Chapter 3

Two-grid Method

In this chapter, we analyze a two-grid method for the Oldroyd model of order one. Two-grid method is a highly efficient, accurate and well-established method for solving a nonlinear system. The main idea of two-grid method is approximate the equation over two meshes of different size. It generally involves two steps; in step one a nonlinear problem is solved over the coarse mesh and then in the second, a linearized equation, based on coarse mesh solution, is solved on a fine mesh. We analyze here a two-grid method based on three steps. The third step is a correction step where the results of the second step are used to solve a linearized problem based on Newton iteration in the same fine mesh but with different right hand side. We obtain optimal velocity and pressure errors for semidiscrete and fully discrete approximations. All the estimates are valid as time goes to infinity. Finally we give some numerical examples to validate our theoretical findings. This work has been published in [11].

3.1 Introduction

For two-grid formulation, we first denote $\omega = H$ or h with $h \ll H$. Now we consider the regular finite triangulation of the domain $\bar{\Omega}$ in two different meshes $\mathcal{T}_H$ and $\mathcal{T}_h$. The first one ($\mathcal{T}_H$) is called coarse mesh and another one ($\mathcal{T}_h$) is called fine mesh. With $\omega = H,\ h$, we assume two family of finite element spaces $\mathbf{H}_\omega$ and L_ω subspaces of $\mathbf{H}_0^1$ and L^2, respectively. For simplicity, we assume that both the spaces consist of piecewise linear polynomial functions (for example, MINI element). We also define the associated divergence free subspace $\mathbf{J}_\omega$ of $\mathbf{H}_\omega$, as

$$\mathbf{J}_\omega = \{\mathbf{v}_\omega \in \mathbf{H}_\omega : (\mathbf{v}_\omega, \nabla \cdot \chi_\omega) = 0, \quad \forall \chi_\omega \in L_\omega\}.$$

The main algorithm for two-grid method for any nonlinear problem is stated as follow:

- **Step I:** Solve the nonlinear problem over the coarse mesh $\mathcal{T}_H$.
- **Step II:** Solve a time-dependent/independent linearized system over the fine mesh $\mathcal{T}_h$.

This method has been analyzed by Xu [147, 148] for the elliptic problems and by Layton *et. al.* [91, 92] for the steady-state NSE. The method changes with the linearized problem that we solve on the fine mesh; for example, Layton et. al. [91, 92] have considered different algorithm for linearizing the Navier-Stokes equation based on the one step Newton method, the discrete steady Oseen problem and the discrete Stokes problem.

Literature is abundant in case of NSE. For example, Girault *et. al.* [55] have obtained the error estimate for steady-state NSEs and these works have been extended for transient NSE in [56]. An extension of this analysis to the fully discretization has been studied by Abboud *et. al.* in [1]. They have also analyzed the second-order finite element (Hood-Taylor element) for the spatial discretizationin in [2].

In [40], de Frutos *et. al.* have analyzed mixed-finite elements of first-order (Mini-element), second-order (quadratic Taylor-Hood element) and third-order (cubic Taylor-Hood element) for spatial discretization along with two-step backward difference scheme and backward Euler method for time discretization. They have obtained optimal $\mathbf{H}^1$-error for the velocity with the choice of $h = H^2$ which is an improvement over the result with $h = H^{3/2}$ obtained in [55]. In addition, they have taken into account "the lack of regularity of the solution" at $t = 0$ and have considered $\mathbf{u}_0$ to be in $\mathbf{H}^2$. We note here that "demanding further regularity requires the data to satisfy nonlocal compatibility conditions unlikely to be fulfilled in practical situations" [79]. The regularity has been further reduced in an article by Goswami *et. al.* [62] where a two-grid method for NSEs has been considered but only for linear approximation. There $\mathbf{u}_0 \in \mathbf{H}_0^1$ has been considered, allowing singularity even in $\mathbf{H}^2$ norm, that is, $\|\mathbf{u}\|_2 \approx \mathcal{O}(t^{-1/2})$.

As far as we know, there is no work on two-grid method for our model except [60, 61]. In [60, 61], the following algorithm has been applied:

"**Step I:** (Solve a nonlinear problem on coarse mesh of size H)

For $t > 0$, seek $(\mathbf{u}_H(t), p_H(t)) \in \mathbf{H}_H \times L_H$ with $\mathbf{u}_H(0) = \mathbf{u}_{0H}$ satisfying

$$\left.\begin{aligned} (\mathbf{u}_{Ht}, \mathbf{v}_H) + \mu a(\mathbf{u}_H, \mathbf{v}_H) + b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{v}_H) + \int_0^t \beta(t-s) a(\mathbf{u}_H(s), \mathbf{v}_H) ds \\ - (p_H, \nabla \cdot \mathbf{v}_H) = (\mathbf{f}, \mathbf{v}_H), \quad \forall \mathbf{v}_H \in \mathbf{H}_H, \\ (\nabla \cdot \mathbf{u}_H, \chi_H) = 0, \quad \forall \chi_H \in L_H. \end{aligned}\right\}$$

Step II: (Solve a linearize Stokes type problem on fine mesh of size h)

For $t > 0$, find $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{H}_h \times L_h$ with $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ satisfying

$$\left.\begin{aligned} (\mathbf{u}_{ht}, \mathbf{v}_h) + \mu a(\mathbf{u}_h, \mathbf{v}_h) + \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{v}_h) ds - (p_h, \nabla \cdot \mathbf{v}_h) \\ = (\mathbf{f}, \mathbf{v}_h) - b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_h, \\ (\nabla \cdot \mathbf{u}_h, \chi_h) = 0, \quad \forall \chi_h \in L_h. \end{aligned}\right\}"$$

Here the problem in the second step is linearized based on discrete Stokes type problem. And the following optimal $\mathbf{H}^1$-velocity and L^2-pressure errors have been obtained for less regular initial data.

$$\|\nabla(\mathbf{u}(t) - \mathbf{u}_h(t))\| + \|p(t) - p_h(t)\| \leq K(t)(h^2 t^{\frac{1}{2}} + H^2 t^{-1}).$$

However, $\mathbf{L}^2$ error estimate obtained there is sub-optimal, which is a drawback of the method applied there. In the framework of NSE also, a sub-optimal $\mathbf{L}^2$ error has been observed, see [40, Remark 2].

A further correcting step enables us to recover optimal estimate. We therefore consider a three steps two-grid method based on the following steps:

- **Step I:** Solve the nonlinear system over the coarse mesh $\mathcal{T}_H$.
- **Step II:** Linearize the problem using the coarse grid solution and solve it over the fine mesh $\mathcal{T}_h$.
- **Step III:** Correct the solution from **Step II** over $\mathcal{T}_h$ which give an updated solution.

This algorithm was proposed by Xu [147] for a nonlinear elliptic problem and then, continued by Dai *et. al.* [38] for steady-state NSE and Pani *et. al.* [7] for transient NSE.

Applying this algorithm for our model, the three steps two-grid semidiscrete formulation for Oldroyd model of order one (1.4)-(1.6) reads as:

Step I ("Nonlinear system on coarse mesh"):

For $t > 0$, find $\mathbf{u}_H(t) \in \mathbf{H}_H$ and $p_H(t) \in L_H$ with $\mathbf{u}_H(0) = P_H\mathbf{u}_0$ satisfying

$$\left.\begin{aligned} &(\mathbf{u}_{Ht}, \mathbf{v}_H) + \mu a(\mathbf{u}_H, \mathbf{v}_H) + b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{v}_H) + \int_0^t \beta(t-s) a(\mathbf{u}_H(s), \mathbf{v}_H) ds \\ &\qquad\qquad - (p_H, \nabla\cdot\mathbf{v}_H) = (\mathbf{f}, \mathbf{v}_H), \quad \forall \mathbf{v}_H \in \mathbf{H}_H, \\ &(\nabla\cdot\mathbf{u}_H, \chi_H) = 0, \quad \forall \chi_H \in L_H. \end{aligned}\right\} \tag{3.1}$$

Step II ("Update on a finer mesh with one Newton iteration"):

For $t > 0$, find $\mathbf{u}_h^*(t) \in \mathbf{H}_h$ and $p_h^*(t) \in L_h$ with $\mathbf{u}_h^*(0) = P_h\mathbf{u}_0$ satisfying

$$\left.\begin{aligned} &(\mathbf{u}_{ht}^*, \mathbf{v}_h) + \mu a(\mathbf{u}_h^*, \mathbf{v}_h) + b(\mathbf{u}_h^*, \mathbf{u}_H, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{u}_h^*, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) \\ &\qquad + \int_0^t \beta(t-s) a(\mathbf{u}_h^*(s), \mathbf{v}_h) ds = (\mathbf{f}, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_h, \\ &(\nabla\cdot\mathbf{u}_h^*, \chi_h) = 0, \quad \forall \chi_h \in L_h. \end{aligned}\right\} \tag{3.2}$$

Step III ("Correction on finer mesh"):

For $t > 0$, find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ with $\mathbf{u}_h(0) = P_h\mathbf{u}_0$ satisfying

$$\left.\begin{aligned} &(\mathbf{u}_{ht}, \mathbf{v}_h) + \mu a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_H, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{u}_h, \mathbf{v}_h) - (p_h^*, \nabla\cdot\mathbf{v}_h) \\ &\qquad + \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{v}_h) ds = (\mathbf{f}, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{u}_h^*, \mathbf{v}_h) \\ &\qquad\qquad + b(\mathbf{u}_h^*, \mathbf{u}_H - \mathbf{u}_h^*, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_h, \\ &(\nabla\cdot\mathbf{u}_h, \chi_h) = 0, \quad \forall \chi_h \in L_h. \end{aligned}\right\} \tag{3.3}$$

We would like to note here that our work is quite close to the work of [7], for transient NSE. There, the authors have obtained optimal $\mathbf{H}^1$-velocity error and L^2-pressure error with choice $h = H^{4-\ell}$. Also they have obtained the $\mathbf{L}^2$ error estimate for velocity with choice of $h = H^{2-\ell}$ for arbitrary small $\ell > 0$. All these analysis have been done taking the initial velocity $\mathbf{u}_0$ in $\mathbf{H}_0^1 \cap \mathbf{H}^2$. We have employed the same method and similar analysis. But our work differs in several instances. First of all, this is the first time that this method has been applied to our model. The presence of the memory term along with the nonlinear term demands new technique and more sophistication. Also we consider nonsmooth initial data, and this loss of regularity presents technical challenges, more notably in fully discrete case. We have considered here a fully discrete approximation employing a first-order backward Euler (BE) scheme in the temporal direction. We list below the main results of this chapter:

(i) Optimal semidiscrete error bounds for each step.

(ii) Uniform in time error bounds under the assumption of smallness condition (3.11).

(iii) Uniform *a priori* estimates for the fully discrete solution giving long term stability under no smallness condition on time step k.

(iv) Unconditional stability of the fully discrete scheme.

(v) Optimal fully discrete error bounds for the velocity and for the pressure.

(vi) Numerical experiments to validate the theoretical findings.

The rest of the chapter contains the following sections. We discuss about the two-grid formulation and *a priori* estimates of semidiscrete solutions in Section 3.2. The semidiscrete error analysis is carried out in Section 3.3 and in Section 3.4 a first-order time discrete scheme is analyzed for the two-grid system. And finally, in Section 3.5, couple of numerical examples are presented which validate our theoretical results.

Throughout this chapter, we will use $C > 0$ as a generic constant that depends on the given data and $K = K(C, t) > 0$ but none depends on h and k. Note that K may grow exponentially with time, rendering error estimates local in nature, but it will grow algebraically with μ^{-1}.

3.2 Two-Grid Formulation

We start this section by discussing about the two-grid formulation of semidiscrete approximation in divergence free spaces. Then, we give *a priori* and regularity bounds of the discrete solutions.

We project the equations (3.1)-(3.3) in appropriate divergence free space, then the equations read as:

Step I: For any $t > 0$, seek $\mathbf{u}_H \in \mathbf{J}_H$ satisfying

$$(\mathbf{u}_{Ht}, \mathbf{v}_H) + \mu a(\mathbf{u}_H, \mathbf{v}_H) + b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{v}_H) + \int_0^t \beta(t-s) a(\mathbf{u}_H(s), \mathbf{v}_H)ds = (\mathbf{f}, \mathbf{v}_H), \quad (3.4)$$

for all $\mathbf{v}_H \in \mathbf{J}_H$ with $\mathbf{u}_H(0) = P_H\mathbf{u}_0$.

Step II: For any $t > 0$ and for all $\mathbf{v}_h \in \mathbf{J}_h$, seek $\mathbf{u}_h^* \in \mathbf{J}_h$ with $\mathbf{u}_h^*(0) = P_h\mathbf{u}_0$ satisfying

$$(\mathbf{u}_{ht}^*, \mathbf{v}_h) + \mu a(\mathbf{u}_h^*, \mathbf{v}_h) + b(\mathbf{u}_h^*, \mathbf{u}_H, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{u}_h^*, \mathbf{v}_h) + \int_0^t \beta(t-s) a(\mathbf{u}_h^*(s), \mathbf{v}_h)ds$$

$$= (\mathbf{f}, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{v}_h). \tag{3.5}$$

Step III: For any $t > 0$ and for all $\mathbf{v}_h \in \mathbf{J}_h$, seek $\mathbf{u}_h \in \mathbf{J}_h$ with $\mathbf{u}_h(0) = P_h\mathbf{u}_0$ satisfying

$$(\mathbf{u}_{ht}, \mathbf{v}_h) + \mu a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_H, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{u}_h, \mathbf{v}_h) + \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{v}_h) ds$$
$$= (\mathbf{f}, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{u}_h^*, \mathbf{v}_h) + b(\mathbf{u}_h^*, \mathbf{u}_H - \mathbf{u}_h^*, \mathbf{v}_h). \tag{3.6}$$

The following estimation of $\mathbf{u}_H$ will be used in our further study. The proof of these estimates are given in Chapter 2, Lemma 2.1 and 2.2 (with $\mathbf{u}_H$ replaced by $\mathbf{u}_h$).

Lemma 3.1. *Suppose the conditions* (**A1**)-(**A2**) *and* (**B1**)-(**B2**) *hold. Also assume that* $\mathbf{u}_H(0) = P_H\mathbf{u}_0$, *then for* $t > 0$, *the solution* $\mathbf{u}_H$ *of* (3.4) *satisfies,*

$$\tau^*(t)\|\mathbf{u}_H(t)\|_2^2 + \|\mathbf{u}_H(t)\|_r^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_H(s)\|_{r+1}^2 ds \leq C, \qquad r \in \{0,1\},$$
$$(\tau^*(t))^{r+1}\|\mathbf{u}_{Ht}(t)\|_r^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau^*(s))^r\|\mathbf{u}_{Hs}(s)\|_r^2 ds \leq C, \qquad r \in \{0,1,2\},$$
$$(\tau^*(t))^3\|\mathbf{u}_{Ht}(t)\|_2^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau^*(s))^{r+1}\|\mathbf{u}_{Hss}(s)\|_{r-1}^2 ds \leq C, \qquad r \in \{-1,0,1\},$$

where, $\tau^*(t) = \min\{1, t\}$ *and* $C > 0$ *is a constant may depends on the given data.*

Remark 3.1. *We would like to note here that the above estimates are still valid if we replace* $\mathbf{u}_H$ *by* $\mathbf{u}_h^*$ *and* $\mathbf{u}_h$. *And the proofs are in fact simpler than ones of Lemma 3.1 since now* $\mathbf{u}_h^*$ *and* $\mathbf{u}_h$ *satisfy linearized versions of the nonlinear problem.*

The inequality below will be frequently used in error estimate [92, Equation (2.10)]:

$$\inf_{\mathbf{w}_h \in \mathbf{J}_h} \sup_{\mathbf{v}_h \in \mathbf{J}_h} \frac{\mu a(\mathbf{w}_h, \mathbf{v}_h) + b(\mathbf{w}_h, \mathbf{u}_H, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{w}_h, \mathbf{v}_h)}{\|\nabla \mathbf{w}_h\|\|\nabla \mathbf{v}_h\|} \geq \mu_1 > 0. \tag{3.7}$$

We finally present some estimates of the nonlinear term $b(\cdot,\cdot,\cdot)$ which are already present in the Chapter 1 except the last two. For a proof of the couple of them, see [92].

Lemma 3.2. *Suppose the conditions* (**A1**), (**B1**) *and* (**B2**) *hold. Then, the following* *bounds hold for arbitrary small $\ell > 0$:*

$$|b(\mathbf{v},\mathbf{w},\boldsymbol{\phi})| = C \begin{cases} \|\nabla\mathbf{v}\|^{\frac{1}{2}}\|\tilde{\Delta}_\omega\mathbf{v}\|^{\frac{1}{2}}\|\nabla\mathbf{w}\|\|\boldsymbol{\phi}\|, & \forall\mathbf{v},\mathbf{w},\boldsymbol{\phi}\in\mathbf{H}_\omega, \\ \|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|^{\frac{1}{2}}\|\tilde{\Delta}_\omega\mathbf{w}\|^{\frac{1}{2}}\|\boldsymbol{\phi}\|, & \forall\mathbf{v},\mathbf{w},\boldsymbol{\phi}\in\mathbf{H}_\omega, \\ \|\mathbf{v}\|\|\nabla\mathbf{w}\|\|\tilde{\Delta}\boldsymbol{\phi}\|, & \forall\mathbf{v},\mathbf{w},\in\mathbf{H}_0^1,\boldsymbol{\phi}\in\mathbf{H}_0^1\cap\mathbf{H}^2, \\ \|\nabla\mathbf{v}\|\|\mathbf{w}\|\|\tilde{\Delta}\boldsymbol{\phi}\|, & \forall\mathbf{v},\mathbf{w},\in\mathbf{H}_0^1,\boldsymbol{\phi}\in\mathbf{H}_0^1\cap\mathbf{H}^2, \\ \|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|\|\nabla\boldsymbol{\phi}\|, & \forall\mathbf{v},\mathbf{w},\boldsymbol{\phi}\in\mathbf{H}_0^1, \\ \|\mathbf{v}\|^{1-\ell}\|\nabla\mathbf{v}\|^{\ell}\|\nabla\mathbf{w}\|\|\nabla\boldsymbol{\phi}\|, & \forall\mathbf{v},\mathbf{w},\boldsymbol{\phi}\in\mathbf{H}_0^1, \\ \|\nabla\mathbf{v}\|\|\mathbf{w}\|^{1-\ell}\|\nabla\mathbf{w}\|^{\ell}\|\nabla\boldsymbol{\phi}\|, & \forall\mathbf{v},\mathbf{w},\boldsymbol{\phi}\in\mathbf{H}_0^1. \end{cases}$$

3.3 Semidiscrete Error Analysis

We analyze here the semidiscrete three step two-grid algorithm applied to the Oldroyd model of order one. Since $\mathbf{J}_\omega \not\subset \mathbf{J}_1$, the weak solution $\mathbf{u}$ must satisfy the following equation

$$(\mathbf{u}_t,\mathbf{v}_\omega) + \mu a(\mathbf{u},\mathbf{v}_\omega) + b(\mathbf{u},\mathbf{u},\mathbf{v}_\omega) + \int_0^t \beta(t-s)\ a(\mathbf{u}(s),\mathbf{v}_\omega)\ ds$$
$$= (p,\nabla\cdot\mathbf{v}_\omega) + (\mathbf{f},\mathbf{v}_\omega), \quad \forall\mathbf{v}_\omega\in\mathbf{J}_\omega, \tag{3.8}$$

where $\omega = H$ or h. We first define the errors in step I by $\mathbf{e}_H = \mathbf{u}-\mathbf{u}_H$, in step II by $\mathbf{e}^* = \mathbf{u}-\mathbf{u}_h^*$ and in step III by $\mathbf{e}_h = \mathbf{u}-\mathbf{u}_h$. Using (3.4)-(3.6) and (3.8), we find the error equations for each step as follows:

Step I: For all $\mathbf{v}_H \in \mathbf{J}_H$

$$(\mathbf{e}_{Ht},\mathbf{v}_H) + \mu a(\mathbf{e}_H,\mathbf{v}_H) + b(\mathbf{u}_H,\mathbf{e}_H,\mathbf{v}_H) + b(\mathbf{e}_H,\mathbf{u}_H,\mathbf{v}_H)$$
$$+\int_0^t \beta(t-s)a(\mathbf{e}_H(s),\mathbf{v}_H)ds = (p,\nabla\cdot\mathbf{v}_H).$$

Step II: For all $\mathbf{v}_h \in \mathbf{J}_h$

$$(\mathbf{e}_t^*,\mathbf{v}_h) + \mu a(\mathbf{e}^*,\mathbf{v}_h) + b(\mathbf{u}_H,\mathbf{e}^*,\mathbf{v}_h) + b(\mathbf{e}^*,\mathbf{u}_H,\mathbf{v}_h) + \int_0^t \beta(t-s)a(\mathbf{e}^*(s),\mathbf{v}_h)ds$$
$$= (p,\nabla\cdot\mathbf{v}_h) - b(\mathbf{e}_H,\mathbf{e}_H,\mathbf{v}_h). \tag{3.9}$$

Step III: For all $\mathbf{v}_h \in \mathbf{J}_h$

$$(\mathbf{e}_{ht},\mathbf{v}_h) + \mu a(\mathbf{e}_h,\mathbf{v}_h) + b(\mathbf{u}_H,\mathbf{e}_h,\mathbf{v}_h) + b(\mathbf{e}_h,\mathbf{u}_H,\mathbf{v}_h) + \int_0^t \beta(t-s)a(\mathbf{e}_h(s),\mathbf{v}_h)ds$$

$$= (p, \nabla \cdot \mathbf{v}_h) - b(\mathbf{e}_H, \mathbf{e}^*, \mathbf{v}_h) - b(\mathbf{e}^*, \mathbf{e}_H, \mathbf{v}_h) + b(\mathbf{e}^*, \mathbf{e}^*, \mathbf{v}_h). \tag{3.10}$$

The error $\mathbf{e}_H$ is the standard Galerkin finite element approximation error and the estimates are presented in Chapter 1 Lemma 1.2. Below, we only give the statement.

