $\tau^*(t) = \min\{1, t\}$ *and* $C > 0$ *is a constant that depends on the given data, but not on time* t.

Since the preceding Lemma's proof is analogous to that of Lemma 1.8 from continuous case, we have avoided a proof.

Lemma 2.2. *Consider the assumptions of the previous Lemma. Then, the semidiscrete solution* $\mathbf{u}_h$, *satisfies the following estimates:*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau^*)^{r+1}(s)\|\mathbf{u}_{hss}\|_{r-1}^2\, ds \le C, \qquad r \in \{-1,0,1\}.$$

Proof. We differentiate the semidiscrete Galerkin formulation on $\mathbf{J}_h$, that is, (2.2) to find that

$$(\mathbf{u}_{htt}, \mathbf{v}_h) + \mu a(\mathbf{u}_{ht}, \mathbf{v}_h) + \beta(0) a(\mathbf{u}_h, \mathbf{v}_h) - \delta \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{v}_h)\, ds$$
$$= -b(\mathbf{u}_h, \mathbf{u}_{ht}, \mathbf{v}_h) - b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{v}_h) + (\mathbf{f}_t, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{J}_h. \tag{2.8}$$

Choose $\mathbf{v}_h = (\tau^*)^2(t) e^{2\alpha t} \mathbf{u}_{htt}$ in (2.8) to obtain

$$(\tau^*)^2(t) e^{2\alpha t} \|\mathbf{u}_{htt}\|^2 + \mu a(\mathbf{u}_{ht}, (\tau^*)^2(t) e^{2\alpha t} \mathbf{u}_{htt})$$
$$= (\tau^*)^2(t) e^{2\alpha t} \Big(-\gamma a(\mathbf{u}_h, \mathbf{u}_{htt}) + \delta \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{u}_{htt})\, ds$$
$$- b(\mathbf{u}_h, \mathbf{u}_{ht}, \mathbf{u}_{htt}) - b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{u}_{htt}) + (\mathbf{f}_t, \mathbf{u}_{htt}) \Big). \tag{2.9}$$

We write the second term on the left of inequality (2.9) as

$$\mu\ a(\mathbf{u}_{ht}, (\tau^*)^2(t) e^{2\alpha t} \mathbf{u}_{htt}) = \frac{\mu}{2} (\tau^*)^2(t) e^{2\alpha t} \frac{d}{dt} \|\mathbf{u}_{ht}\|_1^2$$
$$= \frac{\mu}{2} \frac{d}{dt} \big((\tau^*)^2(t) e^{2\alpha t} \|\mathbf{u}_{ht}\|_1^2 \big)$$
$$- \mu \big(\alpha (\tau^*)^2(t) + \tau^*(t) \frac{d}{dt} (\tau^*)(t) \big) e^{2\alpha t} \|\mathbf{u}_{ht}\|_1^2.$$

Next we use the "Cauchy-Schwarz inequality" in the first, second and last terms on the right of inequality (2.9) and incorporate all these in (2.9) to obtain

$$(\tau^*)^2(t) e^{2\alpha t} \|\mathbf{u}_{htt}\|^2 + \frac{\mu}{2} \frac{d}{dt} \big((\tau^*)^2(t) e^{2\alpha t} \|\mathbf{u}_{ht}\|_1^2 \big) \leq \big(\alpha (\tau^*)^2(t) + \tau^*(t) \big) e^{2\alpha t} \|\mathbf{u}_{ht}\|_1^2$$
$$+ \gamma (\tau^*)^2(t) e^{2\alpha t} \|\mathbf{u}_h\|_2 \|\mathbf{u}_{htt}\| + \delta (\tau^*)^2(t) e^{2\alpha t} \int_0^t \beta(t-s) \|\mathbf{u}_h(s)\|_2 \|\mathbf{u}_{htt}\|\, ds$$
$$+ (\tau^*)^2(t) e^{2\alpha t} \big(|b(\mathbf{u}_h, \mathbf{u}_{ht}, \mathbf{u}_{htt})| + |b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{u}_{htt})| + \|f_t\| \|\mathbf{u}_{htt}\| \big). \tag{2.10}$$

Note that $\tau^*(t) = \min\{1, t\}$ is almost everywhere differentiable and without loss of generality, we may assume that $\frac{d}{dt}\tau^*(t) \leq 1$. Otherwise, we can always first derive these estimate in the interval $(0, 1)$ and then in $(1, t)$, $t > 1$. For the nonlinear terms, we first note that

$$b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{u}_{htt}) = \frac{1}{2} (\mathbf{u}_{ht} \cdot \nabla \mathbf{u}_h, \mathbf{u}_{htt}) - \frac{1}{2} (\mathbf{u}_{ht} \cdot \nabla \mathbf{u}_{htt}, \mathbf{u}_h). \tag{2.11}$$

And we rewrite the second term as follows: (with notations $D_i = \frac{\partial}{\partial x_i}$ and $\mathbf{v} = (v^1, v^2)$)

$$(\mathbf{u}_{ht} \cdot \nabla \mathbf{u}_{htt}, \mathbf{u}_h) = \sum_{i,j=1}^{2} \int_\Omega u_{ht}^i D_i (u_{htt}^j) u_h^j d\mathbf{x} \tag{2.12}$$

$$= -\sum_{i,j=1}^{2}\int_\Omega D_i(u^i_{ht})u^j_{htt}u^j_h d\mathbf{x} - \sum_{i,j=1}^{2}\int_\Omega u^i_{ht}u^j_{htt}D_i(u^j_h)d\mathbf{x}$$
$$= -((\nabla\cdot\mathbf{u}_{ht})\mathbf{u}_{htt},\mathbf{u}_h) - (\mathbf{u}_{ht}\cdot\nabla\mathbf{u}_h,\mathbf{u}_{htt}).$$

Use (2.12) in (2.11) and now use Lemma 1.4 with the "Young's inequality" to bound the nonlinear terms as

$$|b(\mathbf{u}_h,\mathbf{u}_{ht},\mathbf{u}_{htt})| + |b(\mathbf{u}_{ht},\mathbf{u}_h,\mathbf{u}_{htt})| \le \frac{1}{4}\|\mathbf{u}_{htt}\|^2 + C\|\mathbf{u}_{ht}\|_1^2\|\mathbf{u}_h\|_2^2. \tag{2.13}$$

Incorporate (2.13) in (2.10) and apply the "Young's inequality" and kickback argument. Then, we take time integration to obtain

$$\begin{aligned}\mu(\tau^*)^2(t)e^{2\alpha t}\|\mathbf{u}_{ht}\|_1^2 + \int_0^t (\tau^*)^2(s)e^{2\alpha s}\|\mathbf{u}_{hss}(s)\|^2 ds &\le C\int_0^t \tau^*(s)e^{2\alpha s}\|\mathbf{u}_{hs}\|_1^2 ds\\ &+ C\int_0^t (\tau^*)^2(s)e^{2\alpha s}\Big(\big(1+\|\mathbf{u}_{hs}\|^2\big)\|\mathbf{u}_h\|_2^2 + \|\mathbf{f}_t\|^2\Big)ds\\ &+ C\int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\mathbf{u}_h(\tau)\|_2 d\tau\Big)^2 ds.\end{aligned} \tag{2.14}$$

Since $\alpha(\tau^*)^2(t) + \tau^*(t) \le C\tau^*(t)$ and $\tau^*(t) \le 1$. We now estimate the double integral term above by using Hölder's inequality similar to [116, page 761] as

$$\begin{aligned}I &= \int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\mathbf{u}_h(\tau)\|_2 d\tau\Big)^2 ds\\ &= \gamma^2\int_0^t \Big(\int_0^s e^{-(\delta-\alpha)(s-\tau)}e^{\alpha\tau}\|\mathbf{u}_h(\tau)\|_2 d\tau\Big)^2 ds\\ &\le \gamma^2\int_0^t \Big(\int_0^s e^{-(\delta-\alpha)(s-\tau)}d\tau\Big)\Big(\int_0^s e^{-(\delta-\alpha)(s-\tau)}e^{2\alpha\tau}\|\mathbf{u}_h(\tau)\|_2^2 d\tau\Big)ds\\ &\le \frac{\gamma^2}{(\delta-\alpha)}\int_0^t \Big(\int_0^s e^{-(\delta-\alpha)(s-\tau)}e^{2\alpha\tau}\|\mathbf{u}_h(\tau)\|_2^2 d\tau\Big)ds.\end{aligned} \tag{2.15}$$

A use of change of variable in (2.15) yields

$$I \le \frac{\gamma^2}{(\delta-\alpha)}\int_0^t \Big(\int_0^s e^{-(\delta-\alpha)\tau}e^{2\alpha(s-\tau)}\|\mathbf{u}_h(s-\tau)\|_2^2 d\tau\Big)ds. \tag{2.16}$$

First, we apply change of order of integration in (2.16) then use change of variable to obtain

$$\begin{aligned}I &\le \frac{\gamma^2}{(\delta-\alpha)}\int_0^t e^{-(\delta-\alpha)\tau}\Big(\int_\tau^t e^{2\alpha(s-\tau)}\|\mathbf{u}_h(s-\tau)\|_2^2 ds\Big)d\tau\\ &\le \frac{\gamma^2}{(\delta-\alpha)}\int_0^t e^{-(\delta-\alpha)(t-\tau)}\Big(\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_2^2 ds\Big)d\tau\\ &\le \frac{\gamma^2}{(\delta-\alpha)^2}\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_2^2 ds.\end{aligned} \tag{2.17}$$

After using (2.17) in (2.14), we use (2.5)-(2.7) to find

$$\mu\ (\tau^*)^2\|\mathbf{u}_{ht}\|_1^2 + e^{-2\alpha t}\int_0^t (\tau^*)^2(s)e^{2\alpha s}\|\mathbf{u}_{hss}\|^2\ ds \le C.$$

Next, we set $\mathbf{v}_h = -\tau^*(t)e^{2\alpha t}\tilde{\Delta}_h^{-1}\mathbf{u}_{htt}$ in (2.8). From Lemma 1.4 we see that

$$b(\mathbf{u}_{ht}, \mathbf{u}_h, \tilde{\Delta}_h^{-1}\mathbf{u}_{htt}) \le C\|\mathbf{u}_{ht}\|^{1/2}\|\mathbf{u}_{ht}\|_1^{1/2}\|\mathbf{u}_h\|_1\|\mathbf{u}_{htt}\|_{-1},$$

and therefore

$$\begin{aligned}\mu\frac{d}{dt}(\tau^*(t)e^{2\alpha t}\|\mathbf{u}_{ht}\|^2) + \tau^*(t)e^{2\alpha t}\|\mathbf{u}_{htt}\|_{-1}^2 &\le \big(2\alpha\tau^*(t)+1\big)e^{2\alpha t}\|\mathbf{u}_{ht}\|_1^2 \\ &+ C(\mu,\gamma)\tau^*(t)e^{2\alpha t}\|\nabla\mathbf{u}_h\|^2 + 2\|\mathbf{f}_t\|^2 + C(\mu,\delta)e^{2\alpha t}\Big(\int_0^t \beta(t-s)\|\tilde{\Delta}_h\mathbf{u}_h(s)\|\ ds\Big)^2 \\ &+ C(\mu)\tau^*(t)e^{2\alpha t}\Big(\|\nabla\mathbf{u}_h\|^2\|\mathbf{u}_{ht}\|^2 + \|\nabla\mathbf{u}_{ht}\|^2(1+\|\mathbf{u}_h\|\|\nabla\mathbf{u}_h\|)\Big).\end{aligned}$$

After taking time integration, we use (2.17) to estimate the resulting double integration term. Then, we use (2.5)-(2.7) and multiply by $e^{-2\alpha t}$ to conclude

$$\mu\tau^*(t)\|\mathbf{u}_{ht}(t)\|^2 + e^{-2\alpha t}\int_0^t \tau^*(s)e^{2\alpha s}\|\mathbf{u}_{hss}(s)\|_{-1}^2 ds \le C.$$

Finally we set $\mathbf{v}_h = -e^{2\alpha t}\tilde{\Delta}_h^{-2}\mathbf{u}_{htt}$ in (2.8) and proceed as above to arrive at

$$\begin{aligned}\mu\frac{d}{dt}(e^{2\alpha t}\|\mathbf{u}_{ht}\|_{-1}^2) + e^{2\alpha t}\|\mathbf{u}_{htt}\|_{-2}^2 \le & Ce^{2\alpha t}\Big((1+\|\nabla\mathbf{u}_h\|^2)\|\mathbf{u}_{ht}\|^2 + \|\mathbf{f}_t\|^2\Big) \\ &+ Ce^{2\alpha t}\Big(\int_0^t \beta(t-s)\|\mathbf{u}_h(s)\|\ ds\Big)^2.\end{aligned}$$

We now take integration on the both sides and handle the resulting double integral term as above. Then, we use (2.5)-(2.7) and multiply by $e^{-2\alpha t}$ to obtain

$$\mu\|\mathbf{u}_{ht}(t)\|_{-1}^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_{hss}(s)\|_{-2}^2 ds \le C.$$

This concludes the proof. $\square$

We now prove *a priori* results for the fully discrete solutions $\{\mathbf{U}^n\}$.

Lemma 2.3. *Suppose the conditions* (**A1**) *and* (**A2**) *be satisfied. Then, the following results hold:*

$$\|\mathbf{U}^n\|^2 + \frac{3\mu}{2}k\sum_{i=1}^n \|\nabla\mathbf{U}^i\|^2 \le Ct_n, \tag{2.18}$$

$$\|\mathbf{U}^n\|^2 + \mu e^{-2\alpha t_n}k\sum_{i=1}^n e^{2\alpha t_i}\|\nabla\mathbf{U}^i\|^2 \le e^{-2\alpha t_n}\|\mathbf{U}^0\|^2 + \frac{e^{2\alpha k}}{\alpha\mu\lambda_1}\|\mathbf{f}\|_\infty^2 \le C, \tag{2.19}$$

where $\|\mathbf{f}\|_\infty = \|\mathbf{f}\|_{L^\infty(\mathbb{R}_+;\mathbf{L}^2(\Omega))}$ *and* α *is a parameter of our choice satisfying* $0 < \alpha < \min\{\delta, \frac{\mu\lambda_1}{2}\}$ *and*

$$1 + \big(\frac{\mu\lambda_1}{2}\big)k \geq e^{\alpha k}. \tag{2.20}$$

Proof. Although the proof is similar to [117, Lemma 9], we have provided a sketch below for the sake of completeness. Set $\mathbf{v}_h = \mathbf{U}^i$ in the fully discrete equation (2.4) for $n = i$ to obtain

$$(\partial_t \mathbf{U}^i, \mathbf{U}^i) + \mu\|\nabla \mathbf{U}^i\|^2 + a(q_r^i(\mathbf{U}), \mathbf{U}^i) = (\mathbf{f}^i, \mathbf{U}^i) - b(\mathbf{U}^i, \mathbf{U}^i, \mathbf{U}^i). \tag{2.21}$$

Observe that

$$(\partial_t \mathbf{U}^i, \mathbf{U}^i) = \frac{1}{k}(\mathbf{U}^i - \mathbf{U}^{i-1}, \mathbf{U}^i) \geq \frac{1}{2k}(\|\mathbf{U}^i\|^2 - \|\mathbf{U}^{i-1}\|^2) = \frac{1}{2}\partial_t\|\mathbf{U}^i\|^2 \tag{2.22}$$

and that the nonlinear term vanishes, that is, $b(\mathbf{U}^i, \mathbf{U}^i, \mathbf{U}^i) = 0$. A use of "Young's inequality" with (1.7) yields

$$(\mathbf{f}^i, \mathbf{U}^i) \leq \|\mathbf{f}^i\|\|\mathbf{U}^i\| \leq \frac{1}{\sqrt{\lambda_1}}\|\mathbf{f}^i\|\|\nabla\mathbf{U}^i\| \leq \frac{\mu}{4}\|\nabla\mathbf{U}^i\|^2 + \frac{1}{\mu\lambda_1}\|\mathbf{f}^i\|^2. \tag{2.23}$$

A use of (2.22)-(2.23) in (2.21), we deduce that

$$\frac{1}{2}\partial_t\|\mathbf{U}^i\|^2 + \frac{3\mu}{4}\|\nabla\mathbf{U}^i\|^2 + a(q_r^i(\mathbf{U}), \mathbf{U}^i) \leq \frac{1}{\mu\lambda_1}\|\mathbf{f}^i\|^2. \tag{2.24}$$

We now multiply (2.24) by k and take summation over $1 \leq i \leq n$ to obtain

$$\|\mathbf{U}^n\|^2 + \frac{3\mu}{2}k\sum_{i=1}^n \|\nabla\mathbf{U}^i\|^2 + 2k\sum_{i=1}^n a(q_r^i(\mathbf{U}), \mathbf{U}^i) \leq \frac{1}{\mu\lambda_1}k\sum_{i=1}^n \|\mathbf{f}^i\|^2. \tag{2.25}$$

The quadrature term is positive, due to (1.18), and so we drop it. Since $\mathbf{f} \in L^\infty(\mathbf{L}^2)$, then a use of assumption (**A2**) yields

$$\frac{1}{\mu\lambda_1}k\sum_{i=1}^n \|\mathbf{f}^i\|^2 \leq \frac{1}{\mu\lambda_1}\|\mathbf{f}\|_\infty^2 k\sum_{i=1}^n 1 \leq \frac{1}{\mu\lambda_1}M_0 nk \leq Ct_n.$$

Incorporate the above estimate in (2.25), we complete the proof of (2.18).

