Chapter 4

Penalty Method

This chapter deals with the analysis based on a penalty finite element method for the Oldroyd model of order one. Our model being a coupled model, the velocity and the pressure are coupled together by the incompressibility condition, it is difficult to solve the system numerically due to restrictions on the finite element spaces. Penalty method is one of the methods that decouples velocity and pressure by penalizing the continuity equation. We first obtain the *a priori* and regularity estimates for penalized solution with nonsmooth initial data. Then, we obtain optimal error bounds for semidiscrete as well as fully discrete penalized problem. All the results are shown to be uniform in time under the uniqueness condition. Finally, a few numerical examples are considered to validate the theoretical findings. Part of this work has been published in [14].

4.1 Introduction

It is known for a long time that coupling of the velocity $\mathbf{u}$ and the pressure p by means of the "incompressibility condition $div\ u = 0$" in case of incompressible fluid flow model is a hurdle in case of numerical computing. A common way to handle this difficulty is to address the incompressibility condition, in other words, to relax this condition, in an appropriate way. The standard methods that do the job are the projection method, the pressure stabilized method, the artificial compressibility method and the penalty method (see for instance, J. Shen [125] and references therein). We consider to work with the penalty method, it being the simplest and an effective finite element method to address this incompressibility.

The main idea of this method is to approximate the pair of solution $(\mathbf{u}, p)$ of the system (1.4)-(1.6) by the penalized solution $(\mathbf{u}_\varepsilon, p_\varepsilon)$ satisfying the following penalized system:

$$\left.\begin{aligned} \tfrac{\partial \mathbf{u}_\varepsilon}{\partial t} - \mu\Delta\mathbf{u}_\varepsilon - \textstyle\int_0^t \beta(t-s)\Delta\mathbf{u}_\varepsilon(s)\,ds + \mathbf{u}_\varepsilon\cdot\nabla\mathbf{u}_\varepsilon + \tfrac{1}{2}(\nabla\cdot\mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon + \nabla p_\varepsilon& \\ = \mathbf{f}(x,t) \ \text{ in } \Omega,\ t>0& \\ \mu\nabla\cdot\mathbf{u}_\varepsilon + \varepsilon p_\varepsilon = 0, \quad \text{on } \Omega,\ t>0,& \\ \mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_{\varepsilon 0}, \quad \text{in } \Omega, \qquad \mathbf{u}_\varepsilon = \mathbf{0} \ \text{ on } \partial\Omega,\ t\geq 0.& \end{aligned}\right\} \tag{4.1}$$

Note that, we have added a term $\frac{1}{2}(\nabla\cdot\mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon$ to the nonlinear term, introduced by Temam [128], to ensure the dissipativity of the system (4.1). Next, we eliminate the penalized pressure term p_ε from (4.1) to find a system of equations in $\mathbf{u}_\varepsilon$ as:

$$\frac{\partial \mathbf{u}_\varepsilon}{\partial t} + \mu A_\varepsilon\mathbf{u}_\varepsilon + \tilde{B}(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon) - \int_0^t \beta(t-s)\Delta\mathbf{u}_\varepsilon(s)\ ds = \mathbf{f} \tag{4.2}$$

with $\mathbf{u}_\varepsilon(0) = \mathbf{u}_{\varepsilon 0}$, where

$$A_\varepsilon\mathbf{w} := -\Delta\mathbf{w} - \frac{1}{\varepsilon}\nabla(\nabla\cdot\mathbf{w}) \quad \text{ and } \quad \tilde{B}(\mathbf{w},\boldsymbol{\phi}) := (\mathbf{w}\cdot\nabla)\boldsymbol{\phi} + \frac{1}{2}(\nabla\cdot\mathbf{w})\boldsymbol{\phi}.$$

The corresponding weak formulation of penalized Oldroyd model of order one as: Find $(\mathbf{u}_\varepsilon(t), p_\varepsilon(t)),\ t>0$ in $\mathbf{H}_0^1\times L^2$ satisfying

$$\left.\begin{aligned} (\mathbf{u}_{\varepsilon t},\mathbf{v}) + \mu a(\mathbf{u}_\varepsilon,\mathbf{v}) + \textstyle\int_0^t \beta(t-s)a(\mathbf{u}_\varepsilon(s),\mathbf{v})\ ds + \tilde{b}(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{v})& \\ -(p_\varepsilon,\nabla\cdot\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\ \mathbf{v}\in\mathbf{H}_0^1,& \\ \mu(\nabla\cdot\mathbf{u}_\varepsilon,\chi) + \varepsilon(p_\varepsilon,\chi) = 0, \quad \forall\ \chi\in L^2,& \end{aligned}\right\} \tag{4.3}$$

where

$$a(\mathbf{w},\mathbf{v}) = (\nabla\mathbf{w},\nabla\mathbf{v}), \quad \text{ and } \quad \tilde{b}(\mathbf{w},\boldsymbol{\phi},\mathbf{v}) = (\tilde{B}(\mathbf{w},\boldsymbol{\phi}),\mathbf{v}).$$

The equivalent weak form of (4.3) reads as: Find $\mathbf{u}_\varepsilon(t)\in\mathbf{H}_0^1$, $t>0$ such that

$$(\mathbf{u}_{\varepsilon t},\mathbf{v}) + \mu a_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{v}) + \int_0^t \beta(t-s)a(\mathbf{u}_\varepsilon(s),\mathbf{v})\ ds = (\mathbf{f},\mathbf{v}) - \tilde{b}(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{v}), \tag{4.4}$$

for all $\mathbf{v}\in\mathbf{H}_0^1$ with $\mathbf{u}_\varepsilon(0) = \mathbf{u}_{\varepsilon 0}$. Here,

$$a_\varepsilon(\mathbf{u},\mathbf{v}) = a(\mathbf{u},\mathbf{v}) + \frac{1}{\varepsilon}(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v}).$$

For semidiscrete formulation, we consider a finite triangulation $\mathcal{T}_h$ of the domain $\bar{\Omega}$ where $h,\ 0<h<1$ is the space discretization parameter. We also consider the finite element spaces $\mathbf{H}_h$ and L_h that approximate the velocity space $\mathbf{H}_0^1$ and the pressure space L^2, respectively. For simplicity, we assume that both the spaces comprise of piecewise linear polynomial functions like MINI element.

The discrete version of the weak formulations (4.3) reads as: Seek $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h})$ in $\mathbf{H}_h \times L_h$ satisfying

$$\left.\begin{aligned} &(\mathbf{u}_{\varepsilon ht}, \mathbf{v}_h) + \mu a(\mathbf{u}_{\varepsilon h}, \mathbf{v}_h) + \textstyle\int_0^t \beta(t-s) a(\mathbf{u}_{\varepsilon h}(s), \mathbf{v}_h)\; ds + \tilde{b}(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) \\ &\qquad\qquad\qquad -(p_{\varepsilon h}, \nabla\cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\; \mathbf{v}_h \in \mathbf{H}_h, \\ &\mu(\nabla\cdot \mathbf{u}_{\varepsilon h}, \chi_h) + \varepsilon(p_{\varepsilon h}, \chi_h) = 0, \quad \forall\; \chi_h \in L_h. \end{aligned}\right\} \tag{4.5}$$

Choose $\chi_h = \nabla\cdot \mathbf{u}_{\varepsilon h}$ in the second equation of (4.5) and use it to the first equation, then we obtain for all $\mathbf{v}_h \in \mathbf{H}_h$

$$(\mathbf{u}_{\varepsilon ht}, \mathbf{v}_h) + \mu a_\varepsilon(\mathbf{u}_{\varepsilon h}, \mathbf{v}_h) + \int_0^t \beta(t-s) a(\mathbf{u}_{\varepsilon h}(s), \mathbf{v}_h) ds = (\mathbf{f}, \mathbf{v}_h) - \tilde{b}(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon h}, \mathbf{v}_h). \tag{4.6}$$

The semidiscrete formulation(s) mentioned above are still continuous in time and in a fully discrete scheme, we further discretize (it) in the temporal direction. We consider the first-order implicit backward Euler (BE) method to discretize in the time direction. Assuming $[0,T]$ to be the time interval, we proceed as follows: Let $k = \frac{T}{N} > 0$ be the time step with $t_n = nk,\; n \geq 0$ representing the n-th time step. Here N is a positive integer. We next define for a sequence $\{\boldsymbol{\phi}^n\}_{n\geq 0} \subset \mathbf{H}_h$, the backward difference quotient

$$\partial_t \boldsymbol{\phi}^n = \frac{1}{k}(\boldsymbol{\phi}^n - \boldsymbol{\phi}^{n-1}).$$

For any continuous function $\boldsymbol{\phi}(t)$ we set $\boldsymbol{\phi}^n = \boldsymbol{\phi}(t_n)$. We approximate the integral term in (4.5) by right rectangle rule, the BE method being of first-order, with the notation $\beta_{nj} = \beta(t_n - t_j)$:

$$q_r^n(\boldsymbol{\phi}) = k \sum_{j=1}^{n} \beta_{nj} \phi^j \approx \int_0^{t_n} \beta(t_n - s)\boldsymbol{\phi}(s)\; ds. \tag{4.7}$$

Now, the fully discrete formulation after applying backward Euler method for the penalized semidiscrete Oldroyd problem (4.5) read as: Find $\{\mathbf{U}_\varepsilon^n\}_{1\leq n\leq N} \in \mathbf{H}_h$ and $\{P_\varepsilon^n\}_{1\leq n\leq N} \in L_h$ for $1 \leq n \leq N$ satisfying

$$\left.\begin{aligned} &(\partial_t \mathbf{U}_\varepsilon^n, \mathbf{v}_h) + \mu a(\mathbf{U}_\varepsilon^n, \mathbf{v}_h) + a(q_r^n(\mathbf{U}_\varepsilon), \mathbf{v}_h) = (P_\varepsilon^n, \nabla\cdot \mathbf{v}_h) + (\mathbf{f}^n, \mathbf{v}_h) \\ &\qquad\qquad\qquad\qquad\qquad\qquad -\tilde{b}(\mathbf{U}_\varepsilon^n, \mathbf{U}_\varepsilon^n, \mathbf{v}_h), \forall \mathbf{v}_h \in \mathbf{H}_h \\ &\mu(\nabla\cdot \mathbf{U}_\varepsilon^n, \chi_h) + \varepsilon(P_\varepsilon^n, \chi_h) = 0, \quad \forall\; \chi_h \in L_h, \quad n \geq 0, \end{aligned}\right\} \tag{4.8}$$

with $\mathbf{U}_\varepsilon^0 = P_h \mathbf{u}_{\varepsilon 0}$. It can be written in another form for all $\mathbf{v}_h \in \mathbf{H}_h$

$$(\partial_t \mathbf{U}_\varepsilon^n, \mathbf{v}_h) + \mu a_\varepsilon(\mathbf{U}_\varepsilon^n, \mathbf{v}_h) + a(q_r^n(\mathbf{U}_\varepsilon), \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h) - b(\mathbf{U}_\varepsilon^n, \mathbf{U}_\varepsilon^n, \mathbf{v}_h). \tag{4.9}$$

Using variant of Brouwer fixed point theorem and standard uniqueness arguments, one can show the well-posedness of the discrete problem (4.9) as well as (4.8).

The penalty approach was initially introduced by Courant [36] in the context of the calculus of variations, there has been considerable developments in different directions by many researchers. For Oldroyd model of order one, the literature is relatively limited. Only in early 90's, Kotsiolis and Oskolkov [89] and later Oskolkov [112] have studied the penalty method for the Oldroyd model of order one and also of higher orders. After that, Wang et al. [138] have investigated the relations between the penalty parameter and the time step, for the linearized Oldroyd model of order one. In fact, they have obtained optimal error estimate for the penalized system and the time discretized (backward Euler) penalized system. In Wang and He [136], similar results are observed as in [138], except for the fact that the problem is now nonlinear and the estimates are uniform, derived under the uniqueness condition. Subsequently, Wang *et. al.* have extended the analysis in [139, 140] to the finite element approximations of (1.4)-(1.6) and have derived the following optimal error estimates for smooth initial data for all $t_n \in [0,T], T>0$

$$\tau(t_n)\|\mathbf{u}(t_n)-\mathbf{U}_\varepsilon^n\|_{H^1}+\Big(k\sum_{m=0}^{n}\tau^2(t_m)\|p(t_m)-P_\varepsilon^n\|_{L^2}^2\Big)^{\frac{1}{2}}\le C(\varepsilon+h+k),$$

where $\tau(t_n)=\min\{t_n,1\}$.

We would like to point out here that, the optimal error estimate for the spatial discretization of the penalized system in $L^\infty(\mathbf{L}^2)$-norm is not available till now in the literature. Also, for the time discretization, there is hardly any result on optimal error estimate in $L^\infty(\mathbf{L}^2)$-norm. Therefore, in this chapter, an attempt is made to establish $L^\infty(\mathbf{L}^2)$-norm for both spatial and time-discretization schemes. And unlike [139], where the initial data $\mathbf{u}_{\varepsilon 0}$ belongs to $\mathbf{H}_0^1\cap\mathbf{H}^2$, we aim to discuss error analysis for the nonsmooth initial data, that is, the initial data $\mathbf{u}_{\varepsilon 0}$ in $\mathbf{H}_0^1$. The followings are the primary outcomes of this chapter:

(i) Uniform in time regularity bounds are derived for the penalized solution with nonsmooth initial data.

(ii) A *priori* estimates for the semidiscrete as well as fully discrete penalized solution are established for nonsmooth initial data.

(iii) Optimal error estimates for the semidiscrete and fully discrete penalty approximation of the velocity and the pressure are obtained.

(iii) Uniform in time bounds are proved for the discrete solutions.

(iv) Numerical experiments to validate the theoretical findings.

The rest of the part of this chapter comprise of the following sections. Section 4.2 deals with the penalty method and some new *a priori* results for the penalized solution. In Section 4.3, the semidiscrete error analysis is carried out and in Section 4.4, BE method is applied to the penalized system. Finally, in the last Section we give a few numerical experiments that are consistent with our theoretical findings.
Throughout this chapter, $C > 0$ treats as a generic constant which may depend on the given data $\Omega, \mathbf{u}_{\varepsilon 0}, \mathbf{f}, \mu, \delta, \gamma, \lambda_1$ and T but not on h, k and ε.

4.2 Preliminaries

We begin this section by considering the assumptions on the given data. Then, we give few results, which will be used in our later analysis. Next, we study some new *a priori* and regularity bounds of penalized solution. Finally, we state the error due to penalization which is already available in the literature.
We consider the following assumption on the given data for the penalized Oldroyd model.
(**A3**) For a constant $M_0 > 0$, the external force $\mathbf{f}$ and the initial velocity $\mathbf{u}_0$ satisfy

$$\mathbf{u}_0 \in \mathbf{H}_0^1 \ \text{with} \ \|A_\varepsilon^{\frac{1}{2}} \mathbf{u}_0\| \leq M_0, \ \text{and} \ \mathbf{f}, \mathbf{f}_t \in L^\infty([0,\infty]; \mathbf{L}^2) \ \text{with} \ \sup_{t>0} \big\{ \|\mathbf{f}\|, \|\mathbf{f}_t\| \big\} \leq M_0.$$

Note that the operator A_ε, which is associated with the penalty method, is a self-adjoint and positive operator from $\mathbf{H}^2 \cap \mathbf{H}_0^1$ onto $\mathbf{L}^2$, and we can talk of the powers $A_\varepsilon^r,\ r \in \mathbb{R}$. For details, we refer to Temam [18] and Shen [125]. It is observed in [18] that $\|A_\varepsilon \mathbf{v}\|$ is a norm on $\mathbf{H}^2 \cap \mathbf{H}_0^1$ and is, in fact, equivalent to that of $\mathbf{H}^2$, i.e.,

$$\|A_\varepsilon \mathbf{v}\| \approx \|\mathbf{v}\|_2, \tag{4.10}$$

with constants depending on ε. But in [18], one of the inequalities (4.10) is proved to be independent of ε. We present below the following Lemma, to support this. For a proof, see [18] and [125].

Lemma 4.1. *For a positive and sufficiently small ε, the following estimates hold:*

$$\begin{aligned} \|\Delta \mathbf{v}\| &\leq c_0 \|A_\varepsilon \mathbf{v}\| \quad \forall\ \mathbf{v} \in \mathbf{H}^2 \cap \mathbf{H}_0^1 \\ \|\nabla \mathbf{v}\| &\leq c_0 \|A_\varepsilon^{\frac{1}{2}} \mathbf{v}\| \quad \forall\ \mathbf{v} \in \mathbf{H}_0^1 \\ \|A_\varepsilon^{-1} \mathbf{v}\| &\leq c_0 \|\mathbf{v}\|_{-2} \quad \forall\ \mathbf{v} \in \mathbf{H}^{-2}(\Omega), \end{aligned}$$

where $c_0 > 0$ is independent of ε.

We need another estimate, independent of ε, which we state in the next Lemma.

Lemma 4.2. *For $\varepsilon > 0$ sufficiently small, the following holds:*

$$\|A_\varepsilon^{-\frac{1}{2}} \mathbf{v}\| \leq c_0 \|\mathbf{v}\|_{-1} \quad \forall\ \mathbf{v} \in \mathbf{H}^{-1}(\Omega).$$

Proof. With $\mathbf{w} \in \mathbf{H}_0^1$, we use Lemma 4.1 to find that

$$(A_\varepsilon^{-\frac{1}{2}} \mathbf{v}, \mathbf{w}) = (\mathbf{v}, A_\varepsilon^{-\frac{1}{2}} \mathbf{w}) \leq \|\mathbf{v}\|_{-1} \|\nabla (A_\varepsilon^{-\frac{1}{2}} \mathbf{w})\| \leq c_0 \|\mathbf{v}\|_{-1} \|\mathbf{w}\|,$$

and

$$\|A_\varepsilon^{-\frac{1}{2}} \mathbf{v}\| = \sup_{0 \neq \mathbf{w} \in \mathbf{L}^2} \frac{(A_\varepsilon^{-\frac{1}{2}} \mathbf{v}, \mathbf{w})}{\|\mathbf{w}\|} \leq c_0 \|\mathbf{v}\|_{-1}.$$

This completes the proof. Alternative way is to consider the following problem: Let $\mathbf{w}$ be a solution of

$$A_\varepsilon \mathbf{w} = \mathbf{v}, \quad \mathbf{w}|_{\partial\Omega} = 0.$$

Clearly $\|A_\varepsilon \mathbf{w}\| = \|\mathbf{v}\|$ and

$$\|A_\varepsilon^{\frac{1}{2}} \mathbf{w}\|^2 = (A_\varepsilon \mathbf{w}, \mathbf{w}) = (\mathbf{v}, \mathbf{w}) = (A_\varepsilon^{-\frac{1}{2}} \mathbf{v}, A_\varepsilon^{\frac{1}{2}} \mathbf{w}) \leq \|A_\varepsilon^{-\frac{1}{2}} \mathbf{v}\| \|A_\varepsilon^{\frac{1}{2}} \mathbf{w}\|$$

and therefore,

$$\|A_\varepsilon^{\frac{1}{2}} \mathbf{w}\| \leq \|A_\varepsilon^{-\frac{1}{2}} \mathbf{v}\|. \tag{4.11}$$

Now, using (4.11) and Lemma 4.1, we note that

$$\begin{aligned} \|A_\varepsilon^{-\frac{1}{2}} \mathbf{v}\|^2 &= (A_\varepsilon^{-1} \mathbf{v}, \mathbf{v}) = (A_\varepsilon^{-1} \mathbf{v}, A_\varepsilon \mathbf{w}) = (\mathbf{v}, \mathbf{w}) \leq \|\mathbf{v}\|_{-1} \|\nabla \mathbf{w}\| \\ &\leq c_0 \|\mathbf{v}\|_{-1} \| \|A_\varepsilon^{\frac{1}{2}} \mathbf{w}\| \leq c_0 \|\mathbf{v}\|_{-1} \| \|A_\varepsilon^{-\frac{1}{2}} \mathbf{v}\|. \end{aligned}$$

This concludes the desired proof. □

From the definition of $\tilde{b}(\cdot,\cdot,\cdot)$, we can easily check with the help of integration by parts that

$$\tilde{b}(\mathbf{w},\boldsymbol{\phi},\mathbf{v}) = \frac{1}{2}\big\{b(\mathbf{w},\boldsymbol{\phi},\mathbf{v}) - b(\mathbf{w},\mathbf{v},\boldsymbol{\phi})\big\}, \quad \forall \mathbf{v},\mathbf{w},\boldsymbol{\phi} \in \mathbf{H}_0^1,$$

where,

$$b(\mathbf{w},\boldsymbol{\phi},\mathbf{v}) = ((\mathbf{w}\cdot\nabla)\boldsymbol{\phi},\mathbf{v}).$$

Hence,

$$\tilde{b}(\mathbf{w},\boldsymbol{\phi},\boldsymbol{\phi}) = 0, \quad \text{and} \quad \tilde{b}(\mathbf{w},\boldsymbol{\phi},\mathbf{v}) = -\tilde{b}(\mathbf{w},\boldsymbol{\phi},\mathbf{v}), \quad \forall\ \mathbf{v},\mathbf{w},\boldsymbol{\phi} \in \mathbf{H}_0^1. \tag{4.12}$$

We now present below a few bounds for nonlinear term $b(\cdot,\cdot,\cdot)$ which will be used for our later analysis. The proofs go similar to [80] with a use of Lemma 4.1.

Lemma 4.3. *[80] Suppose the condition* (**A1**) *is satisfied. Then, the trilinear form* $b(\cdot,\cdot,\cdot)$ *satisfies the following properties:*

$$|b(\mathbf{v},\mathbf{w},\boldsymbol{\phi})| = C\begin{cases} \|\mathbf{v}\|^{\frac12}\|A_\varepsilon^{\frac12}\mathbf{v}\|^{\frac12}\|A_\varepsilon^{\frac12}\mathbf{w}\|^{\frac12}\|A_\varepsilon\mathbf{w}\|^{\frac12}\|\boldsymbol{\phi}\|, & \forall \mathbf{v},\boldsymbol{\phi}\in\mathbf{H}_0^1, \mathbf{w}\in\mathbf{H}_0^1\cap\mathbf{H}^2,\\ \|A_\varepsilon^{\frac12}\mathbf{v}\|^{\frac12}\|A_\varepsilon\mathbf{v}\|^{\frac12}\|\mathbf{w}\|^{\frac12}\|A_\varepsilon^{\frac12}\mathbf{w}\|^{\frac12}\|\boldsymbol{\phi}\|, & \forall \mathbf{w},\boldsymbol{\phi}\in\mathbf{H}_0^1, \mathbf{v}\in\mathbf{H}_0^1\cap\mathbf{H}^2,\\ \|\mathbf{v}\|^{\frac12}\|A_\varepsilon^{\frac12}\mathbf{v}\|^{\frac12}\|\mathbf{w}\|^{\frac12}\|A_\varepsilon^{\frac12}\mathbf{w}\|^{\frac12}\|A_\varepsilon^{\frac12}\boldsymbol{\phi}\|, & \forall \mathbf{v},\mathbf{w},\boldsymbol{\phi}\in\mathbf{H}_0^1,\\ \|\mathbf{v}\|\|A_\varepsilon^{\frac12}\mathbf{w}\|^{\frac12}\|A_\varepsilon\mathbf{w}\|^{\frac12}\|A_\varepsilon^{\frac12}\boldsymbol{\phi}\|, & \forall \mathbf{v},\boldsymbol{\phi}\in\mathbf{H}_0^1, \mathbf{w}\in\mathbf{H}_0^1\cap\mathbf{H}^2,\\ \|\mathbf{v}\|\|\mathbf{w}\|^{\frac12}\|A_\varepsilon^{\frac12}\mathbf{w}\|^{\frac12}\|A_\varepsilon^{\frac12}\boldsymbol{\phi}\|\|A_\varepsilon\boldsymbol{\phi}\|, & \forall \mathbf{v},\mathbf{w},\in\mathbf{H}_0^1, \boldsymbol{\phi}\in\mathbf{H}_0^1\cap\mathbf{H}^2,\\ \|\mathbf{v}\|^{\frac12}\|A_\varepsilon^{\frac12}\mathbf{v}\|^{\frac12}\|\mathbf{w}\|\|A_\varepsilon^{\frac12}\boldsymbol{\phi}\|\|A_\varepsilon\boldsymbol{\phi}\|, & \forall \mathbf{v},\mathbf{w},\in\mathbf{H}_0^1, \boldsymbol{\phi}\in\mathbf{H}_0^1\cap\mathbf{H}^2. \end{cases}$$

We recall the following result, which is a counter part of the L'Hospital rule.

Theorem 4.1. *("Stolz-Cesaro Theorem") Let us assume two real sequences* $\{\phi^n\}_{n=0}^\infty$ *and* $\{\psi^n\}_{n=0}^\infty$ *with* $\{\psi^n\}_{n=0}^\infty$ *is divergent and strictly monotone. If*

$$\lim_{n\to\infty}\left(\frac{\phi^n-\phi^{n-1}}{\psi^n-\psi^{n-1}}\right) = l,$$

then

$$\lim_{n\to\infty}\left(\frac{\phi^n}{\psi^n}\right) = l,$$

holds.

4.2.1 A Priori and Regularity Estimates for the Penalized Solution

We take a quick glance into the *a priori* estimates of the penalized problem.