Theorem 3.1. *"Let the conditions* (**A1**)-(**A2**) *and* (**B1**)-(**B2**) *hold. Further, let the discrete initial velocity* $\mathbf{u}_{0H} \in \mathbf{J}_H$ *with* $\mathbf{u}_{0H} = P_H\mathbf{u}_0$*, where* $\mathbf{u}_0 \in \mathbf{J}_1$*. Then, there exists a positive constant* C*, that depends only on the given data and the domain* Ω*, such that for* $0 < T < \infty$ *with* $t \in (0, T]$

$$\|\mathbf{e}_H(t)\| + H\|\nabla \mathbf{e}_H(t)\| + H\|(p - p_H)(t)\| \le K(t)H^2 t^{-1/2},$$

where $K(t) = Ce^{Ct}$*. Moreover, under the uniqueness condition, that is,*

$$\frac{N}{\nu^2}\|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\Omega))} < 1 \text{ and } N = \sup_{\mathbf{u},\mathbf{v},\mathbf{w}} \frac{b(\mathbf{u}, \mathbf{v}, \mathbf{w})}{\|\nabla \mathbf{u}\|\|\nabla \mathbf{v}\|\|\nabla \mathbf{w}\|}, \tag{3.11}$$

where $\nu = \mu + \frac{\gamma}{\delta}$*, the above estimates are valid uniformly in time*

$$\|\mathbf{e}_H(t)\| + H\|\nabla \mathbf{e}_H(t)\| + H\|(p - p_H)(t)\| \le CH^2 t^{-1/2}."$$

Remark 3.2. *The negative power of* t *comes in the above estimates due to the singularity of the solution at the initial time. Recall that the regularity of the solution breaks down as* $t \to 0$ *in case of nonsmooth initial data. This, we feel, is a more realistic approach although it hinders normal error estimate for fully discrete case. For example, we generally expect an optimal estimates in Lemma 3.15, whereas we only have managed sub-optimal estimates.*

We also need some additional estimates of Step I error $\mathbf{e}_H$. We now present those in the below lemma. For a proof, see [60].

Lemma 3.3. *Suppose* $0 < \alpha < \min\{\mu_1\lambda_1, \delta\}$ *and the assumption of Theorem* 3.1 *holds true, then the step I error satisfies the following results:*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{e}_H(s)\|^2 ds + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau^*(s))^2\|\mathbf{e}_{Hs}(s)\|^2 ds \le K(t)H^4,$$

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla \mathbf{e}_H(s)\|^2 ds + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau^*(s))^2\|\nabla \mathbf{e}_{Hs}(s)\|^2 ds \le K(t)H^2.$$

Under the uniqueness condition (3.11)*, the above estimates are valid uniformly in time.*

3.3.1 Optimal Velocity Error Estimates in Step II

We first obtain some preliminary results which pave the way for improved estimates.

Lemma 3.4. *Suppose* $0 < \alpha < \min\{\delta, \mu_1\lambda_1\}$ *and let the hypothesis of the Theorem* 3.1 *hold. Then, the step II error satisfies the following result:*

$$e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\|\nabla \mathbf{e}^*(\tau)\|^2 d\tau \leq K(t)(h^2 + H^{6-2\ell}t^{-1}).$$

Moreover, this estimate is valid uniform in time under the smallness condition (3.11)*, that is, the error bound constant* $K(t)$ *becomes to* C*.*

Proof. Put $\mathbf{v}_h = P_h\mathbf{e}^* = \mathbf{e}^* - (\mathbf{u} - P_h\mathbf{u})$ in (3.9) and use (3.7) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{e}^*\|^2 + \mu_1\|\nabla\mathbf{e}^*\|^2 + \int_0^t \beta(t-s)\ a(\mathbf{e}^*(s), \mathbf{e}^*)\ ds \leq (\mathbf{e}^*_t, \mathbf{u} - P_h\mathbf{u}) \\ +\mu a(\mathbf{e}^*, \mathbf{u} - P_h\mathbf{u}) + b(\mathbf{u}_H, \mathbf{e}^*, \mathbf{u} - P_h\mathbf{u}) + b(\mathbf{e}^*, \mathbf{u}_H, \mathbf{u} - P_h\mathbf{u}) + (p, \nabla\cdot P_h\mathbf{e}^*) \\ - b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{e}^*) + \int_0^t \beta(t-s)\ a(\mathbf{e}^*(s), \mathbf{u} - P_h\mathbf{u})\ ds. \quad (3.12)\end{aligned}$$

Using the definition of P_h we rewrite the following as

$$\begin{aligned}(\mathbf{e}^*_t, \mathbf{u} - P_h\mathbf{u}) &= (\mathbf{u}_t - P_h\mathbf{u}_t + P_h\mathbf{u}_t - \mathbf{u}^*_{ht}, \mathbf{u} - P_h\mathbf{u}) \\ &= (\mathbf{u}_t - P_h\mathbf{u}_t, \mathbf{u} - P_h\mathbf{u}) = \frac{1}{2}\frac{d}{dt}\|\mathbf{u} - P_h\mathbf{u}\|^2. \quad (3.13)\end{aligned}$$

The "Cauchy-Schwarz inequality" with (1.15) suggests that

$$|a(\mathbf{e}^*, \mathbf{u} - P_h\mathbf{u})| \leq \|\nabla\mathbf{e}^*\|\|\nabla(\mathbf{u} - P_h\mathbf{u})\| \leq Ch\|\nabla\mathbf{e}^*\|\|\tilde{\Delta}\mathbf{u}\|. \quad (3.14)$$

A use of Lemma 3.2 with Lemma 3.1 allows us to conclude that

$$\begin{aligned}|b(\mathbf{u}_H, \mathbf{e}^*, \mathbf{u} - P_h\mathbf{u}) + b(\mathbf{e}^*, \mathbf{u}_H, \mathbf{u} - P_h\mathbf{u})| &\leq C\|\nabla\mathbf{u}_H\|\|\nabla\mathbf{e}^*\|\|\nabla(\mathbf{u} - P_h\mathbf{u})\| \\ &\leq Ch\|\nabla\mathbf{e}^*\|\|\tilde{\Delta}\mathbf{u}\|. \quad (3.15)\end{aligned}$$

Using the discrete incompressibility condition in conjunction with the approximation property (**B1**) and the "Cauchy-Schwarz inequality" help us to bound the following

$$|(p, \nabla\cdot P_h\mathbf{e}^*)| = |(p - j_h p, \nabla\cdot P_h\mathbf{e}^*)| \leq \|p - j_h p\|\|\nabla\mathbf{e}^*\| \leq Ch\|\nabla\mathbf{e}^*\|\|\nabla p\|. \quad (3.16)$$

Thanks to Lemma 3.2 for helping us to bound the following nonlinear term

$$|b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{e}^*)| \leq C\|\mathbf{e}_H\|^{(1-\ell)}\|\nabla\mathbf{e}_H\|^{(1+\ell)}\|\nabla\mathbf{e}^*\|. \quad (3.17)$$

Incorporating (3.13)-(3.17) in (3.12) with the "Young's inequality", we deduce that

$$\frac{d}{dt}\|\mathbf{e}^*\|^2 + 2\mu_1\|\nabla\mathbf{e}^*\|^2 + 2\int_0^t \beta(t-s)\ a(\mathbf{e}^*(s),\mathbf{e}^*)\ ds$$
$$\leq \frac{d}{dt}\|\mathbf{u}-P_h\mathbf{u}\|^2 + C(\epsilon)\Big(h^2(\|\tilde{\Delta}\mathbf{u}\|^2 + \|\nabla p\|^2) + \|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla\mathbf{e}_H\|^{2(1+\ell)}\Big)$$
$$+ \int_0^t \beta(t-s)\ a(\mathbf{e}^*(s),\mathbf{u}-P_h\mathbf{u})\ ds + \epsilon\|\nabla\mathbf{e}^*\|^2.$$

Multiply both side by $e^{2\alpha t}$ and use (1.7) to obtain

$$\frac{d}{dt}e^{2\alpha t}\|\mathbf{e}^*\|^2 + 2(\mu_1 - \frac{\alpha}{\lambda_1} - \frac{\epsilon}{2})e^{2\alpha t}\|\nabla\mathbf{e}^*\|^2 + 2e^{2\alpha t}\int_0^t \beta(t-s)\ a(\mathbf{e}^*(s),\mathbf{e}^*)\ ds$$
$$\leq \frac{d}{dt}e^{2\alpha t}\|\mathbf{u}-P_h\mathbf{u}\|^2 + Ch^2e^{2\alpha t}\mathcal{K}^2(t) + Ce^{2\alpha t}\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla\mathbf{e}_H\|^{2(1+\ell)}$$
$$+ Che^{2\alpha t}\Big(\int_0^t \beta(t-s)\|\nabla\mathbf{e}^*(s)\|\ ds\Big)\|\tilde{\Delta}\mathbf{u}\| - \alpha e^{2\alpha t}\|\mathbf{u}-P_h\mathbf{u}\|^2, \qquad (3.18)$$

where $\mathcal{K}(t) := \|\nabla p(t)\| + \|\tilde{\Delta}\mathbf{u}(t)\|$. Last term on the right of inequality (3.18) is negative, so we ignore this. We now take time integration on the both sides and use Lemma 3.1 and the fact

$$\|\mathbf{e}^*\| \geq \|\mathbf{u}-P_h\mathbf{u}\|,\ \ \|\mathbf{e}^*(0)\| = \|\mathbf{u}_0 - P_h\mathbf{u}_0\|$$

to arrive at

$$2(\mu_1 - \frac{\alpha}{\lambda_1} - \frac{\epsilon}{2})\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}^*(s)\|^2 ds + \int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)\ a(\mathbf{e}^*(\tau),\mathbf{e}^*(s))\ d\tau\ ds$$
$$\leq Ch^2\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds + C\int_0^t e^{2\alpha s}\|\mathbf{e}_H(s)\|^{2(1-\ell)}\|\nabla\mathbf{e}_H(s)\|^{2(1+\ell)}ds$$
$$+ Ch\int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\nabla\mathbf{e}^*(\tau)\|\ d\tau\Big)\|\tilde{\Delta}\mathbf{u}(s)\|\ ds. \qquad (3.19)$$

For the inequality $\|\mathbf{e}^*\| \geq \|\mathbf{u}-P_h\mathbf{u}\|$, we argue as follows: $\mathbf{u}-P_h\mathbf{u} = \mathbf{e}^* + (\mathbf{u}_h^* - P_h\mathbf{u})$. Now, we use the orthogonal property of P_h to find the following:

$$\|\mathbf{u}-P_h\mathbf{u}\|^2 = (\mathbf{e}^*,\mathbf{u}-P_h\mathbf{u}) + (\mathbf{u}_h^* - P_h\mathbf{u},\mathbf{u}-P_h\mathbf{u})$$
$$= (\mathbf{e}^*,\mathbf{u}-P_h\mathbf{u}) \leq \|\mathbf{e}^*\|\|\mathbf{u}-P_h\mathbf{u}\|. \qquad (3.20)$$

Cancelling $\|\mathbf{u}-P_h\mathbf{u}\|$ from both sides we derive the required result. From the positivity property (Lemma 1.5), we now drop the double integral term from left of inequality (3.19). And we bound the another double integration term similar to I term of Lemma 2.2 of Chapter 2 for some positive ϵ

$$Ch\int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\nabla\mathbf{e}^*(\tau)\|\ d\tau\Big)\|\tilde{\Delta}\mathbf{u}(s)\|\ ds$$

$$\leq C(\epsilon,\gamma,\delta,\alpha)h^2\int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}(s)\|^2ds+\epsilon\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}^*(s)\|^2ds. \tag{3.21}$$

With $0<\alpha<\min\{\delta,\mu_1\lambda_1\}$, we choose $\epsilon=(\mu_1-\alpha/\lambda_1)/2>0$, then, from (3.19) and (3.21), we finally obtain

$$\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}^*(s)\|^2ds\leq Ch^2\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds+C\|\mathbf{e}_H(t)\|^{-2\ell}_{L^\infty(\mathbf{L}^2)}\|\nabla\mathbf{e}_H(t)\|^{2(1+\ell)}_{L^\infty(\mathbf{L}^2)}$$
$$\times\int_0^t e^{2\alpha s}\|\mathbf{e}_H(s)\|^2ds. \tag{3.22}$$

Use Theorem 3.1 and Lemma 3.3 in (3.22) to conclude the result. And under uniqueness condition (3.11), since the estimates of Theorem 3.1 and Lemma 3.3 will be uniform in time, so will be the estimate of (3.22), which concludes the remaining of the proof. □

Remark 3.3. *Although the estimate of above lemma is optimal in nature, we can avoid the singularity in time t^{-1} by going for a sub-optimal result. This will be useful in controlling the singularity in time at a later stage. If we us the bound $\|\nabla\mathbf{e}_H\|\leq C$ in* (3.22)*, then we get*

$$\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}^*(s)\|^2ds\leq Ch^2e^{2\alpha t}+\|\mathbf{e}_H(t)\|^{-2\ell}_{L^\infty(\mathbf{L}^2)}\|\nabla\mathbf{e}_H(t)\|^{2\ell}_{L^\infty(\mathbf{L}^2)}\int_0^t e^{2\alpha s}\|\mathbf{e}_H(s)\|^2ds)$$
$$\leq K(t)e^{2\alpha t}(h^2+H^{4-2\ell}).$$

Remark 3.4. *In the remaining of the chapter, we have shown local error estimates which are dependent on time. But these error bounds can be made independent of time under uniqueness condition* (3.11)*, based on Theorem 3.1 and Lemma 3.3. We have refrained from mentioning this in the statement of each of the following Lemmas to avoid sounding repetitive.*

Lemma 3.5. *Suppose the hypothesis of the Theorem* 3.1 *holds. Then, the step II error satisfies the following result:*

$$\tau^*(t)\|\nabla\mathbf{e}^*(t)\|^2+e^{-2\alpha t}\int_0^t\tau^*(s)e^{2\alpha s}\|\mathbf{e}_s^*(s)\|^2\,ds\leq K(t)(h^2+H^{6-2\ell}t^{-1}).$$

Proof. Substitute $\mathbf{v}_h=\sigma(t)P_h\mathbf{e}_t^*=\sigma(t)(P_h\mathbf{u}_t-\mathbf{u}_t)+\sigma(t)\mathbf{e}_t^*$ with $\sigma(t)=\tau^*(t)e^{2\alpha t}$ in (3.9) to find

$$\sigma\|\mathbf{e}_t^*\|^2+\frac{\mu_1}{2}\frac{d}{dt}\sigma\|\nabla\mathbf{e}^*\|^2\leq\frac{\mu}{2}\sigma_t\|\nabla\mathbf{e}^*\|^2+\sigma(\mathbf{e}_t^*,\mathbf{u}_t-P_h\mathbf{u}_t)+\mu\sigma a(\mathbf{e}^*,\mathbf{u}_t-P_h\mathbf{u}_t)$$

$$- \sigma b(\mathbf{e}^*, \mathbf{u}_H, \mathbf{u}_t - P_h\mathbf{u}_t) - \sigma b(\mathbf{u}_H, \mathbf{e}^*, \mathbf{u}_t - P_h\mathbf{u}_t) + \sigma b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{e}_t^*)$$
$$+ \sigma(p, \nabla \cdot P_h\mathbf{e}_t^*) - \sigma \int_0^t \beta(t-s)\ a(\mathbf{e}^*(s), P_h\mathbf{e}_t^*)\ ds. \qquad (3.23)$$

The “Cauchy-Schwarz inequality” and the “Young's inequality” with (1.15) helps to bound the followings

$$\sigma(\mathbf{e}_t^*, \mathbf{u}_t - P_h\mathbf{u}_t) \le \sigma\|\mathbf{e}_t^*\|\|\mathbf{u}_t - P_h\mathbf{u}_t\| \le Ch^2\sigma\|\nabla\mathbf{u}_t\|^2 + \epsilon\sigma\|\mathbf{e}_t^*\|^2, \qquad (3.24)$$

and

$$\sigma a(\mathbf{e}^*, \mathbf{u}_t - P_h\mathbf{u}_t) \le \sigma\|\nabla\mathbf{e}^*\|\|\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\| \le Ch\sigma\|\nabla\mathbf{e}^*\|\|\tilde{\Delta}\mathbf{u}_t\|$$
$$\le Ch^2\frac{\sigma^2(t)}{\sigma_t(t)}\|\tilde{\Delta}\mathbf{u}_t\|^2 + C\sigma_t\|\nabla\mathbf{e}^*\|^2 \qquad (3.25)$$

We apply Lemma 3.2 with the “Young's inequality” to bound the following as

$$\sigma b(\mathbf{e}^*, \mathbf{u}_H, \mathbf{u}_t - P_h\mathbf{u}_t) + \sigma b(\mathbf{u}_H, \mathbf{e}^*, \mathbf{u}_t - P_h\mathbf{u}_t) \le C\sigma\|\nabla\mathbf{u}_H\|\|\nabla\mathbf{e}^*\|\|\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\|$$
$$\le Ch^2\frac{\sigma^2(t)}{\sigma_t(t)}\|\tilde{\Delta}\mathbf{u}_t\|^2 + C\sigma_t\|\nabla\mathbf{e}^*\|^2. \qquad (3.26)$$

We bound the second to fifth terms on the right of inequality (3.23 similar to (3.13)-(3.15). And, we rewrite the sixth and seventh terms as:

$$\sigma b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{e}_t^*) = \frac{d}{dt}(\sigma b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{e}^*)) - \sigma_t b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{e}^*)$$
$$- \sigma b(\mathbf{e}_{Ht}, \mathbf{e}_H, P_h\mathbf{e}^*) - \sigma b(\mathbf{e}_H, \mathbf{e}_{Ht}, P_h\mathbf{e}^*) \qquad (3.27)$$

and

$$\sigma(p, \nabla \cdot P_h\mathbf{e}_t^*) = \frac{d}{dt}(\sigma(p - j_hp, \nabla \cdot P_h\mathbf{e}^*)) - \sigma_t(p - j_hp, \nabla \cdot P_h\mathbf{e}^*)$$
$$- \sigma(p_t - j_hp_t, \nabla \cdot P_h\mathbf{e}^*) \qquad (3.28)$$

A use of Lemma 3.2 gives

$$|b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{e}^*) \le C\|\mathbf{e}_H\|^{1-\ell}\|\nabla\mathbf{e}_H\|^{1+\ell}\|\nabla\mathbf{e}^*\|, \qquad (3.29)$$

and

$$|b(\mathbf{e}_{Ht}, \mathbf{e}_H, P_h\mathbf{e}^*) + b(\mathbf{e}_H, \mathbf{e}_{Ht}, P_h\mathbf{e}^*)| \le C\|\mathbf{e}_H\|^{1-\ell}\|\nabla\mathbf{e}_H\|^{\ell}\|\nabla\mathbf{e}_{Ht}\|\|\nabla\mathbf{e}^*\|. \qquad (3.30)$$