Note that the bound establised above, that is, Ct_n, depends on time, and hence the estimate is local in nature. This is due to the fact that the forcing term $\mathbf{f}$ is either independent of time or L^∞ in time. The standard technique to overcome this problem is to multiply by a weight $e^{2\alpha t_i}$, resulting in uniform in time bound.

We multiply (2.24) by $ke^{2\alpha t_i}$ and take summation over $1 \leq i \leq n$ to arrive at

$$k\sum_{i=1}^n e^{2\alpha t_i}\partial_t\|\mathbf{U}^i\|^2 + \frac{3\mu}{2}k\sum_{i=1}^n e^{2\alpha t_i}\|\nabla\mathbf{U}^i\|^2 + 2k\sum_{i=1}^n e^{2\alpha t_i}a(q_r^i(\mathbf{U}), \mathbf{U}^i)$$

$$\leq \frac{2}{\mu\lambda_1} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\mathbf{f}^i\|^2. \tag{2.26}$$

As earlier we drop the quadrature term, due to positivity, see (1.18). And using (1.7), the term with ∂_t can be written as

$$\begin{aligned} k\sum_{i=1}^{n} e^{2\alpha t_i}\partial_t\|\mathbf{U}^i\|^2 &= \sum_{i=1}^{n} e^{2\alpha t_i}(\|\mathbf{U}^i\|^2 - \|\mathbf{U}^{i-1}\|^2) \\ &= e^{2\alpha t_n}\|\mathbf{U}^n\|^2 - \|\mathbf{U}^0\|^2 - \sum_{i=1}^{n-1}(e^{2\alpha k}-1)e^{2\alpha t_i}\|\mathbf{U}^i\|^2 \\ &\geq e^{2\alpha t_n}\|\mathbf{U}^n\|^2 - \|\mathbf{U}^0\|^2 - k\sum_{i=1}^{n}\frac{(e^{2\alpha k}-1)}{k\lambda_1}e^{2\alpha t_i}\|\nabla\mathbf{U}^i\|^2 \end{aligned} \tag{2.27}$$

On substituting this in (2.26), we obtain

$$\begin{aligned} e^{2\alpha t_n}\|\mathbf{U}^n\|^2 + \Big(\frac{3\mu}{2} - \frac{e^{2\alpha k}-1}{k\lambda_1}\Big)k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{U}^i\|^2 \\ \leq \|\mathbf{U}^0\|^2 + \frac{2}{\mu\lambda_1}\|\mathbf{f}\|_\infty^2 k\sum_{i=1}^{n} e^{2\alpha t_i}. \end{aligned} \tag{2.28}$$

Note that (2.20) guarantees that $\frac{\mu}{2} \geq \frac{e^{2\alpha k}-1}{k\lambda_1}$. We now use the sum for a finite geometric series and mean value theorem, to find, for some k^* in $(0,k)$ that

$$k\sum_{i=1}^{n} e^{2\alpha t_i} = ke^{2\alpha k}\frac{e^{2\alpha t_n}-1}{e^{2\alpha k}-1} = \frac{1}{2\alpha}e^{2\alpha(k-k^*)}(e^{2\alpha t_n}-1). \tag{2.29}$$

On substituting (2.29) in (2.28), multiply through out by $e^{-2\alpha t_n}$ to conclude the proof.

□

Remark 2.1. *We note here that the assumption (2.20) in Lemma 2.3 is not a smallness condition on time step k. It is in fact a condition we put on α and we may rephrase it as: for $0 < \alpha < \alpha_0$, (2.20) holds. Such a choice of $\alpha_0 > 0$ is possible by choosing $\alpha_0 < \frac{\log(1+\frac{\mu\lambda_1}{2}k)}{k}$. Note that for large $k > 0$, α_0 is small but as $k \to 0$, $\frac{\log(1+\frac{\mu\lambda_1}{2}k)}{k} \to \frac{\mu\lambda_1}{2}$. Thus, with $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu\lambda_1}{2}\}$, we can incorporate both the conditions on α and the second result of Lemma 2.3 is valid.*

Lemma 2.4. *Suppose the conditions* (**A1**) *and* (**A2**) *hold. Further, assume $\alpha_0 > 0$ be such that $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu\lambda_1}{2}\}$, (2.20) be satisfied. Then, the following result holds:*

$$\|\nabla\mathbf{U}^n\|^2 + e^{-2\alpha t_n}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\tilde{\Delta}_h\mathbf{U}^i\|^2 \leq \big(e^{-2\alpha t_n}\|\mathbf{U}^0\|^2 + \frac{e^{2\alpha k}}{\alpha\mu}\|\mathbf{f}\|_\infty^2\big)\exp\{Ct_n\}.$$

Proof. Choose $\mathbf{v}_h = -\tilde{\Delta}_h \mathbf{U}^i$ in the fully discrete equation (2.4) for $n = i$ and use the similar fact of (2.22) to obtain

$$\frac{1}{2}\partial_t \|\nabla \mathbf{U}^i\|^2 + \mu \|\tilde{\Delta}_h \mathbf{U}^i\|^2 + a(q_r^i(\mathbf{U}), -\tilde{\Delta}_h \mathbf{U}^i) \le (\mathbf{f}^i, -\tilde{\Delta}_h \mathbf{U}^i) - b(\mathbf{U}^i, \mathbf{U}^i, -\tilde{\Delta}_h \mathbf{U}^i). \quad (2.30)$$

We apply Lemma 1.4 with the "Young's inequality" to bound the nonlinear term as

$$\begin{aligned} |b(\mathbf{U}^i, \mathbf{U}^i, -\tilde{\Delta}_h \mathbf{U}^i)| &\le C\|\mathbf{U}^i\|^{1/2} \|\nabla \mathbf{U}^i\| \|\tilde{\Delta}_h \mathbf{U}^i\|^{3/2} \\ &\le C(\mu) \|\mathbf{U}^i\|^2 \|\nabla \mathbf{U}^i\|^4 + \frac{\mu}{4} \|\tilde{\Delta}_h \mathbf{U}^i\|^2 \end{aligned} \quad (2.31)$$

Insert (2.31) in (2.30) and use use the "Cauchy-Schwarz inequality" to find

$$\partial_t \|\nabla \mathbf{U}^i\|^2 + \mu \|\tilde{\Delta}_h \mathbf{U}^i\|^2 + 2a(q_r^i(\mathbf{U}), -\tilde{\Delta}_h \mathbf{U}^i) \le \frac{1}{\mu} \|\mathbf{f}^i\|^2 + C(\mu) \|\mathbf{U}^i\|^2 \|\nabla \mathbf{U}^i\|^4). \quad (2.32)$$

Multiply both sides of (2.32) by $ke^{2\alpha t_i}$ and take summation over $1 \le i \le n$. Then, use the similar fact (2.27) to obtain

$$\begin{aligned} e^{2\alpha t_n} \|\nabla \mathbf{U}^n\|^2 &+ \Big(\mu - \frac{e^{2\alpha k} - 1}{k\lambda_1}\Big) k \sum_{i=1}^{n} e^{2\alpha t_i} \|\tilde{\Delta}_h \mathbf{U}^i\|^2 + 2k \sum_{i=1}^{n} e^{2\alpha t_i} (q_r^i(\tilde{\Delta}\mathbf{U}), \tilde{\Delta}_h \mathbf{U}^i) \\ &\le \|\nabla \mathbf{U}^0\|^2 + \frac{1}{\mu} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\mathbf{f}^i\|^2 + C(\mu) k \sum_{i=1}^{n} e^{2\alpha t_i} \|\mathbf{U}^i\|^2 \|\nabla \mathbf{U}^i\|^4. \end{aligned} \quad (2.33)$$

Let $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu\lambda_1}{2}\}$, then we have, $\mu - \frac{e^{2\alpha k} - 1}{k\lambda_1} > 0$. The third term on the left of inequality (2.33) is positive due to (1.18), so we drop it. Finally, we apply the "discrete Gronwall's lemma" to find

$$\begin{aligned} e^{2\alpha t_n} \|\nabla \mathbf{U}^n\|^2 + k \sum_{i=1}^{n} e^{2\alpha t_i} \|\tilde{\Delta}_h \mathbf{U}^i\|^2 &\le \big(\|\nabla \mathbf{U}^0\|^2 + \frac{1}{\mu} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\mathbf{f}^i\|^2\big) \\ &\quad \times \exp\big\{C(\mu) k \sum_{i=1}^{n} \|\mathbf{U}^i\|^2 \|\nabla \mathbf{U}^i\|^2\big\}. \end{aligned} \quad (2.34)$$

Now we use (2.18) in (2.34) and multiply both sides by $e^{-2\alpha t_n}$ to conclude the proof. □

Remark 2.2. *Similar to [63], here, we also observe that the bound of Dirichlet norm of $\mathbf{U}^n$, that is, $\|\nabla \mathbf{U}^n\|,\ 1 \le n \le N$ is exponentially dependent on t_n. In other words, as $t_N \to +\infty$, $\|\nabla \mathbf{U}^n\| \to +\infty,\ 1 \le n \le N$. This is a technical bottle-neck and we do expect a global bound. More importantly, for long-time stability of a implicit scheme, $\|\nabla \mathbf{U}^n\|,\ 1 \le n \le N$ needs to be bounded as $t_N \to +\infty$. In case of Navier-Stokes, the proof of the Dirichlet norm of $\mathbf{U}^n$, which is valid for all time, involves applying discrete version of the "uniform Gronwall's Lemma" (see, [133, Lemma 2.6]). Interestingly, in* *our case, we are not able to apply the "uniform Gronwall's Lemma" directly due to the presence of the quadrature term. Hence, we have adopted a new way of looking into the problem. We have incorporated the ideas behind the "uniform Gronwall's Lemma" to establish our result.*

We start by reformulating our problem as follows: we introduce the following notation:

$$\mathbf{U}_\beta^n = k\sum_{j=1}^{n} \beta_{nj}\mathbf{U}^j, \ n > 0; \quad \mathbf{U}_\beta^0 = 0, \tag{2.35}$$

and rewrite the fully discrete formulation (2.4) as

$$(\partial_t \mathbf{U}^n, \mathbf{v}_h) + \mu a(\mathbf{U}^n, \mathbf{v}_h) + b(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h) + a(\mathbf{U}_\beta^n, \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{J}_h. \tag{2.36}$$

Note that

$$\mathbf{U}_\beta^n = k\gamma\mathbf{U}^n + e^{-\delta k}\mathbf{U}_\beta^{n-1},$$

and therefore,

$$\begin{aligned}\partial_t \mathbf{U}_\beta^n = \frac{1}{k}(\mathbf{U}_\beta^n - \mathbf{U}_\beta^{n-1}) &= \frac{1}{k}\mathbf{U}_\beta^n - \frac{1}{k}e^{\delta k}(\mathbf{U}_\beta^n - k\gamma\mathbf{U}^n) \\ &= \gamma e^{\delta k}\mathbf{U}^n - \frac{(e^{\delta k}-1)}{k}\mathbf{U}_\beta^n. \end{aligned}\tag{2.37}$$

Lemma 2.5. *Suppose the assumptions of Lemma 2.4 hold. Then, the discrete solution* $\mathbf{U}^n$, $1 \le n \le N$, *of (2.4), satisfies the following uniform estimates:*

$$\|\mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma}\|\nabla \mathbf{U}_\beta^n\|^2 \le e^{-\alpha t_n}\|\mathbf{U}^0\|^2 + \left(\frac{1-e^{-\alpha t_n}}{\alpha\mu\lambda_1}\right)\|\mathbf{f}\|_\infty^2 = M_{11}^2, \tag{2.38}$$

and

$$k\sum_{n=m}^{m+l}\big(\mu\|\nabla\mathbf{U}^n\|^2 + \frac{\delta}{\gamma}\|\nabla\mathbf{U}_\beta^n\|^2\big) \le M_{11}^2 + \frac{l}{\mu\lambda_1}\|\mathbf{f}\|_\infty^2 = M_{12}^2(l), \tag{2.39}$$

where $\mathbf{U}_\beta^n$ *is given by (2.35) and* $m, l \in \mathbb{N}$.

Proof. In view of (2.37), we find that

$$a(\mathbf{U}_\beta^n, \mathbf{U}^n) = \frac{e^{-\delta k}}{\gamma}a(\mathbf{U}_\beta^n, \partial_t\mathbf{U}_\beta^n) + \frac{(1-e^{-\delta k})}{k\gamma}\|\nabla\mathbf{U}_\beta^n\|^2.$$

Now take $\mathbf{v}_h = \mathbf{U}^n$ in (2.36), for $n = i$, to find

$$\partial_t\big(\|\mathbf{U}^i\|^2 + \frac{e^{-\delta k}}{\gamma}\|\nabla\mathbf{U}_\beta^i\|^2\big) + 2\mu\|\nabla\mathbf{U}^i\|^2 + 2\Big(\frac{e^{\delta k}-1}{k}\Big)\frac{e^{-\delta k}}{\gamma}\|\nabla\mathbf{U}_\beta^i\|^2 \le 2(\mathbf{f}^i, \mathbf{U}^i). \tag{2.40}$$

An application of the "Poincaré inequality" and the "Cauchy-Schwarz inequality" gives

$$2(\mathbf{f}^i, \mathbf{U}^i) \leq 2\|\mathbf{f}^i\|\|\mathbf{U}^i\| \leq \frac{1}{\mu\lambda_1}\|\mathbf{f}^i\|^2 + \mu\|\nabla\mathbf{U}^i\|^2. \tag{2.41}$$

A use of (2.40) in (2.41) with the kick back "Poincaré inequality" $\mu\|\nabla\mathbf{U}^i\|^2 \geq \mu\lambda_1\|\mathbf{U}^i\|^2$ yields

$$\partial_t\big(\|\mathbf{U}^i\|^2 + \frac{e^{-\delta k}}{\gamma}\|\nabla\mathbf{U}^i_\beta\|^2\big) + \mu\lambda_1\|\mathbf{U}^i\|^2 + 2\Big(\frac{e^{\delta k}-1}{k}\Big)\frac{e^{-\delta k}}{\gamma}\|\nabla\mathbf{U}^i_\beta\|^2 \leq \frac{1}{\mu\lambda_1}\|\mathbf{f}^i\|^2. \tag{2.42}$$

Multiply the inequality (2.42) by $e^{\alpha t_{i-1}}$ and note that

$$\partial_t(e^{\alpha t_i}\boldsymbol{\phi}^i) = e^{\alpha t_{i-1}}\Big\{\partial_t\boldsymbol{\phi}^i + \frac{e^{\alpha k}-1}{k}\boldsymbol{\phi}^i\Big\}. \tag{2.43}$$

Therefore, we obtain from (2.42)

$$\begin{aligned}\partial_t\Big(e^{\alpha t_i}\big(\|\mathbf{U}^i\|^2 + \frac{e^{-\delta k}}{\gamma}\|\nabla\mathbf{U}^i_\beta\|^2\big)\Big) + \Big(\mu\lambda_1 - \frac{e^{\alpha k}-1}{k}\Big)e^{\alpha t_{i-1}}\|\mathbf{U}^i\|^2 \\ + \Big(2(\frac{e^{\delta k}-1}{k}) - (\frac{e^{\alpha k}-1}{k})\Big)\frac{e^{-\delta k}}{\gamma}e^{\alpha t_{i-1}}\|\nabla\mathbf{U}^i_\beta\|^2 \leq \frac{e^{\alpha t_{i-1}}}{\mu\lambda_1}\|\mathbf{f}\|^2_\infty.\end{aligned}$$

With $0 < \alpha < \min\{\alpha_0, \delta, \mu\lambda_1/2\}$, the last two terms on the left of inequality become non-negative and hence, we drop them. Multiply the rest by k and take summation over 1 to n to arrive

$$e^{\alpha t_n}\big(\|\mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma}\|\nabla\mathbf{U}^n_\beta\|^2\big) \leq \|\mathbf{U}^0\|^2 + \frac{e^{-\delta k}}{\gamma}\|\nabla\mathbf{U}^0_\beta\|^2 + \frac{1}{\mu\lambda_1}\|\mathbf{f}\|^2_\infty k\sum_{i=1}^n e^{\alpha t_{i-1}}.$$

From (2.35), $\mathbf{U}^0_\beta = 0$. Now, multiply by $e^{-\alpha t_n}$ to conclude the proof of (2.38). For the second estimate (2.39), we multiply (2.40) by k, sum over m to $m+l$ with $l, m \in \mathbb{N}$ and apply (2.38) to conclude the remaining of the proof. □

Lemma 2.6. *Suppose the assumptions of Lemma 2.4 hold. Then, the discrete solution* $\mathbf{U}^n$, $1 \leq n \leq N$ *of (2.4) satisfies the following uniform estimate:*

$$\|\nabla\mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma}\|\tilde{\Delta}_h\mathbf{U}^n_\beta\|^2 \leq C.$$