Lemma 4.4. *Suppose the condition* (**A1**) *and* (**A3**) *hold. Moreover, assume* $0 < \alpha < \min(\delta, \mu\lambda_1/2c_0^2)$*. Then, the penalized solution* $\mathbf{u}_\varepsilon(t)$ *satisfies the following results for any* $t > 0$*:*

$$\|A_\varepsilon^{\frac{r}{2}}\mathbf{u}_\varepsilon(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|A_\varepsilon^{\frac{r+1}{2}}\mathbf{u}_\varepsilon(s)\|^2 ds \leq C, \quad r \in \{0,1\}.$$

where $C > 0$ *is a constant may depends on given data but not depend on* ε*.*

Proof. Choose $\mathbf{v} = \mathbf{u}_\varepsilon$ in (4.4) and use the "Cauchy-Schwarz inequality", the "Poincaré inequality" with Lemma 4.1 ($\|\mathbf{u}_\varepsilon\|^2 \leq \frac{1}{\lambda_1}\|\nabla\mathbf{u}_\varepsilon\|^2 \leq \frac{c_0^2}{\lambda_1}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2$) for the term on the right left of equality. Then we obtain

$$\frac{d}{dt}\|\mathbf{u}_\varepsilon\|^2 + \mu\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2 + 2\int_0^t \beta(t-s)a(\mathbf{u}_\varepsilon(s), \mathbf{u}_\varepsilon)\, ds \leq \frac{c_0^2}{\mu\lambda_1}\|\mathbf{f}\|^2. \tag{4.13}$$

Note that the nonlinear term vanishes due to (4.12). Multiply (4.13) by $e^{2\alpha t}$, then with $\hat{\mathbf{u}}_\varepsilon = e^{\alpha t}\mathbf{u}_\varepsilon$, we find that

$$\frac{d}{dt}\|\hat{\mathbf{u}}_\varepsilon\|^2 - 2\alpha\|\hat{\mathbf{u}}_\varepsilon\|^2 + \mu\|A_\varepsilon^{\frac{1}{2}}\hat{\mathbf{u}}_\varepsilon\|^2 + 2e^{2\alpha t}\int_0^t \beta(t-s)a(\mathbf{u}_\varepsilon(s), \mathbf{u}_\varepsilon)\, ds = \frac{c_0^2}{\mu\lambda_1}\|\hat{\mathbf{f}}\|^2.$$

After integration, the resulting double integral term drops out, since it is positive (see, Lemma 1.5), since by our assumption $\delta > \alpha > 0$. Using the "Poincaré inequality" with Lemma 4.1, we reach at

$$\|\hat{\mathbf{u}}_\varepsilon(t)\|^2 + (\mu - \frac{2c_0^2\alpha}{\lambda_1})\int_0^t \|A_\varepsilon^{\frac{1}{2}}\hat{\mathbf{u}}_\varepsilon(s)\|^2 ds \leq \|\mathbf{u}_{\varepsilon 0}\|^2 + \frac{c_0^2(e^{2\alpha t}-1)}{2\alpha\mu\lambda_1}\|\mathbf{f}\|_\infty^2.$$

With $0 < \alpha < \min(\delta, \mu\lambda_1/2c_0^2)$, we have $\mu - \frac{2c_0^2\alpha}{\lambda_1} = \beta_1 > 0$. Finally, we multiply throughout by $e^{-2\alpha t}$ to conclude that

$$\|\mathbf{u}_\varepsilon(t)\|^2 + \beta_1 e^{-2\alpha t}\int_0^t e^{2\alpha s}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon(s)\|^2 ds \leq e^{-2\alpha t}\|\mathbf{u}_{\varepsilon 0}\|^2 + \frac{c_0^2(1-e^{-2\alpha t})}{2\alpha\mu\lambda_1}\|\mathbf{f}\|_\infty^2, \tag{4.14}$$

which concludes the proof for the case $r = 0$. For the second estimate, first we integrate (4.13) from t to $t + T_0$, for a fixed T_0 and use (4.14) to find

$$\|\mathbf{u}_\varepsilon(t+T_0)\|^2 + \mu\int_t^{t+T_0} \|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon(s)\|^2 ds \leq \|\mathbf{u}_\varepsilon(t)\|^2 + \frac{c_0^2 T_0}{\mu\lambda_1}\|\mathbf{f}\|_\infty^2. \tag{4.15}$$

Then, put $\mathbf{v} = A_\varepsilon\mathbf{u}_\varepsilon$ in (4.4), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2 + \mu\|A_\varepsilon\mathbf{u}_\varepsilon\|^2 + \int_0^t \beta(t-s)(-\Delta\mathbf{u}_\varepsilon(s), A_\varepsilon\mathbf{u}_\varepsilon)ds$$
$$= (\mathbf{f}, A_\varepsilon\mathbf{u}_\varepsilon) - \tilde{b}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, A_\varepsilon\mathbf{u}_\varepsilon). \tag{4.16}$$

We use the "Cauchy-Schwarz inequality" and the "Young's inequality" to bound the following as

$$
\begin{aligned}
|(\mathbf{f}, A_\varepsilon \mathbf{u}_\varepsilon) - \tilde{b}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, A_\varepsilon \mathbf{u}_\varepsilon)| &\leq \|\mathbf{f}\|\|A_\varepsilon \mathbf{u}_\varepsilon\| + 2^{\frac{1}{2}}\|\mathbf{u}_\varepsilon\|^{\frac{1}{2}}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|\|A_\varepsilon \mathbf{u}_\varepsilon\|^{3/2} \\
&\leq \frac{3}{2\mu}\|\mathbf{f}\|^2 + (\frac{9}{2\mu})^3\|\mathbf{u}_\varepsilon\|^2\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^4 + \frac{\mu}{3}\|A_\varepsilon \mathbf{u}_\varepsilon\|^2. \quad (4.17)
\end{aligned}
$$

A use of (4.17) in (4.16) yields

$$
\begin{aligned}
\frac{d}{dt}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2 + \frac{4\mu}{3}\|A_\varepsilon \mathbf{u}_\varepsilon\|^2 + 2\int_0^t \beta(t-s)(-\Delta \mathbf{u}_\varepsilon(s), A_\varepsilon \mathbf{u}_\varepsilon)ds& \\
\leq \frac{3}{\mu}\|\mathbf{f}\|^2 + 2(\frac{9}{2\mu})^3\|\mathbf{u}_\varepsilon\|^2\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^4.& \quad (4.18)
\end{aligned}
$$

Choose $\gamma_0 > 0$, then a use of "Cauchy-Schwarz inequality" shows

$$
\gamma_0\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\| = \gamma_0(A_\varepsilon \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \leq \gamma_0\|A_\varepsilon \mathbf{u}_\varepsilon\|\|\mathbf{u}_\varepsilon\| \leq \frac{\mu}{3}\|A_\varepsilon \mathbf{u}_\varepsilon\|^2 + \frac{3}{4\mu}\gamma_0^2\|\mathbf{u}_\varepsilon\|^2. \quad (4.19)
$$

Now, add $\gamma_0\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|$ on both sides of (4.18) and use (4.19) with the "Poincaré inequality", and Lemma 4.1 ($\mu\|A_\varepsilon \mathbf{u}_\varepsilon\|^2 \geq \frac{\mu\lambda_1}{c_0^2}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2$) to find

$$
\begin{aligned}
\frac{d}{dt}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2 + \big(\gamma_0 + \frac{\mu\lambda_1}{c_0^2} - 2(\frac{9}{2\mu})^3\|\mathbf{u}_\varepsilon\|^2\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2\big)\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2& \\
+ 2\int_0^t \beta(t-s)(-\Delta \mathbf{u}_\varepsilon(s), A_\varepsilon \mathbf{u}_\varepsilon)ds \leq \frac{3}{\mu}\|\mathbf{f}\|^2 + \frac{3}{4\mu}\gamma_0^2\|\mathbf{u}_\varepsilon\|^2.& \quad (4.20)
\end{aligned}
$$

Setting

$$
h(t) = \gamma_0 + \frac{\mu\lambda_1}{c_0^2} - 2(\frac{9}{2\mu})^3\|\mathbf{u}_\varepsilon\|^2\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2. \quad (4.21)
$$

Then, using (**A3**) and (4.14) in (4.20), we deduce that

$$
\frac{d}{dt}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2 + h(t)\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2 + 2\int_0^t \beta(t-s)(-\Delta \mathbf{u}_\varepsilon(s), A_\varepsilon \mathbf{u}_\varepsilon)ds \leq C.
$$

Multiply both sides by $e^{\int_0^t h(\tau)d\tau}$ and tale time integration to obtain

$$
\begin{aligned}
\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon(t)\|^2 + 2e^{-\int_0^t h(\tau)\,d\tau}\int_0^t e^{\int_0^s h(\tau)\,d\tau}\int_0^s \beta(s-\tau)(-\Delta \mathbf{u}_\varepsilon(\tau), A_\varepsilon \mathbf{u}_\varepsilon(s))d\tau ds& \\
\leq e^{-\int_0^t h(\tau)\,d\tau}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_{\varepsilon 0}\|^2 + C\int_0^t e^{-\int_s^t h(\tau)\,d\tau}\,ds.& \quad (4.22)
\end{aligned}
$$

We integrate (4.21) from t to $t+T_0$, for a fixed T_0 and use (4.14) and (4.15) to find

$$
\begin{aligned}
\int_t^{t+T_0} h(s)ds &= (\gamma_0 + \frac{\mu\lambda_1}{c_0^2})T_0 - 2(\frac{9}{2\mu})^3\int_t^{t+T_0}\|\mathbf{u}_\varepsilon\|^2\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2 ds \\
&\geq (\gamma_0 + \frac{\mu\lambda_1}{c_0^2})T_0 - K_1.
\end{aligned}
$$

Choose γ_0 such that $K_1 = \gamma_0 T_0$ then with $0 < \alpha < \min(\delta, \mu\lambda_1/2c_0^2)$, we obtain

$$\int_t^{t+T_0} h(s)ds \geq T_0 \frac{\mu\lambda_1}{c_0^2} \geq 2\alpha T_0.$$

Next, we choose two non-negative integers l_1 and l_2 such that $l_1 T_0 \leq s \leq (l_1+1)T_0$ and $l_2 T_0 \leq t \leq (l_2+1)T_0$. From (4.21), we have $h(t) \leq (\gamma_0 + \frac{\mu\lambda_1}{c_0^2})$. Then, we find that

$$\begin{aligned}\int_s^t h(\tau)d\tau &= \int_{l_1T_0}^{(l_2+1)T_0} h(\tau)d\tau - \int_{l_1T_0}^{s} h(\tau)d\tau - \int_t^{(l_2+1)T_0} h(\tau)d\tau \\ &\geq (l_2+1-l_1)2\alpha T_0 - (\gamma_0 + \frac{\mu\lambda_1}{c_0^2})T_0 - (\gamma_0 + \frac{\mu\lambda_1}{c_0^2})T_0 \\ &\geq (t-s)2\alpha - 2(\gamma_0 + \frac{\mu\lambda_1}{c_0^2})T_0. \end{aligned} \tag{4.23}$$

A use of (4.23) in the right of inequality (4.22) yields

$$\begin{aligned}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon(t)\|^2 + 2e^{-\int_0^t h(\tau)\,d\tau}\int_0^t e^{\int_0^s h(\tau)\,d\tau}\int_0^s \beta(s-\tau)(-\Delta\mathbf{u}_\varepsilon(\tau), A_\varepsilon\mathbf{u}_\varepsilon(s))d\tau ds \\ \leq e^{2(\gamma_0+\frac{\mu\lambda_1}{c_0^2})T_0}\Big(e^{-2\alpha t}\|A_\varepsilon\mathbf{u}_{\varepsilon 0}\|^2 + \frac{C}{2\alpha}(1-e^{-2\alpha t})\Big). \end{aligned} \tag{4.24}$$

Now, it is enough to show that the double integral term is positive. For this, we use the property $\Delta\mathbf{u}_\varepsilon = \nabla(\nabla\cdot\mathbf{u}_\varepsilon) - \nabla\times\nabla\times\mathbf{u}_\varepsilon$, then we obtain

$$\begin{aligned}(-\Delta\mathbf{u}_\varepsilon, A_\varepsilon\mathbf{u}_\varepsilon) &= (-\Delta\mathbf{u}_\varepsilon, -\Delta\mathbf{u}_\varepsilon - \frac{1}{\varepsilon}\nabla(\nabla\cdot\mathbf{u}_\varepsilon)) \\ &= (\Delta\mathbf{u}_\varepsilon, \Delta\mathbf{u}_\varepsilon) + (\Delta\mathbf{u}_\varepsilon, \frac{1}{\varepsilon}\nabla(\nabla\cdot\mathbf{u}_\varepsilon)) \\ &= (\Delta\mathbf{u}_\varepsilon, \Delta\mathbf{u}_\varepsilon) + \frac{1}{\varepsilon}(\nabla(\nabla\cdot\mathbf{u}_\varepsilon) - \nabla\times\nabla\times\mathbf{u}_\varepsilon, \nabla(\nabla\cdot\mathbf{u}_\varepsilon)) \\ &= (\Delta\mathbf{u}_\varepsilon, \Delta\mathbf{u}_\varepsilon) + \frac{1}{\varepsilon}(\nabla(\nabla\cdot\mathbf{u}_\varepsilon), \nabla(\nabla\cdot\mathbf{u}_\varepsilon)) - \frac{1}{\varepsilon}(\nabla\times\nabla\times\mathbf{u}_\varepsilon, \nabla(\nabla\cdot\mathbf{u}_\varepsilon)). \end{aligned} \tag{4.25}$$

Note that

$$(\nabla\times\nabla\times\mathbf{u}_\varepsilon, \nabla(\nabla\cdot\mathbf{u}_\varepsilon)) = -\langle\nabla\times\mathbf{u}_\varepsilon, \nabla\times(\nabla(\nabla\cdot\mathbf{u}_\varepsilon))\rangle + (\nabla\times\mathbf{u}_\varepsilon, \nabla(\nabla\cdot\mathbf{u}_\varepsilon))_{\partial\Omega}$$

Since curl of gradient of a scalar is zero, so $\nabla\times(\nabla(\nabla\cdot\mathbf{u}_\varepsilon)) = 0$ in the sense of distribution and since $\mathbf{u}_\varepsilon$ vanishes on the boundary, so the last term on the right of inequality (4.25) vanishes. Hence, a use of (4.23) yields

$$\begin{aligned}&\int_0^t e^{\int_0^s h(\tau)\,d\tau}\int_0^s \beta(s-\tau)(-\Delta\mathbf{u}_\varepsilon(\tau), A_\varepsilon\mathbf{u}_\varepsilon(s))d\tau ds \\ &\quad\geq e^{-2(\gamma_0+\frac{\mu\lambda_1}{c_0^2})T_0}\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)\Big((\Delta\mathbf{u}_\varepsilon(\tau), \Delta\mathbf{u}_\varepsilon(s))\end{aligned}$$

$$+\frac{1}{\varepsilon}(\nabla(\nabla\cdot\mathbf{u}_\varepsilon(\tau)),\nabla(\nabla\cdot\mathbf{u}_\varepsilon(s)))\Big)d\tau ds \geq 0.$$

So, we drop the second term from the left of inequality (4.24) to reach at

$$\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon(t)\| \leq C. \tag{4.26}$$

Finally, we multiply (4.18) by $e^{2\alpha t}$ and take time integration and use (4.26) to conclude the remaining of the proof. □

Remark 4.1. *The results in Lemma 4.4 are uniform in time, which are sufficient to prove the existence of a global weak solution of the penalized system. In fact, with the regularity results (that are established below), we are now in a position to establish a unique global strong solution of the penalized system. We refrain from going into the details, as the procedure of establishing existence and uniqueness of solution of the system (4.2) as well as of (4.1) follows similar techniques as done in [63].*

Lemma 4.5. *Suppose the hypothesis of Lemma 4.4 be satisfied. Then, for any $t > 0$, the following results hold:*

$$\begin{aligned}\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_{\varepsilon s}(s)\|^2 ds &\leq C,\\ \tau^*(t)\|A_\varepsilon\mathbf{u}_\varepsilon(t)\|^2 + \tau^*(t)\|\mathbf{u}_{\varepsilon t}(t)\|^2 + e^{-2\alpha t}\int_0^t \sigma(s)\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_{\varepsilon s}(s)\|^2 ds &\leq C,\end{aligned}$$

where $\sigma(t) = e^{2\alpha t}\tau^*(t)$ and $\tau^*(t) = min\{1,t\}$.

Proof. For the first one, we take $\mathbf{v} = e^{2\alpha t}\mathbf{u}_{\varepsilon t}$ in (4.4) to establish

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\|A_\varepsilon^{\frac{1}{2}}\hat{\mathbf{u}}_\varepsilon\|^2 + \|\hat{\mathbf{u}}_{\varepsilon t}\|^2 &= \alpha\mu\|A_\varepsilon^{\frac{1}{2}}\hat{\mathbf{u}}_\varepsilon\|^2 + (\hat{\mathbf{f}},\mathbf{u}_{\varepsilon t}) - e^{2\alpha t}\tilde{b}(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{u}_{\varepsilon t})\\ &\quad + e^{2\alpha t}\int_0^t \beta(t-s)(\Delta\mathbf{u}_\varepsilon(s),\mathbf{u}_{\varepsilon t})\ ds.\end{aligned}$$

After using Lemma 4.3 and the "Cauchy-Schwarz inequality" and Lemma 4.1, we take time integration to obtain

$$\begin{aligned}\mu\|A_\varepsilon^{\frac{1}{2}}\hat{\mathbf{u}}_\varepsilon(t)\|^2 + \int_0^t \|\hat{\mathbf{u}}_{\varepsilon s}(s)\|^2 ds &\leq C\Big[\int_0^t \Big(\|A_\varepsilon^{\frac{1}{2}}\hat{\mathbf{u}}_\varepsilon\|^2 + \|\hat{\mathbf{f}}\|^2 + \|A_\varepsilon\hat{\mathbf{u}}_\varepsilon\|^2\Big)ds\\ &+ \int_0^t \|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2\|A_\varepsilon\hat{\mathbf{u}}_\varepsilon\|^2\ ds + \int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon(\tau)\|\ d\tau\Big)^2 ds\Big].\end{aligned} \tag{4.27}$$

The double integral term in (4.27) can be written in a single integral similar to the estimate (2.17) of Chapter 2. Finally, a use of Lemma 4.4 concludes the proof of the first estimate.

For the second one, we differentiate (4.4) with respect to time to arrive

$$(\mathbf{u}_{\varepsilon tt},\mathbf{v})+\mu a_\varepsilon(\mathbf{u}_{\varepsilon t},\mathbf{v})+\beta(0)a(\mathbf{u}_\varepsilon,\mathbf{v})+\int_0^t \beta_t(t-s)a(\mathbf{u}_\varepsilon(s),\mathbf{v})\,ds$$
$$=-\tilde b(\mathbf{u}_{\varepsilon t},\mathbf{u}_\varepsilon,\mathbf{v})-\tilde b(\mathbf{u}_\varepsilon,\mathbf{u}_{\varepsilon t},\mathbf{v})+(\mathbf{f}_t,\mathbf{v}),\ \forall \mathbf{v}\in\mathbf{H}_0^1. \tag{4.28}$$

Set $\mathbf{v}=\sigma(t)\mathbf{u}_{\varepsilon t}$ in (4.28) and use $\beta_t(t-s)=-\delta\beta(t-s)$ and $\beta(0)=\gamma$ to obtain

$$\frac{1}{2}\frac{d}{dt}(\sigma(t)\|\mathbf{u}_{\varepsilon t}\|^2)+\mu\sigma(t)\|A_\varepsilon^{\frac12}\mathbf{u}_{\varepsilon t}\|^2=\frac{1}{2}\sigma_t(t)\|\mathbf{u}_{\varepsilon t}\|^2-\sigma(t)\tilde b(\mathbf{u}_{\varepsilon t},\mathbf{u}_\varepsilon,\mathbf{u}_{\varepsilon t})$$
$$+\sigma(t)(\mathbf{f}_t,\mathbf{u}_{\varepsilon t})+\gamma\sigma(t)a(\mathbf{u}_\varepsilon,\mathbf{u}_{\varepsilon t})+\delta\sigma(t)\int_0^t\beta(t-s)a(\mathbf{u}_\varepsilon(s),\mathbf{u}_{\varepsilon t})ds. \tag{4.29}$$

A use of Lemma 4.3 with the “Young’s inequality” yields

$$|\tilde b(\mathbf{u}_{\varepsilon t},\mathbf{u}_\varepsilon,\mathbf{u}_{\varepsilon t})|\le C\|\mathbf{u}_{\varepsilon t}\|^{\frac12}\|A_\varepsilon^{\frac12}\mathbf{u}_{\varepsilon t}\|^{\frac12}\|A_\varepsilon^{\frac12}\mathbf{u}_\varepsilon\|\|\mathbf{u}_{\varepsilon t}\|^{\frac12}\|A_\varepsilon^{\frac12}\mathbf{u}_{\varepsilon t}\|^{\frac12}$$
$$\le C(\mu)\|\mathbf{u}_{\varepsilon t}\|^2\|A_\varepsilon^{\frac12}\mathbf{u}_\varepsilon\|^2+\frac{\mu}{2}\|A_\varepsilon^{\frac12}\mathbf{u}_{\varepsilon t}\|^2. \tag{4.30}$$

Using the fact $\sigma(t)\le e^{2\alpha t},\sigma_t(t)\le e^{2\alpha t}$ and (4.30) in (4.29), we take time integration and use the “Cauchy Schwarz inequality” with Lemma 4.1 to derive

$$\sigma(t)\|\mathbf{u}_{\varepsilon t}(t)\|^2+\mu\int_0^t\sigma(s)\|A_\varepsilon^{\frac12}\mathbf{u}_{\varepsilon s}(s)\|^2ds$$
$$\le C\Big[\int_0^t(\|\hat{\mathbf{u}}_{\varepsilon s}(s)\|^2+\|\hat{\mathbf{u}}_{\varepsilon s}(s)\|^2\|A_\varepsilon^{\frac12}\hat{\mathbf{u}}_\varepsilon(s)\|)ds\int_0^t\|\hat{\mathbf{f}}_s\|^2ds$$
$$+\int_0^t\|A_\varepsilon^{\frac12}\hat{\mathbf{u}}_\varepsilon(s)\|^2ds+\int_0^t\Big(\int_0^s\beta(s-\tau)e^{\alpha(s-\tau)}\|A_\varepsilon^{\frac12}\hat{\mathbf{u}}_\varepsilon(\tau)\|\,d\tau\Big)^2ds\Big] \tag{4.31}$$

We estimate the double integral as above and then use the first result of this lemma. Finally multiply by $e^{-2\alpha t}$ to establish the last part of the required result. To estimate $\|A_\varepsilon\mathbf{u}_\varepsilon\|$, we choose $\mathbf{v}=A_\varepsilon\mathbf{u}_\varepsilon$ in (4.4) and apply Lemma 4.3 with Lemma 4.1 to arrive

$$\mu\|A_\varepsilon\mathbf{u}_\varepsilon\|^2\le C\Big(\|\mathbf{u}_{\varepsilon t}\|^2+\|\mathbf{u}_\varepsilon\|^2\|A_\varepsilon^{\frac12}\mathbf{u}_\varepsilon\|^4+\Big(\int_0^t\beta(t-s)\|A_\varepsilon\mathbf{u}_\varepsilon(s)\|ds\Big)^2+\|\mathbf{f}\|^2\Big).$$

Multiply both sides by $\sigma(t)$ and use (4.31) to concludes the remaining of proof. □

Now we present the error due to penalization. For a proof, see [136, Theorem 4.1].

Theorem 4.2. *“Let us assume the hypothesis of Lemma 4.4 be satisfied, then the following holds true*

$$\tau^*(t)\|(\mathbf{u}-\mathbf{u}_\varepsilon)(t)\|^2+(\tau^*)^2(t)\|\nabla(\mathbf{u}-\mathbf{u}_\varepsilon)(t)\|^2+e^{-2\alpha t}\int_0^t\sigma_1(s)\|(p-p_\varepsilon)(s)\|^2ds\le C\varepsilon^2,$$

where the positive constant C depends exponentially on time and $\sigma_1(t) = (\tau^*(t))^2 e^{2\alpha t}$. The above estimate is uniform in time under the uniqueness condition:

$$\frac{N}{\nu^2}\|\mathbf{f}_\infty\|_{-1} < 1 \text{ and } N = \sup_{\mathbf{u},\mathbf{v},\mathbf{w}} \frac{\tilde{b}(\mathbf{u},\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|}, \tag{4.32}$$

where $\nu = \frac{\mu}{2} + \frac{\gamma}{\delta}$ and $\|\mathbf{f}_\infty\|_{-1} = \|\mathbf{f}\|_{L^\infty(0,\infty;\mathbf{H}^{-1}(\Omega))}$."

4.3 Semidiscrete Formulation

In this section, we first define some operators which will be used for our analysis. Then, we concentrate on *a priori* and regularity results of semidiscrete solution. Finally, we discuss about the error analysis due to space discretization.
We begin by defining a discrete analogue $A_{\varepsilon h} : \mathbf{H}_h \to \mathbf{H}_h$ of A_ε satisfying

$$(A_{\varepsilon h}\mathbf{w}_h, \mathbf{v}_h) = a(\mathbf{w}_h, \mathbf{v}_h) + \frac{1}{\varepsilon}(\nabla\cdot\mathbf{w}_h, \nabla\cdot\mathbf{v}_h), \quad \forall \mathbf{w}_h, \mathbf{v}_h \in \mathbf{H}_h. \tag{4.33}$$

Let us now define two linear inverse operators $A_\varepsilon^{-1} : \mathbf{L}^2 \to \mathbf{H}_0^1$ and $A_{\varepsilon h}^{-1} : \mathbf{H}_h \to \mathbf{H}_h$ as follows: For $\mathbf{g} \in \mathbf{L}^2$,

$$a_\varepsilon(A_\varepsilon^{-1}\mathbf{g}, \boldsymbol{\phi}) = (\nabla A_\varepsilon^{-1}\mathbf{g}, \nabla\boldsymbol{\phi}) + \frac{1}{\varepsilon}(\nabla\cdot A_\varepsilon^{-1}\mathbf{g}, \nabla\cdot\boldsymbol{\phi}) = (\mathbf{g}, \boldsymbol{\phi}),\ \forall\boldsymbol{\phi} \in \mathbf{H}_0^1,$$
$$a_\varepsilon(A_{\varepsilon h}^{-1}P_h\mathbf{g}, \boldsymbol{\phi}_h) = (\nabla A_{\varepsilon h}^{-1}P_h\mathbf{g}, \nabla\boldsymbol{\phi}_h) + \frac{1}{\varepsilon}(\nabla\cdot A_{\varepsilon h}^{-1}P_h\mathbf{g}, \nabla\cdot\boldsymbol{\phi}_h) = (P_h\mathbf{g}, \boldsymbol{\phi}_h), \forall\boldsymbol{\phi}_h \in \mathbf{H}_h.$$

Now, arguing in similar lines of [79, Corollary 4.3], one can obtain the followings:

Proposition 4.1. *The map $\Delta_h^{-1}P_h\Delta : \mathbf{H}_0^1 \cap \mathbf{H}^2 \to \mathbf{H}_h$ satisfying*

$$\|\mathbf{v} - \Delta_h^{-1}P_h\Delta\mathbf{v}\| + h\|\nabla(\mathbf{v} - \Delta_h^{-1}P_h\Delta\mathbf{v})\| \le Ch^2\|\Delta\mathbf{v}\|,$$

and the map $A_{\varepsilon h}^{-1}P_hA_\varepsilon : \mathbf{H}_0^1 \cap \mathbf{H}^2 \to \mathbf{H}_h$ satisfying

$$\|\mathbf{v} - A_{\varepsilon h}^{-1}P_hA_\varepsilon\mathbf{v}\| + h\|\nabla(\mathbf{v} - A_{\varepsilon h}^{-1}P_hA_\varepsilon\mathbf{v})\| \le Ch^2\|A_\varepsilon\mathbf{v}\|.$$

We now present the discrete version of the Lemma 4.1 and 4.2.