We use approximation property (**B1**) with (1.15) to bound the terms of (3.28) as

$$(p - j_hp, \nabla \cdot P_h\mathbf{e}^*) \le Ch\|\nabla p\|\|\nabla\mathbf{e}^*\|, \qquad (3.31)$$

and

$$\sigma(p_t - j_h p_t, \nabla \cdot P_h \mathbf{e}^*) \leq Ch\sigma \|\nabla p_t\| \|\nabla \mathbf{e}^*\| \leq C(h^2 \frac{\sigma^2(t)}{\sigma_t(t)} \|\nabla p_t\|^2 + \sigma_t \|\nabla \mathbf{e}^*\|^2). \quad (3.32)$$

Incorporate (3.24)-(3.32) along with (1.15) in (3.23). Then, take time integration on the both sides to obtain

$$\begin{aligned}
&\int_0^t \sigma(s)\|\mathbf{e}_s^*(s)\|^2 ds + \mu_1 \sigma(t)\|\nabla \mathbf{e}^*(t)\|^2 \leq C\int_0^t \sigma_s(s)\|\nabla \mathbf{e}^*(s)\|^2 ds + Ch^2\sigma\|\nabla p\|^2 \\
&\quad + Ch^2\Big(\int_0^t \sigma(s)\|\nabla \mathbf{u}_t\|^2 ds + \int_0^t \sigma_s(s)\|\nabla p\|^2 ds + \int_0^t \frac{\sigma^2(s)}{\sigma_s(s)}(\|\tilde{\Delta}\mathbf{u}_s\|^2 + \|\nabla p_s\|^2) ds\Big) \\
&\quad + C\sigma(t)\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla \mathbf{e}_H\|^{2(1+\ell)} + C\int_0^t \frac{\sigma^2(s)}{\sigma_s(s)}\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla \mathbf{e}_H\|^{2\ell}\|\nabla \mathbf{e}_{Hs}\|^2 ds \\
&\quad + \int_0^t \sigma_s(s)\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla \mathbf{e}_H\|^{2(1+\ell)} ds - \int_0^t \sigma(s)\int_0^s \beta(s-\tau)\ a(\mathbf{e}^*(\tau), P_h\mathbf{e}_s^*(s))\ d\tau\ ds.
\end{aligned}$$

Since $\frac{\sigma^2(t)}{\sigma_t(t)} \leq C(\tau^*(t))^2 e^{2\alpha t}$ and $\sigma_t(t) \leq Ce^{2\alpha t}$, hence

$$\begin{aligned}
&\int_0^t \sigma(s)\|\mathbf{e}_s^*(s)\|^2 ds + \sigma(t)\|\nabla \mathbf{e}^*\|^2 \leq Ch^2\Big(\sigma\|\nabla p\|^2 + \int_0^t e^{2\alpha s}\big(\tau^*(s)\|\nabla \mathbf{u}_s\|^2 + \|\nabla p\|^2\big) ds \\
&\quad + \int_0^t e^{2\alpha s}(\tau^*(s))^2 \mathcal{K}_s^2(s) ds\Big) + \sigma(t)\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla \mathbf{e}_H\|^{2(1+\ell)} + \int_0^t e^{2\alpha s}\|\nabla \mathbf{e}^*\|^2 ds \\
&\quad + \int_0^t e^{2\alpha s}\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla \mathbf{e}_H\|^{2(1+\ell)} ds + \int_0^t \sigma_1(s)\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla \mathbf{e}_H\|^{2\ell}\|\nabla \mathbf{e}_{Hs}\|^2 ds \\
&\quad - \int_0^t \sigma(s)\int_0^s \beta(s-\tau)\ a(\mathbf{e}^*(\tau), P_h\mathbf{e}_s^*(s))\ d\tau\ ds\Big). \qquad (3.33)
\end{aligned}$$

Using integration by parts, we rewrite the following as

$$\begin{aligned}
&\int_0^t \sigma(s)\int_0^s \beta(s-\tau)\ a(\mathbf{e}^*(\tau), \mathbf{e}_s^*(s))\ d\tau\ ds \\
&\quad \leq \sigma(t)\int_0^t \beta(t-\tau)\ a(\mathbf{e}^*(\tau), \mathbf{e}^*)\ d\tau - \int_0^t \sigma_s(s)\int_0^s \beta(s-\tau)\ a(\mathbf{e}^*(\tau), \mathbf{e}^*(s))\ d\tau\ ds \\
&\qquad - \int_0^t \gamma\sigma(s)\ a(\mathbf{e}^*(s), \mathbf{e}^*(s))\ ds - \int_0^t \sigma(s)\int_0^s \beta_s(s-\tau)\ a(\mathbf{e}^*(\tau), \mathbf{e}^*(s))\ d\tau ds \\
&\quad \leq \frac{1}{2}\sigma(t)\|\nabla \mathbf{e}^*\|^2 + C\sigma(t)\Big(\int_0^t \beta(t-\tau)\|\nabla \mathbf{e}^*(\tau)\|\ d\tau\Big)^2 + \int_0^t \sigma_s(s)\|\nabla \mathbf{e}^*(s)\|^2 ds \\
&\qquad + \int_0^t \sigma_s(s)\Big(\int_0^s \beta(s-\tau)\|\nabla \mathbf{e}^*(\tau)\|\ d\tau\Big)^2 ds + \gamma\int_0^t \sigma(s)\|\nabla \mathbf{e}^*(s)\|^2 ds \\
&\qquad + \int_0^t \sigma(s)\|\nabla \mathbf{e}^*(s)\|^2 ds + \int_0^t \sigma(s)\Big(\int_0^s \beta_s(s-\tau)\|\nabla \mathbf{e}^*(\tau)\| d\tau\Big)^2 ds \\
&\quad \leq \frac{1}{2}\sigma(t)\|\nabla \mathbf{e}^*\|^2 + C\int_0^t e^{2\alpha s}\|\nabla \mathbf{e}^*(s)\|^2 ds. \qquad (3.34)
\end{aligned}$$

Note that, in the above inequality, we use the similar estimate of I term of Chapter 2 Lemma 2.2 to write double integration term in single integration. Similar to (3.21), we estimate the another one as

$$\int_0^t \sigma(s) \int_0^s \beta(s-\tau)\ a(\mathbf{e}^*(\tau), (\mathbf{u}_s - P_h\mathbf{u}_s)(s))\ d\tau\ ds$$
$$\leq Ch^2 \int_0^t \frac{\sigma^2(s)}{\sigma(s)} \Big(\int_0^s \beta(s-\tau)\|\tilde{\Delta}\mathbf{u}_\tau(\tau)\|\ d\tau\Big)^2 ds + C\int_0^t e^{2\alpha s}\|\nabla \mathbf{e}^*(s)\|^2 ds$$
$$\leq Ch^2 \int_0^t (\tau^*(s))^2 e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}_s(s)\|^2 ds + C\int_0^t e^{2\alpha s}\|\nabla \mathbf{e}^*(s)\|^2 ds \tag{3.35}$$

Incorporate (3.34)-(3.35) in (3.33). Then, a use of Lemma 3.4 and Theorem 3.1 concludes the remaining of the proof. □

Remark 3.5. *As in Remark* 3.3 *we can resort to a sub-optimal estimate here as well.*

$$\tau^*(t)\|\nabla \mathbf{e}^*(t)\|^2\ +\ e^{-2\alpha t}\int_0^t \sigma(s)\ \|\mathbf{e}_s^*(s)\|^2 ds \leq K(t)(h^2 + H^{4-2\ell}).$$

Lemma 3.6. *Suppose the hypothesis of the Theorem* 3.1 *holds. Then, the step II error satisfies the following result:*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{e}^*(s)\|^2 ds \leq K(t)(h^4 + h^2H^4t^{-1} + H^6t^{-1}).$$

Proof. For L^2 estimate of $\mathbf{e}^*$, we take the following dual problem: For a given T with $0 < t \leq T$ and a given $\mathbf{e}^* \in L^2(\mathbf{L}^2)$, let the pair $(\mathbf{v}(t), \psi(t)) \in \mathbf{J}_1 \times L^2(\Omega)/\mathbb{R}$ with $\mathbf{v}(T) = 0$ satisfy the following

$$(\mathbf{w}, \mathbf{v}_t) - \mu a(\mathbf{w}, \mathbf{v}) - b(\mathbf{u}, \mathbf{w}, \mathbf{v}) - b(\mathbf{w}, \mathbf{u}, \mathbf{v}) - \int_0^t \beta(t-s) a(\mathbf{w}(s), \mathbf{v}) ds$$
$$+(\psi, \nabla\cdot\mathbf{w}) = (e^{2\alpha t}\mathbf{e}^*, \mathbf{w}),\quad \forall \mathbf{w} \in \mathbf{H}_0^1.$$

Then the following bound holds:

$$\int_0^T e^{-2\alpha t}(\|\mathbf{v}\|_2^2 + \|\psi\|_1^2 + \|\mathbf{v}_t\|^2) dt \leq C\int_0^T e^{2\alpha t}\|\mathbf{e}^*\|^2 dt. \tag{3.36}$$

Choose $v = \mathbf{e}^*$ and use (3.9) with $\mathbf{v}_h = P_h\mathbf{v}$ to obtain

$$\begin{aligned} e^{2\alpha t}\|\mathbf{e}^*\|^2 =& \frac{d}{dt}(\mathbf{e}^*, \mathbf{v}) - (\mathbf{e}_t^*, \mathbf{v} - P_h\mathbf{v}) - \mu a(\mathbf{e}^*, \mathbf{v} - P_h\mathbf{v}) - b(\mathbf{u}_H, \mathbf{e}^*, \mathbf{v} - P_h\mathbf{v}) \\ & - b(\mathbf{e}^*, \mathbf{u}_H, \mathbf{v} - P_h\mathbf{v}) - b(\mathbf{e}_H, \mathbf{e}^*, \mathbf{v}) - b(\mathbf{e}^*, \mathbf{e}_H, \mathbf{v}) + b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{v}) \\ & + (\psi - j_h\psi, \nabla\cdot\mathbf{e}^*) - (p - j_hp, \nabla\cdot(P_h\mathbf{v} - \mathbf{v})) \\ & - \int_0^t \beta(t-s)\ a(\mathbf{e}^*(s), \mathbf{v} - P_h\mathbf{v})\ ds. \end{aligned} \tag{3.37}$$

Note that

$$\begin{aligned}(\mathbf{e}^*_t, \mathbf{v} - P_h\mathbf{v}) &= \frac{d}{dt}(\mathbf{e}^*, \mathbf{v} - P_h\mathbf{v}) - (\mathbf{e}^*, \mathbf{v}_t - P_h\mathbf{v}_t)\\ &= \frac{d}{dt}(\mathbf{e}^*, \mathbf{v} - P_h\mathbf{v}) - (\mathbf{u} - P_h\mathbf{u}, \mathbf{v}_t). \qquad (3.38)\end{aligned}$$

Use (3.38) in (3.37), then take time integration on both sides of the resulting equation and use $\mathbf{v}(t) = 0$ to derive

$$\begin{aligned}\int_0^t e^{2\alpha s}\|\mathbf{e}^*(s)\|^2 ds = &- (\mathbf{e}^*(0), P_h\mathbf{v}(0)) + \int_0^t (\mathbf{u} - P_h\mathbf{u}, \mathbf{v}_s) ds - \mu \int_0^t a(\mathbf{e}^*, \mathbf{v} - P_h\mathbf{v}) ds\\ &- \int_0^t \big(b(\mathbf{u}_H, \mathbf{e}^*, \mathbf{v} - P_h\mathbf{v}) + b(\mathbf{e}^*, \mathbf{u}_H, \mathbf{v} - P_h\mathbf{v})\big) ds\\ &- \int_0^t \big(b(\mathbf{e}_H, \mathbf{e}^*, \mathbf{v}) + b(\mathbf{e}^*, \mathbf{e}_H, \mathbf{v})\big) ds + \int_0^t b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{v}) ds\\ &+ \int_0^t \left((\psi - j_h\psi, \nabla\cdot\mathbf{e}^*) - (p - j_hp, \nabla\cdot(P_h\mathbf{v} - \mathbf{v}))\right) ds\\ &- \int_0^t \int_0^s \beta(s-\tau)\ a(\mathbf{e}^*(\tau), (\mathbf{v} - P_h\mathbf{v})(s))\ d\tau\ ds. \qquad (3.39)\end{aligned}$$

The first term on the right of inequality (3.38) vanishes.We use the "Young's inequality" and the "Cauchy-Schwarz inequality" with (1.15) and for some $\epsilon > 0$ to obtain

$$\begin{aligned}\int_0^t |(\mathbf{u} - P_h\mathbf{u}, \mathbf{v}_s) + a(\mathbf{e}^*, \mathbf{v} - P_h\mathbf{v})| ds \le & C(\epsilon) h^2 \int_0^t e^{2\alpha s}\big(h^2\|\tilde{\Delta}\mathbf{u}\|^2 + \|\nabla\mathbf{e}^*\|^2\big) ds\\ &+ \epsilon \int_0^t e^{-2\alpha s}\big(\|\mathbf{v}\|_2^2 + \|\mathbf{v}_s\|^2\big) ds. \qquad (3.40)\end{aligned}$$

We apply Lemma 3.2 and 3.1 and boundedness of $\|\nabla\mathbf{u}_H\|$ with (1.15) to bound the following nonlinear terms as

$$\begin{aligned}&\int_0^t |(b(\mathbf{u}_H, \mathbf{e}^*, \mathbf{v} - P_h\mathbf{v}) + b(\mathbf{e}^*, \mathbf{u}_H, \mathbf{v} - P_h\mathbf{v})) + (b(\mathbf{e}_H, \mathbf{e}^*, \mathbf{v}) + b(\mathbf{e}^*, \mathbf{e}_H, \mathbf{v}))| ds\\ &\quad \le C\int_0^t \big(\|\nabla\mathbf{u}_H\|\|\nabla\mathbf{e}^*\|\|\nabla(\mathbf{v} - P_h\mathbf{v})\| + \|\mathbf{e}_H\|\|\nabla\mathbf{e}^*\|\|\tilde{\Delta}\mathbf{v}\|\big) ds\\ &\quad \le C(\epsilon)\int_0^t e^{2\alpha s}(h^2 + \|\mathbf{e}_H\|^2)\|\nabla\mathbf{e}^*\|^2 ds + \epsilon\int_0^t e^{-2\alpha s}\|\mathbf{v}\|_2^2 d, . \qquad (3.41)\end{aligned}$$

and

$$\begin{aligned}\int_0^t |b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{v})| ds \le & \int_0^t (|b(\mathbf{e}_H, \mathbf{e}_H, P_h\mathbf{v} - \mathbf{v})| + |b(\mathbf{e}_H, \mathbf{e}_H, \mathbf{v})|) ds\\ &\le C(\epsilon)h^2 \int_0^t e^{2\alpha s}\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla\mathbf{e}_H\|^{2(1+\ell)}\ ds\\ &\quad + C(\epsilon)\int_0^t e^{2\alpha s}\|\mathbf{e}_H\|^2\|\nabla\mathbf{e}_H\|^2 ds + \epsilon\int_0^t e^{-2\alpha s}\|\mathbf{v}\|_2^2 ds. \qquad (3.42)\end{aligned}$$

The discrete incompressible condition and (1.15) with approximation property (**B1**) allow us to bound

$$\begin{aligned}
&\int_0^t |(\boldsymbol{\psi} - j_h\boldsymbol{\psi}, \nabla\cdot \mathbf{e}^*)| + |(p - j_hp, \nabla\cdot(P_h\mathbf{v} - \mathbf{v}))|ds \\
&\quad\le C\int_0^t h\|\boldsymbol{\psi}\|_1\|\nabla\mathbf{e}^*\| + h^2\|\nabla p\|\|\mathbf{v}\|_2 ds \\
&\quad\le C(\epsilon)\int_0^t e^{2\alpha s}(h^2\|\nabla\mathbf{e}^*\|^2 ds + h^4\|\nabla p\|^2)ds + \epsilon\int_0^t e^{-2\alpha s}(\|\boldsymbol{\psi}\|_1^2 + \|\mathbf{v}\|_2^2)ds. \qquad (3.43)
\end{aligned}$$

We bound the right hand double integration term of(3.39) similar to (3.21)

$$\begin{aligned}
&\int_0^t \int_0^s \beta(s-\tau)\ a(\mathbf{e}^*(\tau), (\mathbf{v} - P_h\mathbf{v})(s))\ d\tau\ ds \\
&\qquad\le Ch\int_0^t \Big(\int_0^s \beta(s-\tau)\ \|\nabla\mathbf{e}^*(\tau)\|\ d\tau\Big)\|\mathbf{v}\|_2\ ds \\
&\qquad\le C(\epsilon)h^2\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}^*(s)\|^2 ds + \epsilon\int_0^t e^{-2\alpha s}\|\mathbf{v}(s)\|_2^2 ds. \qquad (3.44)
\end{aligned}$$

We use (3.40)-(3.44) in (3.39) with (3.36) and obtain

$$\begin{aligned}
\int_0^t e^{2\alpha s}\|\mathbf{e}^*(s)\|^2 ds \le & C(\epsilon)\big(h^4\int_0^t e^{2\alpha s}(\|\tilde{\Delta}\mathbf{u}\|^2 + \|\nabla p\|^2)ds + h^2\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}^*\|^2 ds \\
&+ \int_0^t e^{2\alpha s}\|\mathbf{e}_H\|^2\|\nabla\mathbf{e}^*\|^2 ds + h^2\int_0^t e^{2\alpha s}\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla\mathbf{e}_H\|^{2(1+\ell)}ds \\
&+ \int_0^t e^{2\alpha s}\|\mathbf{e}_H\|^2\|\nabla\mathbf{e}_H\|^2 ds + \epsilon\int_0^t e^{-2\alpha s}(\|\mathbf{v}\|_2^2 + \|\mathbf{v}_s\|^2 + \|\psi\|_1^2)ds \\
\le & C(\epsilon)\big(h^4\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds + (h^2 + \|\mathbf{e}_H\|^2_{L^\infty(\mathbf{L}^2)})\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}^*\|^2 ds \\
&+ (h^2\|\mathbf{e}_H\|^{-2\ell}_{L^\infty(\mathbf{L}^2)}\|\nabla\mathbf{e}_H\|^{2(1+\ell)}_{L^\infty(\mathbf{L}^2)} + \|\nabla\mathbf{e}_H\|^2_{L^\infty(\mathbf{L}^2)})\int_0^t e^{2\alpha s}\|\mathbf{e}_H\|^2 ds \\
&+ \epsilon\int_0^t e^{-2\alpha s}(\|\mathbf{v}\|_2^2 + \|\mathbf{v}_s\|^2 + \|\psi\|_1^2)ds. \qquad (3.45)
\end{aligned}$$

We use the Remark 3.3 (in order to avoid $O(t^{-2})$ singularity in time) for the second term on the right of inequality (3.45) and for others term we use Theorem 3.1 and Lemma 3.3. Finally, a use of (3.36) with $C\epsilon = 1/2$ concludes the remaining of the proof. □

Remark 3.6. *As in Remark 3.3 we can resort to a sub-optimal estimate here as well.*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{e}^*(s)\|^2 ds \le K(t)(h^4 + H^4).$$

Lemma 3.7. *Suppose the hypothesis of the Theorem* 3.1 *holds. Then, the step II error satisfies the following result:*

$$(\tau^*(t))^2\|\mathbf{e}_t^*(t)\|^2 + e^{-2\alpha t}\int_0^t \sigma_1(s)\|\nabla \mathbf{e}_s^*\|^2 ds \le K(t)(h^2 + H^{6-2\ell}t^{-1}),$$

where $\sigma_1(t) = (\tau^*(t))^2 e^{2\alpha t}$.