Proof. Set $\mathbf{v}_h = -\tilde{\Delta}_h\mathbf{U}^n$ in (2.36) and as in the Lemma 2.5, we now obtain

$$\begin{aligned}\partial_t\big(\|\nabla\mathbf{U}^n\|^2 + \frac{e^{-\delta k}}{\gamma}\|\tilde{\Delta}_h\mathbf{U}^n_\beta\|^2\big) + 2\mu\|\tilde{\Delta}_h\mathbf{U}^n\|^2 + 2\big(\frac{e^{\delta k}-1}{k}\big)\frac{e^{-\delta k}}{\gamma}\|\tilde{\Delta}_h\mathbf{U}^n_\beta\|^2 \\ \leq 2\|\mathbf{f}^n\|\|\tilde{\Delta}_h\mathbf{U}^n\| + 2|b(\mathbf{U}^n, \mathbf{U}^n, -\tilde{\Delta}_h\mathbf{U}^n)|.\end{aligned} \tag{2.44}$$

Use Lemmas 1.4 and 2.5 with the "Young's inequality" to estimate the nonlinear term as

$$
\begin{aligned}
2|b(\mathbf{U}^n,\mathbf{U}^n,-\tilde{\Delta}_h\mathbf{U}^n)| &\le 2C\|\mathbf{U}^n\|^{1/2}\|\nabla\mathbf{U}^n\|^{1/2}\|\nabla\mathbf{U}^n\|^{1/2}\|\tilde{\Delta}_h\mathbf{U}^n\|^{1/2}\|\tilde{\Delta}_h\mathbf{U}^n\| \\
&\le (\frac{9/2}{\mu})^3 M_{11}^2\|\nabla\mathbf{U}^n\|^4+\frac{\mu}{3}\|\tilde{\Delta}_h\mathbf{U}^n\|^2
\end{aligned}
$$

Thus we obtain from (2.44)

$$
\begin{aligned}
\partial_t\big(\|\nabla\mathbf{U}^n\|^2+\frac{e^{-\delta k}}{\gamma}\|\tilde{\Delta}_h\mathbf{U}^n_\beta\|^2\big)+\frac{4\mu}{3}\|\tilde{\Delta}_h\mathbf{U}^n\|^2+2\big(\frac{e^{\delta k}-1}{k}\big)\frac{e^{-\delta k}}{\gamma}\|\tilde{\Delta}_h\mathbf{U}^n_\beta\|^2 \\
\le \frac{3}{\mu}\|\mathbf{f}\|_\infty^2+(\frac{9/2}{\mu})^3 M_{11}^2\|\nabla\mathbf{U}^n\|^4. \qquad (2.45)
\end{aligned}
$$

Choose $\gamma_0>0$ which is to be determined later, and note that

$$
\gamma_0\|\nabla\mathbf{U}^n\|^2=\gamma_0(\mathbf{U}^n,-\tilde{\Delta}_h\mathbf{U}^n)\le\frac{\mu}{3}\|\tilde{\Delta}_h\mathbf{U}^n\|^2+\frac{3}{4\mu}\gamma_0^2\|\mathbf{U}^n\|^2, \qquad (2.46)
$$

and define

$$
g^n=\min\Big\{\gamma_0+\mu\lambda_1-(\frac{9}{2\mu})^3M_{11}^2\|\nabla\mathbf{U}^n\|^2,\ 2\big(\frac{e^{\delta k}-1}{k}\big)\Big\},
$$

and $E^n:=\|\nabla\mathbf{U}^n\|^2+\frac{e^{-\delta k}}{\gamma}\|\tilde{\Delta}_h\mathbf{U}^n_\beta\|^2$ for large enough γ_0 so that $g^n>0$. Now add the two inequalities (2.45) and (2.46) and rewrite the resulting equation as

$$
\partial_tE^n+g^nE^n\le\frac{3}{\mu}\|\mathbf{f}\|_\infty^2+\frac{3}{4\mu}\gamma_0^2\|\mathbf{U}^n\|^2=K_{11}. \qquad (2.47)
$$

Let $\{n_i\}_{i\in\mathbb{N}}$ and $\{\bar{n}_i\}_{i\in\mathbb{N}}$ be two finite subsequences of natural numbers such that

$$
g^{n_i}=\gamma_0+\mu\lambda_1-(\frac{9}{2\mu})^3M_{11}^2\|\nabla\mathbf{U}^{n_i}\|^2,\quad g^{\bar{n}_i}=2\big(\frac{e^{\delta k}-1}{k}\big),\ \forall i.
$$

If for some n,

$$
g^n=\gamma_0+\mu\lambda_1-(\frac{9}{2\mu})^3M_{11}^2\|\nabla\mathbf{U}^n\|^2=2\big(\frac{e^{\delta k}-1}{k}\big)
$$

then without loss of generality, we take the assumption that $n\in\{\bar{n}_i\}$ so as to make the two subsequence $\{n_i\}$ and $\{\bar{n}_i\}$ disjoint. Now for $m,l\in\mathbb{N}$, we write

$$
\begin{aligned}
k\sum_{n=m}^{m+l}g^n&=k\sum_{n=m_1}^{m_{l_1}}g^n+k\sum_{n=\bar{m}_1}^{\bar{m}_{l_2}}g^n \\
&=k\sum_{n=m_1}^{m_{l_1}}\Big(\gamma_0+\mu\lambda_1-(\frac{9}{2\mu})^3M_{11}^2\|\nabla\mathbf{U}^n\|^2\Big)+k\sum_{n=\bar{m}_1}^{\bar{m}_{l_2}}2\big(\frac{e^{\delta k}-1}{k}\big). \qquad (2.48)
\end{aligned}
$$

Here, $\{m_1, m_2, \cdots, m_{l_1}\} \subset \{n_i\}$ and $\{\bar{m}_1, \bar{m}_2, \cdots, \bar{m}_{l_2}\} \subset \{\bar{n}_i\}$ such that $l_1+l_2 = l+1$. Note that l_1 or l_2 could be 0. Using Lemma 2.5, we observe that

$$(\frac{9}{2\mu})^3 k \sum_{n=m}^{m+l} M_{11}^2 \|\nabla \mathbf{U}^n\|^2 \le \frac{9^3 M_{11}^2}{2^3\mu^3} k \sum_{n=m}^{m+l} \|\nabla \mathbf{U}^n\|^2 \le \frac{9^3 M_{11}^2}{2^3\mu^4} M_{12}^2(l) = K_{12}(l).$$

Therefore, from (2.48), we find that

$$k \sum_{n=m}^{m+l} g^n \ge k l_1(\gamma_0 + \mu\lambda_1) - K_{12}(l_1) + 2\big(\frac{e^{\delta k}-1}{k}\big) k l_2.$$

We choose γ_0 such that $kl_1(\gamma_0 + \mu\lambda_1) - K_{12}(l_1) = 2\big(\frac{e^{\delta k}-1}{k}\big)kl_1$, assuming $l_1 \neq 0$, to arrive at

$$k \sum_{n=m}^{m+l} g^n \ge 2\big(\frac{e^{\delta k}-1}{k}\big) t_{l+1}. \tag{2.49}$$

By definition of g^n, we have equality in (2.49) and in fact, $g^n = 2\big(\frac{e^{\delta k}-1}{k}\big)$. Now from (2.47), we obtain

$$\partial_t E^n + 2\big(\frac{e^{\delta k}-1}{k}\big) E^n \le K_{11}.$$

Multiply the above inequality by $e^{\delta t_{n-1}}$ and as in (2.43), we obtain

$$\partial_t(e^{\delta t_n} E^n) + \big(\frac{e^{\delta k}-1}{k}\big) e^{\delta t_{n-1}} E^n \le K_{11} e^{\delta t_{n-1}}.$$

Multiply by k and take summation over 1 to n. Observe that $E^0 = \|\nabla \mathbf{U}^0\|^2$. Finally, multiply by $e^{-\delta t_n}$ to find that

$$E^n \le e^{-\delta t_n} \|\nabla \mathbf{U}^0\|^2 + C.$$

A use of assumption (**A2**) concludes the proof. □

Remark 2.3. *In view of the Lemma 2.6, the following a priori bound is valid:*

$$\tau^*(t_n) \|\tilde{\Delta}_h \mathbf{U}^n\|^2 \le C, \ 1 \le n \le N.$$

2.3 Error Analysis for the Velocity

Here, we find the error bounds for the velocity approximation based on the BE method applied to the semidiscrete Oldroyd model. First, we set, $\mathbf{e}_n = \mathbf{U}^n - \mathbf{u}_h(t_n) = \mathbf{U}^n - \mathbf{u}_h^n$ for fixed $n \in \mathbb{N}, \ 1 \le n \le N$. We now rewrite (2.2) at $t = t_n$ and subtract from (2.4) to obtain

$$(\partial_t \mathbf{e}_n, \mathbf{v}_h) + \mu a(\mathbf{e}_n, \mathbf{v}_h) + a(q_r^n(\mathbf{e}), \mathbf{v}_h) = R_h^n(\mathbf{v}_h) + E_h^n(\mathbf{v}_h) + \Lambda_h^n(\mathbf{v}_h),$$

where,

$$R_h^n(\mathbf{v}_h) = (\mathbf{u}_{ht}^n, \mathbf{v}_h) - (\partial_t \mathbf{u}_h^n, \mathbf{v}_h) = (\mathbf{u}_{ht}^n, \mathbf{v}_h) - \frac{1}{k}\int_{t_{n-1}}^{t_n} (\mathbf{u}_{hs}, \mathbf{v}_h)\, ds$$
$$= \frac{1}{k}\int_{t_{n-1}}^{t_n} (t - t_{n-1})(\mathbf{u}_{htt}, \mathbf{v}_h) dt, \tag{2.50}$$
$$E_h^n(\mathbf{v}_h) = \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{v}_h) ds - a(q_r^n(\mathbf{u}_h), \mathbf{v}_h) = a(\varepsilon_r^n(\mathbf{u}_h), \mathbf{v}_h), \tag{2.51}$$

and

$$\Lambda_h^n(\mathbf{v}_h) = b(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{v}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h)$$
$$= -b(\mathbf{u}_h^n, \mathbf{e}_n, \mathbf{v}_h) - b(\mathbf{e}_n, \mathbf{u}_h^n, \mathbf{v}_h) - b(\mathbf{e}_n, \mathbf{e}_n, \mathbf{v}_h). \tag{2.52}$$

In order to dissociate the effect of nonlinearity, we first linearized the discrete problem (2.4) and introduce $\{\mathbf{V}^n\}_{n\geq 1} \in \mathbf{J}_h$ as solutions of the following linearized problem:

$$(\partial_t \mathbf{V}^n, \mathbf{v}_h) + \mu a(\mathbf{V}^n, \mathbf{v}_h) + a(q_r^n(\mathbf{V}), \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h) - b(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{J}_h, \tag{2.53}$$

given $\mathbf{V}^0$ and $\mathbf{u}_h \in \mathbf{J}_h$ as solution of (2.2). Considering the fact that (2.53) is a linearized version of the nonlinear problem (2.4), it is easy to check the existence and uniqueness of $\{\mathbf{V}^n\}_{n\geq 1} \in \mathbf{J}_h$.
Based on (2.53), we now split the error as follows:

$$\mathbf{e}_n := \mathbf{U}^n - \mathbf{u}_h^n = (\mathbf{U}^n - \mathbf{V}^n) - (\mathbf{u}_h^n - \mathbf{V}^n) =: \boldsymbol{\eta}_n - \boldsymbol{\xi}_n.$$

The following equations are satisfied by $\boldsymbol{\xi}_n$ and $\boldsymbol{\eta}_n$, respectively:

$$(\partial_t \boldsymbol{\xi}_n, \mathbf{v}_h) + \mu a(\boldsymbol{\xi}_n, \mathbf{v}_h) + a(q_r^n(\boldsymbol{\xi}), \mathbf{v}_h) = -R_h^n(\mathbf{v}_h) - E_h^n(\mathbf{v}_h) \tag{2.54}$$

and

$$(\partial_t \boldsymbol{\eta}_n, \mathbf{v}_h) + \mu a(\boldsymbol{\eta}_n, \mathbf{v}_h) + a(q_r^n(\boldsymbol{\eta}), \mathbf{v}_h) = \Lambda_h^n(\mathbf{v}_h). \tag{2.55}$$

We first estimate the errors due to the backward difference operator and the quadrature rule.

Lemma 2.7. *Let $r \in \{0,1\}$ and α as defined in Lemma 2.4. Then with R_h^n and E_h^n defined, respectively, as (2.50) and (2.51), following estimate holds for $n = 1, \cdots, N$ and for $\{\mathbf{v}_h^i\}_i$ in $\mathbf{J}_h$:*

$$2k\sum_{i=1}^{n} e^{2\alpha(t_i - t_n)} \Big(R_h^i(\mathbf{v}_h^i) + E_h^i(\mathbf{v}_h^i) \Big)$$

$$\leq Kk^{(1-r)/2}(1+\log\frac{1}{k})^{(1-r)/2}\left(k\sum_{i=1}^{n}e^{2\alpha(t_i-t_n)}\|\mathbf{v}_h^i\|_{1-r}^2\right)^{1/2}. \quad (2.56)$$

Proof. From (2.50), we observe that

$$2k\sum_{i=1}^{n}e^{2\alpha(t_i-t_n)}R_h^i(\mathbf{v}_h^i)=\frac{1}{k}\int_{t_{n-1}}^{t_n}(t-t_{n-1})(\mathbf{u}_{htt},\mathbf{v}_h)dt$$
$$\leq\left[k^{-1}\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}e^{\alpha(t_i-t_n)}(t-t_{i-1})\|\mathbf{u}_{htt}\|_{r-1}\,dt\Big)^2\right]^{1/2}\left[k\sum_{i=1}^{n}e^{2\alpha(t_i-t_n)}\|\mathbf{v}_h^i\|_{1-r}^2\right]^{1/2}.$$

Note that

$$\left[k^{-1}\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}e^{\alpha(t_i-t_n)}(t-t_{i-1})\|\mathbf{u}_{htt}\|_{r-1}dt\Big)^2\right]^{1/2}$$
$$\leq\Bigg[k^{-1}e^{-2\alpha t_n}\sum_{i=1}^{n}\bigg(\int_{t_{i-1}}^{t_i}\Big((\tau^*)^{-(r+1)/2}(t)(t-t_{i-1})e^{\alpha(t_i-t)}\Big)$$
$$\times\Big((\tau^*)^{(r+1)/2}(t)e^{\alpha t}\|\mathbf{u}_{htt}\|_{r-1}\Big)dt\bigg)^2\Bigg]^{1/2}, \quad (2.57)$$

where, $\tau^*(t)=\min\{1,t\}$. When $r=0$, (2.57) can be bounded by

$$\left[k^{-1}e^{-2\alpha t_n}\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}(\tau^*)^{-1}(t)(t-t_{i-1})^2e^{2\alpha(t_i-t)}dt\Big)\Big(\int_{t_{i-1}}^{t_i}(\tau^*)(t)e^{2\alpha t}\|\mathbf{u}_{htt}\|_{-1}^2dt\Big)\right]^{1/2}$$
$$\leq k^{-1/2}e^{-\alpha t_{n-1}}\Bigg[\Big(\int_0^k t\,dt\Big)\Big(\int_0^k(\tau^*)(t)e^{2\alpha t}\|\mathbf{u}_{htt}\|_{-1}^2dt\Big)$$
$$+k^2\sum_{i=2}^{n}\Big(\int_{t_{i-1}}^{t_i}t^{-1}dt\Big)\Big(\int_{t_{i-1}}^{t_i}(\tau^*)(t)e^{2\alpha t}\|\mathbf{u}_{htt}\|_{-1}^2dt\Big)\Bigg]^{1/2}$$
$$\leq Kk^{1/2}(1+\log\frac{1}{k})^{1/2}.$$

When $r=1$, then, (2.57) can be bounded by

$$\left[k^{-1}e^{-2\alpha t_n}\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}(\tau^*)^{-2}(t)(t-t_{i-1})^2e^{2\alpha(t_i-t)}(\tau^*)^2(t)e^{2\alpha t}\|\mathbf{u}_{htt}\|^2dt\Big)\Big(\int_{t_{i-1}}^{t_i}dt\Big)\right]^{1/2}$$
$$\leq\left[e^{-2\alpha t_n}e^{2\alpha k}\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}(\tau^*)^2(t)e^{2\alpha t}\|\mathbf{u}_{htt}\|^2dt\Big)\right]^{1/2}\leq K.$$

This completes the proof of the first half. For the remaining part, we observe from (2.51) and (1.19) that

$$2k\sum_{i=1}^{n}e^{2\alpha(t_i-t_n)}E_h^i(\mathbf{v}_h^i)\leq\left[k\sum_{i=1}^{n}e^{2\alpha(t_i-t_n)}\|\mathbf{v}_h^i\|_{1-r}^2\right]^{1/2}\times \quad (2.58)$$
$$\left[4k\sum_{i=1}^{n}\Big(\sum_{j=1}^{i}\int_{t_{j-1}}^{t^j}e^{\alpha(t_i-t_n)}(t-t_{j-1})\beta(t_i-t)\{\delta\|\mathbf{u}_h\|_{r+1}+\|\mathbf{u}_{ht}\|_{r+1}\}\,dt\Big)^2\right]^{1/2}.$$

In Lemmas 2.1 and 2.2, we find that the estimates of $\|\mathbf{u}_{htt}\|_{r-1}$ and $\|\mathbf{u}_{ht}\|_{r+1}$ are similar, in fact, the powers of t_i are same. Therefore, the right hand side of (2.58) involving $\|\mathbf{u}_{ht}\|_{r+1}$ can be estimated similarly as in (2.57). The terms involving $\|\mathbf{u}_h\|_{r+1}$ are clearly easy to estimate. But for the sake of completeness, we give the case, when $r = 0$ as

$$
\begin{aligned}
&4\delta^2 k\sum_{i=1}^{n}\Big(\sum_{j=1}^{i}\int_{t_{j-1}}^{t^j} e^{\alpha(t_i-t_n)}(t-t_{j-1})\beta(t_i-t)\|\nabla\mathbf{u}_h\|\;dt\Big)^2\\
&\leq 4\gamma^2\delta^2 e^{-2\alpha t_n}k^3\sum_{i=1}^{n}e^{-2(\delta-\alpha)t_i}\Big(\sum_{j=1}^{i}\int_{t_{j-1}}^{t^j}e^{(\delta-\alpha)t}\|\nabla\tilde{\mathbf{u}}_h\|\;dt\Big)^2\\
&\leq 4\gamma^2\delta^2 e^{-2\alpha t_n}k^3\sum_{i=1}^{n}e^{-2(\delta-\alpha)t_i}\Big(\int_0^{t_i}e^{2(\delta-\alpha)s}ds\Big)\Big(\int_0^{t_i}\|\nabla\tilde{\mathbf{u}}_h(s)\|^2ds\Big)\\
&\leq\frac{2\gamma^2\delta^2}{2(\delta-\alpha)}e^{-2\alpha t_n}k^3\sum_{i=1}^{n}e^{2(\delta-\alpha)k}\big(Ke^{2\alpha t_i}\big)\leq Kk^3e^{2\delta k}.
\end{aligned}
$$

This completes the proof. □

Now, we present the estimates of $\boldsymbol{\xi}$, error due to linear part.