Lemma 4.6. *For $\varepsilon > 0$ sufficiently small, the following estimates hold:*

$$\|\nabla\mathbf{v}_h\| \le c_0\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{v}_h\|, \quad \forall\ \mathbf{v}_h \in \mathbf{H}_h,$$
$$\|\Delta_h\mathbf{v}_h\| \le c_0\|A_{\varepsilon h}\mathbf{v}_h\|, \quad \forall\ \mathbf{v}_h \in \mathbf{H}_h,$$
$$\|A_{\varepsilon h}^{-\frac{r}{2}}\mathbf{v}_h\| \le c_0\|\mathbf{v}_h\|_{-r}, \quad \forall\ \mathbf{v}_h \in \mathbf{H}_h, \quad r \in \{1,2\}.$$

Proof. Let $\mathbf{v}_h \in \mathbf{H}_h$ and $A_{\varepsilon h}\mathbf{v}_h = \mathbf{g}$. With $q_h = -\frac{1}{\varepsilon}\nabla \cdot \mathbf{v}_h$, (4.33) can be written as

$$(\nabla \mathbf{v}_h, \nabla \boldsymbol{\phi}_h) - (q_h, \nabla \cdot \boldsymbol{\phi}_h) = (\mathbf{g}, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h,$$
$$(\nabla \cdot \mathbf{v}_h, \psi_h) + \varepsilon(q_h, \psi_h) = 0, \quad \forall \psi_h \in L_h.$$

From regularity estimate, one can find that (see, [18, (1.20)])

$$\|\Delta_h \mathbf{v}_h\| + \|\nabla q_h\| \leq c_0\|\mathbf{g}\| + \varepsilon c_0\|\nabla q_h\|.$$

Now choose ε sufficiently small such that $c_0\varepsilon < 1$, then we conclude the second result. For the first result, we choose $\boldsymbol{\phi}_h = \mathbf{v}_h$ in (4.33) and arrive at

$$\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{v}_h\|^2 = \|\nabla \mathbf{v}_h\|^2 + \frac{1}{\varepsilon}\|\nabla \cdot \mathbf{v}_h\|^2 \geq \|\nabla \mathbf{v}_h\|^2.$$

For the third one, let $\mathbf{w}_h$ be the solution of $A_{\varepsilon h}^{\frac{r}{2}}\mathbf{w}_h = A_{\varepsilon h}^{-\frac{r}{2}}\mathbf{v}_h$.

$$\|A_{\varepsilon h}^{-\frac{r}{2}}\mathbf{v}_h\|^2 = (A_{\varepsilon h}^{\frac{r}{2}}\mathbf{w}_h, A_{\varepsilon h}^{-\frac{r}{2}}\mathbf{v}_h) = (\mathbf{w}_h, \mathbf{v}_h) \leq \|\mathbf{w}_h\|_r\|\mathbf{v}_h\|_{-r}$$
$$\leq c_0\|A_{\varepsilon h}^{\frac{r}{2}}\mathbf{w}_h\|\|\mathbf{v}_h\|_{-r} \leq c_0\|A_{\varepsilon h}^{-\frac{r}{2}}\mathbf{v}_h\|\|\mathbf{v}_h\|_{-r}.$$

Cancelling one $\|A_{\varepsilon h}^{-\frac{r}{2}}\mathbf{v}_h\|$ from both sides concludes the remaining of the proof. □

4.3.1 *A Priori* Estimates

The *a priori* estimates in the semidiscrete case is similar to those of the continuous case (see Lemma 4.4 and 4.5) and in fact, the proofs are also similar to the continuous case.

Lemma 4.7. *Let us assume the hypothesis of Lemma 4.4 be satisfied. In addition, we assume that* (**B1**) *and* (**B2**) *hold. Then, with* $\mathbf{u}_{\varepsilon h}(0) = P_h\mathbf{u}_{\varepsilon 0}$*, the following result holds for any* $t > 0$*:*

$$\|\mathbf{u}_{\varepsilon h}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(s)\|^2 ds + \|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(t)\|^2 \leq C,$$

where $C > 0$ *is a constant may depends on the given data but not on* ε *and* h*.*

Proof. We take $\mathbf{v}_h = \mathbf{u}_{\varepsilon h}$ in (4.6) and follow the exact sequence of arguments as in the proof of Lemma 4.4 to find

$$\|\mathbf{u}_{\varepsilon h}(t)\|^2 + \beta_1 e^{-2\alpha t}\int_0^t e^{2\alpha s}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(s)\|^2 ds \leq e^{-2\alpha t}\|P_h\mathbf{u}_{\varepsilon 0}\|^2 + \frac{c_0^2}{2\alpha\mu\lambda_1}\|\mathbf{f}\|_\infty^2, \quad (4.34)$$

where $\beta_1 = \mu - \frac{2c_0^2\alpha}{\lambda_1}$. This concludes the first part of the proof.

Now, we obtain two intermediate estimates which will be used in the last part of the proof. A simple modification of the above estimate (4.34) gives

$$\|\mathbf{u}_{\varepsilon h}(t)\|^2 + \mu e^{-2\alpha t}\int_0^t \|A_{\varepsilon h}^{\frac{1}{2}}\hat{\mathbf{u}}_{\varepsilon h}(s)\|^2 ds \le e^{-2\alpha t}\|P_h\mathbf{u}_{\varepsilon 0}\|^2 + \frac{c_0^2(1-e^{-2\alpha t})}{2\alpha\mu\lambda_1}\|\mathbf{f}\|_\infty^2 + 2\alpha e^{-2\alpha t}\int_0^t \|\hat{\mathbf{u}}_{\varepsilon h}(s)\|^2 ds.$$

Take limit supremum as $t \to \infty$, then a use of L'Hospital rule yields

$$\frac{\mu}{2\alpha}\limsup_{t\to\infty}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(t)\|^2 \le \frac{c_0^2}{2\alpha\mu\lambda_1}\|\mathbf{f}\|_\infty^2. \tag{4.35}$$

Again, in the process of obtaining the estimate (4.34), if we avoid multiplying $e^{2\alpha t}$ and simply integrate, we find that

$$\|\mathbf{u}_{\varepsilon h}(t)\|^2 + \mu\int_0^t \|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(s)\|^2 ds \le \|P_h\mathbf{u}_{\varepsilon 0}\|^2 + \frac{c_0^2 t}{\mu\lambda_1}\|\mathbf{f}\|_\infty^2. \tag{4.36}$$

Armed with these estimates, we now proceed for the second part. Choose $\mathbf{v}_h = A_{\varepsilon h}\mathbf{u}_{\varepsilon h}$ and use the similar set of analysis of (4.17) to find

$$\frac{d}{dt}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^2 + \mu\|A_{\varepsilon h}\mathbf{u}_{\varepsilon h}\|^2 \le \frac{3}{\mu}\|\mathbf{f}\|^2 + 2(\frac{9}{2\mu})^3\|\mathbf{u}_{\varepsilon h}\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^4 + \frac{3}{\mu}\Big(\int_0^t \beta(t-s)\|\Delta_h\mathbf{u}_{\varepsilon h}(s)\|ds\Big)^2. \tag{4.37}$$

We rewrite the integral term as

$$\frac{3}{\mu}\Big(\int_0^t \beta(t-s)\|\Delta_h\mathbf{u}_{\varepsilon h}(s)\|ds\Big)^2 \le \frac{3\gamma^2 c_0^2 e^{-2\alpha t}}{2\mu(\delta-\alpha)}\int_0^t \|A_{\varepsilon h}\hat{\mathbf{u}}_{\varepsilon h}(s)\|^2 ds.$$

Use this in (4.37), then multiply by $e^{2\alpha t}$ and take time integration to obtain

$$\|A_{\varepsilon h}^{\frac{1}{2}}\hat{\mathbf{u}}_{\varepsilon h}(t)\|^2 + \mu\int_0^t \|A_{\varepsilon h}\hat{\mathbf{u}}_{\varepsilon h}\|^2 ds \le \|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(0)\|^2 + 2\alpha\int_0^t \|A_{\varepsilon h}^{\frac{1}{2}}\hat{\mathbf{u}}_{\varepsilon h}\|^2 ds + \frac{3}{\mu}\int_0^t \|\hat{\mathbf{f}}\|^2 ds + 2(\frac{9}{2\mu})^3\int_0^t \|\mathbf{u}_{\varepsilon h}\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\hat{\mathbf{u}}_{\varepsilon h}\|^2 ds + \frac{3\gamma^2 c_0^2}{2\mu(\delta-\alpha)}\int_0^t\int_0^s \|A_{\varepsilon h}\hat{\mathbf{u}}_{\varepsilon h}(\tau)\|^2 d\tau ds. \tag{4.38}$$

We set

$$g(t) := \max\Big\{2(\frac{9}{2\mu})^3\|\mathbf{u}_{\varepsilon h}\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^2,\ \frac{3\gamma^2 c_0^2}{2\mu^2(\delta-\alpha)}\Big\},$$

and now from (4.38), we obtain

$$\|A_{\varepsilon h}^{\frac{1}{2}}\hat{\mathbf{u}}_{\varepsilon h}(t)\|^2 + \mu\int_0^t \|A_{\varepsilon h}\hat{\mathbf{u}}_{\varepsilon h}\|^2 ds \le \|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(0)\|^2 + 2\alpha\int_0^t \|A_{\varepsilon h}^{\frac{1}{2}}\hat{\mathbf{u}}_{\varepsilon h}\|^2 ds + \frac{3}{\mu}\int_0^t \|\hat{\mathbf{f}}\|^2 ds + \int_0^t \Big(\|A_{\varepsilon h}^{\frac{1}{2}}\hat{\mathbf{u}}_{\varepsilon h}\|^2 ds + \mu\int_0^s \|A_{\varepsilon h}\hat{\mathbf{u}}_{\varepsilon h}(\tau)\|^2 d\tau\Big) g(s)\ ds.$$

We use the "Gronwall's lemma" to deduce

$$\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(t)\|^2 + \mu e^{-2\alpha t}\int_0^t \|A_{\varepsilon h}\hat{\mathbf{u}}_{\varepsilon h}\|^2 ds \le e^{-2\alpha t}\Big(\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(0)\|^2 + 2\alpha\int_0^t \|A_{\varepsilon h}^{\frac{1}{2}}\hat{\mathbf{u}}_{\varepsilon h}\|^2 ds + \frac{3}{\mu}\int_0^t \|\hat{\mathbf{f}}\|^2 ds\Big)\exp\{\int_0^t g(s)ds\}. \quad (4.39)$$

Note that, for a fixed and finite T_0 with $0 < t \le T_0$, we use (4.34) and (4.36) to find

$$\int_0^{T_0} g(s)ds \le CT_0. \quad (4.40)$$

We now use (4.34) and (4.40) in (4.39) to obtain

$$\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(t)\|^2 + \mu e^{-2\alpha t}\int_0^t \|A_{\varepsilon h}\hat{\mathbf{u}}_{\varepsilon h}\|^2 ds \le C(\alpha, \mu, \lambda_1, c_0^2, M_0, \gamma, \delta, T_0). \quad (4.41)$$

Therefore, the inequality (4.41) is valid for all finite, but fixed time $T_0 > 0$. Also from (4.35), we can say that $\limsup_{t\to\infty} \|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(t)\|$ is bounded, which together leads that $\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(t)\|$ is bounded for all $t > 0$, which concludes the remaining of the proof. □

Lemma 4.8. *Suppose the hypothesis of the Lemma 4.7 be satisfied. Then, for any $t > 0$, the following results hold,*

$$\begin{aligned}\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_{\varepsilon hs}(s)\|^2 ds &\le C,\\ \tau^*(t)\|\mathbf{u}_{\varepsilon ht}(t)\|^2 + e^{-2\alpha t}\int_0^t \sigma(s)\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon hs}(s)\|^2 ds &\le C,\end{aligned}$$

where $\sigma(t) = e^{2\alpha t}\tau^*(t)$ and $\tau^*(t) = min\{1, t\}$.

Proof. First, we differentiate (4.6) with respect to time to deduce

$$(\mathbf{u}_{\varepsilon htt}, \mathbf{v}_h) + \mu a_\varepsilon(\mathbf{u}_{\varepsilon ht}, \mathbf{v}_h) + \beta(0)a(\mathbf{u}_{\varepsilon h}, \mathbf{v}_h) + \int_0^t \beta_t(t-s)a(\mathbf{u}_{\varepsilon h}(s), \mathbf{v}_h)\, ds = -\tilde{b}(\mathbf{u}_{\varepsilon ht}, \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) - \tilde{b}(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon ht}, \mathbf{v}_h) + (\mathbf{f}_t, \mathbf{v}_h),\ \forall \mathbf{v}_h \in \mathbf{H}_h. \quad (4.42)$$

Now set $\mathbf{v}_h = A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht}$ in (4.42) with $\beta_t(t-s) = -\delta\beta(t-s)$ and $\beta(0) = \gamma$ to obtain

$$\frac{1}{2}\frac{d}{dt}(\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}\|^2) + \mu\|\mathbf{u}_{\varepsilon ht}\|^2 \le \delta\int_0^t \beta(t-s)a(\mathbf{u}_{\varepsilon h}(s), A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht})ds - \gamma a(\mathbf{u}_{\varepsilon h}, A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht}) - \tilde{b}(\mathbf{u}_{\varepsilon ht}, \mathbf{u}_{\varepsilon h}, A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht}) - \tilde{b}(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon ht}, A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht}) + (\mathbf{f}_t, A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht}). \quad (4.43)$$

A use of Lemma 4.3 with Lemma 4.6 and the "Young's inequality" yield

$$|-\tilde{b}(\mathbf{u}_{\varepsilon ht}, \mathbf{u}_{\varepsilon h}, A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht}) - \tilde{b}(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon ht}, A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht})| \le C\|\mathbf{u}_{\varepsilon ht}\|^{3/2}\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^{\frac{1}{2}}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|$$

$$\leq C\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^4+\frac{\mu}{4}\|\mathbf{u}_{\varepsilon ht}\|^2. \quad (4.44)$$

We bound the followings using the "Cauchy-Schwarz inequality" and the "Young's inequality" with Lemma 4.6:

$$\int_0^t \beta(t-s)a(\mathbf{u}_{\varepsilon h}(s),A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht})ds-\gamma a(\mathbf{u}_{\varepsilon h},A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht})+(\mathbf{f}_t,A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon ht})$$
$$\leq C\Big(\|\mathbf{u}_{\varepsilon h}\|^2+\|\mathbf{f}_t\|^2+\big(\int_0^t\beta(t-s)\|\mathbf{u}_{\varepsilon h}(s)\|ds\big)^2\Big)+\frac{\mu}{4}\|\mathbf{u}_{\varepsilon ht}\|^2$$
$$\leq C\Big(\|\mathbf{u}_{\varepsilon h}\|^2+\|\mathbf{f}_t\|^2+\int_0^t e^{2\alpha s}\|\mathbf{u}_{\varepsilon h}(s)\|^2ds\Big)+\frac{\mu}{4}\|\mathbf{u}_{\varepsilon ht}\|^2. \quad (4.45)$$

We use (4.44) and (4.45) in (4.43) to find

$$\frac{d}{dt}(\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}\|^2)+\mu\|\mathbf{u}_{\varepsilon ht}\|^2\leq C\Big(\|\mathbf{u}_{\varepsilon h}\|^2+\|\mathbf{f}_t\|^2+\int_0^t e^{2\alpha s}\|\mathbf{u}_{\varepsilon h}(s)\|^2ds\Big)$$
$$+C\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^4. \quad (4.46)$$

Now, we drop the second term from the left of inequality (4.46) and use the "uniform Gronwall's lemma" with Lemma 4.7 to conclude that $\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}(t+T_0)\|^2$ is uniformly bounded with respect to time, which simply says, $\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}(t)\|^2$ is uniformly bounded on $[T_0,\infty)$. Also, a use of the "classical Gronwall's lemma" implies that $\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}(t)\|^2$ is bounded on $(0,T_0)$. Hence, both of these lead to

$$\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}(t)\|^2\leq C,\quad t>0. \quad (4.47)$$

We now multiply (4.46) by $e^{2\alpha t}$ and take time integration on the both sides. Then, we use (4.47) and Lemma 4.7 and multiply by $e^{-2\alpha t}$ to conclude the first result.
For the second proof, we set $\mathbf{v}_h=\sigma(t)\mathbf{u}_{\varepsilon ht}$ in (4.42) to obtain

$$\frac{1}{2}\frac{d}{dt}(\sigma(t)\|\mathbf{u}_{\varepsilon ht}\|^2)+\mu\sigma(t)\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon ht}\|^2=\frac{1}{2}\sigma_t(t)\|\mathbf{u}_{\varepsilon ht}\|^2-\gamma\sigma(t)a(\mathbf{u}_{\varepsilon h},\mathbf{u}_{\varepsilon ht})$$
$$+\delta\sigma(t)\int_0^t\beta(t-s)a(\mathbf{u}_{\varepsilon h}(s),\mathbf{u}_{\varepsilon ht})ds-\sigma(t)\tilde{b}(\mathbf{u}_{\varepsilon ht},\mathbf{u}_{\varepsilon h},\mathbf{u}_{\varepsilon ht})+\sigma(t)(\mathbf{f}_t,\mathbf{u}_{\varepsilon ht}). \quad (4.48)$$

A use of Lemma 4.3 with the "Young's inequality" yields

$$|\tilde{b}(\mathbf{u}_{\varepsilon ht},\mathbf{u}_{\varepsilon h},\mathbf{u}_{\varepsilon ht})|\leq C\|\mathbf{u}_{\varepsilon ht}\|^{\frac{1}{2}}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon ht}\|^{3/2}\|\mathbf{u}_{\varepsilon h}\|^{\frac{1}{2}}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^{\frac{1}{2}}$$
$$\leq C(\mu)\|\mathbf{u}_{\varepsilon ht}\|^2+\frac{\mu}{2}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon ht}\|^2. \quad (4.49)$$

After using the fact $\sigma(t)\leq e^{2\alpha t},\sigma_t(t)\leq e^{2\alpha t}$ and (4.49) in (4.48), we take time integration and apply the "Cauchy-Schwarz inequality" to find

$$\sigma(t)\|\mathbf{u}_{\varepsilon ht}\|^2+\mu\int_0^t\sigma(s)\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon hs}(s)\|^2ds\leq C\Big[\int_0^t e^{2\alpha s}\big(\|\mathbf{u}_{\varepsilon hs}\|^2+\|A_{\varepsilon}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^2+\|\mathbf{f}_s\|^2\big)ds$$

$$+\int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}(\tau)\|\ d\tau\Big)^2 ds\Big]. \quad (4.50)$$

The double integral term in (4.50) can be written in a single integral similar to Lemma 4.4. We use the first estimate of this lemma and use Lemma 4.7. Finally, multiply by $e^{-2\alpha t}$ to establish the second result. □

Lemma 4.9. *Suppose the hypothesis of the Lemma 4.7 be satisfied. Then, for any $t > 0$, the following results hold:*

$$e^{-2\alpha t}\int_0^t \sigma^r(s)\|A_{\varepsilon h}^{(r-2)/2}\mathbf{u}_{\varepsilon hss}\|^2 ds \ \leq\ C, \quad r\in\{0,1,2\}.$$

Proof. The proof is quite similar for the cases $r = 0, 1$ and $r = 2$. Therefore we sketch a proof for the $r = 0$ case. For $r = 1$, we simply point out the extra term and its estimate. And for $r = 2$ case, it follows $r = 1$, and hence is avoided completely.
For $r = 0$, choose $\mathbf{v}_h = e^{2\alpha t}A_{\varepsilon h}^{-2}\mathbf{u}_{\varepsilon htt}$ in (4.42) to obtain

$$\begin{aligned}
e^{2\alpha t}\|A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon htt}\|^2 + \frac{\mu}{2}\frac{d}{dt}(e^{2\alpha t}\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}\|^2 &\leq e^{2\alpha t}\Big(\mu\alpha\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}\|^2 \\
&- \beta(0)a(\mathbf{u}_{\varepsilon h}, A_{\varepsilon h}^{-2}\mathbf{u}_{\varepsilon htt}) - \tilde{b}(\mathbf{u}_{\varepsilon ht}, \mathbf{u}_{\varepsilon h}, A_{\varepsilon h}^{-2}\mathbf{u}_{\varepsilon htt}) - \tilde{b}(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon ht}, A_{\varepsilon h}^{-2}\mathbf{u}_{\varepsilon htt}) \\
&+ (\mathbf{f}_t, A_{\varepsilon h}^{-2}\mathbf{u}_{\varepsilon htt}) - \int_0^t \beta_t(t-s)a(\mathbf{u}_{\varepsilon h}(s), A_{\varepsilon h}^{-2}\mathbf{u}_{\varepsilon htt})ds\Big).
\end{aligned}$$

Use (4.12) and Lemma 4.3 with Lemma 4.6 to bound the followings as

$$\begin{aligned}
&|-\tilde{b}(\mathbf{u}_{\varepsilon ht}, \mathbf{u}_{\varepsilon h}, A_{\varepsilon h}^{-2}\mathbf{u}_{\varepsilon htt}) - \tilde{b}(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon ht}, A_{\varepsilon h}^{-2}\mathbf{u}_{\varepsilon htt})| \\
&\quad \leq |\tilde{b}(\mathbf{u}_{\varepsilon ht}, \mathbf{u}_{\varepsilon h}, A_{\varepsilon h}^{-2}\mathbf{u}_{\varepsilon htt})| + |\tilde{b}(\mathbf{u}_{\varepsilon h}, A_{\varepsilon h}^{-2}\mathbf{u}_{\varepsilon htt}, \mathbf{u}_{\varepsilon ht})| \\
&\quad \leq C\|\mathbf{u}_{\varepsilon ht}\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^2 + \frac{1}{4}\|A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon htt}\|^2.
\end{aligned}$$

Now using above result, we finally obtain

$$\begin{aligned}
e^{2\alpha t}\|A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon htt}\|^2 + \frac{\mu}{2}\frac{d}{dt}(e^{2\alpha t}\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}\|^2) &\leq Ce^{2\alpha t}\Big(\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon ht}\|^2 + \|\mathbf{u}_{\varepsilon ht}\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^2 \\
&+\|\mathbf{u}_{\varepsilon h}\|^2 + \|\mathbf{f}_t\|^2 + (\int_0^t \beta_t(t-s)\|\mathbf{u}_{\varepsilon h}\|ds)^2\Big).
\end{aligned}$$

Integrate both the sides and use Lemmas 4.7 and 4.8 to conclude the proof in the case $r = 0$. For $r = 1$, we choose $\mathbf{v}_h = \sigma(t)A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon htt}$ in (4.42) and proceed as above. Due to the presence of $\sigma(t)$, we see a variational crime, an extra term in the form of $\int_0^t \sigma_t(t)\|\mathbf{u}_{\varepsilon ht}\|^2 ds$, which can be estimated using Lemma 4.8. This completes the case $r = 1$ and the overall proof. □

Lemma 4.10. *Suppose the hypothesis of the Lemma 4.7 be satisfied. Then, the following result holds for* $0 \le t \le T_0$*,*

$$\tau^*(t)\|A_{\varepsilon h}\mathbf{u}_{\varepsilon h}\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|A_{\varepsilon h}\mathbf{u}_{\varepsilon h}\|^2 ds \le C$$

where $\tau^*(t) = min\{1, t\}$*.*

Proof. From (4.41), we obtain the second part. To estimate $\|A_{\varepsilon h}\mathbf{u}_{\varepsilon h}\|$, we choose $\mathbf{v}_h = A_{\varepsilon h}\mathbf{u}_{\varepsilon h}$ in (4.6) and use Lemmas 4.3 and 4.6 and the "Young's inequality" to find

$$\|A_{\varepsilon h}\mathbf{u}_{\varepsilon h}\|^2 \le C\Big(\|\mathbf{u}_{\varepsilon ht}\|^2 + \|\mathbf{u}_{\varepsilon h}\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^4 + \Big(\int_0^t \beta(t-s)\|A_{\varepsilon h}\mathbf{u}_{\varepsilon h}(s)\|ds\Big)^2 + \|\mathbf{f}\|^2\Big).$$

Now multiply both sides by $\sigma(t)$ and use (4.50) to concludes the remaining of the proof. □

Remark 4.2. *Since the estimate (4.41) is valid for* $0 < t \le T_0$*. Hence, the result in Lemma 4.10 is only local and not uniform with respect to time. This is either a technical problem which we have not been able to resolve or a shortcoming of the penalised scheme for our model.*

4.3.2 Error Analysis for the Velocity

In this subsection, we analyze the semidiscrete penalized velocity. Let us denote $\mathbf{e}_\varepsilon = \mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}$, then from (4.4) and (4.6), we find

$$(\mathbf{e}_{\varepsilon t}, \mathbf{v}_h) + \mu a_\varepsilon(\mathbf{e}_\varepsilon, \mathbf{v}_h) + \int_0^t \beta(t-\tau)a(\mathbf{e}_\varepsilon(\tau), \mathbf{v}_h)\, d\tau = \tilde{b}(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) - \tilde{b}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}_h). \quad (4.51)$$

We first introduce an intermediate solution $\mathbf{w}_{\varepsilon h}$ satisfy the following linearized penalized Oldroyd model, that is, $\mathbf{w}_{\varepsilon h}$ is a solution of

$$(\mathbf{w}_{\varepsilon ht}, \mathbf{v}_h) + \mu a_\varepsilon(\mathbf{w}_{\varepsilon h}, \mathbf{v}_h) + \int_0^t \beta(t-s)a(\mathbf{w}_{\varepsilon h}(s), \mathbf{v}_h)\, ds = (\mathbf{f}, \mathbf{v}_h) - \tilde{b}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}_h), \quad (4.52)$$

for all $\mathbf{v}_h \in \mathbf{H}_h$ with $\mathbf{w}_{\varepsilon h}(0) = P_h\mathbf{u}_{\varepsilon 0}$. Now split the semi discrete penalized error as

$$\mathbf{e}_\varepsilon := \mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h} = (\mathbf{u}_\varepsilon - \mathbf{w}_{\varepsilon h}) + (\mathbf{w}_{\varepsilon h} - \mathbf{u}_{\varepsilon h}) = \boldsymbol{\xi} + \boldsymbol{\eta}.$$

Note that the error $\boldsymbol{\xi}$ occurs due to the linearized part and $\boldsymbol{\eta}$ due to the presence of nonlinear part. Below, we obtain a few results for $\boldsymbol{\xi}$. Subtracting (4.52) from (4.4), the equation in $\boldsymbol{\xi}$ is written as

$$(\boldsymbol{\xi}_t, \mathbf{v}_h) + \mu a_\varepsilon(\boldsymbol{\xi}, \mathbf{v}_h) + \int_0^t \beta(t-s)a(\boldsymbol{\xi}(s), \mathbf{v}_h)\, ds = 0, \quad \mathbf{v}_h \in \mathbf{H}_h. \quad (4.53)$$

Lemma 4.11. *Suppose the hypothesis of the Lemma 4.7 be satisfied and* $\mathbf{w}_{\varepsilon h}(t) \in \mathbf{H}_h$ *be a solution of (4.52) with* $\mathbf{w}_{\varepsilon h}(0) = P_h\mathbf{u}_{\varepsilon 0}$ *and* $\mathbf{u}_\varepsilon$ *be a weak solution of (4.2) with* $\mathbf{u}_{\varepsilon 0} \in \mathbf{H}_0^1$. *Then, for any time* $t > 0$, $\boldsymbol{\xi}$ *satisfies*

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|^2\ ds \leq Ch^4 \int_0^t e^{2\alpha s}\|A_\varepsilon\mathbf{u}_\varepsilon(s)\|^2 ds.$$

Proof. We rewrite the equation (4.2) and (4.52) as

$$A_\varepsilon^{-1}\mathbf{u}_{\varepsilon t} + \mu\mathbf{u}_\varepsilon - A_\varepsilon^{-1}\int_0^t \beta(t-s)\Delta\mathbf{u}_\varepsilon(s)\ ds = A_\varepsilon^{-1}(\mathbf{f} - \tilde{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)),$$

and

$$A_{\varepsilon h}^{-1}\mathbf{w}_{\varepsilon ht} + \mu\mathbf{w}_{\varepsilon h} - A_{\varepsilon h}^{-1}\int_0^t \beta(t-s)\Delta_h\mathbf{w}_{\varepsilon h}(s)\ ds = A_{\varepsilon h}^{-1}P_h(\mathbf{f} - \tilde{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)).$$

From the above two equations, we find that

$$\begin{aligned} A_\varepsilon^{-1}\mathbf{u}_{\varepsilon t} - A_{\varepsilon h}^{-1}\mathbf{w}_{\varepsilon ht} + \mu\boldsymbol{\xi} - A_\varepsilon^{-1}\int_0^t \beta(t-s)\Delta\mathbf{u}_\varepsilon(s)\ ds + A_{\varepsilon h}^{-1}\int_0^t \beta(t-s)\Delta_h\mathbf{w}_{\varepsilon h}(s)\ ds \\ = (A_\varepsilon^{-1} - A_{\varepsilon h}^{-1}P_h)(\mathbf{f} - \tilde{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)). \end{aligned}$$