Proof. We take time differentiation on (3.9) to deduce that

$$\begin{aligned}(\mathbf{e}_{tt}^*, \mathbf{v}_h) + \mu a(\mathbf{e}_t^*, \mathbf{v}_h) + b(\mathbf{e}_t^*, \mathbf{u}_H, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{e}_t^*, \mathbf{v}_h) + b(\mathbf{e}^*, \mathbf{u}_{Ht}, \mathbf{v}_h) \\ + b(\mathbf{u}_{Ht}, \mathbf{e}^*, \mathbf{v}_h) + \beta(0)a(\mathbf{e}^*, \mathbf{v}_h) - \delta\int_0^t \beta(t-s)a(\mathbf{e}^*, \mathbf{v}_h)ds \\ = (p_t, \nabla\cdot\mathbf{v}_h) - b(\mathbf{e}_{Ht}, \mathbf{e}_H, \mathbf{v}_h) - b(\mathbf{e}_H, \mathbf{e}_{Ht}, \mathbf{v}_h).\end{aligned} \tag{3.46}$$

Substitute $\mathbf{v}_h = \sigma_1(t)P_h\mathbf{e}_t^* = \sigma_1\mathbf{e}_t^* - \sigma_1(\mathbf{u}_t - P_h\mathbf{u}_t)$ in (3.46). We then use (3.13) in this context and use the fact $\frac{d}{dt}\sigma_1(t) \le C\sigma(t) + 2\alpha\sigma_1(t)$ with (1.7) to arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sigma_1\|\mathbf{e}_t^*\|^2 + (\mu_1 - \frac{\alpha}{\lambda_1})\sigma_1\|\nabla\mathbf{e}_t^*\|^2 - C\sigma\|\mathbf{e}_t^*\|^2 \\ &\quad\le \frac{1}{2}\frac{d}{dt}\sigma_1(t)\|\mathbf{u}_t - P_h\mathbf{u}_t\|^2 - \alpha\sigma_1(t)\|\mathbf{u}_t - P_h\mathbf{u}_t\|^2 - \frac{1}{2}C\sigma(t)\|\mathbf{u}_t - P_h\mathbf{u}_t\|^2 \\ &\quad+ \sigma_1\Big(\mu a(\mathbf{e}_t^*, \mathbf{u}_t - P_h\mathbf{u}_t) - b(\mathbf{u}_H, \mathbf{e}_t^*, \mathbf{u}_t - P_h\mathbf{u}_t) - b(\mathbf{e}_t^*, \mathbf{u}_H, \mathbf{u}_t - P_h\mathbf{u}_t) \\ &\quad- b(\mathbf{e}^*, \mathbf{u}_{Ht}, P_h\mathbf{e}_t^*) - b(\mathbf{u}_{Ht}, \mathbf{e}^*, P_h\mathbf{e}_t^*) + (p_t, \nabla\cdot P_h\mathbf{e}_t^*) - \gamma a(\mathbf{e}^*, P_h\mathbf{e}_t^*) \\ &\quad+ \delta\int_0^t \beta(t-s)\ a(\mathbf{e}^*(s), P_h\mathbf{e}_t^*)\ ds - b(\mathbf{e}_H, \mathbf{e}_{Ht}, P_h\mathbf{e}_t^*) - b(\mathbf{e}_{Ht}, \mathbf{e}_H, P_h\mathbf{e}_t^*)\Big).\end{aligned} \tag{3.47}$$

All the terms on the right of inequality (3.47) except last two term can be bounded by using the similar way of Lemma 3.4. Using Lemma 3.2, we can bound the last two terms as

$$\begin{aligned}|b(\mathbf{e}_H, \mathbf{e}_{Ht}, P_h\mathbf{e}_t^*) + b(\mathbf{e}_{Ht}, \mathbf{e}_H, P_h\mathbf{e}_t^*)| &\le C\|\nabla\mathbf{e}_{Ht}\|\|\mathbf{e}_H\|^{1-\ell}\|\nabla\mathbf{e}_H\|^{\ell}\|\nabla\mathbf{e}_t^*\| \\ &\le C(\epsilon)\|\nabla\mathbf{e}_{Ht}\|^2\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla\mathbf{e}_H\|^{2\ell} + \epsilon\|\nabla\mathbf{e}_t^*\|^2.\end{aligned} \tag{3.48}$$

We integrate (3.47) and use $\|\mathbf{e}_t^*\| \ge \|\mathbf{u}_t - P_h\mathbf{u}_t\|$ and (3.48) to deduce

$$\begin{aligned}\sigma_1(t)\|\mathbf{e}_t^*\|^2 + \int_0^t \sigma_1(s)\|\nabla\mathbf{e}_s^*\|^2 ds &\le C\big(h^2\int_0^t \sigma_1(s)\mathcal{K}_s^2(s)ds + \sup(\tau^*(t)\|\nabla\mathbf{u}_{Ht}\|)^2 \\ &\times\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}^*\|^2 ds + \|\mathbf{e}_H\|^{2(1-\ell)}_{L^\infty(\mathbf{L}^2)}\|\nabla\mathbf{e}_H\|^{2\ell}_{L^\infty(\mathbf{L}^2)}\int_0^t \sigma_1(s)\|\nabla\mathbf{e}_{Hs}\|^2 ds\big).\end{aligned}$$

Now, using Lemmas 3.1 and 3.4 and Theorem 3.1, we conclude the proof. □

Remark 3.7. *As in Remark* 3.3*, we can also obtain the sub-optimal result as below:*

$$(\tau^*(t))^2\|\mathbf{e}_t^*(t)\|^2 + e^{-2\alpha t}\int_0^t \sigma_1(s)\|\nabla \mathbf{e}_s^*\|^2 ds \le K(t)(h^2 + H^{4-2\ell}).$$

For optimal $L^\infty(\mathbf{L}^2)$ error estimate of $\mathbf{e}^*$, we consider a modified stationary Oseen type problem: Find $\mathbf{w}_h(t) \in \mathbf{J}_h$ with $0 < t \le T$ satisfies

$$\mu a(\mathbf{u}-\mathbf{w}_h, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{u}-\mathbf{w}_h, \mathbf{v}_h) + b(\mathbf{u}-\mathbf{w}_h, \mathbf{u}_H, \mathbf{v}_h) + \int_0^t \beta(t-s)a((\mathbf{u}-\mathbf{w}_h)(s), \mathbf{v}_h)ds = (p, \nabla\cdot \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{J}_h. \tag{3.49}$$

We split $\mathbf{e}^*$ as $\mathbf{e}^* = (\mathbf{u}-\mathbf{w}_h) + (\mathbf{w}_h - \mathbf{u}_h^*) = \boldsymbol{\zeta} + \boldsymbol{\rho}$. Then the equation of $\boldsymbol{\zeta}$ is given by

$$\mu a(\boldsymbol{\zeta}, \mathbf{v}_h) + b(\mathbf{u}_H, \boldsymbol{\zeta}, \mathbf{v}_h) + b(\boldsymbol{\zeta}, \mathbf{u}_H, \mathbf{v}_h) + \int_0^t \beta(t-s)a(\boldsymbol{\zeta}(s), \mathbf{v}_h)ds = (p, \nabla\cdot\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{J}_h. \tag{3.50}$$

We below present the estimates of $\boldsymbol{\zeta}$:

Lemma 3.8. *Suppose the hypothesis of the Theorem* 3.1 *be satisfied. Then,* $\boldsymbol{\zeta}(t)$ *satisfies the following result:*

$$\tau^*(t)\|\nabla\boldsymbol{\zeta}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\Big(\|\nabla\boldsymbol{\zeta}(s)\|^2 + (\tau^*(s))^2\|\nabla\boldsymbol{\zeta}_s(s)\|^2\Big)ds \le K(t)h^2,$$
$$\tau^*(t)\|\boldsymbol{\zeta}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\Big(\|\boldsymbol{\zeta}(s)\|^2 + (\tau^*(s))^2\|\boldsymbol{\zeta}_s(s)\|^2\Big)ds \le K(t)(h^4 + h^2H^4t^{-1}).$$

Proof. Choose $\mathbf{v}_h = P_h\boldsymbol{\zeta} = \boldsymbol{\zeta} - (\mathbf{u} - P_h\mathbf{u})$ in (3.50) and use (3.7) to obtain

$$\mu_1\|\nabla\boldsymbol{\zeta}\|^2 + \int_0^t \beta(t-s)a(\boldsymbol{\zeta}(s), \boldsymbol{\zeta})ds \le \mu a(\boldsymbol{\zeta}, \mathbf{u}-P_h\mathbf{u}) + b(\mathbf{u}_H, \boldsymbol{\zeta}, \mathbf{u}-P_h\mathbf{u}) + b(\boldsymbol{\zeta}, \mathbf{u}_H, \mathbf{u}-P_h\mathbf{u}) + (p, \nabla\cdot P_h\boldsymbol{\zeta}) + \int_0^t \beta(t-s)\ a(\boldsymbol{\zeta}(s), \mathbf{u}-P_h\mathbf{u})\ ds. \tag{3.51}$$

Similar to (3.14)-(3.16), we use the “Cauchy-Schwarz inequality” with (1.15), “Young's inequality” and Lemmas 3.2 and 3.1 to find

$$\mu_1\|\nabla\boldsymbol{\zeta}\|^2 + \int_0^t \beta(t-s)a(\boldsymbol{\zeta}(s), \boldsymbol{\zeta})ds \le Ch^2(\|\tilde{\Delta}\mathbf{u}\|^2 + \|\nabla p\|^2) + \frac{\mu_1}{4}\|\nabla\boldsymbol{\zeta}\|^2 + \int_0^t \beta(t-s)a(\boldsymbol{\zeta}(s), \mathbf{u}-P_h\mathbf{u})ds. \tag{3.52}$$

After multiplying both side by $e^{2\alpha t}$, we take time integration and we drop the second term on the left of the resulting inequality due to positivity property Lemma 1.5. And similar to (3.21), we bound the another double integration term as

$$\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)\ a(\boldsymbol{\zeta}(\tau), (\mathbf{u}-P_h\mathbf{u})(s))\ d\tau\ ds$$

$$\leq Ch^2 \int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}(s)\|^2 ds + \frac{\mu_1}{4}\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\zeta}(s)\|^2 ds.$$

Finally, use (3.21) in (3.52) to obtain

$$\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\zeta}(s)\|^2 ds \leq Ch^2 \int_0^t e^{2\alpha s}\mathcal{K}^2(s) ds, \tag{3.53}$$

where $\mathcal{K}(t) := \|\tilde{\Delta}\mathbf{u}(t)\| + \|\nabla p(t)\|$. Now, from (3.52), we easily find that

$$\|\nabla\boldsymbol{\zeta}\|^2 \leq Ch^2\mathcal{K}^2(t) + \int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\zeta}(s)\|^2 ds \leq Ch^2(\mathcal{K}^2(t) + \int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds). \tag{3.54}$$

In order to find L^2 estimate of $\boldsymbol{\zeta}$, we consider the dual problem (3.36) with $\mathbf{e}^*$ is replaced by $\boldsymbol{\zeta}$ and choose $\mathbf{v} = \boldsymbol{\zeta}$ and use (3.50) with $\mathbf{v}_h = P_h\mathbf{v}$ to obtain

$$\begin{aligned} e^{2\alpha t}\|\boldsymbol{\zeta}\|^2 =& (\boldsymbol{\zeta}, \mathbf{v}_t) - \mu a(\boldsymbol{\zeta}, \mathbf{v} - P_h\mathbf{v}) - b(\boldsymbol{\zeta}, \mathbf{u}_H, \mathbf{v} - P_h\mathbf{v}) - b(\mathbf{u}_H, \boldsymbol{\zeta}, \mathbf{v} - P_h\mathbf{v}) \\ & - b(\mathbf{e}_H, \boldsymbol{\zeta}, \mathbf{v}) - b(\boldsymbol{\zeta}, \mathbf{e}_H, \mathbf{v}) + (\psi, \nabla\cdot\boldsymbol{\zeta}) - (p, \nabla\cdot P_h\mathbf{v}) \\ & - \int_0^t \beta(t-s)\ a(\boldsymbol{\zeta}(s), \mathbf{v} - P_h\mathbf{v})\ ds. \end{aligned}$$

After using Lemmas 3.2 and 3.1, we apply the "Young's inequality" with (1.15) to find

$$\begin{aligned} e^{2\alpha t}\|\boldsymbol{\zeta}\|^2 \leq& C(\epsilon)e^{2\alpha t}\Big((h^2 + \|\mathbf{e}_H\|^2)\|\nabla\boldsymbol{\zeta}\|^2 + h^2\big(\int_0^t \beta(t-s)\|\nabla\boldsymbol{\zeta}(s)\|ds\big)^2\Big) \\ & + \frac{1}{2}e^{2\alpha t}\|\boldsymbol{\zeta}\|^2 + \epsilon e^{-2\alpha t}(\|\mathbf{v}\|_2^2 + \|\psi\|_1^2 + \|\mathbf{v}_t\|^2) \end{aligned} \tag{3.55}$$

We take time integration on the both sides from 0 to t and bound the resulting double integration term similar to (3.21). Next use (3.36) with $\mathbf{e}^*$ replaced by $\boldsymbol{\zeta}$ and choose $C\epsilon = \frac{1}{2}$ to arrive at

$$\begin{aligned} \int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2 ds \leq& C(h^2 + \|\mathbf{e}_H(t)\|^2_{L^\infty(\mathbf{L}^2)})\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\zeta}(s)\|^2 ds \\ \leq& Ch^2(h^2 + \|\mathbf{e}_H(t)\|^2_{L^\infty(\mathbf{L}^2)})\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds. \end{aligned} \tag{3.56}$$

Now from (3.55), we easily conclude that

$$e^{2\alpha t}\|\boldsymbol{\zeta}(t)\|^2 \leq C(\epsilon)h^2(h^2 + \|\mathbf{e}_H(t)\|^2_{L^\infty(\mathbf{L}^2)})(\mathcal{K}^2 + \int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds). \tag{3.57}$$

Next differentiate (3.50) with respect to time and substitute $\mathbf{v}_h = P_h\boldsymbol{\zeta}_t = \boldsymbol{\zeta}_t - (\mathbf{u}_t - P_h\mathbf{u}_t)$ with (3.7) to arrive

$$\mu_1\|\nabla\boldsymbol{\zeta}_t\|^2 = \mu_1 a(\boldsymbol{\zeta}_t, \mathbf{u}_t - P_h\mathbf{u}_t) + b(\mathbf{u}_H, \boldsymbol{\zeta}_t, \mathbf{u}_t - P_h\mathbf{u}_t) + b(\boldsymbol{\zeta}_t, \mathbf{u}_H, \mathbf{u}_t - P_h\mathbf{u}_t)$$

$$+ b(\mathbf{u}_{Ht}, \boldsymbol{\zeta}, P_h\boldsymbol{\zeta}_t) + b(\boldsymbol{\zeta}, \mathbf{u}_{Ht}, P_h\boldsymbol{\zeta}_t) + \beta_t(0)a(\boldsymbol{\zeta}, P_h\boldsymbol{\zeta}_t)$$
$$+ \int_0^t \beta_t(t-s)\ a(\boldsymbol{\zeta}(s), P_h\boldsymbol{\zeta}_t)\ ds + (p_t, \nabla \cdot P_h\boldsymbol{\zeta}_t).$$

Then, we use the "Cauchy-Schwarz inequality" and the "Young's inequality" with (1.15) and Lemma 3.23.1. Then, multiply the resulting equation by $\sigma_1(t) = e^{2\alpha t}(\tau^*(t))^2$ and integrate over 0 to t to obtain

$$\mu_1 \int_0^t \sigma_1(s)\|\nabla\boldsymbol{\zeta}_s(s)\|^2 ds \leq Ch^2 \int_0^t \sigma_1(s)(\|\tilde{\Delta}\mathbf{u}_s(s)\| + \|\nabla p_s(s)\|)^2 ds$$
$$+ \int_0^t \sigma_1(s)\|\nabla\mathbf{u}_{Hs}\|^2\|\nabla\boldsymbol{\zeta}\|^2 ds + \int_0^t \sigma_1(s)\|\nabla\boldsymbol{\zeta}(s)\|^2 ds$$
$$+ \int_0^t \sigma_1(s)\Big(\int_0^s \beta(s-\tau)\ \|\nabla\boldsymbol{\zeta}(\tau)\| d\tau\Big)^2 ds.$$

Use Lemma 3.1 and proceed as above, resulting in:

$$\int_0^t \sigma_1(s)\|\nabla\boldsymbol{\zeta}_s(s)\|^2 ds \leq Ch^2 \int_0^t \sigma_1(s)\mathcal{K}_s(s) ds + \sup(\tau^*\|\nabla\mathbf{u}_{Hs}\|)^2 \int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\zeta}\|^2 ds$$
$$\leq Ch^2\Big(\int_0^t \sigma_1(s)\mathcal{K}_s(s) ds + \int_0^t e^{2\alpha s}\mathcal{K}^2(s) ds\Big). \tag{3.58}$$

Here $\mathcal{K}_t(t) := \|\tilde{\Delta}\mathbf{u}_t(t)\| + \|\nabla p_t(t)\|$.

For L^2 estimate of $\boldsymbol{\zeta}_t$, we again consider the dual problem (3.36) with $\mathbf{e}^*$ replaced by $\boldsymbol{\zeta}_t$. Arguing with similar proof techniques, we easily derive the following:

$$\int_0^t \sigma_1(s)\|\boldsymbol{\zeta}_s(s)\|^2 ds \leq Ch^2(h^2 + \|\mathbf{e}_H\|^2_{L^\infty(\mathbf{L}^2)})\Big(\int_0^t \sigma_1(s)\mathcal{K}_s(s) ds + \int_0^t e^{2\alpha s}\mathcal{K}^2(s) ds\Big).$$

Combining the above estimate with (3.53), (3.54), (3.56), (3.57), (3.58) and using Theorem 3.1, we conclude the desired results. □

Recall that $\mathbf{e}^* = \boldsymbol{\zeta} + \boldsymbol{\rho}$ and in the Lemma 3.8, we already find the estimates of $\boldsymbol{\zeta}$. Hence, to estimate $\mathbf{e}^*$, it is enough to derive the estimate of $\boldsymbol{\rho}$. From (3.9) and (3.50), we observe that

$$(\boldsymbol{\rho}_t, \mathbf{v}_h) + \mu a(\boldsymbol{\rho}, \mathbf{v}_h) + b(\boldsymbol{\rho}, \mathbf{u}_H, \mathbf{v}_h) + b(\mathbf{u}_H, \boldsymbol{\rho}, \mathbf{v}_h) + \int_0^t \beta(t-s)a(\boldsymbol{\rho}(s), \mathbf{v}_h) ds$$
$$= -(\boldsymbol{\zeta}_t, \mathbf{v}_h) - b(\mathbf{e}_H, \mathbf{e}_H, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{J}_h. \tag{3.59}$$

Now, we choose $\mathbf{v}_h = \sigma(t)\boldsymbol{\rho}$ in (3.59) and use (3.7). Then, we take time integration on the resulting inequality to find

$$\sigma(t)\|\boldsymbol{\rho}(t)\|^2 + \mu_1 \int_0^t \sigma(s)\|\nabla\boldsymbol{\rho}(s)\|^2\ ds \leq -\int_0^t \sigma(s)(\boldsymbol{\zeta}_s(s), \boldsymbol{\rho}) ds + \int_0^t \sigma_s(s)\|\boldsymbol{\rho}(s)\|^2 ds$$

$$- \int_0^t \sigma(s) b(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\rho})\ ds - \int_0^t \sigma(s) \int_0^s \beta(s-\tau)\ a(\boldsymbol{\rho}(\tau), \boldsymbol{\rho}(s))\ d\tau\ ds. \tag{3.60}$$

We write $\boldsymbol{\rho} = \mathbf{e}^* - \boldsymbol{\zeta}$ and use it to find the following

$$\begin{aligned} |\int_0^t \sigma(s)(\boldsymbol{\zeta}_s(s), \boldsymbol{\rho}) ds| &\leq \int_0^t \frac{\sigma^2(s)}{\sigma_s(s)} \|\boldsymbol{\zeta}_s(s)\|^2 ds + \int_0^t \sigma_s(s) \|\boldsymbol{\rho}(s)\|^2 ds \\ &\leq \int_0^t \sigma_1(s) \|\boldsymbol{\zeta}_s(s)\|^2 ds + \int_0^t e^{2\alpha s} (\|\mathbf{e}^*(s)\|^2 + \|\boldsymbol{\zeta}(s)\|^2) ds. \end{aligned}$$

One can bound the nonlinear terms similar to (3.17). Then, we finally reach at

$$\begin{aligned} \sigma(t)\|\boldsymbol{\rho}(t)\|^2 + \mu_1 \int_0^t \sigma(s) \|\nabla \boldsymbol{\rho}(s)\|^2 ds &\leq C(\epsilon)\Big(\int_0^t \sigma_1(s) \|\boldsymbol{\zeta}_s(s)\|^2 ds \\ &+ \int_0^t e^{2\alpha s} (\|\mathbf{e}^*(s)\|^2 + \|\boldsymbol{\zeta}(s)\|^2)\ ds \Big) \\ &- \int_0^t \sigma(s) \int_0^s \beta(s-\tau)\ a(\boldsymbol{\rho}(\tau), \boldsymbol{\rho}(s))\ d\tau\ ds. \end{aligned} \tag{3.61}$$