Lemma 2.8. *Assume* (**A1**)-(**A2**) *and a space discretization scheme that satisfies conditions* (**B1**)-(**B2**). *Let* $\alpha_0 > 0$ *be such that* $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu\lambda_1}{2}\}$, *(2.20) be satisfied. Further, assume that* $\mathbf{u}_h(t)$ *and* $\mathbf{V}^n$ *satisfy (2.2) and (2.53), respectively. Then, there is a constant* $K > 0$ *such that,* $\boldsymbol{\xi}_n = \mathbf{u}_h^n - \mathbf{V}^n,\ 1 \leq n \leq N$, *satisfy*

$$\|\boldsymbol{\xi}_n\|^2 \;+\; e^{-2\alpha t_n}k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\boldsymbol{\xi}_i\|^2 \leq Kk\big(1+\log\frac{1}{k}\big), \tag{2.59}$$

$$\|\nabla\boldsymbol{\xi}_n\|^2 \;+\; e^{-2\alpha t_n}k\sum_{i=1}^{n}e^{2\alpha t_n}\{\|\tilde{\Delta}_h\boldsymbol{\xi}_i\|^2+\|\partial_t\boldsymbol{\xi}_i\|^2\} \leq K. \tag{2.60}$$

Proof. For $n = i$, we put $\mathbf{v}_h = \boldsymbol{\xi}_i$ in the linerized error equation (2.54) and with the observation $(\partial_t\boldsymbol{\xi}_i, \boldsymbol{\xi}_i) \geq \frac{1}{2}\partial_t\|\boldsymbol{\xi}_i\|^2$, we find that

$$\partial_t\|\boldsymbol{\xi}_i\|^2 + 2\mu\|\nabla\boldsymbol{\xi}_i\|^2 + 2a(q_r^i(\boldsymbol{\xi}), \boldsymbol{\xi}_i) \leq -2R_h^i(\boldsymbol{\xi}_i) - 2E_h^i(\boldsymbol{\xi}_i). \tag{2.61}$$

Multiply (2.61) by $ke^{2\alpha t_i}$ and take summation over $1 \leq i \leq n \leq N$ and use the fact

$$
\begin{aligned}
k\sum_{i=1}^{n}e^{2\alpha t_i}\partial_t\|\boldsymbol{\xi}_i\|^2 &= \sum_{i=1}^{n}e^{2\alpha t_i}(\|\boldsymbol{\xi}_i\|^2-\|\boldsymbol{\xi}_{i-1}\|^2)\\
&= e^{2\alpha t_n}\|\boldsymbol{\xi}_n\|^2-\sum_{i=1}^{n-1}(e^{2\alpha k}-1)e^{2\alpha t_i}\|\boldsymbol{\xi}_i\|^2
\end{aligned}
$$

$$\geq e^{2\alpha t_n}\|\boldsymbol{\xi}_n\|^2 - k\sum_{i=1}^{n}\frac{e^{2\alpha k}-1}{k\lambda_1}e^{2\alpha t_i}\|\nabla\boldsymbol{\xi}_i\|^2,$$

we obtain

$$e^{2\alpha t_n}\|\boldsymbol{\xi}_n\|^2 + \Big(2\mu - \frac{e^{2\alpha k}-1}{k\lambda_1}\Big)k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\boldsymbol{\xi}_i\|^2 + 2k\sum_{i=1}^{n}e^{2\alpha t_i}a(q_r^i(\boldsymbol{\xi}),\boldsymbol{\xi}_i)$$
$$\leq -2k\sum_{i=1}^{n}e^{2\alpha t_i}\Big\{R_h^i(\boldsymbol{\xi}_i)+E_h^i(\boldsymbol{\xi}_i)\Big\}. \tag{2.62}$$

We have dropped the third term from the left of inequality (2.61) as it is non-negative. Now, we use Lemma 2.7 for $r=0$ to observe that

$$|-2k\sum_{i=1}^{n}e^{2\alpha t_i}\Big\{R_h^i(\boldsymbol{\xi}_i)+E_h^i(\boldsymbol{\xi}_i)\Big\}|$$
$$\leq \frac{\mu}{2}k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\boldsymbol{\xi}_i\|^2 + Kk\big(1+\log\frac{1}{k}\big)e^{2\alpha t_n}.$$

Inserting this in (2.62), we obtain

$$e^{2\alpha t_n}\|\boldsymbol{\xi}_n\|^2 + \Big(\frac{3\mu}{2} - \frac{e^{2\alpha k}-1}{k\lambda_1}\Big)k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\boldsymbol{\xi}_i\|^2 \leq Kk\big(1+\log\frac{1}{k}\big)e^{2\alpha t_{n+1}}. \tag{2.63}$$

Similar to Lemma 2.3, one can show that $\Big(\frac{3}{2}\mu - \big(\frac{e^{2\alpha k}-1}{k\lambda_1}\big)\Big) \geq \frac{\mu}{2} > 0$. Now multiply (2.63) by $e^{-2\alpha t_n}$ to establish (2.59). Next, for $n=i$, we put $\mathbf{v}_h = -\tilde{\Delta}_h\boldsymbol{\xi}_i$ in (2.54) and follow as above to obtain the first part of (2.60), that is,

$$\|\nabla\boldsymbol{\xi}_n\|^2 + e^{-2\alpha t_n}k\sum_{i=1}^{n}e^{2\alpha t_i}\|\tilde{\Delta}_h\boldsymbol{\xi}_i\|^2 \leq K.$$

Here, we have used (2.56) for $r=1$ replacing $\mathbf{v}_h^i$ by $\tilde{\Delta}_h\boldsymbol{\xi}_i$. Finally, for $n=i$, we put $\mathbf{v}_h = \partial_t\boldsymbol{\xi}_i$ in (2.54) to find that

$$2\|\partial_t\boldsymbol{\xi}_i\|^2 + \mu\partial_t\|\boldsymbol{\xi}_i\|_1^2 \leq -2a(q_r^i(\boldsymbol{\xi}),\partial_t\boldsymbol{\xi}_i) - 2R_h^i(\partial_t\boldsymbol{\xi}_i) - 2E_h^i(\partial_t\boldsymbol{\xi}_i). \tag{2.64}$$

Multiply (2.64) by $ke^{2\alpha t_i}$ and take summation over $1\leq i\leq n\leq N$. As has been done earlier, using (2.56) for $r=0$, we obtain

$$\sum_{i=1}^{n}ke^{2\alpha t_i}\{2R_h^i(\partial_t\boldsymbol{\xi}_i)+2E_h^i(\partial_t\boldsymbol{\xi}_i)\} \leq \frac{k}{2}\sum_{i=1}^{n}e^{2\alpha t_i}\|\partial_t\boldsymbol{\xi}_i\|^2 + K. \tag{2.65}$$

The only difference is that the resulting double sum (the term involving q_r^i) is no longer non-negative and hence, we need to estimate it. Note that

$$2k\sum_{i=1}^{n}e^{2\alpha t_i}a(q_r^i(\boldsymbol{\xi}),\partial_t\boldsymbol{\xi}_i) = 2\gamma k^2\sum_{i=1}^{n}\sum_{j=1}^{i}e^{-(\delta-\alpha)(t_i-t_j)}e^{2\alpha t_i}a(\boldsymbol{\xi}_j,\partial_t\boldsymbol{\xi}_i)$$

$$\leq \frac{k}{2}\sum_{i=1}^{n} e^{2\alpha t_i}\|\partial_t \boldsymbol{\xi}_i\|^2 + K(\gamma)k\sum_{i=1}^{n}\Big(k\sum_{j=1}^{i} e^{-(\delta-\alpha)(t_i-t_j)}e^{\alpha t_i}\|\tilde{\Delta}_h \boldsymbol{\xi}_j\|\Big)^2. \tag{2.66}$$

Using change of variable and change of order of double sum, we obtain

$$\begin{aligned}
I &:= K(\gamma)k\sum_{i=1}^{n}\Big(k\sum_{j=1}^{i} e^{-(\delta-\alpha)(t_i-t_j)}e^{\alpha t_i}\|\tilde{\Delta}_h \boldsymbol{\xi}_j\|\Big)^2 \\
&\leq K(\alpha,\gamma)e^{(\delta-\alpha)k}k^2\sum_{i=1}^{n} k\sum_{j=1}^{i} e^{-(\delta-\alpha)(t_i-t_j)}e^{2\alpha t_i}\|\tilde{\Delta}_h \boldsymbol{\xi}_j\|^2 \\
&\leq K(\alpha,\gamma)e^{(\delta-\alpha)k}k^2\sum_{i=1}^{n} k\sum_{l=1}^{i} e^{-(\delta-\alpha)t_{l-1}}e^{\alpha t_{i-l+1}}\|\tilde{\Delta}_h \boldsymbol{\xi}_{i-l+1}\|^2 \quad \text{for } l=i-j \\
&\leq K(\alpha,\gamma)e^{(\delta-\alpha)k}k\Big(k\sum_{l=1}^{n-1} e^{-(\delta-\alpha)t_l}\Big)\Big(k\sum_{j=1}^{n} e^{2\alpha t_j}\|\tilde{\Delta}_h \boldsymbol{\xi}_j\|^2\Big) \leq K.
\end{aligned} \tag{2.67}$$

Combining (2.66)-(2.67), we find that

$$2k\sum_{i=1}^{n} e^{2\alpha t_i} a(q_r^i(\boldsymbol{\xi}),\partial_t \boldsymbol{\xi}_i) \leq \frac{k}{2}\sum_{i=1}^{n} e^{2\alpha t_i}\|\partial_t \boldsymbol{\xi}_i\|^2 + K. \tag{2.68}$$

Incorporating (2.65) and (2.68), we obtain

$$k\sum_{i=1}^{n} e^{2\alpha t_i}\|\partial_t \boldsymbol{\xi}_i\|^2 + \mu e^{2\alpha t_n}\|\boldsymbol{\xi}_n\|_1^2 \leq K + \mu k\sum_{i=1}^{n-1}\frac{(e^{2\alpha k}-1)}{k}e^{2\alpha t_i}\|\boldsymbol{\xi}_i\|_1^2.$$

Use (2.59) and the fact that $(e^{2\alpha k}-1)/k \leq K(\alpha)$ to complete the proof. □

Remark 2.4. *Lemma 2.8 provides us with the estimate* $\|\boldsymbol{\xi}_n\| \leq Kk^{1/2}\big(1+\log\frac{1}{k}\big)^{1/2}$, *which is suboptimal and is due to nonsmooth initial data. Similar analysis for smooth initial data would give optimal result, see [117, 137]. Since our analysis is different from these works, we give a sketch for the smooth case, below.*

We start with (2.62), that is,

$$\begin{aligned}
e^{2\alpha t_n}\|\boldsymbol{\xi}_n\|^2 + \Big(2\mu - \frac{e^{2\alpha k}-1}{k\lambda_1}\Big)k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla \boldsymbol{\xi}_i\|^2 \\
\leq -2k\sum_{i=1}^{n} e^{2\alpha t_i}\Big\{R_h^i(\boldsymbol{\xi}_i) + E_h^i(\boldsymbol{\xi}_i)\Big\}.
\end{aligned} \tag{2.69}$$

Note that we have dropped the quadrature term due to positivity. Similar to Lemma 2.7 for $r=0$ (but for smooth initial data) we then obtain

$$\Big| -2k\sum_{i=1}^{n} e^{2\alpha t_i}\Big\{R_h^i(\boldsymbol{\xi}_i) + E_h^i(\boldsymbol{\xi}_i)\Big\}\Big|$$

$$\leq \frac{\mu}{2} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\nabla \boldsymbol{\xi}_i\|^2 + Kk^2 \big(1 + \log \frac{1}{k}\big) e^{2\alpha t_n}.$$

Inserting this in (2.69) and multiplying both sides by $e^{-2\alpha t_n}$, we finally obtain

$$\|\boldsymbol{\xi}_n\| \leq Kk \big(1 + \log \frac{1}{k}\big)^{1/2},$$

which is optimal.

Remark 2.5. *Comparison of the above result with the result of Lemma 2.8 clearly shows the effect of nonsmooth data. This loss of order of convergence is generally compensated by a use of discrete weight function* t_n*. But a straightforward use of this technique fails in our case, due to the quadrature term, without some additional tool. To illustrate this, we introduce a discrete weight function* $\sigma_i = \tau^*(t_i) e^{2\alpha t_i}$ *while using a test function* $(-\tilde{\Delta}_h)^{-1} \partial_t \boldsymbol{\xi}_i$ *in a bid to improve the result of Lemma 2.8.*

We put $\mathbf{v}_h = \sigma_i (-\tilde{\Delta}_h)^{-1} \partial_t \boldsymbol{\xi}_i$ in (2.54) with $n = i$ and obtain

$$\begin{aligned} 2\sigma_i \|\partial_t \boldsymbol{\xi}_i\|_{-1}^2 + \mu \sigma_i \partial_t \|\boldsymbol{\xi}_i\|^2 + 2\sigma_i a(q_r^i(\boldsymbol{\xi}), (-\tilde{\Delta}_h)^{-1} \partial_t \boldsymbol{\xi}_i) \\ \leq -2R_h^i(\sigma_i (-\tilde{\Delta}_h)^{-1} \partial_t \boldsymbol{\xi}_i) - 2E_h^i(\sigma_i (-\tilde{\Delta}_h)^{-1} \partial_t \boldsymbol{\xi}_i). \end{aligned} \tag{2.70}$$

We multiply (2.70) by k and take summation over $1 \leq i \leq n$ and use the fact

$$k \sum_{i=1}^{n} \sigma_i \partial_t \|\boldsymbol{\xi}_i\|^2 \geq \sigma_n \|\boldsymbol{\xi}_n\|^2 - k \sum_{i=1}^{n-1} e^{2\alpha t_i} \|\boldsymbol{\xi}_i\|^2$$

to find that

$$\begin{aligned} \mu \sigma_n \|\boldsymbol{\xi}_n\|^2 + 2k \sum_{i=1}^{n} \sigma_i \|\partial_t \boldsymbol{\xi}_i\|_{-1}^2 + 2k \sum_{i=1}^{n} \sigma_i a(q_r^i(\boldsymbol{\xi}), (-\tilde{\Delta}_h)^{-1} \partial_t \boldsymbol{\xi}_i) \leq k \sum_{i=1}^{n-1} e^{2\alpha t_i} \|\boldsymbol{\xi}_i\|^2 \\ - 2k \sum_{i=1}^{n} R_h^i(\sigma_i (-\tilde{\Delta}_h)^{-1} \partial_t \boldsymbol{\xi}_i) - 2k \sum_{i=1}^{n} E_h^i(\sigma_i (-\tilde{\Delta}_h)^{-1} \partial_t \boldsymbol{\xi}_i). \end{aligned} \tag{2.71}$$