Using (4.2) and rearranging the terms, we arrive at

$$A_{\varepsilon h}^{-1}(P_h\mathbf{u}_{\varepsilon t} - \mathbf{w}_{\varepsilon ht}) + \mu\boldsymbol{\xi} - A_{\varepsilon h}^{-1}\int_0^t \beta(t-s)(P_h\Delta_h\mathbf{u}_\varepsilon - \Delta_h\mathbf{w}_{\varepsilon h})(s)ds = \mu(\mathbf{u}_\varepsilon - A_{\varepsilon h}^{-1}P_hA_\varepsilon\mathbf{u}_\varepsilon).$$

Now use the fact $P_h\mathbf{u}_{\varepsilon t} - \mathbf{w}_{\varepsilon ht} = P_h\boldsymbol{\xi}$, we obtain

$$\begin{aligned} A_{\varepsilon h}^{-1}(P_h\mathbf{u}_{\varepsilon t} - \mathbf{w}_{\varepsilon ht}) + \mu\boldsymbol{\xi} - A_{\varepsilon h}^{-1}\int_0^t \beta(t-s)\Delta_hP_h\boldsymbol{\xi}(s)\ ds = \mu(\mathbf{u}_\varepsilon - A_{\varepsilon h}^{-1}P_hA_\varepsilon\mathbf{u}_\varepsilon) \\ -A_{\varepsilon h}^{-1}\int_0^t \beta(t-s)\Delta_hP_h(\mathbf{u}_\varepsilon - \Delta_h^{-1}P_h\Delta\mathbf{u}_\varepsilon)(s)\ ds. \end{aligned}$$

Now we multiply the above equation by $P_h\boldsymbol{\xi}$ and take integration over Ω to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|A_{\varepsilon h}^{-\frac{1}{2}}P_h\boldsymbol{\xi}\|^2 + \mu\|\boldsymbol{\xi}\|^2 + A_{\varepsilon h}^{-1}\int_0^t \beta(t-s)a(P_h\boldsymbol{\xi}(s), P_h\boldsymbol{\xi})\ ds \\ = \mu(\boldsymbol{\xi}, \mathbf{u}_\varepsilon - P_h\mathbf{u}_\varepsilon) + \mu((\mathbf{u}_\varepsilon - A_{\varepsilon h}^{-1}P_hA_\varepsilon\mathbf{u}_\varepsilon), P_h\boldsymbol{\xi}) \\ + A_{\varepsilon h}^{-1}\int_0^t \beta(t-s)a(P_h(\mathbf{u}_\varepsilon - \Delta_h^{-1}P_h\Delta\mathbf{u}_\varepsilon)(s), P_h\boldsymbol{\xi})\ ds. \end{aligned}$$

Multiplying by $e^{2\alpha t}$ and using the "Cauchy-Schwarz inequality" with Lemma 4.6 we find

$$\begin{aligned} &\frac{d}{dt}\|A_{\varepsilon h}^{-\frac{1}{2}}P_h\hat{\boldsymbol{\xi}}\|^2 - 2\alpha\|A_{\varepsilon h}^{-\frac{1}{2}}P_h\hat{\boldsymbol{\xi}}\|^2 + \mu\|\hat{\boldsymbol{\xi}}\|^2 + 2A_{\varepsilon h}^{-1}\int_0^t \beta(t-s)a(P_h\hat{\boldsymbol{\xi}}(s), P_h\hat{\boldsymbol{\xi}})\ ds \\ &\leq C\big(\mu\|\hat{\mathbf{u}}_\varepsilon - P_h\hat{\mathbf{u}}_\varepsilon\|^2 + \mu\|(\hat{\mathbf{u}}_\varepsilon - A_{\varepsilon h}^{-1}P_hA_\varepsilon\hat{\mathbf{u}}_\varepsilon)\|^2 + \big(\int_0^t \beta(t-s)\|(\hat{\mathbf{u}}_\varepsilon - \Delta_h^{-1}P_h\Delta\hat{\mathbf{u}}_\varepsilon)(s)\|ds\big)^2\big), \end{aligned}$$

where $\hat{\boldsymbol{\xi}} = e^{\alpha t}\boldsymbol{\xi}$. Using the fact $\|A_{\varepsilon h}^{-\frac{1}{2}} P_h \hat{\boldsymbol{\xi}}\|^2 \leq c_0^2 \|P_h \hat{\boldsymbol{\xi}}\|_{-1}^2 \leq \frac{c_0^2}{\lambda_1} \|\hat{\boldsymbol{\xi}}\|^2$, (1.15) and Proposition 4.1, we finally arrive at

$$\frac{d}{dt}\|A_{\varepsilon h}^{-\frac{1}{2}} P_h \hat{\boldsymbol{\xi}}\|^2 + \Big(\mu - \frac{2c_0^2\alpha}{\lambda_1}\Big)\|\hat{\boldsymbol{\xi}}\|^2 + 2A_{\varepsilon h}^{-1}\int_0^t \beta(t-s) a(P_h\hat{\boldsymbol{\xi}}(s), P_h\hat{\boldsymbol{\xi}})\, ds$$
$$\leq Ch^4\Big(\|\Delta \hat{\mathbf{u}}_\varepsilon\|^2 + \|A_\varepsilon \hat{\mathbf{u}}_\varepsilon\|^2 + (\int_0^t \beta(t-s)\|\Delta\hat{\mathbf{u}}_\varepsilon\| ds)^2\Big).$$

Now, we take time integration on the both sides and use the fact $\|A_{\varepsilon h}^{-\frac{1}{2}} P_h \boldsymbol{\xi}(0)\| = 0$ with Lemma 4.6 to obtain

$$\|A_{\varepsilon h}^{-\frac{1}{2}}\hat{\boldsymbol{\xi}}(t)\|^2 + \Big(\mu - \frac{2c_0^2\alpha}{\lambda_1}\Big)\int_0^t \|\hat{\boldsymbol{\xi}}\|^2 ds + 2\int_0^t A_{\varepsilon h}^{-1}\int_0^s \beta(s-\tau) a(P_h\hat{\boldsymbol{\xi}}(\tau), P_h\hat{\boldsymbol{\xi}}(s))\, d\tau ds$$
$$\leq Ch^4 \int_0^t \|A_\varepsilon \hat{\mathbf{u}}_\varepsilon(s)\|^2 ds + Ch^4 \int_0^t \Big(\int_0^s \beta(s-\tau)\|\Delta\hat{\mathbf{u}}_\varepsilon(\tau)\|\, d\tau\, ds\Big)^2. \quad (4.54)$$

We drop the double integration term on the left of inequality, it being positive and the double integration term on the right of inequality is converted to a single integral (see, (2.17)) thereby completing the remaining of the proof. □

In order to find optimal estimate of $\boldsymbol{\xi}$ in $L^\infty(\mathbf{L}^2)$-norm, we consider a projection which we call as penalized Stokes-Volterra projection, which is motivated form the original Stokes-Volterra projection, see [63, 116]. Let $V_h^\varepsilon : [0, T_0] \to \mathbf{H}_h$, for some $T_0 > 0$ satisfy

$$\mu a_\varepsilon(\mathbf{u}_\varepsilon - V_h^\varepsilon \mathbf{u}_\varepsilon, \mathbf{v}_h) + \int_0^t \beta(t-s) a((\mathbf{u}_\varepsilon - V_h^\varepsilon\mathbf{u}_\varepsilon)(s), \mathbf{v}_h)\, ds = 0, \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h, \quad (4.55)$$

for some fixed $\varepsilon > 0$. We note that the above system, similar to the Stokes-Volterra, has a positive definite operator, which in this case is $A_{\varepsilon h}$. Therefore, we can establish the well-posedness of the system (4.55) as in the case of the Stokes-Volterra projection. For details, we refer to [24] and [97].

We now write

$$\boldsymbol{\xi} = (\mathbf{u}_\varepsilon - V_h^\varepsilon \mathbf{u}_\varepsilon) + (V_h^\varepsilon \mathbf{u}_\varepsilon - \mathbf{w}_{\varepsilon h}) =: \boldsymbol{\zeta} + \boldsymbol{\theta}.$$

We are interested in the estimates of $\|\mathbf{u}_\varepsilon - V_h^\varepsilon \mathbf{u}_\varepsilon\|, \|\nabla(\mathbf{u}_\varepsilon - V_h^\varepsilon \mathbf{u}_\varepsilon)\|$, as this is the first step towards obtaining the optimal estimate of $\boldsymbol{\xi}$. With the notation

$$\boldsymbol{\zeta} = \mathbf{u}_\varepsilon - V_h^\varepsilon \mathbf{u}_\varepsilon,$$

we present the following Lemma.

Lemma 4.12. *Suppose the hypothesis of the Lemma 4.7 be satisfied. Then, for any* $t > 0$, *the following results hold:*

$$\|\boldsymbol{\zeta}(t)\|^2 + h^2\|\nabla\boldsymbol{\zeta}(t)\|^2 \le Ch^4\Big(\|A_\varepsilon\mathbf{u}_\varepsilon(t)\|^2 + e^{-2\alpha t}\int_0^t \|A_\varepsilon\hat{\mathbf{u}}_\varepsilon(s)\|^2 ds\Big).$$

Moreover, the following result holds:

$$\|\boldsymbol{\zeta}_t(t)\|^2 + h^2\|\nabla\boldsymbol{\zeta}_t(t)\|^2 \le Ch^4\Big(\|A_\varepsilon\mathbf{u}_\varepsilon(t)\|^2 + \|A_\varepsilon\mathbf{u}_{\varepsilon t}(t)\|^2 + e^{-2\alpha t}\int_0^t \|A_\varepsilon\hat{\mathbf{u}}_\varepsilon(s)\|^2 ds\Big).$$

Proof. We rewrite the equation (4.2) and the equation of $V_h^\varepsilon\mathbf{u}_\varepsilon$ as

$$\mu\mathbf{u}_\varepsilon - A_\varepsilon^{-1}\int_0^t \beta(t-s)\Delta\mathbf{u}_\varepsilon(s)\, ds = A_\varepsilon^{-1}(\mathbf{f} - \mathbf{u}_{\varepsilon t} - \tilde{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)), \tag{4.56}$$

$$\mu V_h^\varepsilon\mathbf{u}_\varepsilon - A_{\varepsilon h}^{-1}\int_0^t \beta(t-s)\Delta_h V_h^\varepsilon\mathbf{u}_\varepsilon(s)\, ds = A_{\varepsilon h}^{-1}P_h(\mathbf{f} - \mathbf{u}_{\varepsilon t} - \tilde{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)). \tag{4.57}$$

Hence, similar to Lemma 4.11 one can take inner product with $P_h\boldsymbol{\zeta}$ and use the "Cauchy-Schwarz inequality" to find that

$$\mu\|\boldsymbol{\zeta}\|^2 + 2A_{\varepsilon h}^{-1}\int_0^t \beta(t-s)a(P_h\boldsymbol{\zeta}(s), P_h\boldsymbol{\zeta})\, ds = C\Big(\mu\|\mathbf{u}_\varepsilon - P_h\mathbf{u}_\varepsilon\|^2 + \mu\|\mathbf{u}_\varepsilon - A_{\varepsilon h}^{-1}P_hA_\varepsilon\mathbf{u}_\varepsilon\|^2 + \big(\int_0^t \beta(t-s)\|(\mathbf{u}_\varepsilon - \Delta_h^{-1}P_h\Delta\mathbf{u}_\varepsilon)(s)\|\, ds\big)^2\Big). \tag{4.58}$$

We now multiply both sides by $e^{2\alpha t}$ and integrate. Then we drop the double integration term from the left of inequality due to positivity and use (1.15) and Proposition 4.1 with Lemma 4.6. Then we write the double integration term on the right of inequality as single integration term and finally we deduce that

$$\mu\int_0^t \|\hat{\boldsymbol{\zeta}}(s)\|^2 ds \le Ch^4\int_0^t \|A_\varepsilon\hat{\mathbf{u}}_\varepsilon(s)\|^2 ds. \tag{4.59}$$

From (4.58), one can obtain

$$\|\boldsymbol{\zeta}(t)\|^2 \le Ch^4\Big(\|A_\varepsilon\mathbf{u}_\varepsilon(t)\|^2 + e^{-2\alpha t}\int_0^t \|A_\varepsilon\hat{\mathbf{u}}_\varepsilon(s)\|^2 ds\Big).$$

A use of triangular inequality concludes the first proof. For estimate involving $\boldsymbol{\zeta}_t$, we differentiate (4.56) and (4.57) with respect to t. Then similar to above one can find the required result. This concludes the proof. □

Armed with the estimates of $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}_t$, we now pursue the estimates of $\boldsymbol{\theta}$ to find the optimal $L^\infty(L^2)$ and $L^\infty(H^1)$-error for $\boldsymbol{\xi}$. From (4.53) and (4.55), the equation in $\boldsymbol{\theta}$ turns out to be

$$(\boldsymbol{\theta}_t, \mathbf{v}_h) + \mu a_\varepsilon(\boldsymbol{\theta}, \mathbf{v}_h) + \int_0^t \beta(t-s)a(\boldsymbol{\theta}(s), \mathbf{v}_h)\, ds = -(\boldsymbol{\zeta}_t, \mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{H}_h. \tag{4.60}$$

Now, we choose $\mathbf{v}_h = \sigma(t)\boldsymbol{\theta}$ in (4.60) to find

$$\frac{d}{dt}\big(\sigma(t)\|\boldsymbol{\theta}\|^2\big) + 2\mu\sigma(t)\|A_{\varepsilon h}^{\frac{1}{2}}\boldsymbol{\theta}\|^2 = -\,2\sigma(t)(\boldsymbol{\zeta}_t,\boldsymbol{\theta}) + \sigma_t(t)\|\boldsymbol{\theta}\|^2 - 2\sigma(t)\int_0^t \beta(t-\tau)a(\boldsymbol{\theta}(\tau),\boldsymbol{\theta})\, d\tau. \tag{4.61}$$

An application of the "Young's inequality" and the "Cauchy-Schwarz inequality" with $\sigma_t(t) \le Ce^{2\alpha t}$ and $\frac{(\sigma(t))^2}{\sigma_t(t)} \le C\sigma_1(t)$ (where $\sigma_1(t) = (\tau^*(t))^2 e^{2\alpha t}$) yields

$$|2\sigma(t)(\boldsymbol{\zeta}_t,\boldsymbol{\theta})| \le \frac{(\sigma(t))^2}{\sigma_t(t)}\|\boldsymbol{\zeta}_t\|^2 + \sigma_t(t)\|\boldsymbol{\theta}\|^2 \le C\sigma_1(t)\|\boldsymbol{\zeta}_t\|^2 + Ce^{2\alpha t}\|\boldsymbol{\theta}\|^2.$$

Incorporate this in (4.61) and integrate from to deduce

$$\sigma(t)\|\boldsymbol{\theta}(t)\|^2 + 2\mu\int_0^t \sigma(s)\|A_{\varepsilon h}^{\frac{1}{2}}\boldsymbol{\theta}(s)\|^2 ds \le C\Big(\int_0^t \sigma_1(s)\|\boldsymbol{\zeta}_s(s)\|^2 ds + \int_0^t e^{2\alpha s}\|\boldsymbol{\theta}(s)\|^2 ds\Big) -2\int_0^t \sigma(s)\int_0^s \beta(s-\tau)a(\boldsymbol{\theta}(\tau),\boldsymbol{\theta}(s))d\tau ds.$$

The double integration term no longer positive. Similar to (3.62), we rewrite this as

$$\begin{aligned}\int_0^t \sigma(s)\int_0^s \beta(s-\tau)a(\boldsymbol{\theta}(\tau),\boldsymbol{\theta}(s))d\tau ds &\le C\int_0^t e^{2\alpha s}\|\nabla\widetilde{\boldsymbol{\theta}}(s)\|^2 ds + \frac{\mu}{2c_0^2}\int_0^t \sigma(s)\|\nabla\boldsymbol{\theta}(s)\|^2 \\ &\le C\int_0^t e^{2\alpha s}\|A_{\varepsilon h}^{\frac{1}{2}}\widetilde{\boldsymbol{\theta}}(s)\|^2 ds + \frac{\mu}{2}\int_0^t \sigma(s)\|A_{\varepsilon h}^{\frac{1}{2}}\boldsymbol{\theta}(s)\|^2,\end{aligned} \tag{4.62}$$

where $\widetilde{\boldsymbol{\theta}} = \int_0^t \boldsymbol{\theta}(s)\, ds$. Combining above two equations and using $\|\boldsymbol{\theta}\| \le \|\boldsymbol{\xi}\| + \|\boldsymbol{\zeta}\|$, we reach at

$$\begin{aligned}\sigma(t)\|\boldsymbol{\theta}(t)\|^2 + \mu\int_0^t \sigma(s)\|A_{\varepsilon h}^{\frac{1}{2}}\boldsymbol{\theta}(s)\|^2 ds &\le C\int_0^t \sigma_1(s)\|\boldsymbol{\zeta}_s(s)\|^2 ds. \\ &+ C\int_0^t e^{2\alpha s}(\|\boldsymbol{\xi}(s)\|^2 + \|\boldsymbol{\zeta}(s)\|^2)ds + C\int_0^t e^{2\alpha s}\|A_{\varepsilon h}^{\frac{1}{2}}\widetilde{\boldsymbol{\theta}}(s)\|^2 ds.\end{aligned} \tag{4.63}$$

In order to find the bound for the term involving 'tilde' operator, we take integration on the both sides of (4.60) and write the double integral term as in (3.64) to obtain

$$(\widetilde{\boldsymbol{\theta}}_t,\mathbf{v}_h) + \mu a_\varepsilon(\widetilde{\boldsymbol{\theta}},\mathbf{v}_h) + \int_0^t \beta(t-\tau)a(\widetilde{\boldsymbol{\theta}}(\tau),\mathbf{v}_h)\, d\tau\, ds = -(\boldsymbol{\zeta},\mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{H}_h. \tag{4.64}$$

Choose $\mathbf{v}_h = e^{2\alpha t}\widetilde{\boldsymbol{\theta}}$ in (4.64) and integrate the resulting equation. Drop the double integral term, as it is non-negative. Using (4.59), we deduce that

$$e^{2\alpha t}\|\widetilde{\boldsymbol{\theta}}(t)\|^2 + (\mu - \frac{2\alpha c_0^2}{\lambda_1})\int_0^t e^{2\alpha s}\|A_{\varepsilon h}^{\frac{1}{2}}\widetilde{\boldsymbol{\theta}}(s)\|^2\, ds \le Ch^4\int_0^t e^{2\alpha s}\|A_\varepsilon \mathbf{u}_\varepsilon\|^2 ds. \tag{4.65}$$

Incorporate (4.65) in (4.63) and use the Lemmas 4.11 and 4.12 and (4.65) to conclude

$$\tau^*(t)\|\boldsymbol{\theta}(t)\|^2 + e^{-2\alpha t}\int_0^t \sigma(s)\|A_{\varepsilon h}^{\frac{1}{2}}\boldsymbol{\theta}(s)\|^2\, ds \;\le\; Ch^4. \tag{4.66}$$

Now withthe triangle inequality, the inverse hypothesis, (4.66) and Lemma 4.12, we conclude the following results:

Lemma 4.13. *Suppose the hypothesis of the Lemma 4.7 be satisfied. Then, the following results hold for any* $t > 0$,

$$\|\boldsymbol{\xi}(t)\| + h\|\nabla\boldsymbol{\xi}(s)\| \;\leq\; Ch^2 t^{-\frac{1}{2}}.$$

With the desired estimate of $\boldsymbol{\xi}$, we aim to achieve the estimates of $\mathbf{e}_\varepsilon$ by means of $\boldsymbol{\eta}$. Note that $\mathbf{e}_\varepsilon = \boldsymbol{\xi} + \boldsymbol{\eta}$.

Lemma 4.14. *Suppose the hypothesis of the Lemma 4.7 be satisfied and* $\mathbf{u}_{\varepsilon h}(t)$ *be a solution of (4.6) with* $\mathbf{u}_{\varepsilon h}(0) = P_h\mathbf{u}_{\varepsilon 0}$. *Then, the following*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{e}_\varepsilon(s)\|^2\,ds \leq K(t)h^4,$$

holds for $0 < t \leq T_0$, *where* $K(t) = Ce^{Ct}$.

Proof. As mentioned above, it suffices to find estimates for $\boldsymbol{\eta}$. From (4.6) and (4.52), we find that

$$(\boldsymbol{\eta}_t, \mathbf{v}_h) + \mu a_\varepsilon(\boldsymbol{\eta}, \mathbf{v}_h) + \int_0^t \beta(t-s)a(\boldsymbol{\eta}(s), \mathbf{v}_h)\,ds = \Lambda_h(\mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{H}_h, \tag{4.67}$$

where

$$\Lambda_h(\mathbf{v}_h) = \tilde{b}(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) - \tilde{b}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}_h) = -\tilde{b}(\mathbf{e}_\varepsilon, \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) - \tilde{b}(\mathbf{u}_\varepsilon, \mathbf{e}_\varepsilon, \mathbf{v}_h). \tag{4.68}$$

Choose $\mathbf{v}_h = e^{2\alpha t}(A_{\varepsilon h}^{-1}\boldsymbol{\eta})$ and use Lemma 4.2 and the "Poincaré inequality" to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A_{\varepsilon h}^{-\frac{1}{2}}\hat{\boldsymbol{\eta}}\|^2 + \Big(\mu - \frac{c_0^2\alpha}{\lambda_1}\Big)\|\hat{\boldsymbol{\eta}}\|^2 + \int_0^t \beta(t-s)e^{\alpha(t-s)}a(A_{\varepsilon h}^{-\frac{1}{2}}\hat{\boldsymbol{\eta}}(s), A_{\varepsilon h}^{-\frac{1}{2}}\hat{\boldsymbol{\eta}})\,ds \\ \leq e^{\alpha t}\Lambda_h(A_{\varepsilon h}^{-1}\hat{\boldsymbol{\eta}}).\end{aligned} \tag{4.69}$$

By writing $\mathbf{e}_\varepsilon = \boldsymbol{\xi} + \boldsymbol{\eta}$ and using Lemma 4.3, we estimate Λ_h as

$$\begin{aligned}|e^{\alpha t}\Lambda_h(A_{\varepsilon h}^{-1}\hat{\boldsymbol{\eta}})| \leq{}& \frac{\mu}{2}\|\hat{\boldsymbol{\eta}}\|^2 + C(\mu)\Big(\|\mathbf{u}_{\varepsilon h}\|\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\| + \|\mathbf{u}_\varepsilon\|\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_\varepsilon\|\Big)\|\hat{\boldsymbol{\xi}}\|^2 \\ &+ C(\mu)\|A_{\varepsilon h}^{-\frac{1}{2}}\hat{\boldsymbol{\eta}}\|^2\Big(\|\mathbf{u}_{\varepsilon h}\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}\|^2 + \|\mathbf{u}_\varepsilon\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_\varepsilon\|^2\Big).\end{aligned}$$

We now integrate (4.69) with respect to time. Remove the resulting double integration term due to positivity property and obtain

$$\|A_{\varepsilon h}^{-\frac{1}{2}}\hat{\boldsymbol{\eta}}\|^2 + \Big(\mu - \frac{2c_0^2\alpha}{\lambda_1}\Big)\int_0^t \|\hat{\boldsymbol{\eta}}\|^2 ds \leq C(K,\mu)\int_0^t \|\hat{\boldsymbol{\xi}}\|^2 ds + C(K,\mu)\int_0^t \|A_{\varepsilon h}^{-\frac{1}{2}}\hat{\boldsymbol{\eta}}\|^2 ds.$$

After using the "Gronwall's Lemma" we use Lemma 4.11 to conclude the remaining of the proof. □

The main result of this section, that is, the finite element Galerkin approximation error estimate for the penalized system is presented now.

Theorem 4.3. *Suppose the conditions* (**A1**),(**A3**), (**B1**) *and* (**B2**) *be satisfied. Also, assume that the* $\mathbf{u}_{\varepsilon h}(0) \in \mathbf{H}_h$ *with* $\mathbf{u}_{\varepsilon h}(0) = P_h \mathbf{u}_{\varepsilon 0}$, *where* $\mathbf{u}_{\varepsilon 0} \in \mathbf{H}_0^1(\Omega)$. *Then, for* $0 < t \le T_0$, *the following holds:*

$$\|(\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h})(t)\| + h\|\nabla(\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h})(t)\| \le K(t) h^2 t^{-\frac{1}{2}},$$

where $K(t) = Ce^{Ct}$*and* $C > 0$ *is a constant not depend on* ε *and* h.