The double integral on the above inequality (3.61) is no longer positive, due to the weight sitting inside the integral. Also a direct estimate of this term would only give sub-optimal result as the time weight is not adequate to handle the singularity of the full optimality. Since integration smoothens a function, we consider $\boldsymbol{\rho}$ under integration:

$$\tilde{\boldsymbol{\rho}} = \int_0^t \boldsymbol{\rho}(s)\ ds.$$

We now integrate by parts the double integral. Keeping in mind that $\beta(t) = \gamma e^{-\delta t}$ and using the "Young's inequality", we deduce that

$$\begin{aligned} &\int_0^t \sigma(s) \int_0^s \beta(s-\tau)\ a(\boldsymbol{\rho}(\tau), \boldsymbol{\rho}(s))\ d\tau\ ds \\ &\quad\leq \gamma \int_0^t \sigma(s) a(\tilde{\boldsymbol{\rho}}(s), \boldsymbol{\rho}(s))\ ds - \delta \int_0^t \sigma(s) \int_0^s \beta(s-\tau) a(\tilde{\boldsymbol{\rho}}(\tau), \boldsymbol{\rho}(s))\ d\tau\ ds \\ &\quad\leq C \int_0^t \sigma(s) \Big(\|\nabla \tilde{\boldsymbol{\rho}}(s)\|^2 + \big(\int_0^s \beta(s-\tau) \|\nabla \tilde{\boldsymbol{\rho}}(\tau)\| d\tau \big)^2 \Big) ds + \epsilon \int_0^t \sigma(s) \|\nabla \boldsymbol{\rho}(s)\|^2 \\ &\quad\leq C(\epsilon) \int_0^t e^{2\alpha s} \|\nabla \tilde{\boldsymbol{\rho}}(s)\|^2 ds + \epsilon \int_0^t \sigma(s) \|\nabla \boldsymbol{\rho}(s)\|^2. \end{aligned} \tag{3.62}$$

In order to find the estimate involving 'tilde' operator in inequality (3.62), we take time integration on the both sides of (3.59) from 0 to t to obtain the equation of $\tilde{\boldsymbol{\rho}}$ as

$$\begin{aligned} (\tilde{\boldsymbol{\rho}}_t, \mathbf{v}_h) + \mu_1 \|\nabla \tilde{\boldsymbol{\rho}}\| \|\nabla \mathbf{v}_h\| + \int_0^t \int_0^s \beta(s-\tau)\ a(\boldsymbol{\rho}(\tau), \mathbf{v}_h)\ d\tau\ ds \\ \leq -(\boldsymbol{\zeta}, \mathbf{v}_h) - \int_0^t b(\mathbf{e}_H(s), \mathbf{e}_H(s), \mathbf{v}_h) ds, \quad \forall \mathbf{v}_h \in \mathbf{J}_h. \end{aligned} \tag{3.63}$$

The resulting double integral is handled similar to [116, Lemma 5.3, page 772]. First we write $\boldsymbol{\rho}(\tau) = \tilde{\boldsymbol{\rho}}_\tau$ and use integrate by parts with respect to τ

$$\begin{aligned}
&\int_0^t \int_0^s \beta(s-\tau)\ a(\boldsymbol{\rho}(\tau), \mathbf{v}_h)\ d\tau\ ds \\
&\qquad = \int_0^t \int_0^s \beta(s-\tau)\ a(\tilde{\boldsymbol{\rho}}_\tau(\tau), \mathbf{v}_h)\ d\tau\ ds \\
&\qquad = \int_0^t \int_0^s \beta(0)\ a(\tilde{\boldsymbol{\rho}}(s), \mathbf{v}_h)\ ds - \int_0^t \int_0^s \beta_\tau(s-\tau)\ a(\tilde{\boldsymbol{\rho}}(\tau), \mathbf{v}_h)\ d\tau\ ds \\
&\qquad = \int_0^t \frac{d}{ds}\Big(\int_0^s \beta(s-\tau)\ a(\tilde{\boldsymbol{\rho}}(\tau), \mathbf{v}_h)\ d\tau\Big) ds \\
&\qquad = \int_0^t \beta(t-s)\ a(\tilde{\boldsymbol{\rho}}(s), \mathbf{v}_h)\ ds. \qquad (3.64)
\end{aligned}$$

We choose $\mathbf{v}_h = e^{2\alpha t}\tilde{\boldsymbol{\rho}}$ in (3.63) and use (1.7). Take time integration on the both sides and drop the double integration term from the left of inequality due to positivity (1.5). Other terms can be handled as done in earlier analysis so as to obtain

$$e^{2\alpha t}\|\tilde{\boldsymbol{\rho}}\|^2 + \int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol{\rho}}\|^2 ds \le C\int_0^t e^{2\alpha s}\big(\|\boldsymbol{\zeta}\|^2 + \|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla\mathbf{e}_H\|^{2(1+\ell)}\big)ds. \qquad (3.65)$$

This result now allows us estimate (3.62) which in turn helps us to estimate (3.61). Now a use of Lemmas 3.8, 3.6 and Theorem 3.1 gives

$$\tau^*(t)\|\boldsymbol{\rho}(t)\|^2 + e^{-2\alpha t}\int_0^t \sigma(s)\|\nabla\boldsymbol{\rho}(s)\|^2 \le K(t)(h^4 + h^2H^4t^{-1} + H^{6-2\ell}t^{-1}).$$

Now, we use the triangle inequality with Lemma 3.6 and 3.5 to derive the following theorem. We would like to note here that the uniform estimates are a direct result of Theorem 3.1, under uniqueness condition (3.11).

Theorem 3.2. *Suppose the hypothesis of Theorem 3.1 holds. Then, with* $\mathbf{u}_h(0) = P_h\mathbf{u}_0 \in \mathbf{J}_h$, *where* $\mathbf{u}_0 \in \mathbf{J}_1$, *the following error bounds are satisfied:*

$$\|\mathbf{u}(t)-\mathbf{u}_h^*(t)\| \le K(t)(h^2t^{-1/2}+H^{3-\ell}t^{-1}),\ \ \|\nabla(\mathbf{u}(t)-\mathbf{u}_h^*(t))\| \le K(t)(ht^{-1/2}+H^{3-\ell}t^{-1})$$

where $\ell > 0$ *is arbitrary small and* $K(t) = Ce^{Ct}$. *Under the smallness condition (3.11), the above estimates are valid uniformly in time, that is,* $K(t) = C$.

3.3.2 Optimal Velocity Error Estimates in Step III

Lemma 3.9. *Suppose the hypothesis of the Theorem 3.1 be satisfied. Then, the following result holds:*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}_h(s)\|^2 ds \le K(t)(h^2 + h^2H^{4-2\ell}t^{-1} + H^{8-4\ell}t^{-1}).$$

Proof. Consider (3.10) with $\mathbf{v}_h = P_h\mathbf{e}_h = (P_h\mathbf{u} - \mathbf{u}) + \mathbf{e}_h$ and use (3.7), (1.7) and (3.13). We then multiply both side by $e^{2\alpha t}$ to arrive

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}e^{2\alpha t}\|\mathbf{e}_h\|^2 + (\mu_1 - \frac{\alpha}{\lambda_1})e^{2\alpha t}\|\nabla\mathbf{e}_h\|^2 + e^{2\alpha t}\int_0^t \beta(t-s)\ a(\mathbf{e}_h(s),\mathbf{e}_h)\ ds \\
\le \frac{1}{2}\frac{d}{dt}e^{2\alpha t}\|\mathbf{u} - P_h\mathbf{u}\|^2 - \alpha e^{2\alpha t}\|\mathbf{u} - P_h\mathbf{u}\|^2 + \mu e^{2\alpha t} a(\mathbf{e}_h, \mathbf{u} - P_h\mathbf{u}) \\
+ e^{2\alpha t}(b(\mathbf{e}_h, \mathbf{u}_H, \mathbf{u} - P_h\mathbf{u}) + b(\mathbf{u}_H, \mathbf{e}_h, \mathbf{u} - P_h\mathbf{u})) + e^{2\alpha t}(p, \nabla\cdot P_h\mathbf{e}_h) \\
+ e^{2\alpha t}(b(\mathbf{e}^*, \mathbf{e}^*, P_h\mathbf{e}_h) - b(\mathbf{e}^*, \mathbf{e}_H, P_h\mathbf{e}_h) - b(\mathbf{e}_H, \mathbf{e}^*, P_h\mathbf{e}_h)) \\
+ e^{2\alpha t}\int_0^t \beta(t-s)\ a(\mathbf{e}_h(s), \mathbf{u} - P_h\mathbf{u})\ ds. \qquad (3.66)
\end{aligned}$$

We drop the second term from the right of inequality (3.66). Similar to (3.13)-(3.16), we can bound the third, fourth, fifth and sixth terms. The remaining nonlinear terms can be bound by using Lemma 3.2 with (1.15) as

$$\begin{aligned}
|b(\mathbf{e}^*, \mathbf{e}^*, P_h\mathbf{e}_h) - b(\mathbf{e}^*, \mathbf{e}_H, P_h\mathbf{e}_h) - b(\mathbf{e}_H, \mathbf{e}^*, P_h\mathbf{e}_h)| \\
\le C(\|\mathbf{e}_H\|^{(1-\ell)}\|\nabla\mathbf{e}_H\|^{\ell}\|\nabla\mathbf{e}^*\| + \|\nabla\mathbf{e}^*\|^2)\|\nabla\mathbf{e}_h\|.
\end{aligned}$$

Taking integration on the both sides of (3.66) from 0 to t and using $\|\mathbf{e}_h(0)\| = \|\mathbf{u}_0 - P_h\mathbf{u}_0\|$, we reach at

$$\begin{aligned}
e^{2\alpha t}\|\mathbf{e}_h(t)\|^2 + (\mu_1 - \frac{2\alpha}{\lambda_1})\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}_h(s)\|^2 ds \le e^{2\alpha t}\|\mathbf{u}(t) - P_h\mathbf{u}(t)\|^2 \\
+ C(\|\mathbf{e}_H(t)\|^{2(1-\ell)}_{L^\infty(\mathbf{L}^2)}\|\nabla\mathbf{e}_H(t)\|^{2\ell}_{L^\infty(\mathbf{L}^2)} + \|\nabla\mathbf{e}^*(t)\|^2_{L^\infty(\mathbf{L}^2)})\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}^*(s)\|^2 ds \\
+ Ch^2\int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds.
\end{aligned}$$

A use of Remark 3.3, Lemma 3.5 and Theorem 3.1, concludes the remaining of the proof. □

Remark 3.8. *To avoid the singularity in time t as in Remark 3.3, we resort to a sub-optimal result as*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}_h(s)\|^2 ds \le K(t)(h^2 + H^{4-2\ell}).$$

Lemma 3.10. *Suppose the hypothesis of the Theorem 3.1 hold. Then, the step III error satisfies the following result:*

$$\tau^*(t)\|\nabla\mathbf{e}_h(t)\|^2 + e^{2\alpha t}\int_0^t \sigma(s)\|\mathbf{e}_{hs}(s)\|^2 ds \le K(t)(h^2 + H^{6-2\ell}t^{-1}).$$

Proof. Substitute $\mathbf{v}_h = \sigma(t)P_h\mathbf{e}_{ht} = \sigma(t)(P_h\mathbf{u}_t - \mathbf{u}_t) + \sigma(t)\mathbf{e}_{ht}$ in (3.10) to deduce

$$\begin{aligned}\sigma\|\mathbf{e}_{ht}\|^2 + \frac{\mu_1}{2}\frac{d}{dt}\sigma\|\nabla\mathbf{e}_h\|^2 = \frac{\mu}{2}\sigma_t\|\nabla\mathbf{e}_h\|^2 + \sigma(\mathbf{e}_{ht}, \mathbf{u}_t - P_h\mathbf{u}_t) + \mu\sigma a(\mathbf{e}_h, \mathbf{u}_t - P_h\mathbf{u}_t)\\ - \sigma(b(\mathbf{e}_h, \mathbf{u}_H, \mathbf{u}_t - P_h\mathbf{u}_t) + b(\mathbf{u}_H, \mathbf{e}_h, \mathbf{u}_t - P_h\mathbf{u}_t)) - \sigma\int_0^t \beta(t-s)a(\mathbf{e}_h, P_h\mathbf{e}_{ht})ds\\ + \sigma(p, \nabla\cdot P_h\mathbf{e}_{ht}) + \sigma(b(\mathbf{e}^*, \mathbf{e}^*, P_h\mathbf{e}_{ht}) - b(\mathbf{e}^*, \mathbf{e}_H, P_h\mathbf{e}_{ht}) - b(\mathbf{e}_H, \mathbf{e}^*, P_h\mathbf{e}_{ht})). \quad (3.67)\end{aligned}$$

Similar to (3.28) and Lemma 3.5, we finally arrive at

$$\begin{aligned}\int_0^t \sigma(s)\|\mathbf{e}_{hs}(s)\|^2 ds + \sigma(t)\|\nabla\mathbf{e}_h\|^2 \le C\Big(\int_0^t \sigma_s(s)\|\nabla\mathbf{e}_h(s)\|^2 ds + \int_0^t \sigma(s)\|\nabla\mathbf{e}_h(s)\|^2 ds\\ + h^2\int_0^t \frac{\sigma^2(s)}{\sigma_s(s)}(\|\tilde{\Delta}\mathbf{u}_s\|^2 + \|\nabla p_s\|^2)ds + h^2\int_0^t (\sigma(s)\|\nabla\mathbf{u}_s\|^2 + \sigma_s(s)\|\nabla p\|^2)ds\\ + \sigma(t)(\|\nabla\mathbf{e}^*\|^2 + \|\nabla\mathbf{e}_H\|^2)\|\nabla\mathbf{e}^*\|^2 + \int_0^t \sigma_s(s)(\|\nabla\mathbf{e}^*\|^2 + \|\nabla\mathbf{e}_H\|^2)\|\nabla\mathbf{e}^*\|^2 ds\\ + h^2\sigma\|\nabla p\|^2 + \int_0^t \frac{\sigma^2(s)}{\sigma_s(s)}((\|\nabla\mathbf{e}_s^*\|^2 + \|\nabla\mathbf{e}_{Hs}\|^2)\|\nabla\mathbf{e}^*\|^2 + \|\nabla\mathbf{e}_s^*\|^2\|\nabla\mathbf{e}_H\|^2)ds\Big).\end{aligned}$$

Use the fact $\sigma_s(s) \le Ce^{2\alpha s}$ and $\frac{\sigma^2(s)}{\sigma_s(s)} \le C\sigma_1(s)$ and a use of Theorem 3.1 and Lemmas 3.9, 3.4, 3.7 and Remarks 3.3, 3.5, 3.7 concludes the remaining of the proof. □

Lemma 3.11. *Suppose the hypothesis of the Theorem* 3.1 *be satisfied. Then, the following result holds:*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{e}_h(s)\|^2 ds \le K(t)(h^4 + h^2H^4t^{-1} + H^{8-2\ell}t^{-1}).$$

Proof. Processing similar idea of the proof of Lemma 3.6, that is, using duality argument we arrive at

$$\begin{aligned}e^{2\alpha t}\|\mathbf{e}_h\|^2 =& \frac{d}{dt}(\mathbf{e}_h, P_h\mathbf{v}) + (\mathbf{u} - P_h\mathbf{u}, \mathbf{v}_t) - \mu a(\mathbf{e}_h, \mathbf{v} - P_h\mathbf{v}) - b(\mathbf{u}_H, \mathbf{e}_h, \mathbf{v} - P_h\mathbf{v})\\ &- b(\mathbf{e}_h, \mathbf{u}_H, \mathbf{v} - P_h\mathbf{v}) - b(\mathbf{e}_H, \mathbf{e}_h, \mathbf{v}) - b(\mathbf{e}_h, \mathbf{e}_H, \mathbf{v}) + b(\mathbf{e}_H, \mathbf{e}^*, P_h\mathbf{v} - \mathbf{v})\\ &+ b(\mathbf{e}^*, \mathbf{e}_H, P_h\mathbf{v} - \mathbf{v}) + b(\mathbf{e}_H, \mathbf{e}^*, \mathbf{v}) + b(\mathbf{e}^*, \mathbf{e}_H, \mathbf{v}) - b(\mathbf{e}^*, \mathbf{e}^*, P_h\mathbf{v} - \mathbf{v})\\ &- b(\mathbf{e}^*, \mathbf{e}^*, \mathbf{v}) + \int_0^t \beta(t-s)a(\mathbf{e}_h, P_h\mathbf{v})ds + (\psi, \nabla\cdot\mathbf{e}_h) - (p, \nabla\cdot P_h\mathbf{v}).\end{aligned} \quad (3.68)$$

An application of Lemma 3.2 with (1.15) help us to bound the following nonlinear terms

$$|b(\mathbf{e}_H, \mathbf{e}^*, P_h\mathbf{v} - \mathbf{v})| + |b(\mathbf{e}^*, \mathbf{e}_H, P_h\mathbf{v} - \mathbf{v})| + |b(\mathbf{e}_H, \mathbf{e}^*, \mathbf{v})| + |b(\mathbf{e}^*, \mathbf{e}_H, \mathbf{v})|$$

$$\leq C(h\|\nabla \mathbf{e}_H\|^{1-\ell}\|\nabla \mathbf{e}_H\|^{\ell} + \|\mathbf{e}_H\|)\|\nabla \mathbf{e}^*\|\|\mathbf{v}\|_2, \tag{3.69}$$

and

$$|b(\mathbf{e}^*, \mathbf{e}^*, P_h\mathbf{v} - \mathbf{v})|+|b(\mathbf{e}^*, \mathbf{e}^*, \mathbf{v})| \leq C(h\|\nabla \mathbf{e}^*\| + \|\mathbf{e}^*\|)\|\nabla \mathbf{e}^*\|\|\mathbf{v}\|_2. \tag{3.70}$$

Integrating (3.68) and using $\mathbf{v}(t) = 0$ and (3.40)-(3.43) with $\mathbf{e}^*$ replaced by $\mathbf{e}_h$ and incorporating (3.69)-(3.70), we arrive at

$$\begin{aligned}\int_0^t e^{2\alpha s}\|\mathbf{e}_h(s)\|^2 ds &\leq C(\epsilon)\big(h^4 \int_0^t e^{2\alpha s}\mathcal{K}^2(s)ds + \int_0^t e^{2\alpha s}(h^2 + \|\mathbf{e}_H\|^2)\|\nabla \mathbf{e}_h\|^2 ds \\ &+ \int_0^t e^{2\alpha s}\big(h^2\|\mathbf{e}_H\|^{2(1-\ell)}\|\nabla \mathbf{e}_H\|^{2\ell} + h^2\|\nabla \mathbf{e}^*\|^2 + \|\mathbf{e}_H\|^2 + \|\mathbf{e}^*\|^2\big)\|\nabla \mathbf{e}^*\|^2 ds\big).\end{aligned}$$

Finally using Lemmas 3.4, 3.6, 3.5, 3.9 and Theorem 3.1 and Remark 3.3, 3.8, we complete the proof. □

For $L^\infty(\mathbf{L}^2)$ error of $\mathbf{e}_h$, we split it as $\mathbf{e}_h =: (\mathbf{u} - \mathbf{w}_h) + (\mathbf{w}_h - \mathbf{u}_h) := \boldsymbol{\zeta} + \boldsymbol{\theta}$. We already have the estimates for $\boldsymbol{\zeta}$, see Lemma 3.8. So it remains to estimate only $\boldsymbol{\theta}$. The equation $\boldsymbol{\theta}$ is given by

$$\begin{aligned}(\boldsymbol{\theta}_t, \mathbf{v}_h) + \mu a(\boldsymbol{\theta}, \mathbf{v}_h) + b(\mathbf{u}_H, \boldsymbol{\theta}, \mathbf{v}_h)b(\boldsymbol{\theta}, \mathbf{u}_H, \mathbf{v}_h) + + \int_0^t \beta(t-s)a(\boldsymbol{\theta}, \mathbf{v}_h)ds \\ = -(\boldsymbol{\zeta}_t, \mathbf{v}_h) + b(\mathbf{e}^*, \mathbf{e}^*, \mathbf{v}_h) - b(\mathbf{e}^*, \mathbf{e}_H, \mathbf{v}_h) - b(\mathbf{e}_H, \mathbf{e}^*, \mathbf{v}_h).\end{aligned} \tag{3.71}$$