We use the "Cauchy-Schwarz inequality" with (2.50) to obtain

$$\begin{aligned} k \sum_{i=1}^{n} \sigma_i R_h^i((-\tilde{\Delta}_h)^{-1} \partial_t \boldsymbol{\xi}_i) &\leq k \sum_{i=1}^{n} \sigma_i \frac{1}{k} \int_{t_{i-1}}^{t_i} (s - t_{i-1}) \|\mathbf{u}_{hss}\|_{-1} ds \|\partial_t \boldsymbol{\xi}_i\|_{-1} \\ &\leq Kk \sum_{i=1}^{n} \sigma_i \Big(\int_{t_{i-1}}^{t_i} \|\mathbf{u}_{hss}\|_{-1} ds \Big)^2 + \epsilon k \sum_{i=1}^{n} \sigma_i \|\partial_t \boldsymbol{\xi}_i\|_{-1}^2 \\ &\leq Kk^2 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sigma_i \|\mathbf{u}_{hss}\|_{-1}^2 ds + \epsilon k \sum_{i=1}^{n} \sigma_i \|\partial_t \boldsymbol{\xi}_i\|_{-1}^2 \end{aligned}$$

$$\leq Kk^2 e^{2\alpha t_n} + \epsilon k \sum_{i=1}^{n} \sigma_i \|\partial_t \boldsymbol{\xi}_i\|_{-1}^2. \tag{2.72}$$

Now from (2.51) and (1.21), we find that

$$\begin{aligned} k\sum_{i=1}^{n} \sigma_i E_h^i((-\tilde{\Delta}_h)^{-1}\partial_t \boldsymbol{\xi}_i) \\ &= k\sum_{i=1}^{n} \sigma_i a(\varepsilon_r^i(\mathbf{u}_h), (-\tilde{\Delta}_h)^{-1}\partial_t \boldsymbol{\xi}_i) = k\sum_{i=1}^{n} \sigma_i (\varepsilon_r^i(\mathbf{u}_h), \partial_t \boldsymbol{\xi}_i) \\ &= \sigma_n(\varepsilon_r^n(\mathbf{u}_h), \boldsymbol{\xi}_n) - k\sum_{i=1}^{n} (\partial_t \sigma_i)(\varepsilon_r^i(\mathbf{u}_h), \boldsymbol{\xi}_i) - k\sum_{i=1}^{n} \sigma_i(\partial_t \varepsilon_r^i(\mathbf{u}_h), \boldsymbol{\xi}_i). \end{aligned} \tag{2.73}$$

Using (1.19) and the "Cauchy-Schwarz inequality", we bound the following as

$$\begin{aligned} \sigma_n(\varepsilon_r^n(\mathbf{u}_h), \boldsymbol{\xi}_n) &\leq Kk^2 \sigma_n \Big(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \beta(t_n - s)\{\delta \|\mathbf{u}_h\| + \|\mathbf{u}_{hs}\|\} ds \Big)^2 + \epsilon \sigma_n \|\boldsymbol{\xi}_n\|^2 \\ &\leq Kk^2 e^{2\alpha t_n} + \epsilon \sigma_n \|\boldsymbol{\xi}_n\|^2. \end{aligned} \tag{2.74}$$

A use of (1.19) with Lemma 2.11 shows

$$\begin{aligned} k\sum_{i=1}^{n} (\partial_t \sigma_i)(\varepsilon_r^i(\mathbf{u}_h), \boldsymbol{\xi}_i) &\leq k\sum_{i=1}^{n} e^{2\alpha t_i}(\varepsilon_r^i(\mathbf{u}_h), \boldsymbol{\xi}_i)) \\ &\leq Kk^2 k \sum_{i=1}^{n} e^{2\alpha t_i} \int_0^{t_i} \{\|\mathbf{u}_h\|^2 + \|\mathbf{u}_{hs}\|^2\} ds + Kk\sum_{i=1}^{n} e^{2\alpha t_i} \|\boldsymbol{\xi}_i\|^2 \\ &\leq Kk^2 e^{2\alpha t_n} + Kk\sum_{i=1}^{n} e^{2\alpha t_i}\|\boldsymbol{\xi}_i\|^2, \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} k\sum_{i=1}^{n} \sigma_i(\partial_t \varepsilon_r^i(\mathbf{u}_h), \boldsymbol{\xi}_i) &\leq Kk^2 k \sum_{i=1}^{n} e^{2\alpha t_i} \int_{t_{i-1}}^{t_i} \{\|\mathbf{u}_h\|^2 + \|\mathbf{u}_{hs}\|^2\} ds + Kk\sum_{i=1}^{n} e^{2\alpha t_i} \|\boldsymbol{\xi}_i\|^2 \\ &\leq Kk^2 e^{2\alpha t_n} + Kk\sum_{i=1}^{n} e^{2\alpha t_i}\|\boldsymbol{\xi}_i\|^2. \end{aligned} \tag{2.76}$$

Incorporate (2.72)-(2.76) in (2.71) and use Lemma 2.11 to deduce

$$\begin{aligned} \sigma_n\|\boldsymbol{\xi}_n\|^2 + k\sum_{i=1}^{n} \sigma_i \|\partial_t \boldsymbol{\xi}_i\|_{-1}^2 + 2k\sum_{i=1}^{n} \sigma_i a(q_r^i(\boldsymbol{\xi}), (\tilde{\Delta}_h)^{-1}\partial_t \boldsymbol{\xi}_i) \\ \leq Kk^2 e^{2\alpha t_n} + Kk\sum_{i=1}^{n} e^{2\alpha t_i}\|\boldsymbol{\xi}_i\|^2. \end{aligned} \tag{2.77}$$

The last term of (2.77) is to be estimated optimally by standard arguments and hence we defer its estimate for now, see Lemma 2.11. However the third term on the left of inequality, which we used to drop due to positivity, is now no longer non-negative due to the presence of σ_i. And we have to estimate it. A use of the "Cauchy-Schwarz inequality" with Lemma 2.8 gives a sub-optimal result.

$$\begin{aligned} 2k\sum_{i=1}^{n}\sigma_i a(q_r^i(\boldsymbol{\xi}),(\tilde{\Delta}_h)^{-1}\partial_t\boldsymbol{\xi}_i) &\leq \epsilon k\sum_{i=1}^{n}\sigma_i\|\partial_t\boldsymbol{\xi}_i\|_{-1}^2 + Kk\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\boldsymbol{\xi}_i\|^2 \\ &\leq \epsilon k\sum_{i=1}^{n}\sigma_i\|\partial_t\boldsymbol{\xi}_i\|_{-1}^2 + Kk\big(1+\log\frac{1}{k}\big). \end{aligned} \tag{2.78}$$

Now, we use (2.78) in (2.77) which gives a suboptimal result for $\|\boldsymbol{\xi}_n\|$ as well.

Remark 2.6. *The above result validates Remark 2.5 that the standard technique does not work in our case. And this forces us to look for some additional tools from the literature of parabolic integro-differential equations. At the same time these technical roadblocks justifies our choice of working with nonsmooth data.*

To improve the result, we introduce the 'hat operator', that is,

$$\widehat{\boldsymbol{\phi}}_h^n := k\sum_{i=1}^{n}\boldsymbol{\phi}_h^i. \tag{2.79}$$

Now using summation by parts (1.20), we can rewrite the last term on the left of inequality (2.77) in terms of 'hat'.

$$\begin{aligned} &2k\sum_{i=1}^{n}\sigma_i a(q_r^i(\boldsymbol{\xi}),(\tilde{\Delta}_h)^{-1}\partial_t\boldsymbol{\xi}_i) \\ &= 2k\sum_{i=1}^{n}\gamma a(\widehat{\boldsymbol{\xi}}_i,\sigma_i(\tilde{\Delta}_h)^{-1}\partial_t\boldsymbol{\xi}_i) - 2k\sum_{i=2}^{n}k\sum_{j=1}^{i-1}\partial_t\beta(t_i-t_j)a(\widehat{\boldsymbol{\xi}}_j,\sigma_i(\tilde{\Delta}_h)^{-1}\partial_t\boldsymbol{\xi}_i). \end{aligned} \tag{2.80}$$

Similar to (2.67), we bound both the terms as follows:

$$2k\sum_{i=1}^{n}\gamma a(\widehat{\boldsymbol{\xi}}_i,\sigma_i(\tilde{\Delta}_h)^{-1}\partial_t\boldsymbol{\xi}_i) \leq \epsilon k\sum_{i=1}^{n}\sigma_i\|\partial_t\boldsymbol{\xi}_i\|_{-1}^2 + c(\epsilon,\mu,\gamma)k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\xi}}_i\|^2. \tag{2.81}$$

and

$$\begin{aligned} &2k\sum_{i=2}^{n}k\sum_{j=1}^{i-1}\partial_t\beta(t_i-t_j)a(\widehat{\boldsymbol{\xi}}_j,\sigma_i(\tilde{\Delta}_h)^{-1}\partial_t\boldsymbol{\xi}_i) \\ &\qquad\leq \epsilon k\sum_{i=1}^{n}\sigma_i\|\partial_t\boldsymbol{\xi}_i\|_{-1}^2 + Kk\sum_{i=2}^{n}\Big(k\sum_{j=1}^{i-1}e^{-\delta(t_i-t_j)}\big(\frac{e^{\delta k-1}}{k}\big)e^{\alpha t_i}\|\nabla\widehat{\boldsymbol{\xi}}_j\|\Big)^2 \\ &\qquad\leq \epsilon k\sum_{i=1}^{n}\sigma_i\|\partial_t\boldsymbol{\xi}_i\|_{-1}^2 + Kk\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\xi}}_i\|^2. \end{aligned} \tag{2.82}$$

With (2.81) and (2.82), we obtain from (2.80)

$$2k\sum_{i=1}^{n}\sigma_i a(q_r^i(\boldsymbol{\xi}),(\tilde{\Delta}_h)^{-1}\partial_t\boldsymbol{\xi}_i)\leq \epsilon k\sum_{i=1}^{n}\sigma_i\|\partial_t\boldsymbol{\xi}_i\|_{-1}^2+Kk\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\xi}}_i\|^2 \tag{2.83}$$

The 'hat' term needs an estimate. For this, we rewrite the equations (2.54) (for $n=i$) as follows:

$$(\partial_t\boldsymbol{\xi}_i,\mathbf{v}_h)+\mu a(\boldsymbol{\xi}_i,\mathbf{v}_h)+\partial_t^i\Big\{k\sum_{j=1}^{i}\beta(t_i-t_j)a(\widehat{\boldsymbol{\xi}}_j,\mathbf{v}_h)\Big\}$$
$$=-R_h^i(\mathbf{v}_h)-E_h^i(\mathbf{v}_h). \tag{2.84}$$

Here, ∂_t^i means the backward difference formula with respect to i and we note that

$$\begin{aligned}k\sum_{j=1}^{i}\beta(t_i-t_j)\boldsymbol{\phi}_j&=\gamma e^{-\delta t_i}k\sum_{j=1}^{i}e^{\delta t_j}\boldsymbol{\phi}_j\\&=\gamma e^{-\delta t_i}\Big\{e^{\delta t_i}\widehat{\boldsymbol{\phi}}_i-k\sum_{j=1}^{i-1}(\frac{e^{\delta t_{j+1}}-e^{\delta t_j}}{k})\widehat{\boldsymbol{\phi}}_j\Big\}\\&=\partial_t^i\Big\{k\sum_{j=1}^{i}\beta(t_i-t_j)\widehat{\boldsymbol{\phi}}_j\Big\}.\end{aligned} \tag{2.85}$$

We now multiply (2.84) by k and take summation over 1 to n. Using the fact that $\partial_t\widehat{\boldsymbol{\xi}}_n=\boldsymbol{\xi}_n$, we observe that

$$(\partial_t\widehat{\boldsymbol{\xi}}_n,\mathbf{v}_h)+\mu a(\widehat{\boldsymbol{\xi}}_n,\mathbf{v}_h)+a(q_r^n(\widehat{\boldsymbol{\xi}}),\mathbf{v}_h)=-k\sum_{i=1}^{n}\big(R_h^i(\mathbf{v}_h)+E_h^i(\mathbf{v}_h)\big). \tag{2.86}$$

Lemma 2.9. *Consider the assumptions of the previous lemma. Then, the following result holds:*

$$\|\widehat{\boldsymbol{\xi}}_n\|^2+e^{-2\alpha t_n}k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\xi}}_i\|^2\leq Kk^2(1+\log\frac{1}{k}),\ \ 1\leq n\leq N.$$

Proof. Choose $\mathbf{v}_h=\widehat{\boldsymbol{\xi}}_i$ in (2.86) for $n=i$, multiply by $ke^{2\alpha t_i}$ and then take summation over $1\leq i\leq n$. We drop the third term on the left of the resulting inequality due to non-negativity.

$$e^{2\alpha t_n}\|\widehat{\boldsymbol{\xi}}_n\|^2+\mu k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\xi}}_i\|^2\leq k\sum_{i=1}^{n}e^{2\alpha t_i}k\sum_{j=1}^{i}\big(|R_h^j(\widehat{\boldsymbol{\xi}}_i)|+|E_h^j(\mathbf{u}_h)(\widehat{\boldsymbol{\xi}}_i)|\big). \tag{2.87}$$

From (2.50) and use the similar technique of Lemma 2.7 to find that

$$k\sum_{j=1}^{i}|R_h^j(\widehat{\boldsymbol{\xi}}_i)|\leq\big(\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\|\mathbf{u}_{hss}\|_{-1}ds\big)\|\nabla\widehat{\boldsymbol{\xi}}_i\|$$

$$\le Kk(1+\log\frac{1}{k})^{1/2}e^{-\alpha k}\|\nabla\widehat{\boldsymbol{\xi}}_i\|.$$

Similar to the proof of Lemma 2.7 , we split the sum in $j=1$ and the rest to obtain

$$k\sum_{j=1}^{i}|R_h^j(\widehat{\boldsymbol{\xi}}_i)| \le Kk(1+\log\frac{1}{k})^{1/2}e^{-\alpha k}\|\nabla\widehat{\boldsymbol{\xi}}_i\|.$$

Therefore

$$k\sum_{i=1}^{n}e^{2\alpha t_i}k\sum_{j=1}^{i}|R_h^j(\widehat{\boldsymbol{\xi}}_i)| \le \frac{\mu}{4}k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\xi}}_i\|^2 + Kk^2(1+\log\frac{1}{k})e^{2\alpha t_n}. \tag{2.88}$$

Similarly

$$k\sum_{i=1}^{n}e^{2\alpha t_i}k\sum_{j=1}^{i}|E_h^j(\widehat{\boldsymbol{\xi}}_i)| \le \frac{\mu}{4}k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\xi}}_i\|^2 + Kk^2(1+\log\frac{1}{k})e^{2\alpha t_n}. \tag{2.89}$$

Incorporate (2.88)-(2.89) in (2.87) to complete the proof. □

Let us remind ourselves that the $l^2(\mathbf{L}^2)$-norm estimate of $\boldsymbol{\xi}_i$ in (2.77) is not yet proved and we complete the task here. Analogous to the semidiscrete case, we resort to duality argument to obtain the same.
For a given $\mathbf{W}_n$ and $\mathbf{g}_i$, let $\mathbf{W}_i,\ n\ge i\ge 1$ satisfy the following backward problem

$$(\mathbf{v}_h,\partial_t\mathbf{W}_i)-\mu a(\mathbf{v}_h,\mathbf{W}_i)-k\sum_{j=i}^{n}\beta(t_j-t_i)a(\mathbf{v}_h,\mathbf{W}_j)=(\mathbf{v}_h,e^{2\alpha t_i}\mathbf{g}_i),\ \ \forall\mathbf{v}_h\in\mathbf{J}_h. \tag{2.90}$$

By denoting $\bar{\mathbf{W}}_i=\mathbf{W}_{n-i}$, we shall obtain a forward problem in $\{\bar{\mathbf{W}}_i\}$, similar to (2.2), but linear. Following the line of argument used to prove Lemma 2.3, it is easy to derive the *a priori* estimates below.