Proof. Since $\mathbf{e}_\varepsilon = \mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h} = (\mathbf{u}_\varepsilon - \mathbf{w}_{\varepsilon h}) + (\mathbf{w}_{\varepsilon h} - \mathbf{u}_{\varepsilon h}) = \boldsymbol{\xi} + \boldsymbol{\eta}$ and the bounds for $\boldsymbol{\xi}$ are already obtained above, it is now enough to estimate $\boldsymbol{\eta}$. Choosing $\mathbf{v}_h = \sigma(t)\boldsymbol{\eta}$ in (4.67), we arrive at

$$\frac{1}{2}\frac{d}{dt}(\sigma(t)\|\boldsymbol{\eta}\|^2) + \mu\sigma(t)\|A_{\varepsilon h}^{\frac{1}{2}}\boldsymbol{\eta}\|^2 = \frac{1}{2}\sigma_t(t)\|\boldsymbol{\eta}\|^2 - \sigma(t)\int_0^t \beta(t-s)a(\boldsymbol{\eta}(s),\boldsymbol{\eta})\, ds + \sigma(t)\Lambda_h(\boldsymbol{\eta}).$$

Use Lemma 1.4 to bound the nonlinear terms as

$$\begin{aligned}\Lambda_h(\boldsymbol{\eta}) &= -\tilde{b}(\mathbf{e}_\varepsilon, \mathbf{w}_{\varepsilon h}, \boldsymbol{\eta}) - \tilde{b}(\mathbf{u}_\varepsilon, \mathbf{e}_\varepsilon, \boldsymbol{\eta}) \\ &\le C(\|\nabla \mathbf{u}_\varepsilon(s)\|\|\Delta \mathbf{u}_\varepsilon(s)\| + \|\nabla \mathbf{w}_{\varepsilon h}(s)\|\|\Delta_h \mathbf{w}_{\varepsilon h}(s)\|)\|\mathbf{e}_\varepsilon\|\|\nabla\boldsymbol{\eta}\|.\end{aligned}$$

A use of the inverse hypothesis and the approximation property yield

$$\begin{aligned}\|\Delta_h \mathbf{w}_{\varepsilon h}\| &\le \|\Delta_h \mathbf{w}_{\varepsilon h} - \Delta_h P_h \mathbf{u}_\varepsilon\| + \|\Delta_h P_h \mathbf{u}_\varepsilon\| \le Ch^{-2}\|\mathbf{w}_{\varepsilon h} - P_h\mathbf{u}_\varepsilon\| + C\|\Delta \mathbf{u}_\varepsilon\| \\ &\le Ch^{-2}(\|\boldsymbol{\xi}\| + \|\mathbf{u}_\varepsilon - P_h\mathbf{u}_\varepsilon\|) + C\|\Delta\mathbf{u}_\varepsilon\| \le C\|\Delta\mathbf{u}_\varepsilon\|. \qquad (4.70)\end{aligned}$$

Combining above three equations and integrating the resulting equation, we find that

$$\begin{aligned}\sigma(t)\|\boldsymbol{\eta}\|^2 + \mu\int_0^t \sigma(s)\|A_{\varepsilon h}^{\frac{1}{2}}\boldsymbol{\eta}(s)\|^2 &\le 2(1+\alpha)\int_0^t \|\hat{\boldsymbol{\eta}}(s)\|^2 ds + C\int_0^t e^{2\alpha s}\|A_{\varepsilon h}^{\frac{1}{2}}\widetilde{\boldsymbol{\eta}}(s)\|^2 ds \\ &+ C\int_0^t \tau^*(s)(\|A_\varepsilon^{\frac{1}{2}}\mathbf{u}_\varepsilon(s)\|\|A_\varepsilon\mathbf{u}_\varepsilon(s)\|)\|\hat{\mathbf{e}_\varepsilon}(s)\|^2. \qquad (4.71)\end{aligned}$$

Note that the resulting double integration term is estimated similar to (4.62) with $\widetilde{\boldsymbol{\eta}}(t) = \int_0^t \boldsymbol{\eta}(s)\, ds$. In order to bound the second term of (4.71), we integrate (4.67) and similar to (4.64), we deduce

$$(\boldsymbol{\eta}, \mathbf{v}_h) + \mu a_\varepsilon(\widetilde{\boldsymbol{\eta}}, \mathbf{v}_h) + \int_0^t \beta(t-s)a(\widetilde{\boldsymbol{\eta}}(s), \mathbf{v}_h)\, ds = \int_0^t \Lambda_h(\mathbf{v}_h)\, ds. \qquad (4.72)$$

Put $\mathbf{v}_h = e^{2\alpha t}\widetilde{\boldsymbol{\eta}}$ in (4.72) and take time integration. Then, drop the double integral term from the left side due to positivity to find

$$e^{2\alpha t}\|\widetilde{\boldsymbol{\eta}}\|^2 + 2(\mu - \frac{c_0^2\alpha}{\lambda_1})\int_0^t e^{2\alpha s}\|A_{\varepsilon h}^{\frac{1}{2}}\widetilde{\boldsymbol{\eta}}(s)\|^2\, ds \le 2\int_0^t e^{2\alpha s}|\int_0^s \Lambda_h(\widetilde{\boldsymbol{\eta}}(s))\, d\tau|\, ds. \quad (4.73)$$

We bound the nonlinear terms using Lemma 4.3 as

$$2\int_0^t e^{2\alpha s}\int_0^s |\Lambda_h(\widetilde{\boldsymbol{\eta}}(s))|d\tau ds \leq Ch^4 t^{\frac{1}{2}} e^{4\alpha t} + \mu \int_0^t e^{2\alpha s}\|A_{\varepsilon h}^{\frac{1}{2}}\widetilde{\boldsymbol{\eta}}(s)\|^2 ds. \tag{4.74}$$

Incorporate (4.74) in (4.73) with $(\mu - \frac{2c_0^2\alpha}{\lambda_1}) > 0$ to obtain

$$\|\widetilde{\boldsymbol{\eta}}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|A_{\varepsilon h}^{\frac{1}{2}}\widetilde{\boldsymbol{\eta}}(s)\|^2\, ds \leq K(t)h^4 t^{\frac{1}{2}}. \tag{4.75}$$

Now, insert (4.75) in (4.71) and apply the Lemmas 4.4, 4.5 and 4.14. Then, multiplying by $e^{-2\alpha t}$, we deduce

$$\tau^*(t)\|\boldsymbol{\eta}(t)\|^2 + e^{-2\alpha t}\mu\int_0^t \sigma(s)\|A_{\varepsilon h}^{\frac{1}{2}}\boldsymbol{\eta}\|^2\, ds \leq K(t)h^4.$$

Since $\boldsymbol{\eta} \in \mathbf{H}_h$, one can apply the inverse hypothesisto find the bounds for $\|\nabla\boldsymbol{\eta}\|$. We apply the triangle inequality with Lemma 4.13 to conclude the remaining of the proof. □

4.3.3 Error Analysis for the Pressure

Below, we present the error estimate for the penalized pressure term, which turns out to be straight forward, given that error estimates of $\mathbf{u}_\varepsilon$ are known. Subtract the second equation of (4.3) from the second equation of (4.5) and obtain

$$(p_\varepsilon - p_{\varepsilon h}, \chi_h) = \frac{\mu}{\varepsilon}(\nabla\cdot\mathbf{e}_\varepsilon, \chi_h). \tag{4.76}$$

Choose $\chi_h = p_{\varepsilon h} - j_h p_\varepsilon = e_p - (p_\varepsilon - j_h p_\varepsilon)$ with $e_p = p_\varepsilon - p_{\varepsilon h}$ in (4.76) to find that

$$\begin{aligned}\|e_p\|^2 &= (e_p, p_\varepsilon - j_h p_\varepsilon) + \frac{\mu}{\varepsilon}(\nabla\cdot\mathbf{e}_\varepsilon, e_p) - \frac{\mu}{\varepsilon}(\nabla\cdot\mathbf{e}_\varepsilon, p_\varepsilon - j_h p_\varepsilon)\\ &\leq Ch^2\|p_\varepsilon\|_1^2 + \frac{C}{\varepsilon^2}\|\nabla\cdot\mathbf{e}_\varepsilon\|^2 + \frac{1}{2}\|e_p\|^2.\end{aligned}$$

If we use a direct bound like $\|\nabla\cdot\mathbf{v}\| \leq C\|\nabla\mathbf{v}\|$, then the error bound for pressure will depend on $1/\varepsilon$. Alternatively, if we estimate the divergence form, say from Lemma 4.12, as $\|\nabla\cdot(\mathbf{u}_\varepsilon - V_h^\varepsilon\mathbf{u}_\varepsilon)\| \leq C\sqrt{\varepsilon}ht^{-\frac{1}{2}}$, then we will find that $\frac{1}{\varepsilon}\|\nabla\cdot\mathbf{e}_\varepsilon\|$ depends on $1/\sqrt{\varepsilon}$ and so does e_p. Therefore it is clear that the error bound for the pressure always depends on $1/\varepsilon$ or $1/\sqrt{\varepsilon}$ if we find it directly using velocity error. A similar discussion for penalized NSE can be seen in [100] where the authors concluded that higher regularity on the data allow them to derive ε independent result. In our case, if we choose the finite element spaces $\mathbf{H}_h$ and L_h in such a way that satisfy the discrete inf-sup condition (**B2**), then we can find the ε-uniform pressure error estimate as given below.

First we split the pressure error as

$$\|e_p\| = \|p_\varepsilon - p_{\varepsilon h}\| \le \|p_\varepsilon - j_h p_\varepsilon\| + \|j_h p_\varepsilon - p_{\varepsilon h}\|. \tag{4.77}$$

From (**B2**), we observe that

$$\begin{aligned} \|j_h p_\varepsilon - p_{\varepsilon h}\| &\le C \sup_{\mathbf{v}_h \in \mathbf{H}_h \setminus \{0\}} \left\{ \frac{|(j_h p_\varepsilon - p_{\varepsilon h}, \nabla \cdot \mathbf{v}_h)|}{\|\nabla \mathbf{v}_h\|} \right\} \\ &\le C \bigg(\|j_h p_\varepsilon - p_\varepsilon\| + \sup_{\mathbf{v}_h \in \mathbf{H}_h \setminus \{0\}} \left\{ \frac{|(p_\varepsilon - p_{\varepsilon h}, \nabla \cdot \mathbf{v}_h)|}{\|\nabla \mathbf{v}_h\|} \right\} \bigg). \end{aligned} \tag{4.78}$$

Using (**B1**), we bound the first term of (4.78). For the second term, we subtract (4.5) from (4.3) to obtain for all $\mathbf{v}_h \in \mathbf{H}_h$

$$(e_p, \nabla \cdot \mathbf{v}_h) = (\mathbf{e}_{\varepsilon t}, \mathbf{v}_h) + \mu a(\mathbf{e}_\varepsilon, \mathbf{v}_h) + \int_0^t \beta(t-s) a(\mathbf{e}_\varepsilon, \mathbf{v}_h) ds - \Lambda_h(\mathbf{v}). \tag{4.79}$$

To estimate Λ_h, we use $\tilde{b}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{v}_h) \le C\|\nabla \mathbf{u}_h\| \|\nabla \mathbf{v}_h\| \|\nabla \mathbf{v}_h\|$ to bound

$$|\Lambda_h(\mathbf{v}_h)| \le C(\|\nabla \mathbf{u}_\varepsilon\| + \|\nabla \mathbf{u}_{\varepsilon h}\|) \|\nabla \mathbf{e}_\varepsilon\| \|\nabla \mathbf{v}_h\|. \tag{4.80}$$

After inserting (4.80) in (4.79), we apply the “Cauchy-Schwarz inequality” to arrive at

$$(e_p, \nabla \cdot \mathbf{v}_h) \le \Big[C\|\mathbf{e}_{\varepsilon t}\|_{-1;h} + C\|\nabla \mathbf{e}_\varepsilon\| + \int_0^t \beta(t-s) \|\nabla \mathbf{e}_\varepsilon\| ds \Big] \|\nabla \mathbf{v}_h\| \tag{4.81}$$

where,

$$\|\mathbf{e}_{\varepsilon t}\|_{-1;h} = \sup \left\{ \frac{\langle \mathbf{e}_{\varepsilon t}, \mathbf{v}_h \rangle}{\|\nabla \mathbf{v}_h\|} : \mathbf{v}_h \in \mathbf{H}_h, \mathbf{v}_h \neq 0 \right\}. \tag{4.82}$$

Since all the estimate on the right of inequality in (4.81) are known except $\|\mathbf{e}_{\varepsilon t}\|_{-1;h}$, and since $\|\mathbf{e}_{\varepsilon t}\|_{-1;h} \le \|\mathbf{e}_{\varepsilon t}\|_{-1} := \sup \left\{ \frac{\langle \mathbf{e}_{\varepsilon t}, \mathbf{v} \rangle}{\|\nabla \mathbf{v}\|} : \mathbf{v} \in \mathbf{H}_0^1, \mathbf{v} \neq 0 \right\}$, it is sufficient to derive the following estimate.

Lemma 4.15. *Suppose the hypothesis of Theorem 4.3 be satisfied. Then, the following negative error estimate holds for* $0 < t < T$*:*

$$\|\mathbf{e}_{\varepsilon t}\|_{-1} \le C\Big(h(\|\mathbf{u}_{\varepsilon t}\| + \|\nabla \mathbf{e}_\varepsilon\| + \int_0^t \beta(t-s) \|\nabla \mathbf{e}_\varepsilon\| ds \Big).$$

Proof. For any $\boldsymbol{\psi} \in \mathbf{H}_0^1$, we use the orthogonal projection $P_h : \mathbf{L}^2 \to \mathbf{H}_h$ and (4.51) with $\mathbf{v}_h = P_h\boldsymbol{\psi}$ to obtain

$$\begin{aligned}(\mathbf{e}_{\varepsilon t}, \boldsymbol{\psi}) &= (\mathbf{e}_{\varepsilon t}, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) + (\mathbf{e}_{\varepsilon t}, P_h\boldsymbol{\psi}) \\ &= (\mathbf{e}_{\varepsilon t}, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) - \mu a_\varepsilon(\mathbf{e}_\varepsilon, P_h\boldsymbol{\psi}) - \int_0^t \beta(t-s)a(\mathbf{e}_\varepsilon, P_h\boldsymbol{\psi}_h)ds - \Lambda_h(P_h\boldsymbol{\psi}).\end{aligned} \tag{4.83}$$

We use the approximation property of P_h to bound the following as

$$(\mathbf{e}_{\varepsilon t}, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) = (\mathbf{u}_{\varepsilon t} - P_h\mathbf{u}_{\varepsilon t}, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) \leq Ch\|\mathbf{u}_{\varepsilon t}\|\|\nabla\boldsymbol{\psi}\|. \tag{4.84}$$

Also, using Lemma 4.3 with boundedness of $\mathbf{u}_\varepsilon$ and $\mathbf{u}_{\varepsilon h}$ to bound

$$\Lambda_h(P_h\boldsymbol{\psi}) \leq C(\|\nabla\mathbf{u}_\varepsilon\| + \|\nabla\mathbf{u}_{\varepsilon h}\|)\|\nabla\mathbf{e}_\varepsilon\|\|\nabla\boldsymbol{\psi}\| \leq C\|\nabla\mathbf{e}_\varepsilon\|\|\nabla\boldsymbol{\psi}\|. \tag{4.85}$$

Now substitute (4.84)-(4.85) in (4.83) to obtain

$$(\mathbf{e}_{\varepsilon t}, \boldsymbol{\psi}) \leq C\Big(h(\|\mathbf{u}_{\varepsilon t}\| + C\|\nabla\mathbf{e}_\varepsilon\| + \int_0^t \beta(t-s)\|\nabla\mathbf{e}_\varepsilon\|ds\Big)\|\nabla\boldsymbol{\psi}\|.$$

and therefore,

$$\begin{aligned}\|\mathbf{e}_{\varepsilon t}\|_{-1} &\leq \sup\Big\{\frac{<\mathbf{e}_{\varepsilon t}, \mathbf{v}>}{\|\nabla\mathbf{v}\|} : \mathbf{v} \in \mathbf{H}_0^1, \mathbf{v} \neq 0\Big\} \\ &\leq C\Big(h(\|\mathbf{u}_{\varepsilon t}\| + C\|\nabla\mathbf{e}_\varepsilon\| + \int_0^t \beta(t-s)\|\nabla\mathbf{e}_\varepsilon\|ds\Big).\end{aligned}$$

This completes the proof. □

A use of (4.77)-(4.81) and Lemma 4.15 with Lemma 4.5 and Theorem 4.4 will now result in the following:

Lemma 4.16. *Let us assume the hypothesis of the Lemma* 4.11 *be satisfied. Then, for* $0 < t < T_0$, *it holds:*

$$\|(p_\varepsilon - p_{\varepsilon h})(t)\| \leq K(t)ht^{-\frac{1}{2}},$$

where $K(t) = Ce^{Ct}$.

4.3.4 Uniform in Time Bounds

The estimates derived in Theorem 4.3 are not uniform in time due to the exponential in time behaviour of the error bounds. But under the uniqueness condition (4.32), we find the following uniform (in time) estimates.

Theorem 4.4. *Suppose the assumptions of Theorem 4.3 and the uniqueness condition (4.32) hold. Then, for any $t>0$ following holds :*

$$\|(\mathbf{u}_\varepsilon-\mathbf{u}_{\varepsilon h})(t)\|+h\|\nabla(\mathbf{u}_\varepsilon-\mathbf{u}_{\varepsilon h})(t)\|+h\|(p_\varepsilon-p_{\varepsilon h})(t)\|\le Ch^2t^{-\frac{1}{2}}.$$

Proof. Recall $\mathbf{e}_\varepsilon=\boldsymbol{\xi}+\boldsymbol{\eta}$ and the bounds of $\boldsymbol{\xi}$ are uniformly in time (see, Lemma 4.13), but the estimates of $\boldsymbol{\eta}$ are not uniform (see, Lemma 4.14) due to use of the 'wall's lemma. Hence, it is enough to make the estimates of $\boldsymbol{\eta}$ are uniform in time. The idea is to estimate nonlinear term in a different manner using uniqueness condition (4.32) such that we can avoid the use of "Gronwall's lemma". For this, we choose $\mathbf{v}_h=e^{2\alpha t}\boldsymbol{\eta}$ in (4.67) to obtain

$$\frac{1}{2}\frac{d}{dt}(e^{2\alpha t}\|\boldsymbol{\eta}\|^2)+\mu e^{2\alpha t}\|A_{\varepsilon h}^{\frac{1}{2}}\boldsymbol{\eta}\|^2+e^{2\alpha t}\int_0^t\beta(t-s)a(\boldsymbol{\eta}(s),\boldsymbol{\eta})\,ds=e^{2\alpha t}(\alpha\|\boldsymbol{\eta}\|^2+\Lambda_h(\boldsymbol{\eta})).\quad(4.86)$$

From (4.68) and (4.12), we rewrite the nonlinear terms as

$$\Lambda_h(\boldsymbol{\eta})=-\tilde{b}(\mathbf{e}_\varepsilon,\mathbf{u}_{\varepsilon h},\boldsymbol{\eta})-\tilde{b}(\mathbf{u}_\varepsilon,\mathbf{e}_\varepsilon,\boldsymbol{\eta})=\tilde{b}(\boldsymbol{\xi},\mathbf{w}_{\varepsilon h},\boldsymbol{\eta})-\tilde{b}(\boldsymbol{\eta},\mathbf{u}_{\varepsilon h},\boldsymbol{\eta})-\tilde{b}(\mathbf{u}_\varepsilon,\boldsymbol{\xi},\boldsymbol{\eta}).$$

A use of (4.32) help us to bound the second nonlinear term as

$$|\tilde{b}(\boldsymbol{\eta},\mathbf{u}_{\varepsilon h},\boldsymbol{\eta})|\le N\|\nabla\mathbf{u}_{\varepsilon h}\|\|\nabla\boldsymbol{\eta}\|^2.$$

We apply Lemma 1.4 with (4.70) and the "Cauchy-Schwarz inequality" to find

$$\begin{aligned}|\tilde{b}(\boldsymbol{\xi},\mathbf{w}_{\varepsilon h},\boldsymbol{\eta})-\tilde{b}(\mathbf{u}_\varepsilon,\boldsymbol{\xi},\boldsymbol{\eta})|&\le C(\|\nabla\mathbf{u}_\varepsilon\|^{\frac{1}{2}}\|\Delta\mathbf{u}_\varepsilon\|^{\frac{1}{2}}+\|\nabla\mathbf{w}_{\varepsilon h}\|^{\frac{1}{2}}\|\Delta_h\mathbf{w}_{\varepsilon h}\|^{\frac{1}{2}})\|\boldsymbol{\xi}\|\|\nabla\boldsymbol{\eta}\|\\&\le C(\|\nabla\mathbf{u}_\varepsilon\|^2+\|\Delta\mathbf{u}_\varepsilon\|^2)\|\boldsymbol{\xi}\|^2+\frac{\mu}{2}\|\nabla\boldsymbol{\eta}\|^2.\end{aligned}$$

Substitute the above two in (4.86) and integrate to find

$$\begin{aligned}e^{2\alpha t}\|\boldsymbol{\eta}(t)\|^2&+2\int_0^t e^{2\alpha s}(\frac{\mu}{2}-N\|\nabla\mathbf{u}_{\varepsilon h}\|)\|\nabla\boldsymbol{\eta}\|^2ds+\frac{2\mu}{\varepsilon}\int_0^t e^{2\alpha s}\|\nabla\cdot\boldsymbol{\eta}\|^2ds\\&+2\int_0^t e^{2\alpha s}\int_0^s\beta(s-\tau)a(\boldsymbol{\eta}(\tau),\boldsymbol{\eta}(s))\,d\tau\,ds\le\|\boldsymbol{\eta}(0)\|^2+2\alpha\int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(s)\|^2ds\\&+C\int_0^t e^{2\alpha s}(\|\nabla\mathbf{u}_\varepsilon(s)\|^2+\|\Delta\mathbf{u}_\varepsilon(s)\|^2)\|\boldsymbol{\xi}(s)\|^2\,ds.\quad(4.87)\end{aligned}$$

We rewrite the last term as

$$\begin{aligned}&\int_0^t e^{2\alpha s}(\|\nabla\mathbf{u}_\varepsilon(s)\|^2+\|\Delta\mathbf{u}_\varepsilon(s)\|^2)\|\boldsymbol{\xi}(s)\|^2ds\\&\le\|\boldsymbol{\xi}(t)\|^2_{L^\infty(\mathbf{L}^2)}\int_0^t e^{2\alpha s}(\|\nabla\mathbf{u}_\varepsilon(s)\|^2+\|\Delta\mathbf{u}_\varepsilon(s)\|^2)ds\quad(4.88)\end{aligned}$$

Use (4.88) with Lemma 4.4 in (4.87) and multiply both sides by $e^{-2\alpha t}$ to find

$$\|\boldsymbol{\eta}(t)\|^2 + 2e^{-2\alpha t}\int_0^t e^{2\alpha s}(\frac{\mu}{2} - N\|\nabla \mathbf{u}_{\varepsilon h}\|)\|\nabla\boldsymbol{\eta}\|^2 ds + \frac{2\mu}{\varepsilon}e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\cdot\boldsymbol{\eta}\|^2 ds$$
$$+ 2e^{-2\alpha t}\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)a(\boldsymbol{\eta}(\tau),\boldsymbol{\eta}(s))\, d\tau\, ds$$
$$\leq e^{-2\alpha t}\|\boldsymbol{\eta}(0)\|^2 + 2\alpha e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(s)\|^2 ds + C\|\boldsymbol{\xi}(t)\|^2_{L^\infty(\mathbf{L}^2)}.$$

Now, take limit supremum as $t\to\infty$ and L'Hospital rule with the followings from [63]

$$\lim_{t\to\infty}\sup e^{-2\alpha t}\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)a(\boldsymbol{\eta}(\tau),\boldsymbol{\eta}(s))\, d\tau\, ds = \frac{\gamma}{2\alpha\delta}\lim_{t\to\infty}\sup\|\nabla\boldsymbol{\eta}\|^2,$$
$$\lim_{t\to\infty}\sup\|\nabla\mathbf{u}_{\varepsilon h}\| \leq \nu^{-1}\|\mathbf{f}_\infty\|_{-1},$$

to conclude

$$[\frac{\mu}{2} - N\nu^{-1}\|\mathbf{f}_\infty\|_{-1} + \frac{\gamma}{\delta}]\lim_{t\to\infty}\sup\|\nabla\boldsymbol{\eta}\|^2 \leq C\lim_{t\to\infty}\sup\|\boldsymbol{\xi}(t)\|^2_{L^\infty(\mathbf{L}^2)}.$$

With $1 - N\nu^{-2}\|\mathbf{f}_\infty\|_{-1} > 0$, we have $[\frac{\mu}{2} - N\nu^{-1}\|\mathbf{f}_\infty\|_{-1} + \frac{\gamma}{\delta}] = \frac{1}{\nu}[1 - N\nu^{-2}\|\mathbf{f}_\infty\|_{-1}] > 0$ and we obtain the following

$$\lim_{t\to\infty}\sup\|\boldsymbol{\eta}\| \leq \lim_{t\to\infty}\sup\|\nabla\boldsymbol{\eta}\| \leq C\lim_{t\to\infty}\sup\|\boldsymbol{\xi}(t)\|_{L^\infty(\mathbf{L}^2)}.$$

Combine with the estimates of $\boldsymbol{\xi}$, we conclude the first two parts of the proof. For the pressure estimate, we use these uniform results in (4.77)-(4.81) and Lemma 4.15. □

4.4 Fully Discrete Formulation

We begin this section with short discussion about *a priori* bounds of the fully discrete solution. And then we move on to the error estimates due to time discretization.

We prove *a priori* bounds for the discrete solutions $\{\mathbf{U}^n\}_{1\leq n\leq N}$.

Lemma 4.17. *Suppose the conditions* (**A1**) *and* (**A3**) *be satisfied. Then, for* $1\leq n\leq N$, *the following results hold:*

$$\Big(\|\mathbf{U}^n_\varepsilon\|^2 + \mu e^{-2\alpha t_n}k\sum_{i=1}^n e^{2\alpha t_i}\|A^{\frac{1}{2}}_{\varepsilon h}\mathbf{U}^i_\varepsilon\|^2\Big) + \|A^{\frac{1}{2}}_{\varepsilon h}\mathbf{U}^n_\varepsilon\|^2 \leq C, \tag{4.89}$$

where we choose $\alpha > 0$ *such that* $0 < \alpha < \min\{\delta, \frac{\mu\lambda_1}{2c_0^2}\}$ *and the following holds*

$$1 + (\frac{\mu\lambda_1}{2c_0^2})k \geq e^{2\alpha k}. \tag{4.90}$$

Remark 4.3. *We would like to note here that the assumption (4.90) above can be rephrased as: for* $0 < \alpha < \alpha_0$*, (4.90) holds. And therefore both the conditions of Lemma 4.17 can be incorporated in one:* $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu\lambda_1}{2c_0^2}\}$*. See Reamrk 2.1, Chapter 2 for details.*

Proof of Lemma 4.17: For $n = i$, we substitute $\mathbf{v}_h = \mathbf{U}_\varepsilon^i$ (4.9) and use the fact $(\partial_t \mathbf{U}_\varepsilon^i, \mathbf{U}_\varepsilon^i) \geq \frac{1}{2}\partial_t \|\mathbf{U}_\varepsilon^i\|^2$ and $\tilde{b}(\mathbf{U}_\varepsilon^i, \mathbf{U}_\varepsilon^i, \mathbf{U}_\varepsilon^i) = 0$. Then we use the "Cauchy-Schwarz inequality" and the "Poincaré inequality" with Lemma 4.6 ($\|\mathbf{U}_\varepsilon^i\|^2 \leq \frac{1}{\lambda_1}\|\nabla \mathbf{U}_\varepsilon^i\|^2 \leq \frac{c_0^2}{\lambda_1}\|A_{\varepsilon h}^{\frac{1}{2}} \mathbf{U}_\varepsilon^i\|^2$) to deduce

$$\partial_t \|\mathbf{U}_\varepsilon^i\|^2 + \frac{3\mu}{2}\|A_{\varepsilon h}^{\frac{1}{2}} \mathbf{U}_\varepsilon^i\|^2 + 2a(q_r^i(\mathbf{U}_\varepsilon), \mathbf{U}_\varepsilon^i) \leq \frac{2c_0^2}{\mu\lambda_1}\|\mathbf{f}^i\|^2. \tag{4.91}$$

Multiply by $ke^{2\alpha t_i}$ and take summation over $1 \leq i \leq n$ and then use the following fact

$$k\sum_{i=1}^{n} e^{2\alpha t_i}\partial_t \|\mathbf{U}_\varepsilon^i\|^2 \geq e^{2\alpha t_n}\|\mathbf{U}_\varepsilon^n\|^2 - \|\mathbf{U}_\varepsilon^0\|^2 - k\sum_{i=1}^{n-1} c_0^2\Big(\frac{e^{2\alpha k}-1}{k\lambda_1}\Big)e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}} \mathbf{U}_\varepsilon^i\|^2$$

to obtain

$$\begin{aligned} e^{2\alpha t_n}\|\mathbf{U}_\varepsilon^n\|^2 + \Big(\frac{3\mu}{2} - c_0^2\big(\frac{e^{2\alpha k}-1}{k\lambda_1}\big)\Big)k\sum_{i=1}^{n} e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}} \mathbf{U}_\varepsilon^i\|^2 + 2k\sum_{i=1}^{n} e^{2\alpha t_i} a(q_r^i(\mathbf{U}_\varepsilon), \mathbf{U}_\varepsilon^i) \\ \leq \|\mathbf{U}_\varepsilon^0\|^2 + \frac{2c_0^2}{\mu\lambda_1}\|\mathbf{f}\|_\infty^2 k\sum_{i=1}^{n} e^{2\alpha t_i}, \end{aligned} \tag{4.92}$$

where we denote $\|\mathbf{f}\|_\infty = \|\mathbf{f}\|_{L^\infty(\mathbb{R}_+;\mathbf{L}^2(\Omega))}$. Third term of the left of inequality (4.92) is positive due to (1.18), hence we drop it. With $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu\lambda_1}{2c_0}\}$, we have $\frac{\mu}{2} \geq c_0^2\big(\frac{e^{2\alpha k}-1}{k\lambda_1}\big)$. Hence, multiply both sides by $e^{-2\alpha t_n}$ to conclude

$$\|\mathbf{U}_\varepsilon^n\|^2 + \mu e^{-2\alpha t_n}k\sum_{i=1}^{n} e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}} \mathbf{U}_\varepsilon^i\|^2 \leq e^{-2\alpha t_n}\|\mathbf{U}_\varepsilon^0\|^2 + \frac{c_0^2 e^{2\alpha k}}{\alpha\mu\lambda_1}\|\mathbf{f}\|_\infty^2 = M_{11}, \tag{4.93}$$

which concludes the first part of the proof.