We observe that the only difference between (3.59) and (3.71) is the nonlinear terms and we can bound the nonlinear terms as earlier. So if we choose $\mathbf{v}_h = \sigma(t)\boldsymbol{\theta}$ in (3.71), then similar to (3.60)-(3.65), we find

$$\begin{aligned}\frac{d}{dt}\sigma\|\boldsymbol{\theta}\|^2 - \sigma_t\|\boldsymbol{\theta}\|^2 + 2\mu_1\sigma\|\nabla\boldsymbol{\theta}\|^2 + 2\sigma\int_0^t \beta(t-s)a(\boldsymbol{\theta}(s), \boldsymbol{\theta})ds \\ = -2\sigma(\boldsymbol{\zeta}_t, \boldsymbol{\theta}) + 2\sigma\big(b(\mathbf{e}^*, \mathbf{e}^*, \boldsymbol{\theta}) - b(\mathbf{e}^*, \mathbf{e}_H, \boldsymbol{\theta}) - b(\mathbf{e}_H, \mathbf{e}^*, \boldsymbol{\theta})\big).\end{aligned}$$

Take integration on the both sides to deduce

$$\begin{aligned}\sigma(t)\|\boldsymbol{\theta}(t)\|^2 + 2\mu_1\int_0^t \sigma(s)\|\nabla\boldsymbol{\theta}(s)\|^2 ds + 2\int_0^t \sigma(s)\int_0^s \beta(s-\tau)\ a(\boldsymbol{\theta}(\tau), \boldsymbol{\theta}(s))\ d\tau\ ds \\ \leq -2\int_0^t \sigma(s)\Big((\boldsymbol{\zeta}_s, \boldsymbol{\theta}) - b(\mathbf{e}^*, \mathbf{e}^*, \boldsymbol{\theta}) + b(\mathbf{e}^*, \mathbf{e}_H, \boldsymbol{\theta}) + b(\mathbf{e}_H, \mathbf{e}^*, \boldsymbol{\theta})\Big)ds \\ + 2\int_0^t \sigma_s(s)\|\boldsymbol{\theta}(s)\|^2 ds.\end{aligned} \tag{3.72}$$

Thanks to Lemma 3.2 and the "Young's inequality" to bound the following

$$2\int_0^t \sigma(s)\ |b(\mathbf{e}^*, \mathbf{e}^*, \boldsymbol{\theta}) - b(\mathbf{e}^*, \mathbf{e}_H, \boldsymbol{\theta}) - b(\mathbf{e}_H, \mathbf{e}^*, \boldsymbol{\theta})|\ ds$$

$$\leq C(\mu)\int_0^t \sigma(s)(\|\nabla \mathbf{e}_H\|^2+\|\nabla \mathbf{e}^*\|^2)\|\nabla \mathbf{e}^*\|^2 ds+\frac{\mu_1}{2}\int_0^t \sigma(s)\|\nabla \boldsymbol{\theta}\|^2 ds. \quad (3.73)$$

Using Lemma 3.8 with the "Cauchy-Schwarz inequality" and the "Young's inequality", we estimate the first term on the right of inequality (3.72) as

$$\int_0^t \sigma(s)\ (\boldsymbol{\zeta}_s(s),\boldsymbol{\theta})\ ds \leq \int_0^t \frac{\sigma^2(s)}{\sigma_s(s)}\|\boldsymbol{\zeta}_s(s)\|^2 ds+\int_0^t \sigma_s(s)\|\boldsymbol{\theta}(s)\|^2 ds. \quad (3.74)$$

We now write $\boldsymbol{\theta}=\mathbf{e}_h-\boldsymbol{\zeta}$ and use $\sigma_t(t)\leq Ce^{2\alpha t}$ to obtain

$$\int_0^t \sigma_s(s)\|\boldsymbol{\theta}\|^2 ds \leq C\int_0^t e^{2\alpha s}\|\mathbf{e}_h(s)\|^2 ds+C\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2 ds. \quad (3.75)$$

Use (3.73), (3.74) and (3.75) in (3.72) and use the fact $\frac{\sigma^2(t)}{\sigma_t(t)}\leq C\sigma_1(t)$ to arrive at

$$\begin{aligned}\sigma(t)\|\boldsymbol{\theta}(t)\|^2&+\frac{3\mu_1}{2}\int_0^t \sigma(s)\|\nabla\boldsymbol{\theta}(s)\|^2 ds+2\int_0^t\sigma(s)\int_0^s \beta(s-\tau)\ a(\boldsymbol{\theta}(\tau),\boldsymbol{\theta}(s))\ d\tau\ ds\\ &\leq C\Big(\int_0^t \sigma_1(s)\|\boldsymbol{\zeta}_s(s)\|^2 ds+\int_0^t e^{2\alpha s}\|\mathbf{e}_h(s)\|^2 ds+\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2 ds\\ &\quad+\int_0^t \sigma(s)\big(\|\nabla \mathbf{e}_H\|^2+\|\nabla \mathbf{e}^*\|^2\big)\|\nabla \mathbf{e}^*\|^2 ds\Big). \quad (3.76)\end{aligned}$$

The double integration term on the left of inequality (3.76) can be written in exactly the same way of (3.62) as

$$\begin{aligned}2\int_0^t &\sigma(s)\int_0^s \beta(s-\tau)\ a(\boldsymbol{\theta}(\tau),\boldsymbol{\theta}(s))\ d\tau\ ds\\ &\leq C(\mu_1)\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol{\theta}}(s)\|^2 ds+\frac{\mu_1}{2}\int_0^t \sigma(s)\|\nabla\boldsymbol{\theta}(s)\|^2 ds, \quad (3.77)\end{aligned}$$

where $\tilde{\boldsymbol{\theta}}(t)=\int_0^t \boldsymbol{\theta}(s)ds$. Similar to (3.63)-(3.65), one can find the estimate for $\tilde{\boldsymbol{\theta}}$ as:

$$\begin{aligned}e^{2\alpha t}\|\tilde{\boldsymbol{\theta}}_t(t)\|^2+\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol{\theta}}(s)\|^2 ds \leq & C\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2+C\int_0^t e^{2\alpha s}\|\nabla \mathbf{e}_H\|^2\|\nabla \mathbf{e}^*\|^2 ds\\ &+C\int_0^t e^{2\alpha s}\|\nabla \mathbf{e}^*(s)\|^4 ds. \quad (3.78)\end{aligned}$$

Inserting (3.77)-(3.78) in (3.76) and applying the Lemmas 3.8, 3.11, 3.5 and Theorem 3.1, we finally arrive

$$\tau^*(t)\|\boldsymbol{\theta}(t)\|^2+e^{-2\alpha t}\int_0^t \sigma(s)\|\nabla\boldsymbol{\theta}(s)\|^2 ds \leq K(t)(h^4+h^2H^4t^{-1}+H^{8-2\ell}t^{-1}). \quad (3.79)$$

Now, applying the triangle inequality with (3.79) and, Lemma 3.4 and Lemma 3.6, we find our main result:

Theorem 3.3. *Suppose the hypothesis of Theorem 3.1 holds. Then, with* $\mathbf{u}_h(0) = P_h\mathbf{u}_0 \in \mathbf{J}_h$*, where* $\mathbf{u}_0 \in \mathbf{J}_1$*, the following error bounds hold for any* $t > 0$

$$\|(\mathbf{u}(t) - \mathbf{u}_h(t)\| \leq K(t)(h^2 t^{-1/2} + H^{4-\ell} t^{-1}),$$
$$\|\nabla(\mathbf{u}(t) - \mathbf{u}_h(t))\| \leq K(t)(h t^{-1/2} + H^{3-\ell} t^{-1}),$$

where $\ell > 0$ *is arbitrary small and* $K(t) = Ce^{Ct}$*. Under the smallness condition (3.11), the above estimates are valid uniformly in time, that is,* $K(t) = C$*.*

3.3.3 Optimal Pressure Error Estimates

We first find the pressure error estimate for step II. First, we split $p - p_h^*$ as

$$\|p - p_h^*\| \leq \|p - j_h p\| + \|j_h p - p_h^*\|. \tag{3.80}$$

using (**B2**), we can write the following

$$\begin{aligned}\|j_h p - p_h^*\|_{L^2/N_h} &\leq C \sup_{\mathbf{v}_h \in \mathbf{H}_h \setminus \{0\}} \left\{ \frac{|(j_h p - p_h^*, \nabla \cdot \mathbf{v}_h)|}{\|\nabla \mathbf{v}_h\|} \right\} \\ &\leq C\Bigg(\|j_h p - p\| + \sup_{\mathbf{v}_h \in \mathbf{H}_h \setminus \{0\}} \left\{ \frac{|(p - p_h^*, \nabla \cdot \mathbf{v}_h)|}{\|\nabla \mathbf{v}_h\|} \right\} \Bigg).\end{aligned} \tag{3.81}$$

The first term of the above inequality can be evaluated by using (**B1**). For the remaining term, we subtract (3.2) from (1.8), then use the "Cauchy-Schwarz inequality" with Lemma 3.2 to find

$$\begin{aligned}(p - p_h^*, \nabla \cdot \mathbf{v}_h) =&(\mathbf{e}_t^*, \mathbf{v}_h) + \mu a(\mathbf{e}^*, \mathbf{v}_h) + b(\mathbf{u}_H, \mathbf{e}^*, \mathbf{v}_h) + b(\mathbf{e}^*, \mathbf{u}_H, \mathbf{v}_h) \\ &+ \int_0^t \beta(t-s)\ a(\mathbf{e}^*(s), \mathbf{v}_h)\ ds + b(\mathbf{e}_H, \mathbf{e}_H, \mathbf{v}_h) \\ \leq& C\Big(\|\mathbf{e}_t^*\|_{-1;h} + \mu\|\nabla \mathbf{e}^*\| + \|\nabla \mathbf{u}_H\|\|\nabla \mathbf{e}^*\| + \|\mathbf{e}_H\|^{1-\ell}\|\nabla \mathbf{e}_H\|^{1+\ell} \\ &+ \int_0^t \beta(t-s)\|\nabla \mathbf{e}^*(s)\|\ ds\Big)\ \|\nabla \mathbf{v}_h\|.\end{aligned} \tag{3.82}$$

For the discrete negative norm, we have

$$\|\mathbf{e}_t^*\|_{-1;h} = \sup\left\{ \frac{\langle \mathbf{e}_t^*, \mathbf{v}_h \rangle}{\|\nabla \mathbf{v}_h\|} : \mathbf{v}_h \in \mathbf{H}_h, \mathbf{v}_h \neq 0 \right\} \leq \|\mathbf{e}_t^*\|_{-1}. \tag{3.83}$$

We use similar technique as in [63, Lemma 6.2, page 348] to bound the negative norm.

Lemma 3.12. *For* $0 < t < T$*, the following negative norm estimate holds:*

$$\|\mathbf{e}_t^*\|_{-1} \leq K(t)(h t^{-1/2} + H^{3-\ell} t^{-1}).$$

Proof. For any $\boldsymbol{\psi} \in \mathbf{H}_0^1$, a use of the properties of P_h and (3.9) with $\mathbf{v}_h = P_h\boldsymbol{\psi}$ yield

$$\begin{aligned}
(\mathbf{e}_t^*, \boldsymbol{\psi}) &= (\mathbf{e}_t^*, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) + (\mathbf{e}_t^*, P_h\boldsymbol{\psi}) \\
&= (\mathbf{e}_t^*, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) - \mu a(\mathbf{e}^*, P_h\boldsymbol{\psi}) - b(\mathbf{e}^*, \mathbf{u}_H, P_h\boldsymbol{\psi}) - b(\mathbf{u}_H, \mathbf{e}^*, P_h\boldsymbol{\psi}) \\
&\quad - \int_0^t \beta(t-s)\, a(\mathbf{e}^*(s), P_h\boldsymbol{\psi})\, ds + (p, \nabla \cdot P_h\boldsymbol{\psi}) + b(\mathbf{e}_H, \mathbf{e}_H, P_h\boldsymbol{\psi}). \qquad (3.84)
\end{aligned}$$

We use the approximation property of P_h to bound the following as

$$(\mathbf{e}_t^*, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) = (\mathbf{u}_t^*, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) \le Ch\|\mathbf{u}_t^*\|\|\nabla\boldsymbol{\psi}\|. \qquad (3.85)$$

Also, using H^1-stability of P_h and the "discrete incompressibility condition", we find that

$$(p, \nabla \cdot P_h\boldsymbol{\psi}) \le (p - j_h p, \nabla \cdot P_h\boldsymbol{\psi}) \le Ch\|\nabla p\|\|\nabla\boldsymbol{\psi}\|. \qquad (3.86)$$

Apply Lemma 3.2 with the "Cauchy-Schwarz inequality" to deduce

$$\begin{aligned}
&|b(\mathbf{e}_H, \mathbf{e}_H, P_h\boldsymbol{\psi}) + b(\mathbf{e}^*, \mathbf{u}_H, P_h\boldsymbol{\psi}) + b(\mathbf{u}_H, \mathbf{e}^*, P_h\boldsymbol{\psi})| \\
&\qquad \le C(\|\nabla\mathbf{e}^*\|\|\nabla\mathbf{u}_H\| + \|\mathbf{e}_H\|^{1-\ell}\|\nabla\mathbf{e}_H\|^{1+\ell})\|\nabla\boldsymbol{\psi}\|. \qquad (3.87)
\end{aligned}$$

Incorporate (3.85)-(3.87) in (3.84) and use $\|\nabla\mathbf{u}_H\| \le C$ to obtain

$$\begin{aligned}
(\mathbf{e}_t^*, \boldsymbol{\psi}) &\le C\Big(h\|\mathbf{u}_t^*\| + \|\nabla\mathbf{e}^*\| + \|\nabla\mathbf{e}^*\|\|\nabla\mathbf{u}_H\| + \|\mathbf{e}_H\|^{1-\ell}\|\nabla\mathbf{e}_H\|^{1+\ell} + h\|\nabla p\| \\
&\quad - \int_0^t \beta(t-s)\|\nabla\mathbf{e}^*\|\, ds\Big)\, \|\nabla\boldsymbol{\psi}\| \\
&\le C\Big(h\|\mathbf{u}_t^*\| + \|\nabla\mathbf{e}^*\| + \|\mathbf{e}_H\|^{1-\ell}\|\nabla\mathbf{e}_H\|^{1+\ell} + h\|\nabla p\| \\
&\quad - \int_0^t \beta(t-s)\|\nabla\mathbf{e}^*\|\, ds\Big)\, \|\nabla\boldsymbol{\psi}\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathbf{e}_t^*\|_{-1} &\le \sup\Big\{\frac{\langle\mathbf{e}_t^*, \mathbf{v}\rangle}{\|\nabla\mathbf{v}\|} : \mathbf{v} \in \mathbf{H}_0^1, \mathbf{v} \ne 0\Big\} \\
&\le C(h(\|\mathbf{u}_t^*\| + \|\nabla p\|) + \|\nabla\mathbf{e}^*\| + \|\mathbf{e}_H\|^{1-\ell}\|\nabla\mathbf{e}_H\|^{1+\ell} - \int_0^t \beta(t-s)\|\nabla\mathbf{e}^*\|\, ds).
\end{aligned}$$

Incorporating with Lemma 3.4 and Theorem 3.1, we concludes the remaining of the proof. □

Lemma 3.12 along with (3.82) in (3.80) results in the following:

Theorem 3.4. *Suppose the hypothesis of the Theorem* 3.1 *be satisfied. Then, the following result holds for all* $t > 0$*:*

$$\|(p - p_h^*)\|_{L^2/N_h} \leq K(t)(ht^{-1/2} + H^{3-\ell}t^{-1}),$$

where $K(t) = Ce^{Ct}$ *and* $\ell > 0$ *is arbitrary small. Under the smallness condition (3.11), the above estimates are valid uniformly in time, that is,* $K(t) = C$.

Proceeding in the similar way as above, we can obtain the pressure estimate for step 3, that is,

Theorem 3.5. *Suppose the hypothesis of the Theorem* 3.1 *be satisfied. Then, for any* $t > 0$*, the following result holds:*

$$\|(p - p_h)\|_{L^2/N_h} \leq K(t)(ht^{-1/2} + H^{3-\ell}t^{-1}).$$

where $K(t) = Ce^{Ct}$ *and* $\ell > 0$ *is arbitrary small. Under the smallness condition (3.11), the above estimates are valid uniformly in time, that is,* $K(t) = C$.

3.4 Fully Discrete Error Estimates

In this section we study the backward Euler (BE) time discretization scheme for the three step two-grid finite element approximation of Oldroyd model of order one (1.4)-(1.6). First we discretize the time interval $[0, T]$ in a uniform partition $\{t_n\}_{n=0}^N$ with equal time stem k and take $t_n = nk,\ 0 \leq n \leq N$. We approximate the time derivative term by $\partial_t \phi^n = (\phi^n - \phi^{n-1})/k$, where $\phi^n = \phi(t_n)$, ϕ is defined on $[0, T]$. The BE method applied to (3.4)-(3.6) results in the following algorithm:

Step I ("Solve the nonlinear system on a coarse grid $\mathcal{T}_H$"):

For any $t > 0$ and for all $\mathbf{v}_H \in \mathbf{J}_H$, seek $\mathbf{U}_H^n \in \mathbf{J}_H$ with $\mathbf{U}_H^0 = P_H \mathbf{u}_0$ satisfying

$$(\partial_t \mathbf{U}_H^n, \mathbf{v}_H) + \mu a(\mathbf{U}_H^n, \mathbf{v}_H) + b(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{v}_H) + a(q_r^n(\mathbf{U}_H), \mathbf{v}_H) = (\mathbf{f}^n, \mathbf{v}_H). \tag{3.88}$$

Step II ("Update on a finer mesh $\mathcal{T}_h$ with one Newton iteration"):

For any $t > 0$ and for all $\mathbf{v}_h \in \mathbf{J}_h$, seek $\mathbf{U}_h^{*n} \in \mathbf{J}_h$ with $\mathbf{U}_h^{*0} = P_h \mathbf{u}_0$ satisfying

$$\begin{aligned}(\partial_t \mathbf{U}_h^{*n}, \mathbf{v}_h) + \mu a(\mathbf{U}_h^{*n}, \mathbf{v}_h) + b(\mathbf{U}_h^{*n}, \mathbf{U}_H^n, \mathbf{v}_h) + b(\mathbf{U}_H^n, \mathbf{U}_h^{*n}, \mathbf{v}_h) + a(q_r^n(\mathbf{U}_h^*), \mathbf{v}_h) \\ = (\mathbf{f}^n, \mathbf{v}_h) + b(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{v}_h).\end{aligned} \tag{3.89}$$

Step III ("Correction on finer mesh $\mathcal{T}_h$"):

For any $t > 0$ and for all $\mathbf{v}_H \in \mathbf{J}_H$, seek $\mathbf{U}_h^n \in \mathbf{J}_h$ with $\mathbf{U}_h^0 = P_h\mathbf{u}_0$ satisfying

$$(\partial_t \mathbf{U}_h^n, \mathbf{v}_h) + \mu a(\mathbf{U}_h^n, \mathbf{v}_h) + b(\mathbf{U}_h^n, \mathbf{U}_H^n, \mathbf{v}_h) + b(\mathbf{U}_H^n, \mathbf{U}_h^n, \mathbf{v}_h) + a(q_r^n(\mathbf{U}_h), \mathbf{v}_h)$$
$$= (\mathbf{f}^n, \mathbf{v}_h) + b(\mathbf{U}_H^n, \mathbf{U}_h^{*n}, \mathbf{v}_h) + b(\mathbf{U}_h^{*n}, \mathbf{U}_H^n - \mathbf{U}_h^{*n}, \mathbf{v}_h). \tag{3.90}$$

where we have used the right rectangle rule to approximate the integral term:

$$q_r^n(\mathbf{v}) = k \sum_{j=1}^{n} \beta(t_n - t_j)\mathbf{v}^j \approx \int_0^t \beta(t_n - s)\mathbf{v}(s)ds.$$

We would like to note here that the integral term satisfies certain positivity property (see (1.18)).

3.4.1 *A Priori* Bounds

Below, we first present the *a priori* bounds for the fully discrete solutions for step I, II and III. We have proved the *a priori* estimates for step I in Chapter 2, Lemma 2.3 and 2.6 and Remark 2.3, so we only recall them.