Lemma 2.10. *Let the conditions* (**A1**), (**A2**), (**B1**) *and* (**B2**) *hold. Then, following estimates hold under appropriate assumptions on* $\mathbf{W}_n$ *and* g *for* $r\in\{0,1\}$*:*

$$\|\mathbf{W}_0\|_r^2+k\sum_{i=1}^{n}e^{-2\alpha t_i}\{\|\mathbf{W}_i\|_{r+1}+\|\partial_t\mathbf{W}_i\|_{r-1}\}\le K\big\{\|\mathbf{W}_n\|_r^2+k\sum_{i=1}^{n}e^{2\alpha t_i}\|\mathbf{g}_i\|_{r-1}^2\big\}.$$

Lemma 2.11. *Consider the assumptions of the previous lemma. Then, the following result holds:*

$$\|\boldsymbol{\xi}_n\|_{-1}^2+e^{-2\alpha t_n}k\sum_{i=1}^{n}e^{2\alpha t_i}\|\boldsymbol{\xi}_i\|^2\le Kk^2.$$

Proof. With $\mathbf{W}_n=(-\tilde{\Delta}_h)^{-1}\boldsymbol{\xi}_n,\quad \mathbf{g}_i=\boldsymbol{\xi}_i\ \forall i$, we choose $\mathbf{v}_h=\boldsymbol{\xi}_i$ in (2.90) and use (2.54) to obtain

$$e^{2\alpha t_i}\|\boldsymbol{\xi}_i\|^2=(\boldsymbol{\xi}_i,\partial_t\mathbf{W}_i)-\mu a(\boldsymbol{\xi}_i,\mathbf{W}_i)-k\sum_{j=i}^{n}\beta(t_j-t_i)a(\boldsymbol{\xi}_i,\mathbf{W}_j)$$

$$= \partial_t(\boldsymbol{\xi}_i, \mathbf{W}_i) - (\partial_t \boldsymbol{\xi}_i, \mathbf{W}_{i-1}) - \mu a(\boldsymbol{\xi}_i, \mathbf{W}_i) - k \sum_{j=i}^{n} \beta(t_j - t_i) a(\boldsymbol{\xi}_i, \mathbf{W}_j)$$

$$= \partial_t(\boldsymbol{\xi}_i, \mathbf{W}_i) + k(\partial_t \boldsymbol{\xi}_i, \partial_t \mathbf{W}_i) + k \sum_{j=1}^{i} \beta(t_i - t_j) a(\boldsymbol{\xi}_j, \mathbf{W}_i) + R_h^i(\mathbf{W}_i)$$

$$+ E_h^i(\mathbf{W}_i) - k \sum_{j=i}^{n} \beta(t_j - t_i) a(\boldsymbol{\xi}_i, \mathbf{W}_j). \tag{2.91}$$

Multiply (2.91) by k and take summation over $1 \leq i \leq n$. Observe that the resulting two double sums cancel out (change of order of double sum). Therefore, we find that

$$k \sum_{i=1}^{n} e^{2\alpha t_i} \|\boldsymbol{\xi}_i\|^2 + \|\boldsymbol{\xi}_n\|_{-1}^2 = k \sum_{i=1}^{n} \big[k(\partial_t \boldsymbol{\xi}_i, \partial_t \mathbf{W}_i) + R_h^i(\mathbf{W}_i) + E_h^i(\mathbf{W}_i) \big]. \tag{2.92}$$

From (2.50), we observe that

$$k \sum_{i=1}^{n} R_h^i(\mathbf{W}_i) \leq k \sum_{i=1}^{n} \frac{1}{k} \int_{t_{i-1}}^{t_i} (s - t_{i-1}) \|\mathbf{u}_{hss}\|_{-2} \|\mathbf{W}_i\|_2$$

$$\leq k e^{\alpha k} \Big(\int_0^{t_n} e^{2\alpha s} \|\mathbf{u}_{hss}\|_{-2}^2 ds \Big)^{1/2} \Big(k \sum_{i=1}^{n} e^{-2\alpha t_i} \|\mathbf{W}_i\|_2^2 \Big)^{1/2}. \tag{2.93}$$

Similar to (2.63), we obtain

$$k \sum_{i=1}^{n} E_h^i(\mathbf{W}_i) \leq K \Big(k^3 \sum_{i=1}^{n} \int_0^{t_i} e^{2\alpha s} (\|\mathbf{u}_h\|^2 + \|\mathbf{u}_{hs}\|^2) \; ds \Big)^{1/2} \Big(k \sum_{i=1}^{n} e^{-2\alpha t_i} \|\mathbf{W}_i\|_2^2 \Big)^{1/2},$$

and

$$k \sum_{i=1}^{n} k(\partial_t \boldsymbol{\xi}_i, \partial_t \mathbf{W}_i) \leq k \Big(k \sum_{i=1}^{n} e^{2\alpha t_i} \|\partial_t \boldsymbol{\xi}_i\|^2 \Big)^{1/2} \Big(k \sum_{i=1}^{n} e^{-2\alpha t_i} \|\partial_t \mathbf{W}_i\|^2 \Big)^{1/2}. \tag{2.94}$$

Incorporating (2.93)-(2.94) in (2.92), and using Lemmas 2.1, 2.8 and 2.10, we find that

$$\|\boldsymbol{\xi}_n\|_{-1}^2 + e^{-2\alpha t_n} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\boldsymbol{\xi}_i\|^2 \leq K k^2.$$

This concludes the proof. □

This in fact completes the proof of an important result: optimal estimate of $\boldsymbol{\xi}_n$ in $l_2(\mathbf{L}^2)$-norm. We present the result below.

Lemma 2.12. *Consider the assumptions of the Lemma 2.8. Then, the following result holds:*

$$t_n \|\boldsymbol{\xi}_n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^{n} \sigma_i \|\partial_t \boldsymbol{\xi}_i\|_{-1}^2 \leq K k^2 (1 + \log \frac{1}{k}), \quad 1 \leq n \leq N, \tag{2.95}$$

where $\sigma_i = \tau^*(t_i) e^{2\alpha t_i}$.

Proof. We use (2.83) with the Lemmas 2.9 and 2.10 in (2.77) to obtain the desired result. □

Remark 2.7. *The generic error constant $K > 0$, involving the estimates of $\boldsymbol{\xi}_n$ in various norms, established above, is independent of n and hence the estimates of $\boldsymbol{\xi}_n$ are uniform in time. In other words, these estimates are still valid as $t_N \to +\infty$.*

We now obtain estimates of $\boldsymbol{\eta}$ below. Hence forward, K_n means $K(e^{t_n})$.

Lemma 2.13. *Suppose the assumptions of Lemma 2.8 hold. Further, let $\mathbf{U}^n$ and $\mathbf{V}^n$ satisfy (2.4) and (2.53), respectively. Then, $\boldsymbol{\eta}_n = \mathbf{U}^n - \mathbf{V}^n$, $1 \le n \le N$, satisfy the following:*

$$\|\boldsymbol{\eta}_n\|_r^2 + e^{-2\alpha t_n} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\boldsymbol{\eta}_i\|_{r+1}^2 \le K_n [k(1+\log \frac{1}{k})]^{(1-r)}, \quad r \in \{0,1\}.$$

Proof. For $n = i$, we put $\mathbf{v}_h = \boldsymbol{\eta}_i$ in (2.55), multiply by $ke^{2\alpha t_i}$ and take summation over $1 \le i \le n \le N$ to obtain as in (2.62)

$$e^{2\alpha t_n} \|\boldsymbol{\eta}_n\|^2 + 2\mu k \sum_{i=1}^{n} e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2 \le 2k \sum_{i=1}^{n-1} \frac{(e^{2\alpha k} - 1)}{k} e^{2\alpha t_i} \|\boldsymbol{\eta}_i\|^2 + 2k \sum_{i=1}^{n} e^{2\alpha t_i} \Lambda_h^i(\boldsymbol{\eta}_i). \tag{2.96}$$

We recall from (2.52) that

$$\Lambda_h^i(\boldsymbol{\eta}_i) = -b(\mathbf{u}_h^i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i) - b(\mathbf{e}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i) - b(\mathbf{e}_i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i). \tag{2.97}$$

Using Lemma 1.4 and 2.8, we obtain the following estimates:

$$\begin{aligned} b(\mathbf{e}_i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i) &\le \|\boldsymbol{\xi}_i\|^{1/2} \|\nabla \boldsymbol{\xi}_i\|^{3/2} \|\nabla \boldsymbol{\eta}_i\| + \|\nabla \boldsymbol{\xi}_i\| \|\boldsymbol{\eta}_i\| \|\nabla \boldsymbol{\eta}_i\| \\ &\le \epsilon \|\nabla \boldsymbol{\eta}_i\|^2 + K\|\boldsymbol{\eta}_i\|^2 + Kk^{1/2}(1+\log \frac{1}{k})^{1/2} \|\nabla \boldsymbol{\xi}_i\|^2, \qquad (2.98) \\ b(\mathbf{u}_h^i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i) &\le \epsilon \|\nabla \boldsymbol{\eta}_i\|^2 + K\|\nabla \boldsymbol{\xi}_i\|^2, \qquad (2.99) \\ b(\mathbf{e}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i) &\le \epsilon \|\nabla \boldsymbol{\eta}_i\|^2 + K\big(\|\nabla \boldsymbol{\xi}_i\|^2 + \|\boldsymbol{\eta}_i\|^2\big). \qquad (2.100) \end{aligned}$$

Incorporate (2.98)-(2.100) in (2.97) and then in (2.96)and choose $\epsilon = \mu/6$ to obtain

$$\begin{aligned} e^{2\alpha t_n} \|\boldsymbol{\eta}_n\|^2 + \frac{3}{2}\mu k \sum_{i=1}^{n} e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2 \le & 2k \sum_{i=1}^{n-1} \frac{(e^{2\alpha k} - 1)}{k} e^{2\alpha t_i} \|\boldsymbol{\eta}_i\|^2 + Kk \sum_{i=1}^{n} e^{2\alpha t_i} \|\nabla \boldsymbol{\xi}_i\|^2 \\ & + Kk \sum_{i=1}^{n} e^{2\alpha t_i} \|\boldsymbol{\eta}_i\|^2. \end{aligned}$$

Using the boundedness of $\boldsymbol{\eta}_n$, last term above can be bounded as

$$Kk\sum_{i=1}^{n} e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|^2 = Kke^{2\alpha t_n}\|\boldsymbol{\eta}_n\|^2 + Kk\sum_{i=1}^{n-1} e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|^2$$
$$\leq Kke^{2\alpha t_n} + Kk\sum_{i=1}^{n-1} e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|^2.$$

A use of the Lemma 2.8 with the "discrete Gronwall's Lemma" completes the proof for the case $r = 0$. For the case $r = 1$, the estimate follows similarly. □

Remark 2.8. *Combining Lemmas 2.8 and 2.13, we find suboptimal order of convergence for* $\|\mathbf{e}_n\|$*:*

$$\|\mathbf{e}_n\|^2 + e^{-2\alpha t_n}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}_i\|_1^2 \leq K_n k(1+\log\frac{1}{k}),\ 1 \leq n \leq N, \tag{2.101}$$

$$\|\mathbf{e}_n\|_1^2 + e^{-2\alpha t_n}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}_i\|_2^2 \leq K_n,\ 1 \leq n \leq N. \tag{2.102}$$

Below, we shall prove optimal estimate of $\|\mathbf{e}_n\|$ with the help of a set of Lemmas.

Lemma 2.14. *Consider the assumptions of the Lemma 2.13. Then, the following result holds:*

$$\|\boldsymbol{\eta}_n\|_{-1}^2 + e^{-2\alpha t_n}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|^2 \leq K_n k^2(1+\log\frac{1}{k}),\ 1 \leq n \leq N.$$

Proof. Put $\mathbf{v}_h = e^{2\alpha t_i}(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_i$ in (2.55) for $n = i$. Then, we multiply by $ke^{2\alpha ik}$ and take summation over $1 \leq i \leq n \leq N$ to arrive at

$$e^{2\alpha t_n}\|\boldsymbol{\eta}_n\|_{-1}^2 + 2\mu k\sum_{i=1}^{n} e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|^2 \leq \sum_{i=1}^{n-1}(e^{2\alpha k}-1)e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|_{-1}^2$$
$$+ 2k\sum_{i=1}^{n} e^{2\alpha t_i}\Lambda_h^i(e^{2\alpha t_i}(-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_i). \tag{2.103}$$

From (2.52), we find that

$$|2\Lambda_h^i((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_i)|$$
$$\leq |2b(\mathbf{e}_i, \mathbf{u}_h^i, (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_i) + b(\mathbf{u}_h^i, \mathbf{e}_i, (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_i) - b(\mathbf{e}_i, \mathbf{e}_i, (-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_i)|. \tag{2.104}$$

We use Lemma 1.4 and similar technique as in (2.12) to find that

$$|2b(\mathbf{e}_i, \mathbf{u}_h^i, -\tilde{\Delta}_h^{-1}\boldsymbol{\eta}_i)| + |2b(\mathbf{u}_h^i, \mathbf{e}_i, -\tilde{\Delta}_h^{-1}\boldsymbol{\eta}_i)| \leq c\|\mathbf{e}_i\|\|\mathbf{u}_h^i\|_1\|\boldsymbol{\eta}_i\|_{-1}^{1/2}\|\boldsymbol{\eta}_i\|^{1/2}, \tag{2.105}$$

And use Lemma 1.4 to observe that

$$|2b(\mathbf{e}_i, \mathbf{e}_i, -\tilde{\Delta}_h^{-1}\mathbf{e}_i)| \leq c\|\mathbf{e}_i\|\big(\|\mathbf{e}_i\|_1 + \|\mathbf{e}_i\|^{1/2}\|\mathbf{e}_i\|_1^{1/2}\big)\|\boldsymbol{\eta}_i\|_{-1}^{1/2}\|\boldsymbol{\eta}_i\|^{1/2}. \tag{2.106}$$

Now, combine (2.104)-(2.106) and use the fact that $\|\mathbf{e}_i\|_1 \leq \|\mathbf{u}_h^i\|_1 + \|\mathbf{U}^i\|_1 \leq K$ to conclude from (2.104) that

$$\begin{aligned}|2\Lambda_h^i((-\tilde{\Delta}_h)^{-1}\boldsymbol{\eta}_i)| &\leq K\|\mathbf{e}_i\|\|\boldsymbol{\eta}_i\|_{-1}^{1/2}\|\boldsymbol{\eta}_i\|^{1/2} \\ &\leq K\|\boldsymbol{\xi}_i\|\|\boldsymbol{\eta}_i\|_{-1}^{1/2}\|\boldsymbol{\eta}_i\|^{1/2} + K\|\boldsymbol{\eta}_i\|_{-1}^{1/2}\|\boldsymbol{\eta}_i\|^{3/2}. \end{aligned} \tag{2.107}$$

Incorporate (2.107) in (2.103) and use kickback argument to obtain

$$e^{2\alpha t_n}\|\boldsymbol{\eta}_n\|_{-1}^2 + \mu k\sum_{i=1}^n e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|^2 \leq Kk\sum_{i=1}^n e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|_{-1}^2 + Kk\sum_{i=1}^n e^{2\alpha t_i}\|\boldsymbol{\xi}_i\|^2.$$

Similar to the proof of Lemma 2.13, we have

$$\begin{aligned}Kk\sum_{i=1}^n e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|_{-1}^2 &= Kke^{2\alpha t_n}\|\boldsymbol{\eta}_n\|_{-1}^2 + Kk\sum_{i=1}^{n-1} e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|_{-1}^2 \\ &\leq Kke^{2\alpha t_n}\|\boldsymbol{\eta}_n\|^2 + Kk\sum_{i=1}^{n-1} e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|_{-1}^2.\end{aligned}$$

An application of Lemmas 2.11 and 2.13 with the "discrete Gronwall's lemma" yields after multiplication by $e^{-2\alpha t_i}$ to the desired estimate. This concludes the proof. □

Remark 2.9. *From Lemmas 2.11 and 2.14, we have the following estimate*

$$\|\mathbf{e}_n\|_{-1}^2 + e^{-2\alpha t_n}k\sum_{i=1}^n e^{2\alpha t_i}\|\mathbf{e}_i\|^2 \leq K_n k^2(1 + \log\frac{1}{k}). \tag{2.108}$$

We present below a lemma with optimal estimate for $\boldsymbol{\eta}_n$.

Lemma 2.15. *Consider the assumptions of the Lemma 2.13. Then, the following result holds:*

$$t_n\|\boldsymbol{\eta}_n\|^2 + e^{-2\alpha t_n}k\sum_{i=1}^n \sigma_i\|\boldsymbol{\eta}_i\|_1^2 \leq K_n k^2(1 + \log\frac{1}{k}),\ \ 1 \leq n \leq N. \tag{2.109}$$

Proof. We choose $\mathbf{v}_h = \sigma_i\boldsymbol{\eta}_i$ in (2.55) for $n = i$ and then multiply by k and take summation over $1 \leq i \leq n$ to find that

$$\sigma_n\|\boldsymbol{\eta}_n\|^2 + 2\mu k\sum_{i=1}^n \sigma_i\|\nabla\boldsymbol{\eta}_i\|^2$$

$$\le K(\alpha)k\sum_{i=2}^{n-1} e^{2\alpha t_i}\|\boldsymbol{\eta}_i\|^2 - 2k\sum_{i=1}^{n} a(q_r^i(\boldsymbol{\eta}),\sigma_i\boldsymbol{\eta}_i) + 2k\sum_{i=1}^{n}\Lambda_h^i(\sigma_i\boldsymbol{\eta}_i). \quad (2.110)$$

Following the estimates of (2.80-(2.83), we obtain

$$2k\sum_{i=1}^{n}\sigma_i a(q_r^i(\boldsymbol{\eta}),\boldsymbol{\eta}_i) \le \epsilon k\sum_{i=1}^{n}\sigma_i\|\nabla\boldsymbol{\eta}_i\|^2 + Kk\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\eta}}_i\|^2. \quad (2.111)$$

We recall (2.52) and using Lemma 1.4 and similar argument as (2.12), to obtain the estimates for nonlinear terms as:

$$2k\sum_{i=1}^{n}\Lambda_h^i(\sigma_i\boldsymbol{\eta}_i) \le \epsilon k\sum_{i=1}^{n}\sigma_i\|\nabla\boldsymbol{\eta}_i\|^2 + Kk\sum_{i=1}^{n}\sigma_i\big(\|\tilde{\Delta}_h\mathbf{u}_h^i\|^2 + \|\tilde{\Delta}_h\mathbf{U}^i\|^2\big)\|\mathbf{e}_i\|^2. \quad (2.112)$$

Substitute (2.111)-(2.112) in (2.110) to obtain

$$\sigma_n\|\boldsymbol{\eta}_n\|^2 + \mu k\sum_{i=1}^{n}\sigma_i\|\nabla\boldsymbol{\eta}_i\|^2 \le K_n k^2(1+\log\frac{1}{k}) + Kk\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\eta}}_i\|^2. \quad (2.113)$$