For the remaining part, first we obtain two intermediate estimates. We drop the first term on the left of inequality (4.93) and let

$$\phi^n = \mu k\sum_{i=1}^{n} e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}} \mathbf{U}_\varepsilon^i\|^2 \quad \text{and} \quad \psi^n = e^{2\alpha t_n}.$$

Note that here, ψ^n is a monotonically increasing sequence with $\psi \to \infty$ as $n \to \infty$ and

$$\limsup_{n\to\infty}\left(\frac{\phi^n - \phi^{n-1}}{\psi^n - \psi^{n-1}}\right) = \frac{\mu k}{1 - e^{-2\alpha k}}\limsup_{n\to\infty}\|A_{\varepsilon h}^{\frac{1}{2}} \mathbf{U}_\varepsilon^n\|^2,$$

and from (4.93), it is clear that

$$\limsup_{n\to\infty}\left(\frac{\phi^n}{\psi^n}\right) \le \frac{c_0^2}{\alpha\mu\lambda_1}\|\mathbf{f}\|_\infty^2 e^{2\alpha k}.$$

Then, an application of the Theorem 4.1 and the mean value theorem yield

$$\limsup_{n\to\infty}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^n\|^2 \le \frac{c_0^2}{\alpha\mu^2\lambda_1}\|\mathbf{f}\|_\infty^2\Big(\frac{e^{2\alpha k}-1}{k}\Big) \le \frac{2c_0^2}{\mu^2\lambda_1}\|\mathbf{f}\|_\infty^2 e^{2\alpha k}. \tag{4.94}$$

Again, we take sum in (4.91) over $i = m$ to $m+l$, for $m, l \ge 0$ and drop the third term on the left of the resulting inequality due to positivity property to obtain

$$\|\mathbf{U}_\varepsilon^{m+l}\|^2 + \frac{3\mu}{2}k\sum_{i=m}^{m+l}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^i\|^2 \le \|\mathbf{U}_\varepsilon^m\|^2 + \frac{2c_0^2 l}{\mu\lambda_1}\|\mathbf{f}\|_\infty^2 \le M_{11} + \frac{2c_0^2 l}{\mu\lambda_1}\|\mathbf{f}\|_\infty^2. \tag{4.95}$$

Now, we choose $\mathbf{v}_h = A_{\varepsilon h}\mathbf{U}_\varepsilon^i$ in (4.9) and argue with the similar set of analysis of (4.37) to arrive

$$\partial_t\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^i\|^2 + \mu\|A_{\varepsilon h}\mathbf{U}_\varepsilon^i\|^2 \le \frac{3}{\mu}\|\mathbf{f}^i\|^2 + 2(\frac{9}{2\mu})^3\|\mathbf{U}_\varepsilon^i\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^i\|^4 + \frac{3}{\mu}\|q_r^i(\Delta_h\mathbf{U}_\varepsilon)\|^2.$$

Using (4.7), the last term can be written as

$$\frac{3}{\mu}\|q_r^i(\Delta_h\mathbf{U}_\varepsilon)\|^2 \le \frac{3}{\mu}\Big(k\sum_{j=1}^{i}\beta(t_n-t_j)\|\Delta_h\mathbf{U}_\varepsilon^j\|\Big)^2 \le \frac{3\gamma^2c_0^2e^{-2\alpha t_i}}{2\mu(\delta-\alpha)}k\sum_{j=1}^{i}e^{2\alpha t_j}\|A_{\varepsilon h}\mathbf{U}_\varepsilon^j\|^2$$

Combine above two equations and multiply by $e^{2\alpha t_i}$ and take sum $i = 1$ to n to find

$$\begin{aligned} e^{2\alpha t_n}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^n\|^2 + \mu k\sum_{i=1}^{n}e^{2\alpha t_i}\|A_{\varepsilon h}\mathbf{U}_\varepsilon^i\|^2 &\le \|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^0\|^2 + \Big(\frac{e^{2\alpha k}-1}{k}\Big)k\sum_{i=1}^{n}e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^i\|^2 \\ &\quad + \frac{3}{\mu}k\sum_{i=1}^{n}e^{2\alpha t_i}\|\mathbf{f}^i\|^2 + 2(\frac{9}{2\mu})^3 k\sum_{i=1}^{n}e^{2\alpha t_i}\|\mathbf{U}_\varepsilon^i\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^i\|^4 \\ &\quad + \frac{3\gamma^2c_0^2}{2\mu(\delta-\alpha)}k\sum_{i=1}^{n}k\sum_{j=1}^{i}e^{2\alpha t_j}\|A_{\varepsilon h}\mathbf{U}_\varepsilon^j\|^2. \end{aligned}$$

We now set,

$$g^i = \max\{2(\frac{9}{2\mu})^3\|\mathbf{U}_\varepsilon^i\|^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^i\|^2, \frac{3\gamma^2c_0^2}{2\mu^2(\delta-\alpha)}\}. \tag{4.96}$$

Then

$$\begin{aligned} e^{2\alpha t_n}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^n\|^2 + \mu k\sum_{i=1}^{n}e^{2\alpha t_i}\|A_{\varepsilon h}\mathbf{U}_\varepsilon^i\|^2 &\le \|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^0\|^2 + \Big(\frac{e^{2\alpha k}-1}{k}\Big)k\sum_{i=1}^{n}e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^i\|^2 \\ &\quad + \frac{3}{\mu}k\sum_{i=1}^{n}e^{2\alpha t_i}\|\mathbf{f}^i\|^2 + k\sum_{i=1}^{n}\Big[e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^i\|^2 + \mu k\sum_{j=1}^{i}e^{2\alpha t_j}\|A_{\varepsilon h}\mathbf{U}_\varepsilon^j\|^2\Big]g^i. \end{aligned}$$

Now, an application of the "discrete Gronwall's lemma" yields

$$e^{2\alpha t_n}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^n\|^2+\mu k\sum_{i=1}^n e^{2\alpha t_i}\|A_{\varepsilon h}\mathbf{U}_\varepsilon^i\|^2\le\Big(\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^0\|^2+\frac{3}{\mu}k\sum_{i=1}^n e^{2\alpha t_i}\|\mathbf{f}^i\|^2$$
$$+\Big(\frac{e^{2\alpha k}-1}{k}\Big)k\sum_{i=1}^n e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^i\|^2\Big)\exp\{k\sum_{i=1}^n g^i\}. \tag{4.97}$$

For a finite but fixed N and $1\le n\le N$, from (4.96) and (4.95), it follows that

$$k\sum_{i=1}^N g^i\le CN. \tag{4.98}$$

Now, a use of (4.93) and (4.98) in (4.97) gives

$$\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^n\|^2+\mu e^{-2\alpha t_n}k\sum_{i=1}^n e^{2\alpha t_i}\|A_{\varepsilon h}\mathbf{U}_\varepsilon^i\|^2\le C(\alpha,\mu,\lambda_1,c_0,\gamma,\delta,M_0,N) \tag{4.99}$$

Therefore, the inequality (4.99) is valid for all finite but fixed N. Also, from (4.94), we can say that $\limsup_{n\to\infty}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^n\|$ is bounded, which together leads the uniform in time bound for $\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^n\|$ for all $n>0$. This concludes the remaining of the proof. □

Remark 4.4. *We note here that since the bounds proved above are independent of n, $1\le n\le N$, these bounds are uniform in time, that is, they are still valid as the final time $t_N\to+\infty$.*

4.4.1 Fully Discrete Error Estimates

Define $\mathbf{u}_{\varepsilon h}(t_n)=\mathbf{u}_{\varepsilon h}^n$ and $\mathbf{e}_\varepsilon^n=\mathbf{U}_\varepsilon^n-\mathbf{u}_{\varepsilon h}^n$. Consider (4.6) at $t=t_n$ and subtract from (4.9) to arrive at

$$(\partial_t\mathbf{e}_\varepsilon^n,\mathbf{v}_h)+\mu a_\varepsilon(\mathbf{e}_\varepsilon^n,\mathbf{v}_h)+a(q_r^n(\mathbf{e}_\varepsilon),\mathbf{v}_h)=R_h^n(\mathbf{v}_h)+\Lambda_h^n(\mathbf{v}_h)+E_h^n(\mathbf{v}_h) \tag{4.100}$$

where,

$$R_h^n(\mathbf{v}_h)=(\mathbf{u}_{\varepsilon ht}^n,\mathbf{v}_h)-(\partial_t\mathbf{u}_{\varepsilon h}^n,\mathbf{v}_h)=\frac{1}{k}\int_{t_{n-1}}^{t_n}(t-t_{n-1})(\mathbf{u}_{\varepsilon hss},\mathbf{v}_h)\,ds, \tag{4.101}$$

$$\begin{aligned}\Lambda_h^n(\mathbf{v}_h)&=\tilde b(\mathbf{u}_{\varepsilon h}^n,\mathbf{u}_{\varepsilon h}^n,\mathbf{v}_h)-\tilde b(\mathbf{U}_\varepsilon^n,\mathbf{U}_\varepsilon^n,\mathbf{v}_h)\\&=\tilde b(\mathbf{e}_\varepsilon^n,\mathbf{e}_\varepsilon^n,\mathbf{v}_h)-\tilde b(\mathbf{e}_\varepsilon^n,\mathbf{u}_{\varepsilon h}^n,\mathbf{v}_h)-\tilde b(\mathbf{u}_{\varepsilon h}^n,\mathbf{e}_\varepsilon^n,\mathbf{v}_h),\end{aligned} \tag{4.102}$$

$$\begin{aligned}E_h^n(\mathbf{v}_h)&=\int_0^{t_n}\beta(t_n-s)a(\mathbf{u}_{\varepsilon h}(s),\mathbf{v}_h)\,ds-a(q_r^n(\mathbf{u}_{\varepsilon h}),\mathbf{v}_h)\\&\le C\sum_{i=1}^n\int_{t_{i-1}}^{t_i}(s-t_{i-1})\Big(\beta_s(t_n-s)a(\mathbf{u}_{\varepsilon h},\mathbf{v}_h)+\beta(t_n-s)a(\mathbf{u}_{\varepsilon hs},\mathbf{v}_h)\Big)\,ds.\end{aligned} \tag{4.103}$$

Below, we discuss the error analysis and prove optimal error estimates through a series of lemmas.

Lemma 4.18. *Suppose the conditions of Lemma 4.17 hold true. Further, assume that* (**B1**) *and* (**B2**) *be satisfied. Then, for* $0 < n < N$*, the following results hold:*

$$\|A_{\varepsilon h}^{r/2}\mathbf{e}_\varepsilon^n\|^2 + ke^{-2\alpha t_n}\sum_{i=1}^n e^{2\alpha t_i}\|A_{\varepsilon h}^{(1+r)/2}\mathbf{e}_\varepsilon^i\|^2 \le K_n k^{1-r}, \quad r = -1, 0.$$

Proof. For $r = 0$, we choose $\mathbf{v}_h = \mathbf{e}_\varepsilon^i$ in (4.100) with $n = i$, and multiply by $ke^{2\alpha t_i}$ and take sum from $i = 1$ to n. Then using the fact

$$k\sum_{i=1}^n e^{2\alpha t_i}\partial_t\|\mathbf{e}_\varepsilon^i\|^2 \ge e^{2\alpha t_n}\|\mathbf{e}_\varepsilon^n\|^2 - k\sum_{i=1}^{n-1} c_0^2\Big(\frac{e^{2\alpha k}-1}{k\lambda_1}\Big)e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac12}\mathbf{e}_\varepsilon^i\|^2, \tag{4.104}$$

we arrive at

$$e^{2\alpha t_n}\|\mathbf{e}_\varepsilon^n\|^2 + \Big(2\mu - c_0^2\big(\frac{e^{2\alpha k}-1}{k\lambda_1}\big)\Big)k\sum_{i=1}^n e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac12}\mathbf{e}_\varepsilon^i\|^2 + 2k\sum_{i=1}^n e^{2\alpha t_i}a(q_r^i(\mathbf{e}_\varepsilon), \mathbf{e}_\varepsilon^i)$$
$$\le 2k\sum_{i=1}^n e^{2\alpha t_i}\Big(R_h^i(\mathbf{e}_\varepsilon^i) + \Lambda_h^i(\mathbf{e}_\varepsilon^i) + E_h^i(\mathbf{e}_\varepsilon^i)\Big). \tag{4.105}$$

Third term on the left of inequality vanishes due to the positivity property (1.18). We use (4.101) with the "Cauchy-Schwarz inequality" and $t - t_{i-1} \le t, t \in [t_{i-1}, t_i]$ to find

$$\begin{aligned}
2k\sum_{i=1}^n e^{2\alpha t_i}R_h^i(\mathbf{e}_\varepsilon^i) &\le 2k\sum_{i=1}^n e^{2\alpha t_i}\Big(\frac1k\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|A_{\varepsilon h}^{-\frac12}\mathbf{u}_{\varepsilon hss}\|ds\Big)\|A_{\varepsilon h}^{\frac12}\mathbf{e}_\varepsilon^i\| \\
&\le Ck\sum_{i=1}^n e^{2\alpha t_i}\Big(\frac1k\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|A_{\varepsilon h}^{-\frac12}\mathbf{u}_{\varepsilon hss}\|ds\Big)^2 + \frac{\mu}{3}k\sum_{i=1}^n e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac12}\mathbf{e}_\varepsilon^i\|^2 \\
&\le \frac{C}{k}\sum_{i=1}^n e^{2\alpha t_i}\Big(\int_{t_{i-1}}^{t_i}(s-t_{i-1})ds\Big)\Big(\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|A_{\varepsilon h}^{-\frac12}\mathbf{u}_{\varepsilon hss}\|^2ds\Big) \\
&\quad + \frac{\mu}{3}k\sum_{i=1}^n e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac12}\mathbf{e}_\varepsilon^i\|^2 \\
&\le Ck\int_0^{t_n}\sigma(s)\|A_{\varepsilon h}^{-\frac12}\mathbf{u}_{\varepsilon hss}\|^2ds + \frac{\mu}{3}k\sum_{i=1}^n e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac12}\mathbf{e}_\varepsilon^i\|^2.
\end{aligned} \tag{4.106}$$

Using (4.103) and similar argument as (4.106), we arrive at

$$\begin{aligned}
2k\sum_{i=1}^n e^{2\alpha t_i}E_h^i(\mathbf{e}_\varepsilon^i) &\le Ck\sum_{i=1}^n e^{2\alpha t_i}\Big(\sum_{j=1}^i\int_{t_{j-1}}^{t_j}(s-t_{j-1})\beta(t_i-s)\{\delta\|\nabla\mathbf{u}_{\varepsilon h}\| + \|\nabla\mathbf{u}_{\varepsilon hs}\|\}\Big)\|\nabla\mathbf{e}_\varepsilon^i\| \\
&\le Cke^{2\alpha t_n} + \frac{\mu}{3}k\sum_{i=1}^n e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac12}\mathbf{e}_\varepsilon^i\|^2.
\end{aligned} \tag{4.107}$$

We use (4.12) and Lemma 4.3 with the "Cauchy-Schwarz inequality" in (4.102) to bound the nonlinear terms as

$$2k\sum_{i=1}^n e^{2\alpha t_i}\Lambda_h^i(\mathbf{e}_\varepsilon^i) \le Ck\sum_{i=1}^n e^{2\alpha t_i}\|\mathbf{e}_\varepsilon^i\|^{\frac12}\|A_{\varepsilon h}^{\frac12}\mathbf{e}_\varepsilon^i\|^{\frac12}\|A_{\varepsilon h}^{\frac12}\mathbf{u}_\varepsilon^i\|\|\mathbf{e}_\varepsilon^i\|^{\frac12}\|A_{\varepsilon h}^{\frac12}\mathbf{e}_\varepsilon^i\|^{\frac12}$$

$$\le Ck\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}_\varepsilon^i\|^2 + \frac{\mu}{3}k\sum_{i=1}^{n} e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2$$
$$\le Cke^{2\alpha t_n} + Ck\sum_{i=1}^{n-1} e^{2\alpha t_i}\|\mathbf{e}_\varepsilon^i\|^2 + \frac{\mu}{3}k\sum_{i=1}^{n} e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2. \qquad (4.108)$$

Note that, in the last line of (4.108), we use $\|\mathbf{e}_\varepsilon^n\| \le \|\mathbf{u}_\varepsilon^n\| + \|\mathbf{U}_\varepsilon^n\| \le C$. Now, inserting (4.106)-(4.108) in (4.105), we find that

$$e^{2\alpha t_n}\|\mathbf{e}_\varepsilon^n\|^2 + \Big(\mu - c_0^2\big(\frac{e^{2\alpha k}-1}{k\lambda_1}\big)\Big)k\sum_{i=1}^{n} e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2 \le Cke^{2\alpha t_n} + Ck\sum_{i=1}^{n-1} e^{2\alpha t_i}\|\mathbf{e}_\varepsilon^i\|^2.$$

With $0 < \alpha < \min\{\alpha_0, \delta, \frac{\mu\lambda_1}{2c_0^2}\}$, we have $\mu - c_0^2\big(\frac{e^{2\alpha k}-1}{k\lambda_1}\big) > 0$. Then, we use the "discrete Gronwall's lemma" to conclude the proof for the case $r = 0$.

For $r = -1$, we take $\mathbf{v}_h = A_{\varepsilon h}^{-1}\mathbf{e}_\varepsilon^n$ in (4.100) with $n = i$. Then multiply by $ke^{2\alpha t_i}$ and sum from $i = 1$ to n to arrive at

$$e^{2\alpha t_n}\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{e}_\varepsilon^n\|^2 + \Big(2\mu - c_0^2\big(\frac{e^{2\alpha k}-1}{k\lambda_1}\big)\Big)k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}_\varepsilon^i\|^2 + 2k\sum_{i=1}^{n} e^{2\alpha t_i}a(q_r^i(\mathbf{e}_\varepsilon), A_{\varepsilon h}^{-1}\mathbf{e}_\varepsilon^i)$$
$$\le 2k\sum_{i=1}^{n} e^{2\alpha t_i}\Big(R_h^i(A_{\varepsilon h}^{-1}\mathbf{e}_\varepsilon^i) + \Lambda_h^i(A_{\varepsilon h}^{-1}\mathbf{e}_\varepsilon^i) + E_h^i(A_{\varepsilon h}^{-1}\mathbf{e}_\varepsilon^i)\Big). \qquad (4.109)$$

We use the positivity property (1.18) to drop the quadrature term from the left of inequality. A use of the "Cauchy-Schwarz inequality" and Lemma 4.9 with (4.101), we bound the R_h^i term as:

$$2k\sum_{i=1}^{n} e^{2\alpha t_i}R_h^i(A_{\varepsilon h}^{-1}\mathbf{e}_\varepsilon^i) \le 2k\sum_{i=1}^{n} e^{2\alpha t_i}\Big(\frac{1}{k}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon hss}\|ds\Big)\|\mathbf{e}_\varepsilon^i\|$$
$$\le 2k\sum_{i=1}^{n} e^{2\alpha t_i}\frac{1}{k}\Big(\int_{t_{i-1}}^{t_i}(s-t_{i-1})^2ds\Big)^{\frac{1}{2}}\Big(\int_{t_{i-1}}^{t_i}\|A_{\varepsilon h}^{-1}\mathbf{u}_{\varepsilon hss}\|^2ds\Big)^{\frac{1}{2}}\|\mathbf{e}_\varepsilon^i\|$$
$$\le Ck^2e^{2\alpha t_n} + \frac{\mu}{3}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}_\varepsilon^i\|^2. \qquad (4.110)$$

Using (4.103) and similar argument as (4.110), we estimate E_h^i as:

$$2k\sum_{i=1}^{n} e^{2\alpha t_i}E_h^i(A_{\varepsilon h}^{-1}\mathbf{e}_\varepsilon^i) \le Ck\sum_{i=1}^{n} e^{2\alpha t_i}\Big(\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\beta(t_i-s)\{\delta\|\mathbf{u}_{\varepsilon h}\| + \|\mathbf{u}_{\varepsilon hs}\|\}\Big)\|\mathbf{e}_\varepsilon^i\|$$
$$\le Ck^2e^{2\alpha t_n} + \frac{\mu}{3}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}_\varepsilon^i\|^2. \qquad (4.111)$$

Next we use Lemma 4.3 and "Young's inequality" to bound Λ_h^i as

$$2k\sum_{i=1}^{n} e^{2\alpha t_i}\Lambda_h^i(A_{\varepsilon h}^{-1}\mathbf{e}_\varepsilon^i) \le 2k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}_\varepsilon^i\|^{3/2}(\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}^i\| + \|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{U}_\varepsilon^i\|)\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^{\frac{1}{2}}$$

$$\leq Ck\sum_{i=1}^{n} e^{2\alpha t_i}\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{e}_{\varepsilon}^i\|^2 + \frac{\mu}{3}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}_{\varepsilon}^i\|^2$$

$$\leq Cke^{2\alpha t_n}\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{e}_{\varepsilon}^n\|^2 + Ck\sum_{i=1}^{n-1} e^{2\alpha t_i}\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{e}_{\varepsilon}^i\|^2 + \frac{\mu}{3}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}_{\varepsilon}^i\|^2. \qquad (4.112)$$

Use (4.110)-(4.112) in (4.109) with $\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{e}_{\varepsilon}^n\|^2 \leq c_0^2\|\mathbf{e}_{\varepsilon}^n\|_{-1}^2 \leq C\|\mathbf{e}_{\varepsilon}^n\|^2 \leq Ck$ to obtain

$$e^{2\alpha t_n}\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{e}_{\varepsilon}^n\|^2 + \big(\mu - c_0^2(\frac{e^{2\alpha k}-1}{k\lambda_1})\big)k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}_{\varepsilon}^i\|^2 \leq Ck^2e^{2\alpha t_n} + Ck\sum_{i=1}^{n-1} e^{2\alpha t_i}\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{e}_{\varepsilon}^i\|^2.$$

we apply the "discrete Gronwall's Lemma" to conclude the proof for the case $r = -1$. □

Remark 4.5. *Due to use of the "discrete Gronwall's lemma", the generic constant $K_n > 0$, which is the form of Ce^{Ct_n}, $C > 0$, depends on n, hence the above estimates are local in time.*

Note that the error estimates obtained in Lemma 4.18 are sub-optimal. But based on these, we derive our optimal results. We first present below an optimal error in $L^\infty(\mathbf{L}^2)$-norm.

Lemma 4.19. *Let the assumptions of Lemma 4.18 be satisfied. Then, for $0 < n < N$, the following holds*

$$\tau_n\|\mathbf{e}_{\varepsilon}^n\|^2 + ke^{-2\alpha t_n}\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_{\varepsilon}^i\|^2 \leq K_nk^2,$$

where $\sigma_i = \tau_i e^{2\alpha t_i}$ and $\tau_i = \min\{1, t_i\}$.

Proof. Take $n = i$ and $\mathbf{v}_h = \sigma_i\mathbf{e}_{\varepsilon}^i$ in (4.100) to arrive at

$$\sigma_i\partial_t\|\mathbf{e}_{\varepsilon}^i\|^2 + 2\mu\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_{\varepsilon}^i\|^2 + 2a(q_r^i(\mathbf{e}_{\varepsilon}), \sigma_i\mathbf{e}_{\varepsilon}^i) \leq 2R_h^i(\sigma_i\mathbf{e}_{\varepsilon}^i) + 2\Lambda_h^i(\sigma_i\mathbf{e}_{\varepsilon}^i) + 2E_h^i(\sigma_i\mathbf{e}_{\varepsilon}^i).$$

Now multiply by k and take summation over $1 \leq i \leq n$ and use the fact

$$k\sum_{i=1}^{n}\sigma_i\partial_t\|\mathbf{e}_{\varepsilon}^i\|^2 \geq \sigma_n\|\mathbf{e}_{\varepsilon}^n\|^2 - e^{2\alpha k}k\sum_{i=1}^{n-1} e^{2\alpha t_i}\|\mathbf{e}_{\varepsilon}^i\|^2 - c_0^2\Big(\frac{e^{2\alpha k}-1}{k\lambda_1}\Big)k\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_{\varepsilon}^i\|^2,$$

to obtain

$$\sigma_n\|\mathbf{e}_{\varepsilon}^n\|^2 + \Big(2\mu - c_0^2(\frac{e^{2\alpha k}-1}{k\lambda_1})\Big)k\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_{\varepsilon}^i\|^2 \leq e^{2\alpha k}k\sum_{i=1}^{n-1} e^{2\alpha t_i}\|\mathbf{e}_{\varepsilon}^i\|^2$$

$$-2k\sum_{i=1}^{n} a(q_r^i(\mathbf{e}_{\varepsilon}), \sigma_i\mathbf{e}_{\varepsilon}^i) + 2k\sum_{i=1}^{n}\sigma_i\big(R_h^i(\mathbf{e}_{\varepsilon}^i) + \Lambda_h^i(\mathbf{e}_{\varepsilon}^i) + E_h^i(\mathbf{e}_{\varepsilon}^i)\big). \qquad (4.113)$$

With $\widehat{\mathbf{v}}_h^n = k\sum_{i=1}^n \mathbf{v}_h^i$, a use of (1.21) with the "Poincaré inequality" and Lemma 4.6 gives

$$
\begin{aligned}
|2k\sum_{i=1}^{n} a(q_r^i(\mathbf{e}_\varepsilon),\sigma_i\mathbf{e}_\varepsilon^i)| &= k\sum_{i=1}^{n}\sigma_i k\sum_{j=1}^{i}\beta(t_i-t_j)a(\mathbf{e}_\varepsilon^j,\mathbf{e}_\varepsilon^i)\\
&= k\sum_{i=1}^{n}\sigma_i\Big(\gamma a(\widehat{\mathbf{e}}_\varepsilon^i,\mathbf{e}_\varepsilon^i) - k\sum_{j=1}^{i-1}\partial_t\beta(t_i-t_j)a(\widehat{\mathbf{e}}_\varepsilon^j,\mathbf{e}_\varepsilon^i)\Big)\\
&\leq \frac{\mu}{8}k\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2 + Ck\sum_{i=1}^{n}e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\widehat{\mathbf{e}}_\varepsilon^i\|^2. \qquad (4.114)
\end{aligned}
$$

A use of (4.101) with the "Cauchy-Schwarz inequality" yields

$$
\begin{aligned}
k\sum_{i=1}^{n}\sigma_i R_h^i(\mathbf{e}_\varepsilon^i) &\leq k\sum_{i=1}^{n}\sigma_i\Big(\frac{1}{k}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon hss}\|\;ds\Big)\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|\\
&\leq Ck\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}ds\Big)\Big(\int_{t_{i-1}}^{t_i}\sigma(s)\|A_{\varepsilon h}^{-\frac{1}{2}}\mathbf{u}_{\varepsilon hss}\|^2 ds\Big) + \frac{\mu}{8}k\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2\\
&\leq Ck^2e^{2\alpha t_n} + \frac{\mu}{8}k\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2. \qquad (4.115)
\end{aligned}
$$

We use (4.103) with the "Cauchy-Schwarz inequality" and Lemma 4.6 to bound

$$
k\sum_{i=1}^{n}\sigma_i E_h^i(\mathbf{e}_\varepsilon^i) \leq Ck^2e^{2\alpha t_n} + \frac{\mu}{8}k\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2. \qquad (4.116)
$$

From Lemma 4.3 and 4.18, we bound the nonlinear terms as

$$
\begin{aligned}
k\sum_{i=1}^{n}\sigma_i\Lambda_h^i(\mathbf{e}_\varepsilon^i) &\leq Ck\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}^i\|^2\|\mathbf{e}_\varepsilon^i\|^2 + \frac{\mu}{8}k\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2\\
&\leq Ce^{Ct_n}k^2e^{2\alpha t_n} + \frac{\mu}{8}k\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2. \qquad (4.117)
\end{aligned}
$$

Incorporating (4.114)-(4.117) in (4.113) and assuming

$$
ke^{-2\alpha t_n}\sum_{i=1}^{n}e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\widehat{\mathbf{e}}_\varepsilon^i\|^2 \leq K_n k^2, \qquad (4.118)
$$

completes the remaining of the proof. □

We now prove (4.118) in the following lemma.