Lemma 3.13. *(A priori estimate for $\mathbf{U}_H^n$) Let $\alpha_0 > 0$ be such that for $0 < \alpha < \alpha_0$*

$$1 + (\frac{\mu_1 \lambda_1}{2})k \geq e^{2\alpha k}. \tag{3.91}$$

Then, with $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu_1\lambda_1}{2}\}$ and $\mathbf{U}_H^0 = \mathbf{u}_{0H} = P_H\mathbf{u}_0$, where $\mathbf{u}_0 \in \mathbf{J}_1$, the following bounds hold:

$$\|\mathbf{U}_H^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\nabla \mathbf{U}_H^i\|^2 \leq C,$$
$$\tau_n^* \|\tilde{\Delta}_H \mathbf{U}_H^n\|^2 + \|\nabla \mathbf{U}_H^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\tilde{\Delta}_H \mathbf{U}_H^i\|^2 \leq C,$$

where $\tau_n^* = \min\{1, t_n\}$.

We prove below *a priori* bounds for $\mathbf{U}_h^{*n}$ and $\mathbf{U}_h^n$. The proof techniques are quite similar and easier than the proof of Lemma 3.13, so we only sketch the proofs.

Lemma 3.14. *(A priori estimate for $\mathbf{U}_h^{*n}$ and $\mathbf{U}_h^n$) Under the hypothesis of Lemma 3.13 with $\mathbf{U}_h^{*0} = \mathbf{u}_{0h}^* = P_h\mathbf{u}_0$ and $\mathbf{U}_h^0 = \mathbf{u}_{0h} = P_h\mathbf{u}_0$, the following bounds hold:*

$$\|\mathbf{U}^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\nabla \mathbf{U}^i\|^2 \leq C,$$

$$\tau_n^*\|\tilde{\Delta}_h \mathbf{U}^n\|^2 + \|\nabla \mathbf{U}^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^{n} e^{2\alpha t_i} \|\tilde{\Delta}_h \mathbf{U}^i\|^2 \le C,$$

with $\mathbf{U}^n = \mathbf{U}_h^{*n}$ *and* $\mathbf{U}_h^n$*, respectively.*

Proof. From (3.89) and (3.90), we observe that both the equations are linearized and differ by nonlinear terms. We here only give a sketch of the proof for $\mathbf{U}_h^{*n}$. For this, we choose $\mathbf{v}_h = \mathbf{U}_h^{*i}$ with $n = i$ in (3.89) and use the fact $(\partial_t \mathbf{U}_h^{*i}, \mathbf{U}_h^{*i}) \ge \frac{1}{2}\partial_t \|\mathbf{U}_h^{*i}\|^2$ with (3.7)(replace $\mathbf{u}_H$ by $\mathbf{U}_H^i$) to obtain

$$\frac{1}{2}\partial_t \|\mathbf{U}_h^{*i}\|^2 + \mu_1 \|\nabla \mathbf{U}_h^{*i}\|^2 + a(q_r^i(\mathbf{U}_h^*), \mathbf{U}_h^{*i}) \le (\mathbf{f}^i, \mathbf{U}_h^{*i}) + b(\mathbf{U}_H^i, \mathbf{U}_H^i, \mathbf{U}_h^{*i}). \tag{3.92}$$

After using the "Cauchy-Schwarz inequality", the "Young's inequality" and the "Poincaré inequality" with Lemma 3.2, we multiply by $ke^{2\alpha t_i}$ and take summation over $1 \le i \le n$ to obtain

$$\begin{aligned} k\sum_{i=1}^{n} e^{2\alpha t_i}\partial_t \|\mathbf{U}_h^{*i}\|^2 + \mu_1 k \sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla \mathbf{U}_h^{*i}\|^2 + 2k\sum_{i=1}^{n} e^{2\alpha t_i} a(q_r^i(\mathbf{U}_h^*), \mathbf{U}_h^{*i}) \\ \le C(\lambda_1, \mu_1) k \sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{f}^i\|^2 + C(\mu_1) k \sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla \mathbf{U}_H^i\|^4. \end{aligned} \tag{3.93}$$

Third term on the left of inequality (3.93) is positive, hence, we drop it. We rewrite the first term as

$$k\sum_{i=1}^{n} e^{2\alpha t_i}\partial_t \|\mathbf{U}_h^{*i}\|^2 \ge e^{2\alpha t_n}\|\mathbf{U}_h^{*n}\|^2 - \|\mathbf{U}_h^{*0}\|^2 - \Big(\frac{e^{2\alpha k}-1}{k\lambda_1}\Big) k \sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla \mathbf{U}_h^{*i}\|^2 \tag{3.94}$$

Then, we finally obtain

$$\begin{aligned} e^{2\alpha t_n}\|\mathbf{U}_h^{*n}\|^2 + \Big(\mu_1 - (\frac{e^{2\alpha k}-1}{k\lambda_1})\Big) k \sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla \mathbf{U}_h^{*i}\|^2 \\ \le \|\mathbf{U}_h^{*0}\|^2 + C(\lambda_1, \mu_1) k \sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{f}^i\|^2 + C(\mu_1) k \sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla \mathbf{U}_H^i\|^4. \end{aligned}$$

With $0, \alpha \le \min\{\alpha_0, \delta, \frac{\mu_1\lambda_1}{2}\}$, we have $1 + \frac{\mu_1\lambda_1}{2}k \ge e^{2\alpha k}$, which guarantees that $\big(\mu_1 - (\frac{e^{2\alpha k}-1}{k\lambda_1})\big) > 0$. Now, use Lemma 3.13 and multiply both side by $e^{-2\alpha t_n}$ to conclude the first proof.

Now substitute $\mathbf{v}_h = -\tilde{\Delta}_h \mathbf{U}_h^{*i}$ in (3.89) and use the fact $(\partial_t \mathbf{U}_h^{*i}, -\tilde{\Delta}_h \mathbf{U}_h^{*i}) \ge \frac{1}{2}\partial_t \|\nabla \mathbf{U}_h^{*i}\|^2$ with (3.7)(replace $\mathbf{u}_H$ by $\mathbf{U}_H^i$) to obtain

$$\frac{1}{2}\partial_t \|\nabla \mathbf{U}_h^{*i}\|^2 + \mu_1 \|\tilde{\Delta}_h \mathbf{U}_h^{*i}\|^2 + a(q_r^i(\mathbf{U}), -\tilde{\Delta}_h \mathbf{U}_h^{*i})$$

$$\leq (\mathbf{f}^i, -\tilde{\Delta}_h \mathbf{U}_h^{*i}) + b(\mathbf{U}_H^i, \mathbf{U}_H^i, -\tilde{\Delta}_h \mathbf{U}_h^{*i}). \tag{3.95}$$

The nonlinear term can be bounded using Lemma 3.2 and the "Young's inequality" as

$$\begin{aligned} |b(\mathbf{U}_H^i, \mathbf{U}_H^i, -\tilde{\Delta}_h \mathbf{U}_h^{*i})| &\leq C \|\nabla \mathbf{U}_H^i\| \|\tilde{\Delta}_h \mathbf{U}_H^i\| \|\tilde{\Delta}_h \mathbf{U}_h^{*i}\| \\ &\leq C(\mu_1) \|\nabla \mathbf{U}_H^i\|^2 \|\tilde{\Delta}_H \mathbf{U}_H^i\|^2 + \frac{\mu_1}{4} \|\tilde{\Delta}_h \mathbf{U}_h^{*i}\|^2. \end{aligned} \tag{3.96}$$

Inserting (3.96) in (3.95) and multiplying both side by $ke^{2\alpha t_i}$. Then, sum over $i = 1$ to n, we find that

$$\begin{aligned} e^{2\alpha t_n} \|\nabla \mathbf{U}_h^{*n}\|^2 + \Big(\mu_1 - (\frac{e^{2\alpha k} - 1}{k\lambda_1})\Big) k \sum_{i=1}^{n} e^{2\alpha t_i} \|\tilde{\Delta}_h \mathbf{U}_h^{*i}\|^2 + k \sum_{i=1}^{n} e^{2\alpha t_i} a(q_r^i(\mathbf{U}), -\tilde{\Delta}_h \mathbf{U}_h^{*i}) \\ \leq \|\nabla \mathbf{U}_h^{*0}\|^2 + Ck \sum_{i=1}^{n} e^{2\alpha t_i} \big(\|\mathbf{f}^i\|^2 + \|\nabla \mathbf{U}_H^i\|^2 \|\tilde{\Delta}_H \mathbf{U}_H^i\|^2\big). \end{aligned} \tag{3.97}$$

We drop the third term from the left side due to positivity. Finally, we multiply by $e^{-2\alpha t_n}$ and use Lemma 3.13 to concludes the remaining of the proof. □

Remark 3.9. *Since, the bounds in Lemma 3.13 are uniform in time, hence, all the bounds obtained in Lemma 3.14 are still uniform in time.*

3.4.2 Fully Discrete Error Estimates

Define $\mathbf{u}_H(t_n) = \mathbf{u}_H^n, \mathbf{u}_h^*(t_n) = \mathbf{u}_h^{*n}, \mathbf{u}_h(t_n) = \mathbf{u}_h^n$ and set $\mathbf{e}_H^n = \mathbf{U}_H^n - \mathbf{u}_H^n, \mathbf{e}_h^{*n} = \mathbf{U}_h^{*n} - \mathbf{u}_h^{*n}, \mathbf{e}_h^n = \mathbf{U}_h^n - \mathbf{u}_h^n$. We also define a few notations below for ease of presentation.

$$R_\omega^i(\mathbf{v}) = (\mathbf{u}_{\omega t}^i, \mathbf{v}) - (\partial_t \mathbf{u}_\omega^i, \mathbf{v}), \tag{3.98}$$

$$E_\omega^i(\mathbf{v}) = \int_0^{t_i} \beta(t_i - s) a(\mathbf{u}_\omega(s), \mathbf{v}) ds - a(q_r^i(\mathbf{u}_\omega), \mathbf{v}). \tag{3.99}$$

For the error equations, we consider (3.4)-(3.6) at $t = t_n$ and subtract from (3.88)-(3.90), respectively, to obtain the following:

Step 1: For all $\mathbf{v}_H \in \mathbf{J}_H$

$$\begin{aligned} (\partial_t \mathbf{e}_H^n, \mathbf{v}_H) + \mu a(\mathbf{e}_H^n, \mathbf{v}_H) + b(\mathbf{u}_H^n, \mathbf{e}_H^n, \mathbf{v}_H) + b(\mathbf{e}_H^n, \mathbf{u}_H^n, \mathbf{v}_H) + a(q_r^n(\mathbf{e}_H), \mathbf{v}_H) \\ = R_H^n(\mathbf{v}_H) + \Lambda_H^n(\mathbf{v}_H) + E_H^n(\mathbf{v}_H), \end{aligned} \tag{3.100}$$

where $\Lambda_H^n(\mathbf{v}_H) = b(\mathbf{u}_H^n, \mathbf{e}_H^n, \mathbf{v}_H) - b(\mathbf{U}_H^n, \mathbf{e}_H^n, \mathbf{v}_H)$.

Step 2: For all $\mathbf{v}_h \in \mathbf{J}_h$

$$(\partial_t \mathbf{e}_h^{*n}, \mathbf{v}_h) + \mu a(\mathbf{e}_h^{*n}, \mathbf{v}_h) + b(\mathbf{e}_h^{*n}, \mathbf{u}_H^n, \mathbf{v}_h) + b(\mathbf{u}_H^n, \mathbf{e}_h^{*n}, \mathbf{v}_h) + a(q_r^n(\mathbf{e}_h^*), \mathbf{v}_h)$$

$$= R_h^{*n}(\mathbf{v}_h) + \Lambda_h^{*n}(\mathbf{v}_h) + E_h^{*n}(\mathbf{v}_h).$$

where $\Lambda_h^{*n}(\mathbf{v}_h) = -b(\mathbf{U}_h^{*n}, \mathbf{e}_H^n, \mathbf{v}_h) - b(\mathbf{e}_H^n, \mathbf{U}_h^{*n}, \mathbf{v}_h) + b(\mathbf{U}_H^n, \mathbf{e}_H^n, \mathbf{v}_h) + b(\mathbf{e}_H^n, \mathbf{u}_H^n, \mathbf{v}_h)$.

Step 3: For all $\mathbf{v}_h \in \mathbf{J}_h$

$$\begin{aligned}(\partial_t \mathbf{e}_h^n, \mathbf{v}_h) + \mu a(\mathbf{e}_h^n, \mathbf{v}_h) + b(\mathbf{e}_h^n, \mathbf{u}_H^n, \mathbf{v}_h) + b(\mathbf{u}_H^n, \mathbf{e}_h^n, \mathbf{v}_h) + a(q_r^n(\mathbf{e}_h), \mathbf{v}_h)\\ = R_h^n(\mathbf{v}_h) + \Lambda_h(\mathbf{v}_h) + E_h^n(\mathbf{v}_h),\end{aligned}$$

where

$$\begin{aligned}\Lambda_h(\mathbf{v}_h) = -b(\mathbf{U}_h^n, \mathbf{e}_H^n, \mathbf{v}_h) - b(\mathbf{e}_H^n, \mathbf{U}_h^n, \mathbf{v}_h) + b(\mathbf{U}_H^n, \mathbf{U}_h^{*n}, \mathbf{v}_h) - b(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \mathbf{v}_h)\\ + b(\mathbf{U}_h^{*n}, \mathbf{U}_H^n - \mathbf{U}_h^{*n}, \mathbf{v}_h) - b(\mathbf{u}_h^{*n}, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \mathbf{v}_h).\end{aligned}$$

Error Estimate in Step I

From (3.98),(3.99) and (3.100), we make the following observations.

$$\begin{aligned}R_H^n(\mathbf{v}_H) &= (\mathbf{u}_{Ht}^n, \mathbf{v}_H) - (\partial_t \mathbf{u}_H^n, \mathbf{v}_H) = (\mathbf{u}_{Ht}^n, \mathbf{v}_H) - \frac{1}{k}\int_{t_{n-1}}^{t_n} (\mathbf{u}_{Hs}, \mathbf{v}_H)\, ds\\ &= \frac{1}{k}\int_{t_{n-1}}^{t_n} (t - t_{n-1})(\mathbf{u}_{Hss}, \mathbf{v}_H)\, ds, \qquad (3.101)\\ E_H^n(\mathbf{v}_H) &= \int_0^{t_n} \beta(t-s)\; a(\mathbf{u}_H(s), \mathbf{v}_H)\, ds - k\sum_{i=1}^n \beta(t_n - t_i)\; a(\mathbf{u}_H^i, \mathbf{v}_H) \qquad (3.102)\\ &\le K\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (s - t_{i-1})\big(\beta_s(t_n - s)\; a(\mathbf{u}_H(s), \mathbf{v}_H) + \beta(t_n - s) a(\mathbf{u}_{Hs}(s), \mathbf{v}_H)\big)\, ds.\end{aligned}$$

We have already proved the $L^\infty(\mathbf{L}^2)$ for the velocity for Step I in Chapter 2. So, we recollect the necessary estimates from Chapter 2, which will be used for further analysis. From Remarks 2.8 and 2.9 and Lemmas 2.12 and 2.15, we find the followings:

Lemma 3.15. *Suppose the conditions* (**A1**)-(**A2**) *and* (**B1**)-(**B2**) *hold true. Assume* $\alpha_0 > 0$ *such that for* $0 < \alpha < \alpha_0$, *(3.91) is satisfied. Then, with* $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu\lambda_1}{2}\}$, *there exist some positive constant* K_n, *that depends on* T, *the followings hold for* $r = \{-1, 0, 1\}$

$$\|\mathbf{e}_H^n\|_r^2 + ke^{-2\alpha t_n}\sum_{i=1}^n e^{2\alpha t_i}\|\mathbf{e}_H^i\|_{r+1}^2 \le K_n k^{1-r}(1 + \log\frac{1}{k})^{1-r^2},$$

$$\tau_n^*\|\mathbf{e}_H^n\|^2 + ke^{-2\alpha t_n}\sum_{i=1}^n \sigma_i\|\partial_t \mathbf{e}_H^i\|_{-1}^2 \le K_n k^2(1 + \log\frac{1}{k}),$$

where $\sigma_i = \tau_i^* e^{2\alpha t_i}$ *and* $\tau_i^* = \min\{1, t_i\}$. *Moreover, under the uniqueness condition (3.11) the estimates are uniform in time that is,* $K_n = K$.

We are now going to obtain the $\mathbf{H}^1$-velocity error and L^2-pressure error.

Lemma 3.16. *Let the assumption of Lemma 3.15 be satisfied. Then, for* $0 < n < N$*, the following holds*

$$\tau_n^*\|\nabla \mathbf{e}_H^n\|^2 + ke^{-2\alpha t_n}\sum_{i=1}^{n}\sigma_i\|\partial_t \mathbf{e}_H^i\|^2 \le K_n k(1+\log\frac{1}{k}),$$

Proof. Take $n = i$ and $\mathbf{v}_h = \sigma_i\partial_t \mathbf{e}_H^i$ in (3.100) and use (3.7). Then multiply by k and take summation over $1 \le i \le n$. Finally, using the following fact

$$k\sum_{i=1}^{n}\sigma_i\partial_t\|\mathbf{e}_H^i\|^2 \ge \sigma_n\|\nabla \mathbf{e}_H^n\|^2 - k\sum_{i=1}^{n-1}e^{2\alpha t_i}\|\nabla \mathbf{e}_H^i\|^2,$$

we obtain

$$\begin{aligned}\mu_1\sigma_n\|\nabla \mathbf{e}_H^n\|^2 + 2k\sum_{i=1}^{n}\sigma_i\|\partial_t \mathbf{e}_H^i\|^2 \le{}& k\sum_{i=1}^{n-1}e^{2\alpha t_i}\|\nabla \mathbf{e}_H^i\|^2 - 2k\sum_{i=1}^{n}a(q_r^i(\mathbf{e}_H),\sigma_i\partial_t \mathbf{e}_H^i)\\ &+ 2k\sum_{i=1}^{n}\sigma_i\big(R_H^i(\partial_t \mathbf{e}_H^i) + \Lambda_H^i(\partial_t \mathbf{e}_H^i) + E_H^i(\partial_t \mathbf{e}_H^i)\big). \qquad (3.103)\end{aligned}$$

We use (1.21) and (1.7) with the Lemmas 3.15 with $\widehat{\mathbf{v}}^n = k\sum_{i=1}^{n}\mathbf{v}^i$ to bound

$$\begin{aligned}&k\sum_{i=1}^{n}a(q_r^i(\mathbf{e}_H),\sigma_i\partial_t \mathbf{e}_H^i)\\ &= \sigma_n a(q_r^n(\mathbf{e}_H),\mathbf{e}_H^n) - k\sum_{i=1}^{n}(\partial_t\sigma_i)a(q_r^i(\mathbf{e}_H),\mathbf{e}_H^i) - k\sum_{i=1}^{n}\sigma_i a(\partial_t q_r^i(\mathbf{e}_H),\mathbf{e}_H^i)\\ &\le \frac{\mu_1}{4}\sigma_n\|\nabla \mathbf{e}_H^n\|^2 + K\sigma_n\|\nabla \hat{\mathbf{e}}_H^n\|^2 + k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla \mathbf{e}_H^i\|^2 + k\sum_{i=1}^{n}e^{2\alpha t_i}\|\nabla \hat{\mathbf{e}}_H^i\|^2\\ &\le Kk(1+\log\frac{1}{k})e^{2\alpha t_n} + \frac{\mu_1}{4}\sigma_n\|\nabla \mathbf{e}_H^n\|^2 + K\sigma_n\|\nabla \hat{\mathbf{e}}_H^n\|^2. \qquad (3.104)\end{aligned}$$

From (3.101), we can find the following bound

$$\begin{aligned}k\sum_{i=1}^{n}\sigma_i R_H^i(\partial_t \mathbf{e}_H^i) \le{}& k\sum_{i=1}^{n}\sigma_i\frac{1}{k}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|\mathbf{u}_{Hss}\|ds\|\partial_t \mathbf{e}_H^i\|\\ \le{}& Kk\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}\frac{\tau^i}{s^2}ds\Big)\Big(\int_{t_{i-1}}^{t_i}e^{2\alpha t_i}s^2\|\mathbf{u}_{Hss}\|^2ds\Big) + \frac{1}{2}k\sum_{i=1}^{n}\sigma_i\|\partial_t \mathbf{e}_H^i\|^2\\ \le{}& Kk(1+\log\frac{1}{k})e^{2\alpha t_n} + \frac{1}{2}k\sum_{i=1}^{n}\sigma_i\|\partial_t \mathbf{e}_H^i\|^2. \qquad (3.105)\end{aligned}$$