For the 'hat' term, we multiply (2.55) by k and take summation over $1 \le i \le n$ and similar to (2.86), we find

$$(\partial_t\widehat{\boldsymbol{\eta}}_n,\mathbf{v}_h) + \mu a(\widehat{\boldsymbol{\eta}}_n,\mathbf{v}_h) + a(q_r^n(\widehat{\boldsymbol{\eta}}),\mathbf{v}_h) = k\sum_{i=1}^{n}\Lambda_h^i(\mathbf{v}_h). \quad (2.114)$$

Choose $\mathbf{v}_h = \widehat{\boldsymbol{\eta}}_i$ in (2.114) for $n = i$, multiply by $ke^{2\alpha t_i}$ and then take summation over $1 \le i \le n$ to observe as in (2.87) that

$$e^{2\alpha t_n}\|\widehat{\boldsymbol{\eta}}_n\|^2 + 2\mu k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\eta}}_i\|^2 \le 2k\sum_{i=1}^{n} e^{2\alpha t_i}k\sum_{j=1}^{i}|\Lambda_h^i(\widehat{\boldsymbol{\eta}}_i)|. \quad (2.115)$$

In (2.52), use Lemma 1.4 with (2.101), (2.102) and (2.108) to obtain

$$\begin{aligned} k\sum_{i=1}^{n} e^{2\alpha t_i}k\sum_{j=1}^{i}|\Lambda_h^j(\widehat{\boldsymbol{\eta}}_i)| &= k\sum_{i=1}^{n} e^{2\alpha t_i}k\sum_{j=1}^{i}\Big|b(\mathbf{u}_h^j,\mathbf{e}_j,\widehat{\boldsymbol{\eta}}_i) + b(\mathbf{e}_j,\mathbf{u}_h^j,\widehat{\boldsymbol{\eta}}_i) + b(\mathbf{e}_j,\mathbf{e}_j,\widehat{\boldsymbol{\eta}}_i)\Big| \\ &\le Kk\sum_{i=1}^{n} e^{2\alpha t_i}\Big(k\sum_{j=1}^{i}\big(\|\mathbf{e}_j\|\|\tilde{\Delta}\mathbf{u}_h^j\| + \|\mathbf{e}_j\|^{1/2}\|\nabla\mathbf{e}_j\|^{3/2}\big)\Big)\|\nabla\widehat{\boldsymbol{\eta}}_i\| \\ &\le K_n k^2(1+\log\frac{1}{k}) + \frac{\mu}{2}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\eta}}_i\|^2. \end{aligned}$$

Incorporate this in (2.115), we find that

$$e^{2\alpha t_n}\|\widehat{\boldsymbol{\eta}}_n\|^2 + \mu k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\widehat{\boldsymbol{\eta}}_i\|^2 \le K_n k^2(1+\log\frac{1}{k})e^{2\alpha t_n} \quad (2.116)$$

Finally, we use (2.116) in (2.113) and multiply both sides by $e^{-2\alpha t_n}$ to complete the proof. □

We now combine the Lemmas 2.12 and 2.15 to conclude the final result of this chapter as follows:

Theorem 2.1. *Suppose the assumptions of Lemma 2.13 hold. Then, the fully discrete error* $\mathbf{e}_n$ *satisfies the following result:*

$$\|\mathbf{e}_n\| \le K_n t_n^{-1/2} k(1+\log\frac{1}{k})^{1/2},\ 1 \le n \le N.$$

Remark 2.10. *We note that the error analysis has been carried out by splitting the error into two parts; error due to linearized part and error due to nonlinear part, and both parts have been analysed separately. The same analysis will go through even without splitting and would have been concise. Such an approach has been employed in our next chapter to obtain optimal* $\mathbf{H}^1$*-velocity error and* L^2 *pressure error, as a part of analysing a two-grid method.*

However, our current approach has two advantages. Firstly, we notice that the exponential increase of the error bound (as $t \to +\infty$*) is due to the nonlinear part, since the other part is uniformly bounded as* $t \to +\infty$*, see, Lemma 2.12. Secondly, for nonsmooth initial data, the uniform error estimate can only be obtained by splitting the error likewise, which, in fact, we present in the Section 2.4 below.*

2.4 Uniform Error Estimates

In this section, we prove the estimate in Theorem 2.1 to be uniform under the uniqueness condition, that is,

$$\mu - 2N\nu^{-1}\|\mathbf{f}\|_\infty > 0 \quad \text{and} \quad N = \sup_{\mathbf{v},\mathbf{w},\phi\in\mathbf{H}_0^1} \frac{b(\mathbf{v},\mathbf{w},\boldsymbol{\phi})}{\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|\|\nabla\boldsymbol{\phi}\|}, \tag{2.117}$$

where $\nu = \mu + \frac{\gamma}{\delta}$ and $\|\mathbf{f}\|_\infty := \|\mathbf{f}\|_{L^\infty(\mathbb{R}_+;\mathbf{L}^2(\Omega))}$. We observe that the estimate (2.95) involving $\boldsymbol{\xi}_n$, that is, $\|\boldsymbol{\xi}_n\| \le Kt_n^{-1/2}k(1+\log\frac{1}{k})^{1/2}$ is uniform in nature. Hence, we are left to deal with $\mathbf{L}^2$ estimate of $\boldsymbol{\eta}_n$.

Lemma 2.16. *Assume* (**A1**)-(**A2**) *and a space discretization scheme that satisfies conditions* (**B1**)-(**B2**)*. Let* $\alpha_0 > 0$ *be such that* $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu\lambda_1}{2}\}$*, (2.20) be satisfied. Also let the uniqueness condition (2.117) be holds, then the following uniform estimate holds:*

$$\|\boldsymbol{\eta}_n\| \le Kt_n^{-1/2}k(1+\log\frac{1}{k})^{1/2},$$

where $K > 0$ is a constant may depends on the given data but not on h and k.

Proof. Choose $\mathbf{v}_h = \boldsymbol{\eta}_i$ in (2.55) for $n = i$ and multiply by $ke^{2\alpha t_i}$ and take summation over $i_0 + 1$ to n to obtain

$$k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} \partial_t \|\boldsymbol{\eta}_i\|^2 + 2\mu k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2 + 2k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} a(q_r^i(\boldsymbol{\eta}), \boldsymbol{\eta}_i) \leq 2k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} \Lambda_h^i(\boldsymbol{\eta}_i). \tag{2.118}$$

A use of the "Poincaré inequality" gives

$$\begin{aligned} k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} \partial_t \|\boldsymbol{\eta}_i\|^2 &= k \sum_{i=i_0+1}^{n} \Big[\partial_t \{ e^{2\alpha t_i} \|\boldsymbol{\eta}_i\|^2 \} - (\frac{e^{2\alpha k} - 1}{k}) e^{2\alpha t_{i-1}} \|\boldsymbol{\eta}_{i-1}\|^2 \Big] \\ &= e^{2\alpha t_n} \|\boldsymbol{\eta}_n\|^2 - \sum_{i=i_0+1}^{n-1} e^{2\alpha t_i} (e^{2\alpha k} - 1) \|\boldsymbol{\eta}_i\|^2 - e^{2\alpha t_{i_0}} \|\boldsymbol{\eta}_{i_0}\|^2 \\ &\geq e^{2\alpha t_n} \|\boldsymbol{\eta}_n\|^2 - e^{2\alpha t_{i_0}} \|\boldsymbol{\eta}_{i_0}\|^2 - \Big(\frac{e^{2\alpha k} - 1}{k\lambda_1} \Big) k \sum_{i=i_0+1}^{n-1} e^{2\alpha t_i} \|\nabla \boldsymbol{\eta}_i\|^2. \end{aligned} \tag{2.119}$$

We recall from (2.97) that

$$\begin{aligned} \Lambda_h^i(\boldsymbol{\eta}_i) &= -b(\mathbf{u}_h^i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i) - b(\mathbf{e}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i) - b(\mathbf{e}_i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i) \\ &= -b(\mathbf{u}_h^i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i) - b(\boldsymbol{\xi}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i) - b(\boldsymbol{\eta}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i) - b(\mathbf{e}_i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i). \end{aligned} \tag{2.120}$$

We use Lemma 1.4 to bound the first and second terms as

$$| - b(\mathbf{u}_h^i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i) - b(\boldsymbol{\xi}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i)| \leq C \|\tilde{\Delta}_h \mathbf{u}_h^i\| \|\boldsymbol{\xi}_i\| \|\nabla \boldsymbol{\eta}_i\|. \tag{2.121}$$

The last term can be estimated using $\|\tilde{\Delta}_h \mathbf{e}_i\| \leq C(\|\tilde{\Delta}_h \mathbf{u}_h^i\| + \|\tilde{\Delta}_h \mathbf{U}^i\|)$ as

$$| - b(\mathbf{e}_i, \boldsymbol{\xi}_i, \boldsymbol{\eta}_i)| \leq C \|\tilde{\Delta}_h \mathbf{e}^i\| \|\boldsymbol{\xi}_i\| \|\nabla \boldsymbol{\eta}_i\| \leq C(\|\tilde{\Delta}_h \mathbf{u}_h^i\| + \|\tilde{\Delta}_h \mathbf{U}^i\|) \|\boldsymbol{\xi}_i\| \|\nabla \boldsymbol{\eta}_i\|. \tag{2.122}$$

And the third term of Λ_h can be estimated via uniqueness condition (2.117) and using the fact that $\limsup_{t \to +\infty} \|\nabla \mathbf{u}_h(t)\| \leq \nu^{-1} \|\mathbf{f}\|_\infty$, as

$$| - b(\boldsymbol{\eta}_i, \mathbf{u}_h^i, \boldsymbol{\eta}_i)| \leq N \|\nabla \mathbf{u}_h^i\| \|\nabla \boldsymbol{\eta}_i\|^2 \leq N \nu^{-1} \|\mathbf{f}\|_\infty \|\nabla \boldsymbol{\eta}_i\|^2. \tag{2.123}$$

We rewrite the quadrature term as

$$2k \sum_{i=i_0+1}^{n} e^{2\alpha t_i} a(q_r^i(\boldsymbol{\eta}), \boldsymbol{\eta}_i) = 2k \sum_{i=1}^{n} e^{2\alpha t_i} a(q_r^i(\boldsymbol{\eta}), \boldsymbol{\eta}_i) - 2k \sum_{i=1}^{i_0} e^{2\alpha t_i} a(q_r^i(\boldsymbol{\eta}), \boldsymbol{\eta}_i). \tag{2.124}$$

Inserting the above five estimates in (2.118), we finally obtain

$$
\begin{aligned}
e^{2\alpha t_n}\|\boldsymbol{\eta}_n\|^2 - e^{2\alpha t_{i_0}}\|\boldsymbol{\eta}_{i_0}\|^2 + \big(\frac{3\mu}{4} - \frac{e^{2\alpha k}-1}{k\lambda_1}\big)k\sum_{i=i_0+1}^{n} e^{2\alpha t_i}\|\nabla\boldsymbol{\eta}_i\|^2 \\
+ \big(\mu - N\nu^{-1}\|\mathbf{f}\|_\infty\big)k\sum_{i=i_0+1}^{n} e^{2\alpha t_i}\|\nabla\boldsymbol{\eta}_i\|^2 + 2k\sum_{i=1}^{n} e^{2\alpha t_i} a(q_r^i(\boldsymbol{\eta}),\boldsymbol{\eta}_i) \\
\leq Ck\sum_{i=i_0+1}^{n} e^{2\alpha t_i}(\|\tilde{\Delta}_h\mathbf{u}_h^i\|^2 + \|\tilde{\Delta}_h\mathbf{U}^i\|^2)\|\boldsymbol{\xi}_i\|^2 + 2k\sum_{i=1}^{i_0} e^{2\alpha t_i} a(q_r^i(\boldsymbol{\eta}),\boldsymbol{\eta}_i). \quad (2.125)
\end{aligned}
$$

With $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu\lambda_1}{2}\}$, we have $(\frac{3\mu}{4} - \frac{e^{2\alpha k}-1}{k\lambda_1}) > 0$ and the uniqueness condition confirms $(\mu - N\nu^{-1}\|\mathbf{f}\|_\infty) > 0$ and due to positivity property, the last term on the left side is positive, so we drop all three terms. Following the proof techniques of (2.66)-(2.67), we bound the quadrature term of (2.125) as

$$
2k\sum_{i=1}^{i_0} e^{2\alpha t_i} q_r^i(\|\nabla\boldsymbol{\eta}_i\|)\|\nabla\boldsymbol{\eta}_i\| \leq Kk\sum_{i=1}^{i_0} e^{2\alpha t_i}\|\nabla\boldsymbol{\eta}_i\|^2, \quad (2.126)
$$

And finally we arrive at

$$
e^{2\alpha t_n}\|\boldsymbol{\eta}_n\|^2 \leq Ce^{2\alpha t_n}\|\boldsymbol{\xi}_n\|^2 + e^{2\alpha t_{i_0}}\|\boldsymbol{\eta}_{i_0}\|^2 + Kk\sum_{i=1}^{i_0} e^{2\alpha t_i}\|\nabla\boldsymbol{\eta}_i\|^2. \quad (2.127)
$$

Multiply by $e^{-2\alpha t_i}$ and assuming the last sum is bounded appropriately (see Lemma 2.17 below) we conclude that

$$
\|\boldsymbol{\eta}_n\| \leq K_{i_0} t_n^{-1/2} k(1 + \log\frac{1}{k})^{1/2}, n > i_0.
$$

Combining this with (2.109) for $n \leq i_0$, we obtain the desired result, since $i_0 > 0$ is fixed. This completes the proof. □

We now bound the last sum from (2.127).

Lemma 2.17. *Suppose the assumptions of Lemma 2.16 hold. Then, the following error result is valid:*

$$
e^{-2\alpha t_i} k\sum_{i=1}^{i_0} e^{2\alpha t_i}\|\nabla\boldsymbol{\eta}_i\|^2 \leq K_{i_0} t_{i_0}^{-1} k^2(1 + \log\frac{1}{k}).
$$

Proof. In (2.96), we use

$$
\Lambda_h^i(\boldsymbol{\eta}_i) = -b(\mathbf{u}_h^i, \mathbf{e}_i, \boldsymbol{\eta}_i) - b(\mathbf{e}_i, \mathbf{U}^i, \boldsymbol{\eta}_i) \leq \frac{\mu}{4}\|\nabla\boldsymbol{\eta}_i\|^2 + K\|\mathbf{e}_i\|^2(\|\tilde{\Delta}_h\mathbf{u}_h^i\| + \|\tilde{\Delta}_h\mathbf{U}^i\|),
$$

along with Lemma 2.14 and Theorem 2.1 to obtain the desired result. □

Theorem 2.2. *Suppose the assumptions of Lemma 2.16 hold. Then, the fully discrete error* $\mathbf{e}_n$ *satisfies the following results:*

$$\|\mathbf{e}_n\| \leq K t_n^{-1/2} k (1 + \log \frac{1}{k})^{1/2}, \quad 1 \leq n \leq N.$$

Proof. Combine the Lemmas 2.12 and 2.16 to complete the proof. □

Now, combine the Theorem 1.2, 2.1 and 2.2 to conclude the following:

Theorem 2.3. *Suppose the assumptions of Theorem 1.2 and Lemma 2.8 be satisfied. Then, the error holds the following result:*

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \leq K_n t_n^{-1/2} \Big(h^2 + k(1 + \log \frac{1}{k})^{1/2} \Big), \quad 1 \leq n \leq N.$$

Moreover, under the uniqueness condition (2.117), the above result is valid uniformly in time.

Remark 2.11. *We have obtained an improved estimate of Theorem 2.1 in chapter 4 for the penalized fully discrete solution. In the similar way, we can also find the following improved result,*

$$\|\mathbf{u}_h(t_n) - \mathbf{U}^n\| \leq K_n t_n^{-1/2} k, \; 1 \leq n \leq N.$$

2.5 Numerical Experiments

In this section, we present some numerical experiments that verify the results of the previous sections, namely; the order of convergence of the error estimates. For simplicity, we use examples with known solutions. All the numerical computations have been done in MATLAB. We consider the Oldroyd model of order one subject to homogeneous Dirichlet boundary conditions in the domain $\Omega = [0, 1] \times [0, 1]$. We approximate the equation using $(P_1 b, P_1)$ and (P_2, P_0) elements over a regular triangulation of Ω. We decompose the domain into triangles with size $h = 2^{-i}, \; i = 2, 3, \ldots, 6$. Now, we consider the following example:

Example 2.1. *For initial data* $\mathbf{u}_0 \in \mathbf{H}^2$*, we choose* $f(x,t)$ *so as to have the following solutions of the problem*

$$u_1(x,t) = 2e^t x_1^2 (x_1 - 1)^2 x_2 (x_2 - 1)(2x_2 - 1),$$

$$u_2(x,t) = -2e^t x_1(x_1-1)(2x_1-1)x_2^2(x_2-1)^2,$$
$$p(x,t) = 2e^t(x_1 - x_2).$$

Table 2.1: Errors and convergence rates (C.R.) for Example 2.1 for (P_2, P_0) element

h	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P^n - p(t_n)\|_{L^2}$	C.R.
1/8	0.00386700	-	0.15057567	-	0.17021691	-
1/16	0.00104657	1.8855	0.07849371	0.9398	0.08591565	0.9864
1/32	0.00026335	1.9906	0.03939885	0.9944	0.04246851	1.0165
1/64	0.00006623	1.9913	0.01976541	0.9952	0.02115282	1.0055

Table 2.2: Errors and convergence rates (C.R.) for Example 2.1 for (P_1b, P_1) element

h	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P^n - p(t_n)\|_{L^2}$	C.R.
1/8	0.00172068		0.04302980		0.17416894	
1/16	0.00045020	1.9344	0.02212674	0.9595	0.10199069	0.7720
1/32	0.00009954	2.1771	0.01037882	1.0921	0.04131507	1.3037
1/64	0.00002414	2.0436	0.00489803	1.0834	0.01289942	1.6794

In Table 2.1 and 2.2, we present the numerical errors and rates of convergence obtained on successive meshes using (P_2, P_0) and (P_1b, P_1) elements, respectively, for BE method applied to the system (1.4)-(1.6) with $\mu = 1, \gamma = 0.1, \delta = 0.1$ and $T = 1$. The numerical analysis shows that the rates of convergence are $\mathcal{O}(h^2)$ in $\mathbf{L}^2$-norm and $\mathcal{O}(h)$ in $\mathbf{H}^1$-norm for the velocity. And the rate of convergence for the pressure is $\mathcal{O}(h)$ in $\mathbf{L}^2$-norm. We choose the time step $k = \mathcal{O}(h^2)$ for all the experiments. The error graphs are presented in Fig 2.1 and Fig 2.2. These numerical results assist the optimal rates convergence obtained in Theorem 2.3.