Lemma 4.20. *Let the assumptions of Lemma 4.18 be satisfied. Then, for* $0 < n < N$, *the following holds*

$$
\|\widehat{\mathbf{e}}_\varepsilon^n\|^2 + ke^{-2\alpha t_n}\sum_{i=1}^{n}e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\widehat{\mathbf{e}}_\varepsilon^i\|^2 \leq K_n k^2.
$$

Proof. First we multiply (4.100) by k and take summation over $1 \le i \le n$ to obtain

$$(\partial_t \widehat{\mathbf{e}}_\varepsilon^n, \mathbf{v}_h) + \mu a(\widehat{\mathbf{e}}_\varepsilon^n, \mathbf{v}_h) + a(q_r^n(\widehat{\mathbf{e}}_\varepsilon), \mathbf{v}_h) = k \sum_{i=1}^n (R_h^i(\mathbf{v}_h) + \Lambda_h^i(\mathbf{v}_h) + E_h^i(\mathbf{v}_h)). \qquad (4.119)$$

For $n = i$ and choose $\mathbf{v}_h = \widehat{\mathbf{e}}_\varepsilon^i$ in (4.119) then we find that

$$\partial_t \|\widehat{\mathbf{e}}_\varepsilon^i\|^2 + 2\mu \|A_{\varepsilon h}^{\frac{1}{2}} \widehat{\mathbf{e}}_\varepsilon^i\|^2 + 2a(q_r^i(\widehat{\mathbf{e}}_\varepsilon), \widehat{\mathbf{e}}_\varepsilon^i) = 2k \sum_{j=1}^i (R_h^j(\widehat{\mathbf{e}}_\varepsilon^i) + \Lambda_h^j(\widehat{\mathbf{e}}_\varepsilon^i) + E_h^j(\widehat{\mathbf{e}}_\varepsilon^i)).$$

Now multiply by $ke^{2\alpha t_i}$ and take summation over $1 \le i \le n$ and use the similar fact as like (4.104). Then, we use the "Poincaré inequality" and Lemma 4.1 to arrive at

$$\begin{aligned} e^{2\alpha t_n} \|\widehat{\mathbf{e}}_\varepsilon^n\|^2 + \Big(2\mu - \frac{c_0^2(e^{2\alpha k} - 1)}{k\lambda_1}\Big) k \sum_{i=1}^n e^{2\alpha t_i} \|A_{\varepsilon h}^{\frac{1}{2}} \widehat{\mathbf{e}}_\varepsilon^i\|^2 + 2k \sum_{i=1}^n e^{2\alpha t_i} a(q_r^i(\widehat{\mathbf{e}}_\varepsilon), \widehat{\mathbf{e}}_\varepsilon^i) \\ \le 2k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{j=1}^i (R_h^j(\widehat{\mathbf{e}}_\varepsilon^i) + \Lambda_h^j(\widehat{\mathbf{e}}_\varepsilon^i) + E_h^j(\widehat{\mathbf{e}}_\varepsilon^i)). \quad (4.120) \end{aligned}$$

The quadrature term vanishes due to positivity property (1.18). From (4.101), we obtain

$$\begin{aligned} 2k \sum_{i=1}^n e^{2\alpha t_i} k \sum_{j=1}^i R_h^j(\widehat{\mathbf{e}}_\varepsilon^i) &= 2k \sum_{i=1}^n e^{2\alpha t_i} \Big(k \sum_{j=1}^i \frac{1}{k} \int_{t_{j-1}}^{t_j} (s - t_{j-1})(\mathbf{u}_{\varepsilon hss}, \widehat{\mathbf{e}}_\varepsilon^i) ds\Big) \\ &\le 2k \sum_{i=1}^n e^{2\alpha t_i} \Big(k \sum_{j=1}^i \frac{1}{k} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) \|A_{\varepsilon h}^{-\frac{1}{2}} \mathbf{u}_{\varepsilon hss}\| ds\Big) \|A_{\varepsilon h}^{\frac{1}{2}} \widehat{\mathbf{e}}_\varepsilon^i\|. \qquad (4.121) \end{aligned}$$

We use the "Cauchy-Schwarz inequality" with $\tau_n \le \tau_{n-1} + k \le C\tau(t), t \in [t_{n-1}, t_n]$ to estimate the term in the bracket on the right of inequality (4.121) as

$$\begin{aligned} \sum_{j=1}^i \int_{t_{j-1}}^{t_j} (s - t_{j-1}) \|A_{\varepsilon h}^{-\frac{1}{2}} \mathbf{u}_{\varepsilon hss}\| ds &\le k\Big(\sum_{j=1}^i \int_{t_{j-1}}^{t_j} \frac{e^{-2\alpha s}}{\tau_j} ds\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^i \int_{t_{j-1}}^{t_j} \tau_j e^{2\alpha s} \|A_{\varepsilon h}^{-\frac{1}{2}} \mathbf{u}_{\varepsilon hss}\|^2 ds\Big)^{\frac{1}{2}} \\ &\le Ck\Big(\sum_{j=1}^i \frac{k}{\tau_j}\Big)^{\frac{1}{2}} \Big(\int_0^{t_i} \tau(s) e^{2\alpha s} \|A_{\varepsilon h}^{-\frac{1}{2}} \mathbf{u}_{\varepsilon hss}\|^2 ds\Big)^{\frac{1}{2}}. \quad (4.122) \end{aligned}$$

If $0 < t_j < 1$, then $\tau_j = t_j = jk$ and hence

$$\sum_{j=1}^i \frac{k}{\tau_j} = \sum_{j=1}^i \frac{k}{t_j} = \sum_{j=1}^i \frac{1}{j} = \log(i) + r,$$

where r is Euler constant. And when $t_j \ge 1$, then $\tau_j = 1$ and hence

$$\sum_{j=1}^i \frac{k}{\tau_j} = \sum_{j=1}^i k = t_i,$$

Now use (4.122) in (4.121) to obtain

$$2k\sum_{i=1}^{n} e^{2\alpha t_i} k\sum_{j=1}^{i} R_h^j(\widehat{\mathbf{e}}_\varepsilon^i) \le Ck^2 e^{2\alpha t_n} + \frac{\mu}{3} k\sum_{i=1}^{n} e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\widehat{\mathbf{e}}_\varepsilon^i\|^2. \tag{4.123}$$

Using (4.103) and similar argument as (4.122), we arrive at

$$2k\sum_{i=1}^{n} e^{2\alpha t_i} k\sum_{j=1}^{i} E_h^j(\widehat{\mathbf{e}}_\varepsilon^i) \le Ck^2 e^{2\alpha t_n} + \frac{\mu}{3} k\sum_{i=1}^{n} e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\widehat{\mathbf{e}}_\varepsilon^i\|^2. \tag{4.124}$$

Also using the Lemma 4.3 to bound the nonlinear term as:

$$\begin{aligned}
&2k\sum_{i=1}^{n} e^{2\alpha t_i} k\sum_{j=1}^{i} \Lambda_h^j(\widehat{\mathbf{e}}_\varepsilon^i) = 2k\sum_{i=1}^{n} e^{2\alpha t_i} k\sum_{j=1}^{i}(\tilde{b}(\mathbf{e}_\varepsilon^j,\mathbf{e}_\varepsilon^j,\widehat{\mathbf{e}}_\varepsilon^i) - \tilde{b}(\mathbf{e}_\varepsilon^j,\mathbf{u}_{\varepsilon h}^j,\widehat{\mathbf{e}}_\varepsilon^i) - \tilde{b}(\mathbf{u}_{\varepsilon h}^j,\mathbf{e}_\varepsilon^j,\widehat{\mathbf{e}}_\varepsilon^i)\\
&\le Ck\sum_{i=1}^{n} e^{2\alpha t_i}\Big(k\sum_{j=1}^{i}(\|\mathbf{e}_\varepsilon^j\|^{\frac{1}{2}}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^j\|^{3/2} + \|\mathbf{e}_\varepsilon^j\|\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}^j\|^{\frac{1}{2}}\|A_{\varepsilon h}\mathbf{u}_{\varepsilon h}^j\|^{\frac{1}{2}})\Big)\|A_{\varepsilon h}^{\frac{1}{2}}\widehat{\mathbf{e}}_\varepsilon^i\|\\
&\le Ck\sum_{i=1}^{n} e^{2\alpha t_i}\Bigg[\Big(k\sum_{j=1}^{i} e^{2\alpha t_j}\|\mathbf{e}_\varepsilon^j\|^2\Big)^{1/4}\Big(k\sum_{j=1}^{i} e^{2\alpha t_j}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^j\|^2\Big)^{\frac{1}{2}}\Big(k\sum_{j=1}^{i} e^{-6\alpha t_j}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^j\|^2\Big)^{1/4}\\
&\qquad + \Big(k\sum_{j=1}^{i} e^{2\alpha t_j}\|\mathbf{e}_\varepsilon^j\|^2\Big)^{\frac{1}{2}}\Big(k\sum_{j=1}^{i} e^{2\alpha t_j}\|A_{\varepsilon h}\mathbf{u}_{\varepsilon h}^j\|^2\Big)^{\frac{1}{2}}\Big(k\sum_{j=1}^{i} e^{-6\alpha t_j}\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{u}_{\varepsilon h}^j\|^2\Big)^{\frac{1}{4}}\Bigg]\|A_{\varepsilon h}^{\frac{1}{2}}\widehat{\mathbf{e}}_\varepsilon^i\|\\
&\le Ce^{Ct_n}k^2 e^{2\alpha t_n} + \frac{\mu}{3}k\sum_{i=1}^{n} e^{2\alpha t_i}\|A_{\varepsilon h}^{\frac{1}{2}}\widehat{\mathbf{e}}_\varepsilon^i\|^2.
\end{aligned} \tag{4.125}$$

A use of (4.121)-(4.125) in (4.120) concludes the remaining of the proof. □

Arguing with the similar way, we can derive the optimal $\mathbf{H}^1$-velocity error.

Lemma 4.21. *Suppose the hypothesis of Lemma 4.18 be satisfied. Then, for* $0 < n < N$*, the following holds*

$$(\tau_n)^2\|\nabla\mathbf{e}_\varepsilon^n\|^2 + ke^{-2\alpha t_n}\sum_{i=1}^{n}(\tau_i)^2 e^{2\alpha t_i}\|A_{\varepsilon h}\mathbf{e}_\varepsilon^i\|^2 \le K_n k^2.$$

Proof. The proof is very close to the previous lemma's proof. So we only give a sketch of the proof. Let $\sigma_i^2 = (\tau_i)^2 e^{2\alpha t_i}$ and choose $\mathbf{v}_h = \sigma_i^2 A_{\varepsilon h}\mathbf{e}_\varepsilon^n$ with $n = i$ in (4.100) and multiplying by k and summing over $1 \le i \le n$, we reach at

$$\begin{aligned}
\sigma_n^2\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^n\|^2 + 2\mu k\sum_{i=1}^{n}\sigma_i^2\|A_{\varepsilon h}\mathbf{e}_\varepsilon^i\|^2 + 2k\sum_{i=1}^{n}\sigma_i^2 a(q_r^i(\mathbf{e}_\varepsilon), A_{\varepsilon h}\mathbf{e}_\varepsilon^i) \le k\sum_{i=1}^{n-1}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2\\
+ 2k\sum_{i=1}^{n}\sigma_i^2\big(R_h^i(A_{\varepsilon h}\mathbf{e}_\varepsilon^i) + \Lambda_h^i(A_{\varepsilon h}\mathbf{e}_\varepsilon^i) + E_h^i(A_{\varepsilon h}\mathbf{e}_\varepsilon^i)\big).
\end{aligned} \tag{4.126}$$

As in (4.117), an application of (1.21) with the "Poincaré inequality" and Lemma 4.6 gives

$$|2k\sum_{i=1}^{n} a(q_r^i(\mathbf{e}_\varepsilon), \sigma_i^2 A_{\varepsilon h}\mathbf{e}_\varepsilon^i)| \le \frac{\mu}{8}k\sum_{i=1}^{n}\sigma_i^2\|A_{\varepsilon h}\mathbf{e}_\varepsilon^i\|^2 + Ck\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}\widehat{\mathbf{e}}_\varepsilon^i\|^2. \tag{4.127}$$

We use the "Cauchy-Schwarz inequality" and the "Young's inequality" with (4.101) and Lemma 4.2 to bound

$$\begin{aligned} k\sum_{i=1}^{n}\sigma_i^2 R_h^i(A_{\varepsilon h}\mathbf{e}_\varepsilon^i) &\le k\sum_{i=1}^{n}\sigma_i^2\Big(\frac{1}{k}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|\mathbf{u}_{\varepsilon hss}\|\ ds\Big)\ \|A_{\varepsilon h}\mathbf{e}_\varepsilon^i\| \\ &\le Ck\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i} ds\Big)\Big(\int_{t_{i-1}}^{t_i}\sigma_i^2(s)\|\mathbf{u}_{\varepsilon hss}\|^2 ds\Big) + \frac{\mu}{4}k\sum_{i=1}^{n}\sigma_i^2\|A_{\varepsilon h}\mathbf{e}_\varepsilon^i\|^2 \\ &\le Ck^2\int_0^{t_n}\sigma^2(s)\|\mathbf{u}_{\varepsilon hss}\|^2 ds + \frac{\mu}{8}k\sum_{i=1}^{n}\sigma_i^2\|A_{\varepsilon h}\mathbf{e}_\varepsilon^i\|^2. \end{aligned} \tag{4.128}$$

Incorporating with the Lemmas 4.3, 4.6 and 4.18, we can bound the nonlinear terms as

$$\begin{aligned} k\sum_{i=1}^{n}\sigma_i^2\Lambda_h(A_{\varepsilon h}\mathbf{e}_\varepsilon^i) &\le Ck\sum_{i=1}^{n}\sigma_i^2\big(\|\Delta_h\mathbf{u}_{\varepsilon h}^i\|\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\| + \|\mathbf{e}_\varepsilon^i\|^2\|\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2\big)\|A_{\varepsilon h}\mathbf{e}_\varepsilon^i\| \\ &\le Ck\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}^{\frac{1}{2}}\mathbf{e}_\varepsilon^i\|^2 + \frac{\mu}{8}k\sum_{i=1}^{n}\sigma_i^2\|A_{\varepsilon h}\mathbf{e}_\varepsilon^i\|^2. \end{aligned} \tag{4.129}$$

A use of (4.103) with the "Cauchy-Schwarz inequality" and Lemma 4.6 gives

$$k\sum_{i=1}^{n}\sigma_i^2 E_h^i(A_{\varepsilon h}\mathbf{e}_\varepsilon^i) \le Ck^2 e^{2\alpha t_n} + \frac{\mu}{8}k\sum_{i=1}^{n}\sigma_i^2\|A_{\varepsilon h}\mathbf{e}_\varepsilon^i\|^2. \tag{4.130}$$

Inserting (4.127)-(4.130) in (4.126), then using the Lemmas 4.18 and 4.21 and the following assumption

$$k\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}\widehat{\mathbf{e}}_\varepsilon^i\|^2 \le K_n k^2, \tag{4.131}$$

we conclude the rest of the proof. □

In order to proof the estimate (4.131), we choose $\mathbf{v}_h = \widehat{\mathbf{e}}_\varepsilon^i$ in (4.119) with $n = i$ and exactly similar to Lemma 4.20, we prove the following result.

Lemma 4.22. *Suppose the hypothesis of Lemma 4.18 be satisfied. Then, for* $0 < n < N$*, the following holds*

$$\tau_n\|\nabla\widehat{\mathbf{e}}_\varepsilon^n\|^2 + ke^{-2\alpha t_n}\sum_{i=1}^{n}\sigma_i\|A_{\varepsilon h}\widehat{\mathbf{e}}_\varepsilon^i\|^2 \le K_n k^2.$$

We now also find the error bounds for the pressure term. In fact, similar to the above analysis, we can easily prove that $\tau_n\|\partial_t \mathbf{e}_\varepsilon^n\| \leq K_n k$. Now using this and the available estimates for $\mathbf{e}_\varepsilon^n$, we can easily prove the following result:

Lemma 4.23. *Suppose the hypothesis of Lemma 4.18 be satisfied. Then, for* $0 < n < N$*, the following holds*

$$\tau_n\|P_\varepsilon^n - p_{\varepsilon h}(t_n)\| \leq K_n k.$$

Proof. From (4.5) and (4.8), we find the following equation

$$(P_\varepsilon^n - p_{\varepsilon h}^n, \nabla\cdot\mathbf{v}_h) = (\partial_t \mathbf{e}_\varepsilon^n, \mathbf{v}_h) + \nu a(\mathbf{e}_\varepsilon^n, \mathbf{v}_h) + a(q_r^n(\mathbf{e}_\varepsilon), \mathbf{v}_h) - R_h^n(\mathbf{v}_h) - E_h^n(\mathbf{v}_h) - \Lambda_h^n(\mathbf{v}_h),$$

where R_h^n, E_h^n and Λ_h^n are defined by (4.101), 4.103 and (4.102), respectively. A use of Lemma 1.4 gives

$$\begin{aligned}(P_\varepsilon^n - p_{\varepsilon h}^n, \nabla\cdot\mathbf{v}_h) =& \Big(\|\partial_t \mathbf{e}_\varepsilon^n\|_{-1,h} + \nu\|\nabla\mathbf{e}_\varepsilon^n\| + \|q_r^n(\nabla\mathbf{e}_\varepsilon)\| + C(\|\nabla\mathbf{u}_{\varepsilon h}^n\| + \|\nabla\mathbf{U}_\varepsilon^n\|)\|\nabla\mathbf{e}_\varepsilon^n\| \\ &+ C\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (s - t_{i-1})\beta(t_n - s)(\delta\|\nabla\mathbf{u}_{\varepsilon h}\| + \|\nabla\mathbf{u}_{\varepsilon hs}\|)\, ds \\ &+ \frac{1}{k}\int_{t_{n-1}}^{t_n} (t - t_{n-1})\|\mathbf{u}_{\varepsilon hss}\|_{-1}\, ds\Big)\|\nabla\mathbf{v}_h\|,\end{aligned}$$

where $\|\cdot\|_{-1,h}$ is defined in (4.82) and clearly $\|\cdot\|_{-1,h} \leq \|\cdot\|_{-1} \leq C\|\cdot\|$. Finally, we use the Lemmas 4.7, 4.17 and 4.21 to conclude the remaining of the proof. □

Finally, combining Theorem 4.2, 4.3, 4.16 and Lemma 4.19, 4.21, we conclude our main result of this chapter.

Theorem 4.5. *Suppose the conditions* (**A1**),(**A3**), (**B1**) *and* (**B2**) *be satisfied. Then, for* $0 < n < N$*, the followings hold:*

$$\begin{aligned}\sqrt{\tau_n}\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\| &\leq K_n(\varepsilon + h^2 + k), \\ \tau_n\|\nabla(\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n)\| &\leq K_n(\varepsilon + h + k), \\ \tau_n\|p(t_n) - P_\varepsilon^n\| &\leq K_n(\varepsilon + h + k),\end{aligned}$$

where $K_n = Ce^{Ct_n}$ *and* $C > 0$ *is a constant may depends on the given data but not on* ε, h *and* k*. Moreover, the above results are uniform in time under the uniqueness condition (4.32).*

Optimal Penalty Error Estimate for the Navier-Stokes Equations (NSEs)

We would like to point out that the optimal $L^\infty(\mathbf{L}^2)$-error for the velocity, in case of NSEs, is not available in the literature to the best of our knowledge. The result of [73] is sub-optimal in nature. We have in fact studied the penalty method for the NSEs to begin with and have analyzed the system with nonsmooth initial data. Using the time weighted estimates, the negative norm estimates and the inverse of the penalized Stokes operator, we have obtained optimal error estimates for higher order finite element approximations. Our work on penalized Oldroyd model of order one is an extension of this work. Although worked out for linear polynomial approximation, the presence of the integral term makes things more technical.

Instead of a detailed presentation in the Navier-Stokes' case, we simply present the main results here. For details, see [12].

Theorem 4.6. *Suppose the conditions* (**A1**),(**A3**)*,* (**B1**) *and* (**B2**) *be satisfied. Then, for* $0 < n < N$*, the followings hold:*

$$\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\| \le K_n\Big((\varepsilon + k)t^{-\frac{1}{2}} + h^{m+1}t^{-\frac{m}{2}}\Big),$$
$$\|\nabla(\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n)\| \le K_n\Big((\varepsilon + k)t^{-1} + h^{m}t^{-\frac{m}{2}}\Big),$$
$$\|p(t_n) - P_\varepsilon^n\| \le K_n\Big((\varepsilon + k)t^{-1} + h^{m}t^{-\frac{m}{2}}\Big),$$

where the positive constant $K_n = Ce^{Ct_n}$ *depends exponentially on time. The estimates are uniform in time under the uniqueness condition (4.32), that is, the constant* K_n *becomes* C*.*

4.5 Numerical Experiments

This section is devoted for numerical verification of our theoretical findings, mainly verify the order of convergence of the error estimates.

4.5.1 Oldroyd Model of Order One

We consider the Oldroyd model of order one subject to homogeneous Dirichlet boundary conditions. We approximate the equation using (P_2, P_0) and (P_1b, P_1) elements over a regular triangulation of Ω. We take the domain $\Omega = [0,1]\times[0,1]$, which is partitioned into triangles with size $h = 2^{-i}$, $i = 2, 3, \ldots, 6$. To verify the theoretical result, we first consider Example 2.1 from Chapter 2 and perform the following numerical simulations.

Table 4.1: Errors and convergence rates (C.R.) for Example 2.1 using (P_2, P_0) element

h	$\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\|_{\mathbf{H}^1}$	C.R.	$\|p(t_n) - P_\varepsilon^n\|_{L^2}$	C.R.
1/4	0.12555664		3.00836657		0.36152403	
1/8	0.03021356	2.0551	1.50243340	1.0017	0.16959213	1.0920
1/16	0.00833015	1.8588	0.78440706	0.9376	0.08582330	0.9826
1/32	0.00208880	1.9957	0.39392201	0.9937	0.04245084	1.0156
1/64	0.00052683	1.9877	0.19764538	0.9950	0.02114583	1.0054

Table 4.2: Errors and convergence rates (C.R.) for Example 2.1 using (P_1b, P_1) element

h	$\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\|_{\mathbf{H}^1}$	C.R.	$\|p(t_n) - P_\varepsilon^n\|_{L^2}$	C.R.
1/4	0.05078143		0.32553170		0.18223676	
1/8	0.01200307	2.0809	0.07571734	2.0141	0.02962798	2.6208
1/16	0.00296632	2.0167	0.02670858	1.5033	0.01127635	1.3937
1/32	0.00073988	2.0033	0.01102859	1.2761	0.00430807	1.3882
1/64	0.00018605	1.9916	0.00498786	1.1448	0.00137334	1.6494

In Tables 4.1 and 4.2, we give the numerical errors and rates of convergence derived on successive meshes using (P_2, P_0) and (P_1b, P_1) elements for BE scheme applied to the penalized system (4.1) with $\mu = 0.1, \gamma = 0.01, \delta = 0.1$ and time $t = [0, 1]$. The numerical results show that the rates of convergence are $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$ for the velocity in $\mathbf{L}^2$ and energy norms, respectively. And the rate of convergence for the pressure in L^2-norm is $\mathcal{O}(h)$. We choose the time step $k = \mathcal{O}(h^2)$, penalty parameter $\varepsilon = \mathcal{O}(h^2)$ and the final time $T = 1$. The optimal rates of convergence derived in previous sections are supported by these numerical findings. The error graphs are presented in Fig 4.1.

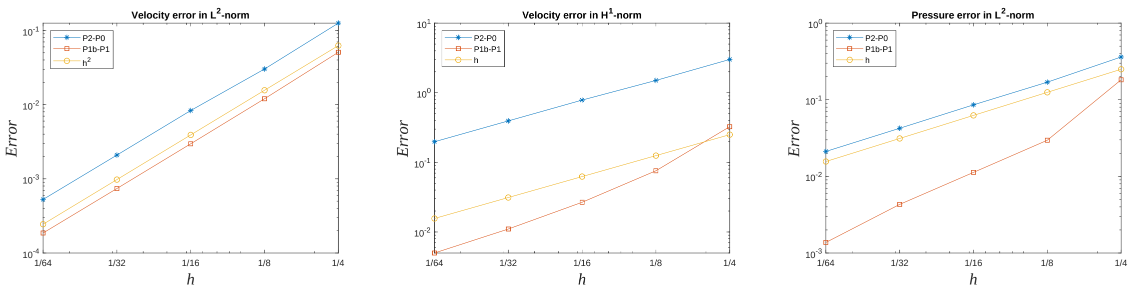

Figure 4.1: Velocity and pressure errors for example 2.1.

In order to verify the rate of convergence for nonsmooth data, we consider the following example [150].

Example 4.1. *For initial data* $\mathbf{u}_0 \in \mathbf{H}_0^1$*, we consider the forcing term* $f(x,t)$ *so as to get the following exact solutions*

$$u_1(x,t) = 5e^t x^{5/2}(x-1)^2 y^{3/2}(y-1)(9y-5),$$

$$u_2(x,t) = -5e^t x^{3/2}(x-1)(9x-5)y^{5/2}(y-1)^2,$$

$$p(x,t) = 2e^t(x-y).$$

Tables 4.3 and 4.4 represent the numerical errors and rates of convergence for nonsmooth initial data. In this case, we take $\mu = 0.1, \gamma = 0.01, \delta = 0.1,$ $k = \mathcal{O}(h^2)$ and $\varepsilon = \mathcal{O}(h^2)$. The error graphs are presented in Fig 4.2. The Tables 4.3 and 4.4 as well as the Fig 4.2 show that the rates of convergence for the velocity are 2 and 1 in $\mathbf{L}^2$ and $\mathbf{H}^1$-norms, respectively. And it is linear rate in case of pressure in L^2-norm.