A use of (1.21) gives

$$k\sum_{i=1}^{n}\sigma_i E_H^i(\partial_t \mathbf{e}_H^i) = \sigma_n E_H^n(\mathbf{e}_H^n) - k\sum_{i=1}^{n}(\partial_t\sigma_i)E_H^i(\mathbf{e}_H^i) - k\sum_{i=1}^{n}\sigma_i\partial_t E_H^i(\mathbf{e}_H^i)$$

We apply the "Young's inequality" with (3.102) to bound the following:

$$\sigma_n E_H^n(\mathbf{e}_H^n) \leq K\sigma_n\Big(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (s-t_{i-1})\beta(t_n-s)\{\delta\|\nabla \mathbf{u}_H\| + \|\nabla \mathbf{u}_{Hs}\|\}ds\Big)\|\nabla \mathbf{e}_H^n\|^2$$
$$\leq Kk(1+\log\frac{1}{k})e^{2\alpha t_n} + \frac{\mu_1}{4}\sigma_n\|\nabla \mathbf{e}_H^n\|^2, \tag{3.106}$$

Similarly, we can obtain

$$k\sum_{i=1}^n(\partial_t\sigma_i)E_H^i(\mathbf{e}_H^i) + k\sum_{i=1}^n \sigma_i\partial_t E_H^i(\mathbf{e}_H^i) \leq Kk(1+\log\frac{1}{k})e^{2\alpha t_n}. \tag{3.107}$$

We use Lemma 3.2 with Lemmas 3.13 and 3.15 to bound the nonlinear terms as

$$k\sum_{i=1}^n \sigma_i\Lambda_H^i(\partial_t \mathbf{e}_H^i) \leq k\sum_{i=1}^n \sigma_i(\|\tilde{\Delta}_H \mathbf{u}_H^i\|^2 + \|\tilde{\Delta}_H \mathbf{U}_H^i\|^2)\|\nabla \mathbf{e}_H^i\|^2 + \frac{1}{2}k\sum_{i=1}^n \sigma_i\|\partial_t \mathbf{e}_H^i\|^2$$
$$\leq k(1+\log\frac{1}{k})e^{2\alpha t_n} + \frac{1}{2}k\sum_{i=1}^n \sigma_i\|\partial_t \mathbf{e}_H^i\|^2. \tag{3.108}$$

We use (3.104)-(3.108) in (3.103). Finally, we multiply both side by $e^{2\alpha t_n}$ and under the assumption, which is proved below

$$\tau_n^*\|\nabla \hat{\mathbf{e}}_H^n\|^2 \leq K_n k(1+\log\frac{1}{k}), \tag{3.109}$$

we conclude the remaining of the proof. □

We are now left with the proof (3.109). For this, we multiply (3.100) by k and take summation over $1 \leq i \leq n$ and use the similar fact (2.85) to obtain

$$(\partial_t \hat{\mathbf{e}}_H^n, \mathbf{v}_H) + \mu a(\hat{\mathbf{e}}_H^n, \mathbf{v}_H) + a(q_r^n(\hat{\mathbf{e}}_H), \mathbf{v}_H) + k\sum_{i=1}^n b(\mathbf{e}_H^i, \mathbf{u}_H^i, \mathbf{v}_H) + b(\mathbf{u}_H^i, \mathbf{e}_H^i, \mathbf{v}_H)$$
$$= k\sum_{i=1}^n (R_H^i(\mathbf{v}_H) + \Lambda_H^i(\mathbf{v}_H) + E_H^i(\mathbf{v}_H)). \tag{3.110}$$

Lemma 3.17. *Let the assumption of Lemma 3.15 be satisfied. Then, for* $0 < n < N$*, the following holds*

$$\tau_n^*\|\nabla \hat{\mathbf{e}}_H^n\|^2 + ke^{-2\alpha t_n}\sum_{i=1}^n \sigma_i\|\partial_t \hat{\mathbf{e}}_H^i\|^2 \leq K_n k(1+\log\frac{1}{k}).$$

Proof. For $n = i$ and choose $\mathbf{v}_H = \sigma_i\partial_t \hat{\mathbf{e}}_H^i$ in (3.110) and similar to above lemma, we conclude the proof. □

We will use the following estimate to obtain the pressure error.

Lemma 3.18. *Let the assumption of Lemma 3.15 be satisfied. Then, for* $0 < n < N$, *the following holds*

$$\tau_n^* \|\partial_t \mathbf{e}_H^n\|_{-1}^2 \leq Kk(1 + \log \frac{1}{k}).$$

Proof. Take $n = i$ and $\mathbf{v}_h = \sigma_i(-\tilde{\Delta}_H)^{-1}\partial_t \mathbf{e}_H^i$ in (3.100) and use (3.7) with $\mathbf{u}_H$ replaced by $\mathbf{u}_H^i$, then we arrive at

$$2\sigma_i\|\partial_t \mathbf{e}_H^i\|_{-1}^2 + \mu_1\sigma_i\partial_t\|\mathbf{e}_H^i\|^2 + 2(q_r^i(\mathbf{e}_H), \sigma_i\partial_t\mathbf{e}_H^i) \leq 2R_H^i(\sigma_i(-\tilde{\Delta}_H)^{-1}\partial_t\mathbf{e}_H^i)$$
$$+2\Lambda_H(\sigma_i(-\tilde{\Delta}_H)^{-1}\partial_t\mathbf{e}_H^i) + 2E_H^i(\sigma_i(-\tilde{\Delta}_H)^{-1}\partial_t\mathbf{e}_H^i).$$

We apply the "Cauchy-Schwarz inequality" to obtain

$$\|\partial_t \mathbf{e}_H^n\|_{-1}^2 \leq C\Bigg(\|\nabla \mathbf{e}_H^n\| + k\sum_{i=1}^{n} \beta(t_n - t_i)\|\nabla \mathbf{e}_H^i\| + \frac{1}{k}\int_{t_{n-1}}^{t_n} (s - t_{n-1})\|\mathbf{u}_{Hss}\|_{-1} ds$$
$$+ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (s - t_{i-1})\beta(t_i - s)(\delta\|\nabla \mathbf{u}_H\| + \|\nabla \mathbf{u}_{Hs})ds\Bigg) \|\partial_t \mathbf{e}_H^n\|_{-1}.$$

Incorporating with the "Young's inequality" and Lemma 3.16, we conclude the remaining of the proof. □

To derive pressure errors, we first recall the mixed finite element formulation of semidiscrete approximations (3.1) as: Find $\mathbf{u}_H(t) \in \mathbf{H}_H$ and , $p_H(t) \in L_H$ satisfying

$$(\mathbf{u}_{Ht}, \mathbf{v}_H) + \mu a(\mathbf{u}_H, \mathbf{v}_H) + b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{v}_H) + \int_0^t \beta(t - s)a(\mathbf{u}_H(s), \mathbf{v}_H)ds$$
$$= (p_H, \nabla \cdot \mathbf{v}_H) + (\mathbf{f}, \mathbf{v}_H), \quad \forall \mathbf{v}_H \in \mathbf{H}_H, \qquad (3.111)$$

and $(\nabla \cdot \mathbf{u}_H, \chi_H) = 0,\ \forall \chi \in L_H$. Also the mixed formulation of fully discrete approximations (3.100) is as below: Find $(\mathbf{U}_H^n, P_H^n) \in \mathbf{H}_H \times L_H$ such that

$$(\partial_t \mathbf{U}_H^n, \mathbf{v}_H) + \mu a(\mathbf{U}_H^n, \mathbf{v}_H) + b(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{v}_H) + a(q_r^n(\mathbf{U}_H), \mathbf{v}_H)$$
$$= (P_H^n, \nabla \cdot \mathbf{v}_H) + (\mathbf{f}^n, \mathbf{v}_H), \quad \forall \mathbf{v}_H \in \mathbf{H}_H, \qquad (3.112)$$

with $(\nabla \cdot \mathbf{U}_H^n, \chi_H) = 0,\ \ \forall \chi \in L_H$.
Take $t = t_n$ in (3.111) and subtract it from (3.112) to obtain

$$(P_H^n - p_H^n, \nabla \cdot \mathbf{v}_H) = (\partial_t \mathbf{e}_H^n, \mathbf{v}_H) + \mu a(\mathbf{e}_H^n, \mathbf{v}_H) + b(\mathbf{u}_H^n, \mathbf{e}_H^n, \mathbf{v}_H) + b(\mathbf{e}_H^n, \mathbf{u}_H^n, \mathbf{v}_H)$$
$$+a(q_r^n(\mathbf{e}_H), \mathbf{v}_H) - (R_H^n(\mathbf{v}_H) + \Lambda_H(\mathbf{v}_H) + E_H^n(\mathbf{v}_H)),$$

where $p_H^n = p_H(t_n)$. Now proceed similar as semidiscrete pressure error estimate along with the "Cauchy-Schwarz inequality" and Lemma 3.16, we derive the following result:

Lemma 3.19. *Suppose the assumption of Lemma 3.15 be satisfied. Then, for* $0 < n < N$*, the following holds*

$$\tau_n^* \|P_H^n - p_H^n\|^2 \leq K_n k (1 + \log \frac{1}{k}).$$

Error Estimates in Step 2 and 3

We present below the optimal error estimates for steps II and III. We refrain from the proofs of the estimates involving steps 2 and 3, since they are quite similar to step 1. If we compare the error equations at the three steps, they differ only in terms of the nonlinear terms. And these terms can be handled by means of Lemma 3.2.

Lemma 3.20. *Let the assumption of Lemma 3.15 be satisfied. Then, for* $0 < n < N$*, the following holds*

$$\tau_n^* \|\mathbf{e}_h^{*n}\|^2 + k e^{-2\alpha t_n} \sum_{i=1}^{n} \sigma_i \|\partial_t \mathbf{e}^i\|_{-1}^2 \leq K_n k^2 (1 + \log \frac{1}{k}),$$
$$\tau_n^* \|\nabla \mathbf{e}_h^{*n}\|^2 + k e^{-2\alpha t_n} \sum_{i=1}^{n} \sigma_i \|\partial_t \mathbf{e}^i\|^2 \leq K_n k (1 + \log \frac{1}{k}),$$
$$\tau_n^* \|P_h^{*n} - p_h^{*n}\|^2 \leq K_n k (1 + \log \frac{1}{k}).$$

Lemma 3.21. *Let the assumption of Lemma 3.15 be satisfied. Then, for* $0 < n < N$*, the following holds*

$$\tau_n^* \|\mathbf{e}_h^{n}\|^2 + k e^{-2\alpha t_n} \sum_{i=1}^{n} \sigma_i \|\partial_t \mathbf{e}_h^i\|_{-1}^2 \leq K_n k^2 (1 + \log \frac{1}{k}),$$
$$\tau_n^* \|\nabla \mathbf{e}_h^{n}\|^2 + k e^{-2\alpha t_n} \sum_{i=1}^{n} \sigma_i \|\partial_t \mathbf{e}_h^i\|^2 \leq K_n k (1 + \log \frac{1}{k}),$$
$$\tau_n^* \|P_h^{n} - p_h^{n}\|^2 \leq K_n k (1 + \log \frac{1}{k}).$$

Now from the Theorem 3.3 and Lemma 3.21, we conclude our main results of this chapter:

Theorem 3.6. *Suppose the assumptions of Theorem 3.3 and Lemma 3.15 be satisfied. Then, for* $0 < n < N$*, the following holds:*

$$\|\mathbf{u}(t_n) - \mathbf{U}_h^n\| \leq K_n (h^2 t_n^{-1/2} + H^{4-\ell} t_n^{-1} + k (1 + \log \frac{1}{k})^{\frac{1}{2}} t_n^{-1/2}),$$
$$\|\nabla (\mathbf{u}(t_n) - \mathbf{U}_h^n)\| \leq K_n (h t_n^{-1/2} + H^{3-\ell} t_n^{-1} + k^{\frac{1}{2}} (1 + \log \frac{1}{k})^{\frac{1}{2}} t_n^{-1/2}),$$

$$\|p(t_n) - P_h^n)\| \leq K_n(ht_n^{-1/2} + H^{3-\ell}t_n^{-1} + k^{\frac{1}{2}}(1 + \log\frac{1}{k})^{\frac{1}{2}}t_n^{-1/2}).$$

Moreover, under the uniqueness condition (3.11) the above estimates are uniform in time, that is, $K_n = K$.

3.5 Numerical Experiments

In this section, we present some numerical experiments that conforms with the results from the previous section, namely, verify the order of convergence of the fully discrete errors. For simplicity, we will use examples with known solutions. All the numerical computations have been done in MATLAB.

We consider the Oldroyd model of order one subject to zero boundary conditions. We approximate the equation using (P_1b, P_1) over a regular triangulation of Ω. We take $[0,1] \times [0,1]$ as the domain which is partitioned into triangles with size $h = 2^{-i}$, $i = 2, 3, \ldots, 6$. To verify the theoretical result, we consider the following examples:

Example 3.1. *In our first example, we consider the forcing term $f(x,t)$ so as to get the following exact solutions*

$$u_1(x,t) = 2e^t x^2(x-1)^2 y(y-1)(2y-1),$$
$$u_2(x,t) = -2e^t x(x-1)(2x-1)y^2(y-1)^2,$$
$$p(x,t) = e^t y.$$

Table 3.1: Numerical results for the three step two-grid method with $k = h^2, \mu = 1, \gamma = 0.1, \delta = 0.1$ at time $T = 1$ for Example 3.1

h	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{U}^n - \mathbf{u}(t_n)\|_{\mathbf{H}^1}$	C.R.	$\|P^n - p(t_n)\|_{L^2}$	C.R.
1/4	0.01022974		0.10009952		0.03204744	
1/8	0.00332044	1.6233	0.05339255	0.9067	0.01724230	0.8943
1/16	0.00088486	1.9079	0.02700925	0.9832	0.00536430	1.6845
1/32	0.00022110	2.0007	0.01353995	0.9962	0.00146472	1.8728
1/64	0.00005159	2.0995	0.00677458	0.9990	0.00039991	1.8729

Example 3.2. *In our second example, we consider the forcing term $f(x,t)$ so as to get the following exact solutions*

$$u_1(x,t) = 5e^t x^{\frac{5}{2}}(x-1)^2 y^{\frac{3}{2}}(y-1)(9y-5),$$

$$u_2(x,t) = -5e^t x^{\frac{3}{2}}(x-1)(9x-5)y^{\frac{5}{2}}(y-1)^2,$$
$$p(x,t) = e^t y.$$

Table 3.2: Numerical results for the three step two-grid method with $\mu = 2, \gamma = 0.1, \delta = 0.1, k = \mathcal{O}(h^2)$ at time $T = 1$ for Example 3.2

h	$\\|\mathbf{U}^n - \mathbf{u}(t_n)\\|_{\mathbf{L}^2}$	C.R.	$\\|\mathbf{U}^n - \mathbf{u}(t_n)\\|_{\mathbf{H}^1}$	C.R.	$\\|P^n - p(t_n)\\|_{L^2}$	C.R.
1/4	0.04956891		0.54851138		0.19727316	
1/8	0.01527122	1.6986	0.30612031	0.8414	0.10400004	0.9236
1/16	0.00412764	1.8874	0.16005459	0.9355	0.03013737	1.7870
1/32	0.00107618	1.9394	0.08206322	0.9638	0.00811187	1.8934
1/64	0.00027926	1.9463	0.04182879	0.9722	0.00318134	1.3504

The theoretical analysis shows that the rates of convergence are of $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$ for the velocity in $\mathbf{L}^2$-norm and $\mathbf{H}^1$-norm, respectively. And the rate of convergence for the pressure in L^2 norm is $\mathcal{O}(h)$. We take the time step $k = \mathcal{O}(h^2)$ and the final $T = 1$ for our experiments. Tables 3.1 and 3.2 give the numerical results for example 3.1 and 3.2, respectively. The optimal rates of convergence derived in Theorem 3.6 are supported by these numerical findings. In Figures 3.1 and 3.2 below we present the error graphs of Examples 3.1 and 3.2, respectively.

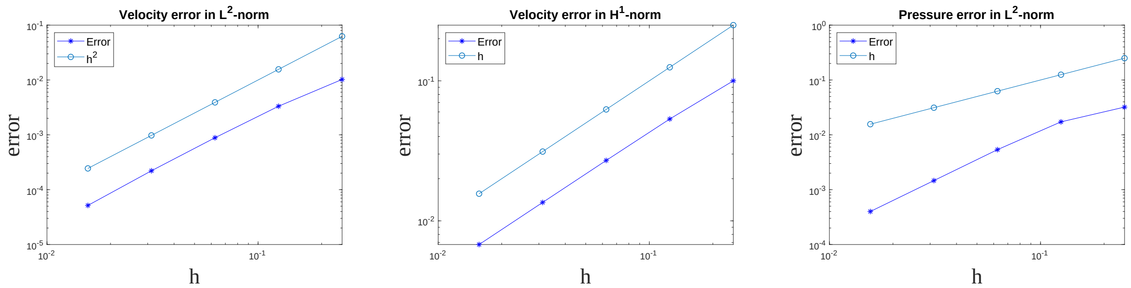

Figure 3.1: Velocity and pressure error for Example 3.1.

At final time $t = 1$ and $h = 2^{-i}$, $i = 2, 3, \ldots, 5$ with $H = \mathcal{O}(h^{\frac{1}{2}})$ for a choice of $k = h^2$, we have shown a comparison of computing time (CPU time) between the “direct solution” and the solution produced by the “three step two-grid method” in Table 3.3. In this table, we can see that the three-step two-grid method takes almost half the time that the direct method take. As we add more mesh refreshments, the computational time gap between the three-step two-grid method and the direct method grows.

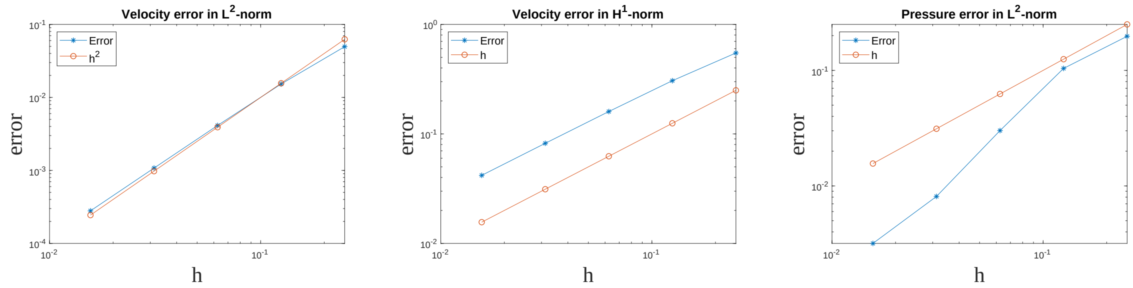

Figure 3.2: Velocity and pressure error for Example 3.2.

Table 3.3: Comparison of the "direct solution" versus the "three step two-grid solution" in terms of computing time (in second) for Example 3.2

h	"Direct solution"	"Two-grid solution"
1/4	1.97	2.30
1/8	7.68	5.74
1/16	105.16	63.10
1/32	2512.28	1167.20

3.6 Conclusion

In the first part of our work, we have discussed the semidiscrete error analysis for three step two-grid finite element method applied to the Oldroyd model of order one. We have proved that the largest scaling between the coarse mesh size H and the fine mesh size h are $h = \mathcal{O}(H^{2-\ell})$ and $h = \mathcal{O}(H^{3-\ell})$ for velocity in $L^\infty(\mathbf{L}^2)$-norm and in $L^\infty(\mathbf{H}^1)$-norm. It is $h = \mathcal{O}(H^{3-\ell})$ for the pressure in $L^\infty(\mathbf{L}^2)$-norm, for arbitrary small $\ell > 0$. In the second part, we have applied BE method in the temporal direction and have obtained optimal $\mathbf{L}^2$-error estimates for velocity of order $k(1+\log\frac{1}{k})^{\frac{1}{2}}$. Also, we have found $\mathbf{H}^1$-error for the velocity and L^2-error for the pressure of order $k^{\frac{1}{2}}(1+\log\frac{1}{k})^{\frac{1}{2}}$. The error analysis has been carried out for nonsmooth initial data (that is, $\mathbf{u}_0 \in \mathbf{H}_0^1$), which tells us that the singular behaviour of the (discrete) solutions as $t \to 0$ is more prominent in case of fully discrete error analysis and hence is more involved than usual.