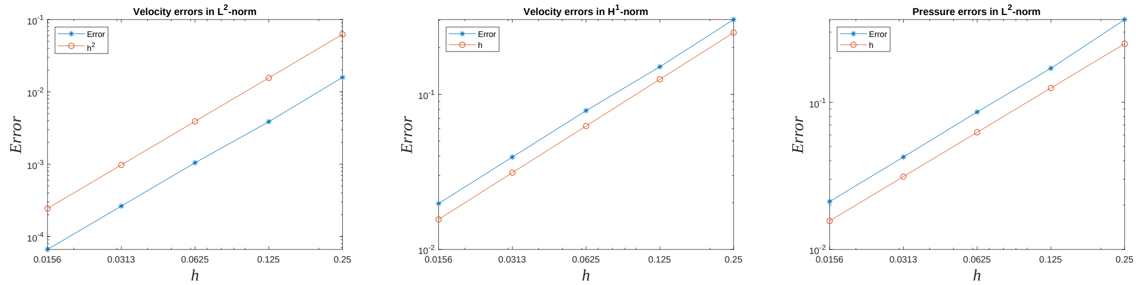

Figure 2.1: Velocity and pressure errors based on (P_2, P_0) element for Example 2.1.

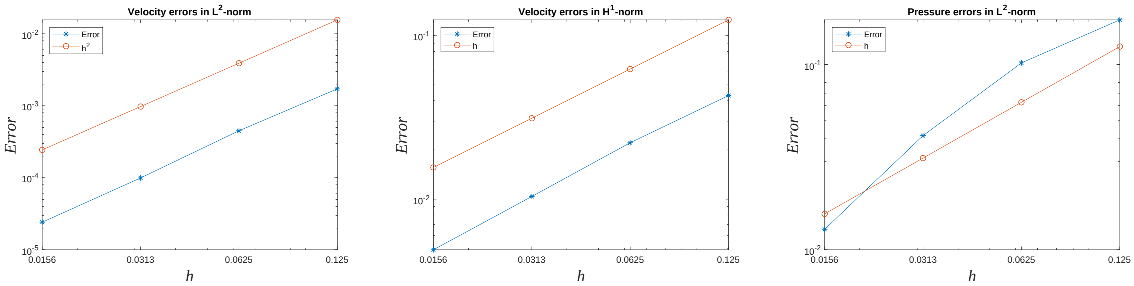

Figure 2.2: Velocity and pressure errors based on $(P_1 b, P_1)$ element for Example 2.1.

In order to verify the rates of convergence in both spatial and temporal directions and the uniform convergence in time for nonsmooth data, we consider the following example [77, 150].

Example 2.2. *For initial data* $\mathbf{u}_0 \in \mathbf{H}_0^1$*, we choose* $f(x,t)$ *so as to have the following solutions of the problem*

$$u_1(x,t) = 5x_1^{5/2}(x_1-1)^2 x_2^{3/2}(x_2-1)(9x_2-5)\cos t,$$
$$u_2(x,t) = -5x_1^{3/2}(x_1-1)(9x_1-5)x_2^{5/2}(x_2-1)^2\cos t,$$
$$p(x,t) = 2(x_1-x_2)\cos t.$$

Table 2.3: Errors and convergence rates (C.R.) for Example 2.2 for (P_2, P_0) element

h	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P^n - p(t_n)\|_{L^2}$	C.R.
1/4	0.00295597	-	0.05958679	-	0.07233700	-
1/8	0.00071240	2.0529	0.02832958	1.0727	0.03383893	1.0960
1/16	0.00019314	1.8830	0.01456592	0.9597	0.01708781	0.9857
1/32	0.00004903	1.9780	0.00726227	1.0041	0.00845973	1.0143
1/64	0.00001294	1.9217	0.00363780	0.9973	0.00423842	0.9971

Table 2.4: Errors and convergence rates (C.R.) for Example 2.2 for $(P_1 b, P_1)$ element

h	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P^n - p(t_n)\|_{L^2}$	C.R.
1/8	0.00208654	-	0.05026012	-	0.20559044	-
1/16	0.00054627	1.9334	0.02563557	0.9713	0.10681639	0.9446
1/32	0.00014985	1.8660	0.01262894	1.0214	0.05064130	1.0767
1/64	0.00004098	1.8705	0.00614613	1.0390	0.01583420	1.6772

Table 2.5: L^2-errors and convergence rates (C.R.) in time direction for Example 2.2

	(P_2, P_0) **element**		(P_1b, P_1) **element**	
k	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.
1/4	0.00755768	-	0.02197941	-
1/16	0.00290626	0.6894	0.00802012	0.7272
1/64	0.00070503	1.0217	0.00207538	0.9751
1/256	0.00019182	0.9389	0.00053884	0.9727
1/1024	0.00004856	0.9909	0.00014555	0.9442

In Table 2.3 and 2.4, we have shown the numerical errors and the rates of convergence for the BE method using (P_2, P_0) and (P_1b, P_1) elements, respectively, with $\mu = 1, \gamma = 0.1, \delta = 1$ and final time $T = 1$. The numerical results confirm the optimal $\mathbf{L}^2$-convergence rates of the velocity error as in Theorem 2.3. The error graphs are given in Fig 2.3 and Fig 2.4. In Table 2.5, we present the numerical results in temporal direction for (P_2, P_0) and (P_1b, P_1) elements, respectively. Here, we take $k = 2^{-2i}$, $i = 1, 2, \dots, 5$, $\mu = 1, \delta = 0.1, \gamma = 0.1, h = \mathcal{O}(\sqrt{k})$ and $T = 1$. The error graph is given in Fig 2.5. We observe that the rate of convergence confirms the theoretical findings.

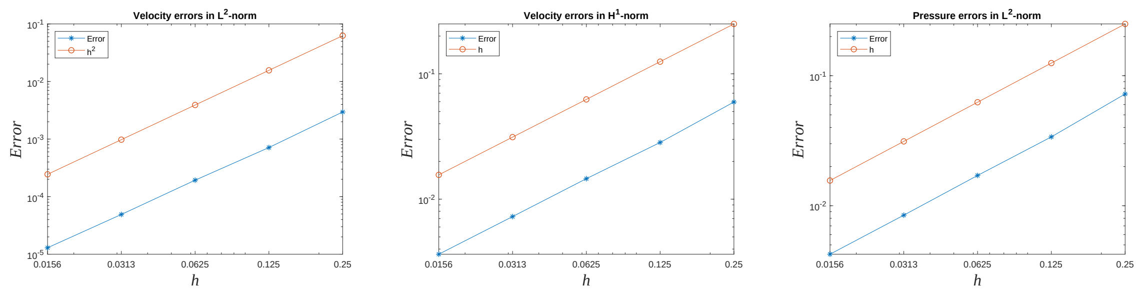

Figure 2.3: Velocity and pressure errors based on (P_2, P_0) element for Example 2.2.

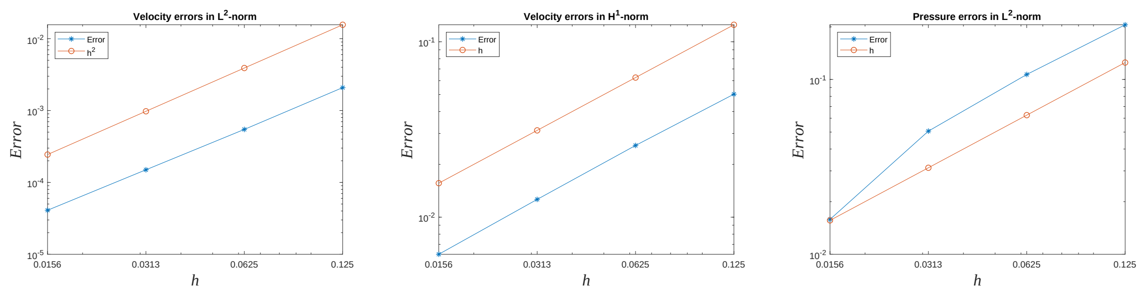

Figure 2.4: Velocity and pressure errors based on (P_1b, P_1) element for Example 2.2.

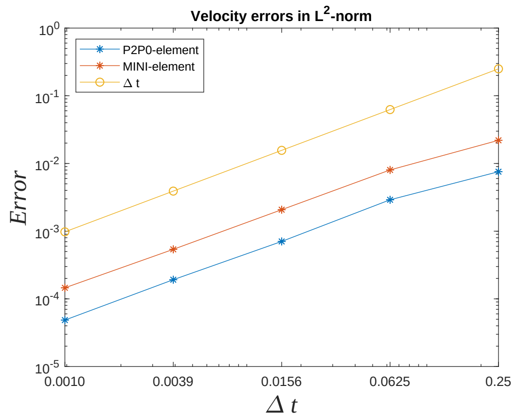

Figure 2.5: Velocity errors in L^2- norm with respect to time for Example 2.2.

For the example 2.2, the numerical results are shown for final time $T = 10$, 20, 30, 40 and 50 with $\mu = 1, \gamma = 0.1, \delta = 1$, $k = 0.1$ and $h = 2^{-i}$, $i = 2, 3, \ldots, 6$. We represent the numerical errors and the rate of convergences for the velocity in $\mathbf{L}^2$-norm for (P_2, P_0) and (P_1b, P_1) elements in Table 2.6 and Fig 2.6. The numerical experiments show that for a large time the convergence rates remain same.

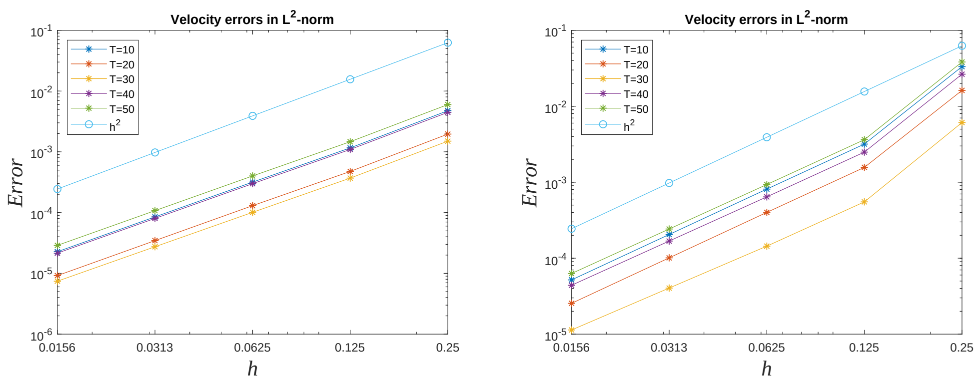

Figure 2.6: Uniform in time errors for (P_2, P_0) and (P_1b, P_1) elements for Example 2.2.

In tables 2.7 to 2.9, we have shown the maximal L^2-norm of the pressure and maximal $\mathbf{L}^2$- and $\mathbf{H}^1$-norm of the velocity among several time steps $k = 0.1, 0.5, 1, 1.3$ again for the Example 2.2. The results indicate that the scheme can run well for the values of the time steps going from $k = 0.1$ to $k = 1.3$, but there is a deterioration of the convergence rate for $k = 1$ and $k = 1.3$.

Table 2.6: L^2-Errors and convergence rates for Example 2.2

		(P_2, P_0) **element**		(P_1b, P_1) **element**	
Final time	h	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	Rate	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	Rate
T=10	1/4	0.00475212	-	0.03323502	-
	1/8	0.00116079	2.0334	0.00317788	3.3866
	1/16	0.00031742	1.8706	0.00081112	1.9701
	1/32	0.00008496	1.9015	0.00020500	1.9843
	1/64	0.00002272	1.9030	0.00005176	1.9857
T=20	1/4	0.00195842	-	0.01616627	-
	1/8	0.00047565	2.0417	0.00156824	3.3657
	1/16	0.00013005	1.8708	0.00040039	1.9697
	1/32	0.00003459	1.9105	0.00010100	1.9870
	1/64	0.00000919	1.9120	0.00002545	1.9885
T=30	1/4	0.00149863	-	0.00610584	-
	1/8	0.00036956	2.0197	0.00055113	3.4697
	1/16	0.00010101	1.8713	0.00014395	1.9368
	1/32	0.00002742	1.8810	0.00004044	1.8318
	1/64	0.00000744	1.8824	0.00001135	1.8333
T=40	1/4	0.00446125	-	0.02641260	-
	1/8	0.00109409	2.0277	0.00249141	3.4062
	1/16	0.00029908	1.8711	0.00064118	1.9582
	1/32	0.00008028	1.8973	0.00016808	1.9316
	1/64	0.00002153	1.8988	0.00004402	1.9330
T=50	1/4	0.00599158	-	0.03821832	-
	1/8	0.00146697	2.0301	0.00363026	3.3961
	1/16	0.00040103	1.8710	0.00093218	1.9614
	1/32	0.00010770	1.8967	0.00024209	1.9451
	1/64	0.00002889	1.8982	0.00006281	1.9465

Table 2.7: The norm $\sup_{0\leq t_n\leq 5}\|\mathbf{U}^n\|_{\mathbf{L}^2}$ with nonsmooth data

	h \ k	0.1	0.5	1	1.3
(P_2, P_0)	1/10	0.04066058	0.04013758	0.03994627	0.03431507
	1/20	0.04060207	0.04007989	0.03988995	0.03426480
	1/30	0.04059567	0.04007359	0.03988379	0.03425928
	1/40	0.04059327	0.04007121	0.03988147	0.03425720
(P_1b, P_1)	1/10	0.04164312	0.03825401	0.03808147	0.03272622
	1/20	0.04310498	0.03967860	0.03949458	0.03393053
	1/30	0.04334520	0.03991041	0.03972447	0.03412640
	1/40	0.04343595	0.03999863	0.03981201	0.03420103

Table 2.8: The norm $\sup_{0\leq t_n\leq 5}\|\mathbf{U}^n\|_{\mathbf{H}^1}$ with nonsmooth data

	h \ k	0.1	0.5	1	1.3
(P_2, P_0)	1/10	0.31297840	0.30901528	0.30752906	0.26430325
	1/20	0.31034508	0.30641145	0.30500030	0.26201939
	1/30	0.30987407	0.30594553	0.30454751	0.26161070
	1/40	0.30970478	0.30577810	0.30438480	0.26146391
(P_1b, P_1)	1/10	0.32158956	0.29632714	0.29503887	0.25354515
	1/20	0.32860122	0.30352433	0.30217202	0.25959799
	1/30	0.32979560	0.30474032	0.30337746	0.26062074
	1/40	0.33024171	0.30519856	0.30383212	0.26100694

Table 2.9: The norm $\sup_{0\leq t_n\leq 5}\|P^n\|_{L^2}$ of with nonsmooth data

	h \ k	0.1	0.5	1	1.3
(P_2, P_0)	1/10	0.80976358	0.80204569	0.80174532	0.69388452
	1/20	0.81411956	0.80642016	0.80616249	0.69778546
	1/30	0.81499218	0.80729509	0.80704511	0.69856265
	1/40	0.81529345	0.80759726	0.80735004	0.69883114
(P_1b, P_1)	1/10	0.86979545	0.85337708	0.85455721	0.73827331
	1/20	0.82750272	0.81637519	0.81639044	0.70638134
	1/30	0.82289194	0.81236294	0.81225804	0.70293783
	1/40	0.81951684	0.80935013	0.80914669	0.70034738

2.6 Conclusion

In this chapter, optimal $\mathbf{L}^2$- velocity error estimate is derived for the backward Euler method employed to the Oldroyd model with nonsmooth initial data, that is, $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$. For the complete discrete scheme, uniform a priori bounds are shown for the discrete solution. Both optimal and uniform error estimate for the velocity are proved. Uniform estimates are derived under the uniqueness condition. The analysis has been carried out for the nonsmooth initial data and the proofs are more involved in comparison to the smooth case. Our numerical results confirms our theoretical results.