Table 4.3: Numerical results for Example 4.1 using (P_2, P_0) element

h	$\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\|_{\mathbf{H}^1}$	C.R.	$\|p(t_n) - P_\varepsilon^n\|_{L^2}$	C.R.
1/4	0.12541167		3.00855024		0.36172525	
1/8	0.03022854	2.0527	1.50246564	1.0017	0.17009449	1.0886
1/16	0.00837611	1.8516	0.78454574	0.9374	0.08690876	0.9688
1/32	0.00221660	1.9179	0.39406891	0.9934	0.04458552	0.9629
1/64	0.00057676	1.9423	0.19787680	0.9938	0.02515063	0.8260

Table 4.4: Numerical results for Example 4.1 using $(P_1 b, P_1)$ element

h	$\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\|_{\mathbf{L}^2}$	C.R.	$\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\|_{\mathbf{H}^1}$	C.R.	$\|p(t_n) - P_\varepsilon^n\|_{L^2}$	C.R.
1/4	0.08746297		0.86228925		0.64287436	
1/8	0.01646827	2.4090	0.27053709	1.6723	0.12551514	2.3567
1/16	0.00407032	2.0165	0.13114574	1.0447	0.05406822	1.2150
1/32	0.00102293	1.9924	0.06421120	1.0303	0.02652004	1.0277
1/64	0.00026565	1.9451	0.03184465	1.0118	0.01379871	0.9425

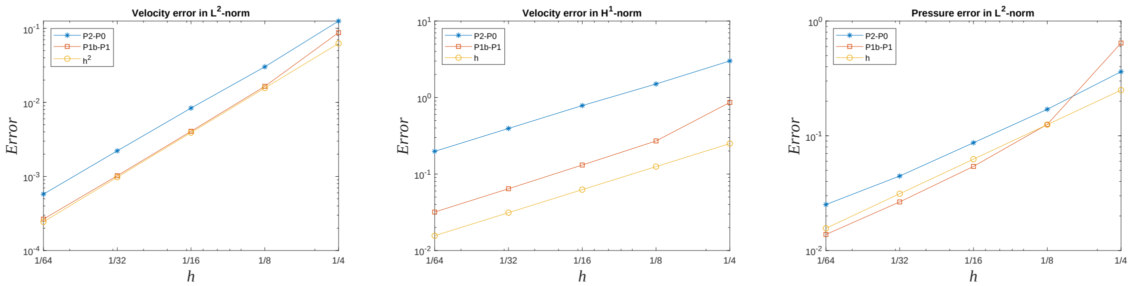

Figure 4.2: Velocity and pressure errors for example 4.1.

The next example is related to "2D Lid driven Cavity Flow Benchmark problem".

Example 4.2. *"We consider a benchmark problem related to a 2D lid driven cavity flow on a unit square with zero body force. Also, no slip boundary condition are considered everywhere except the non zero velocity* $\mathbf{u} = (1, 0)^T$ *on upper boundary."*

For numerical simulations, we choose the lines $(x, 0.5)$ and $(0.5, y)$. In Figure 4.3, we present the values of the velocity and the pressure of unsteady problem (4.1) and it's steady version at final time $T = 75$, and $\nu = 1, 0.01, 0.0025, 0.001$ with the choice of time step $k = 0.01$, $h = 1/64$, $\delta = 0.1$ and $\gamma = 0.1\mu$. From the graphs, it is observed that the unsteady velocity and pressure profiles coincide with the steady profiles very well for a large time.

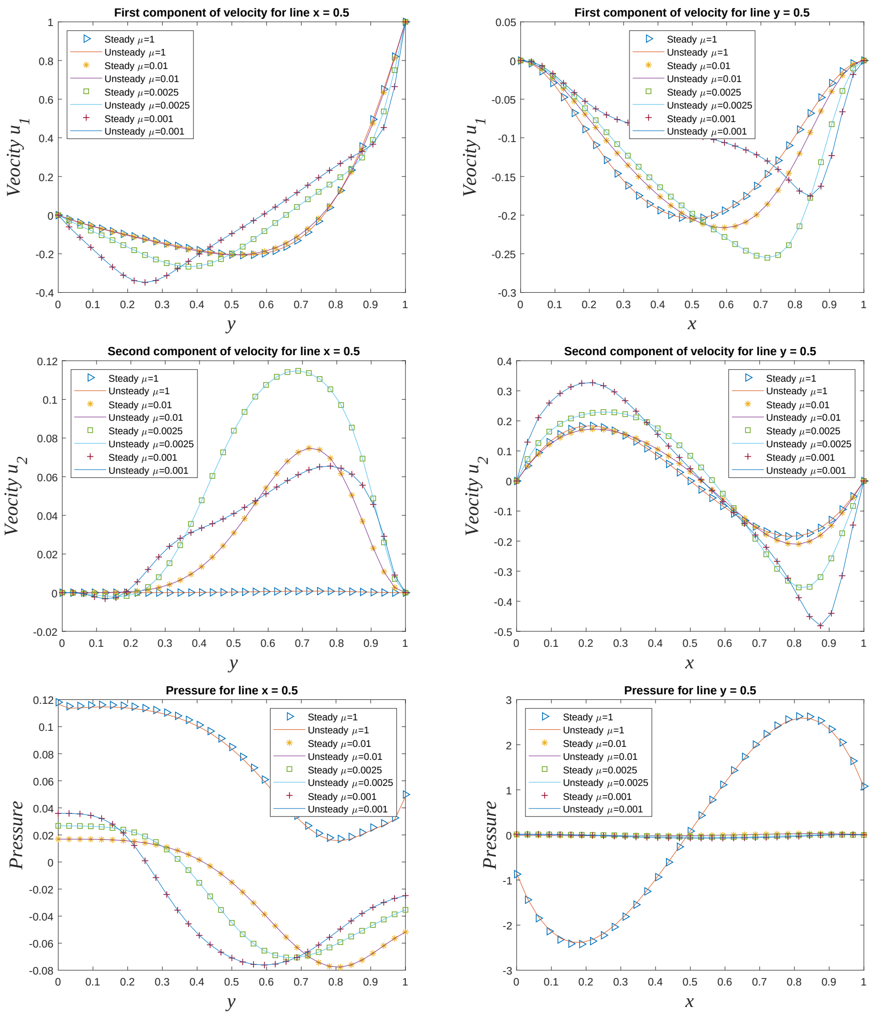

Figure 4.3: Velocity and pressure profiles for different values of μ for Example 4.2.

If $\gamma = 0$, then the system reduces to the well-known Navier-Stokes equations. In Fig 4.4, we present the velocity along vertical line and horizontal line through the geometric center of the cavity for different values of $\mu = 0.01, 0.0025, 0.001$ and each of the values of μ, we vary γ from 0.001 to 10 with fixed $\delta = 0.01$, $k = 0.01$, $h = 1/32$, $\varepsilon = \mu h^2$ and final time $T = 10$. From the graphs, first we observe that when $\gamma = 0$, then the velocity profiles coincide with well-known Ghia's [54] results. Secondly, as μ decreases, the difference between the velocity profiles of Navier-Stokes equations and Oldroyd model of order one become larger as γ increases. Now, we fixed $\gamma = 0.1$ and vary δ from 0.001 to 10 for each values of μ and the results are presented in Fig 4.5. In Fig 4.5, we can see that as δ changes the velocity profiles remain almost same for any values of μ that is the parameter δ has a little influence to the solution. It is clear from the above graphs that the influence of γ on numerical solution is larger than δ.

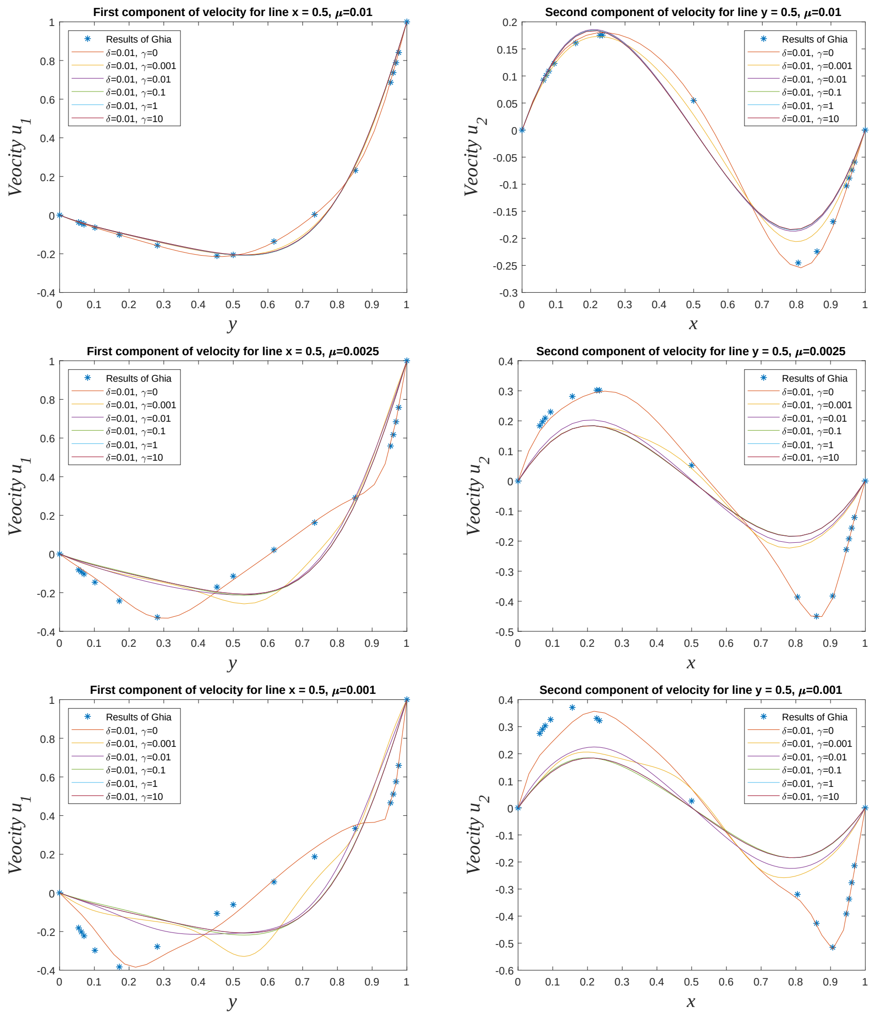

Figure 4.4: Velocity and pressure profiles for different values of γ for Example 4.2.

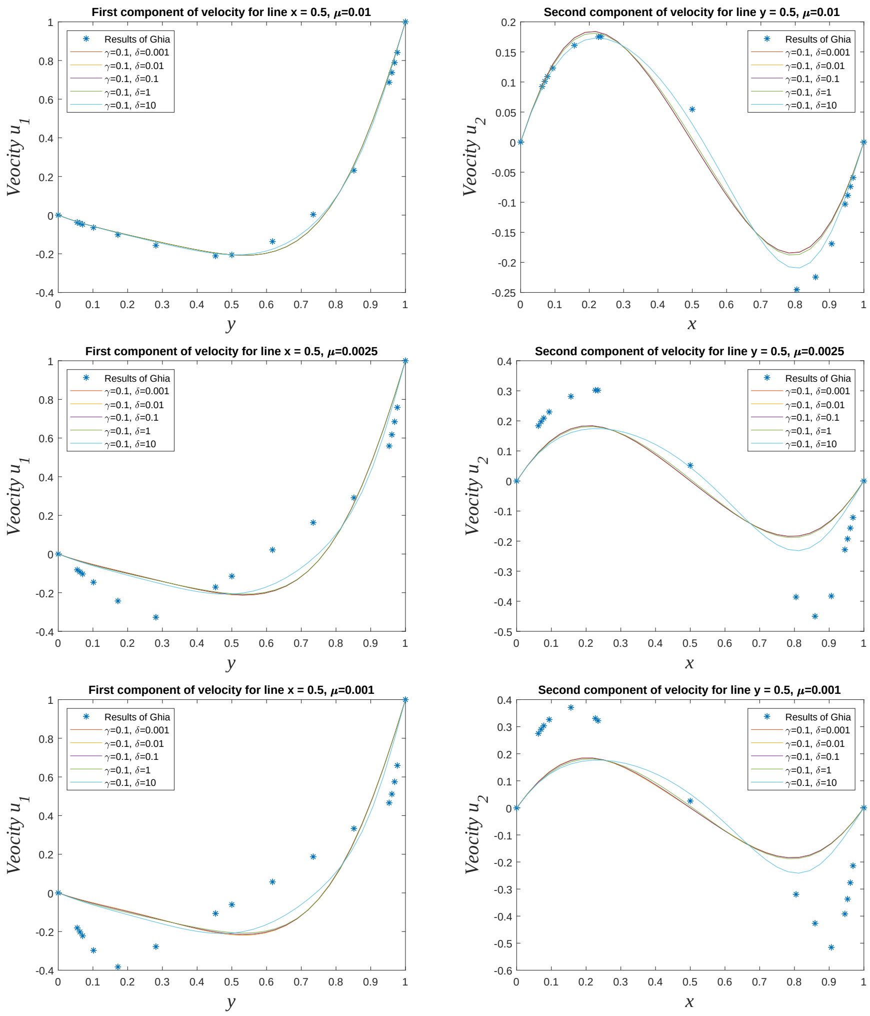

Figure 4.5: Velocity and pressure profiles for different values of δ for Example 4.2.

4.5.2 Navier-Stokes Equations

We approximate NSEs using (P_2, P_1), (P_3, P_2) and (P_1^{NC}, P_0) elements over a triangulation of Ω. Here also we discretize the domain with mesh size $h = 2^{-i}, i = 1, 2, \ldots, 6$. To verify the theoretical result, we consider Example 2.1

In Table 4.5 and 4.6, we present the numerical errors and rates of convergence derived for the fully discrete penalized NSEs using (P_m, P_{m-1}) elements for $m = 2, 3$, respectively. The numerical analysis shows that the rates of convergence are $\mathcal{O}(h^{m+1})$ and $\mathcal{O}(h^m)$ for the velocity in $\mathbf{L}^2$-norm and $\mathbf{H}^1$-norm, respectively. The rate of convergence for the pressure is $\mathcal{O}(h^m)$ in $\mathbf{L}^2$-norm. We choose the time step and penalty parameter as $k = \varepsilon = \mathcal{O}(h^{m+1})$ and $T = 1$ and $\nu = 1$ for our experiments. These findings support the results found in Theorem 4.6. The error graphs are presented below in Fig 4.6-4.7. In Table 4.7, we give the numerical results for (P_1^{NC}, P_0) element. It is observed in Table 4.7 as well as Fig 4.8 that the rates of convergence for the velocity in $\mathbf{L}^2$-norm and $\mathbf{H}^1$-norm are 2 and 1, respectively. Moreover it is linear in pressure in L^2-norm.

Table 4.5: Numerical results for Example 2.1 using (P_2, P_1) element for NSEs

h	$\\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\\|_{\mathbf{L}^2}$	C.R.	$\\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\\|_{\mathbf{H}^1}$	C.R.	$\\|p(t_n) - P_\varepsilon^n\\|_{L^2}$	C.R.
1/2	3.30633896e-03	-	2.96918336e-02	-	3.29192462e-02	-
1/4	5.11077157e-04	2.6936	8.36078857e-03	1.8284	6.96272136e-03	2.2412
1/8	5.25170055e-05	3.2826	1.99271639e-03	2.0689	9.29102388e-04	2.9057
1/16	6.32080598e-06	3.0546	5.32350596e-04	1.9042	2.78763334e-04	1.7368
1/32	7.91350016e-07	2.9977	1.35504728e-04	1.9740	7.93302459e-05	1.8131
1/64	9.89942025e-08	2.9989	3.39036941e-05	1.9988	2.01622898e-05	1.9762

Table 4.6: Numerical results for Example 2.1 using (P_3, P_2) element for NSEs

h	$\\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\\|_{\mathbf{L}^2}$	C.R.	$\\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\\|_{\mathbf{H}^1}$	C.R.	$\\|p(t_n) - P_\varepsilon^n\\|_{L^2}$	C.R.
1/2	8.72519439e-04	-	9.64515599e-03	-	2.42266708e-02	-
1/4	7.86457181e-05	3.4717	2.53372020e-03	1.9285	3.77601493e-03	2.6816
1/8	5.40305188e-06	3.8635	3.35107263e-04	2.9156	4.04997117e-04	3.2209
1/16	3.65287201e-07	3.8866	4.34389795e-05	2.9476	3.73502090e-05	3.4387
1/32	2.40020778e-08	3.9278	5.55651552e-06	2.9667	3.34112960e-06	3.4827

Table 4.7: Numerical results for Example 2.1 using (P_1^{NC}, P_0) element for NSEs

h	$\\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\\|_{\mathbf{L}^2}$	C.R.	$\\|\mathbf{u}(t_n) - \mathbf{U}_\varepsilon^n\\|_{\mathbf{H}^1}$	C.R.	$\\|p(t_n) - P_\varepsilon^n\\|_{L^2}$	C.R.
1/4	6.46328013e-02	-	4.53947780e-01	-	4.68679354e-01	-
1/8	2.01782694e-02	1.6795	2.39739250e-01	0.9211	1.74839152e-01	1.4226
1/16	5.43929542e-03	1.8913	1.21753766e-01	0.9775	5.78140714e-02	1.5965
1/32	1.39082972e-03	1.9675	6.12053289e-02	0.9922	1.98791319e-02	1.5401
1/64	3.49954196e-04	1.9907	3.06603901e-02	0.9973	7.92197390e-03	1.3273

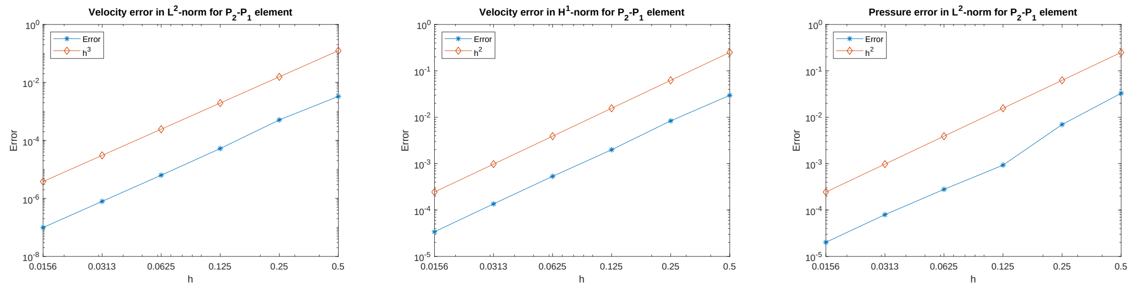

Figure 4.6: Velocity and pressure error for Example 2.1 using (P_2, P_1) element.

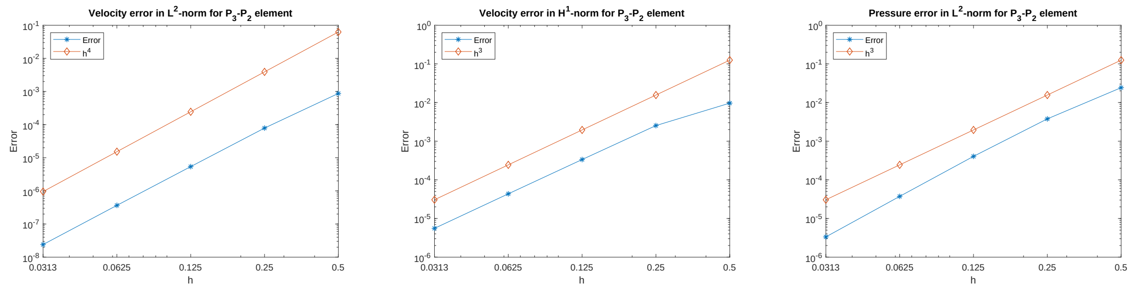

Figure 4.7: Velocity and pressure error for Example 2.1 using (P_3, P_2) element.

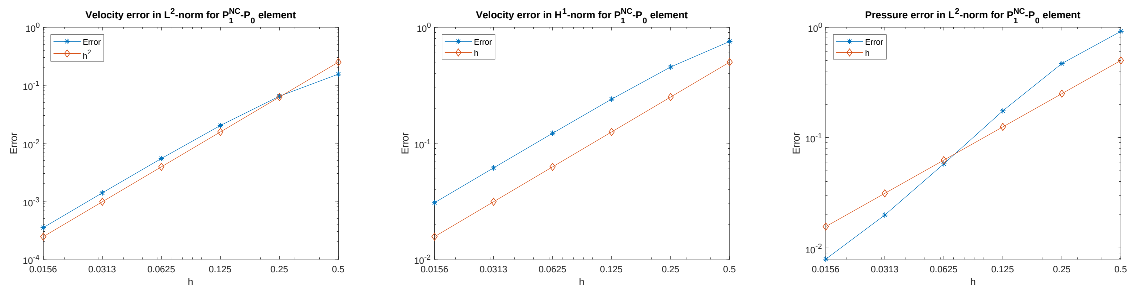

Figure 4.8: Velocity and pressure error for Example 2.1 using (P_1^{NC}, P_0) element.

Next we consider Example 4.2, that is,“ 2D Lid driven Cavity Flow Benchmark problem”. In Figure 4.9, we present the comparison between velocity obtained by penalty method and velocity obtained by Ghia et. al. [54] of NSEs for final time $t = 75$, for $\nu = 10^{-2}, 10^{-3}$ and $t = 150$, for $\nu = 10^{-4}$, respectively, with the choice of time step $k = 0.01$. From the graphs, it is observed that the velocity profiles coincide with those of Ghia's results very well for a large time and that for ν small.

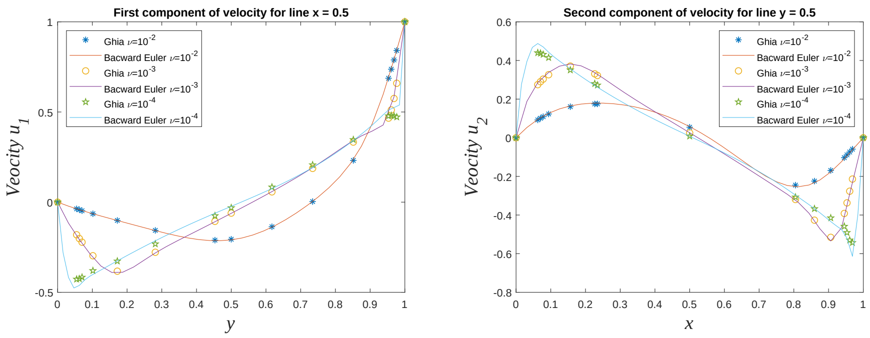

Figure 4.9: Velocity components for Example 4.1.

Finally, in Fig. 4.10, we present contours of pressure and the velocity vector of the NSEs and the Oldroyd model of order one with different values of $\mu = 1/100, 1/400$, $1/1000$. And we observe that the swirls in the corners of the cavity in the NSEs are larger than those in the Oldroyd model. This is happened due to the presence of the integral term, and the integral term plays the vital role of stabling the velocity field. All the computation for Examples 2.1 and 4.1 were done in MATLAB and the others were done in FreeFem++ [78].

4.6 Conclusion

In this chapter, a penalized Oldroyd model of order one has been analysed for nonsmooth initial data, that is, $\mathbf{u}_{\varepsilon 0} \in \mathbf{H}_0^1$. Based on penalized Stokes operator, and appropriate application of weighted time estimates with positivity of the memory term, uniform in time regularity results are established for the penalized problem which are valid as the penalty parameter ε tends to zero. This is followed by semidiscrete analysis of the model based on conforming finite element method. With the help of discrete penalty Stokes operator and "uniform Gronwall's Lemma", uniform in time bound for the discrete velocity in the Dirichlet norm is derived. Subsequently, optimal velocity error in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}^1)$-norms and pressure in $L^\infty(L^2)$-norm have been established, and these are uniform in time. Our analysis relies on the application of the inverse penalized Stokes operator with its discrete version, the penalized Stokes-Volterra projection, weighted time estimates and positivity of the memory term. Then, based on BE method, a fully discrete penalized system has been analyzed with nonsmooth initial data. We have shown the first-order rate of convergence in time direction for the velocity and the pressure. Finally, we have considered some numerical examples to validate our theoretical findings. Also, several numerical experiments are conducted on benchmark problems and for various small values of μ and γ.

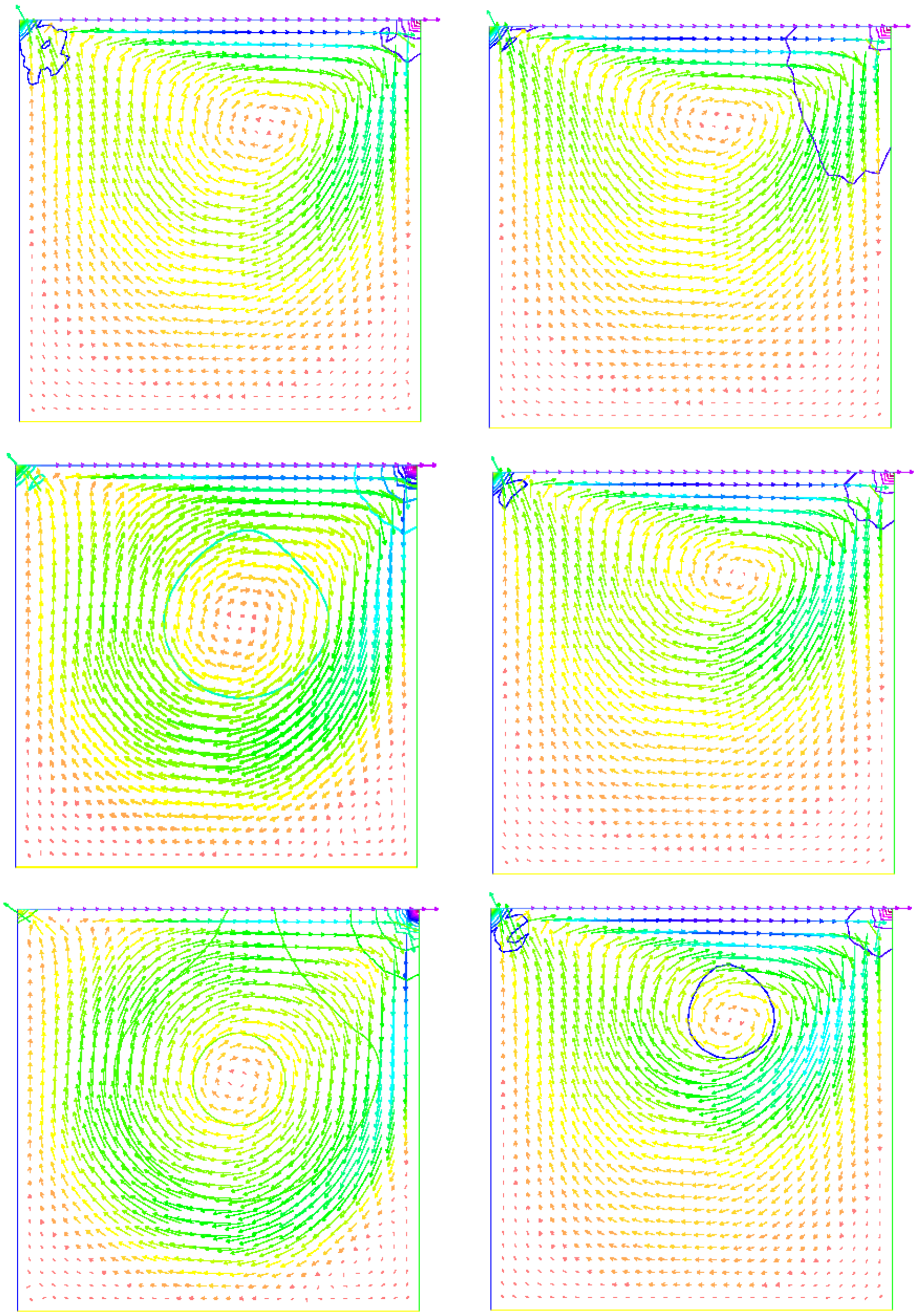

Figure 4.10: The velocity vector and contour of the pressure obtained from Navier-Stokes equations (first column) and Oldroyd model (second column) at final time $T = 10$, $\delta = 0.001$ and $\mu = 0.01, 0.0025, 0.001$ from top to bottom.