Chapter 5

Grad-div Stabilization

Here, we study a stabilized finite element method namely, grad-div stabilization for Oldroyd model of order one. This type of stabilization gives a stable simulation when the coefficient of viscosity is very small. We derive optimal error estimates for the velocity and for the pressure for semidiscrete as well as for fully discrete scheme with the error bound constants independent of inverse power viscosity. We present a few numerical examples to find a suitable choice of grad-div parameter for the Oldroyd model of order one and give some numerical results which validate our theoretical findings.

5.1 Introduction

Galerkin mixed finite element for the model has been analyzed on a few occasions [63, 76] with optimal error estimates. However, it is well known that, similar to NSEs, the coupling of velocity and pressure, through the divergence free term, is in fact not desirable. Methods for decoupling by various means, like penalty method, artificial compressibility method, pressure correction method, projection method etc. attempt to overcome this difficulty by means of artificial conditions. Work in these directions for the Oldroyd model can be found in [99, 136, 139, 152]. Unfortunately, these methods are less efficient when Reynolds number is high. This is due to the domination of the convection term on the viscous term, which typically arises for small values of viscosity. It is handled via methods based on stabilization techniques; like streamline upwind/Petrov-Galerkin(SUPG) method, interior-penalty methods, local projection stabilization and residual-free bubbles enrichment method see, [16, 20–22].

In particular, in SUPG method, a grad-div stabilization is included which allows to achieve the stability and accuracy for small values of viscosity.

Here, we study a stabilized finite element method for our model, namely grad-div stabilization method. It is known that this stabilized scheme is more efficient for high Reynolds number (or small viscosity). The main idea is that we add a stabilization term to the momentum equation with respect to the continuity equation. Now, the semidiscrete stabilized weak formulations of our model read as: Find $(\mathbf{u}_h, p_h)$ in $\mathbf{H}_h \times L_h$ satisfying

$$\begin{aligned}(\mathbf{u}_{ht}, \mathbf{v}_h) + \mu a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{v}_h)\, ds \\ -(p_h, \nabla \cdot \mathbf{v}_h) + \rho(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\ \mathbf{v}_h \in \mathbf{H}_h, \end{aligned} \tag{5.1}$$

with $(\nabla \cdot \mathbf{u}_h, \chi_h) = 0,\ \forall\ \chi_h \in L_h$, where $\rho \geq 0$ is the stabilization parameter and $\mathbf{u}_{0h} \in \mathbf{H}_h$ is appropriate approximation of the initial velocity $\mathbf{u}_0 \in \mathbf{J}_1$ and $\mathbf{H}_h$ and L_h are the discrete spaces that approximate the velocity and pressure spaces. It is assumed that the spaces ($\mathbf{H}_h$,L_h) are of the form (P_m, P_{m-1}) where P_m comprises of piecewise polynomial functions of degree at most $m, m > 1$. [However for $m = 1$, we consider the mini element $(P_1 b, P_1)$ where $P_1 b$ is the P_1 space with bubble enrichment.] We recall the weekly divergence free subspace $\mathbf{J}_h$ of the discrete space $\mathbf{H}_h$ as

$$\mathbf{J}_h = \{\mathbf{w}_h \in \mathbf{H}_h : (z_h, \nabla \cdot \mathbf{w}_h) = 0, \quad \forall z_h \in L_h\}.$$

It is noted that the space $\mathbf{J}_h \not\subset \mathbf{J}_1$. Now, we also recall the equivalent Galerkin approximation in $\mathbf{J}_h$ as: For $t > 0$, seek $\mathbf{u}_h(t) \in \mathbf{J}_h$ satisfying

$$\begin{aligned}(\mathbf{u}_{ht}, \mathbf{v}_h) + \mu a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \int_0^t \beta(t-\tau) a(\mathbf{u}_h(\tau), \mathbf{v}_h)\, d\tau \\ +\rho(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{J}_h. \end{aligned} \tag{5.2}$$

The semidiscrete formulation(s) mentioned above are still continuous in time. For fully discrete formulation, we further discretize (it) in the temporal direction. We consider the first-order implicit backward Euler method to discretize in the temporal direction. Assuming $[0, T]$ to be the time interval, we proceed as follows: Let $k = \frac{T}{N} > 0$ be the time step with $t_n = nk,\ n \geq 0$ representing the n-th time step. Here N is a positive integer. We next define for a sequence $\{\boldsymbol{\phi}^n\}_{n\geq 0} \subset \mathbf{H}_h$, the backward difference quotient

$$\partial_t \boldsymbol{\phi}^n = \frac{1}{k}(\boldsymbol{\phi}^n - \boldsymbol{\phi}^{n-1}).$$

For any continuous function $\boldsymbol{\phi}(t)$ we set $\boldsymbol{\phi}^n = \boldsymbol{\phi}(t_n)$. We approximate the integral term in (5.1) by right rectangle rule, the BE method being of first-order, with the notation $\beta_{nj} = \beta(t_n - t_j)$:

$$q_r^n(\boldsymbol{\phi}) = k\sum_{j=1}^{n} \beta_{nj}\phi^j \approx \int_0^{t_n} \beta(t_n - s)\boldsymbol{\phi}(s)\ ds.$$

Now the backward Euler method applied in (5.1) is stated as below: For $0 < n < N$, find $\mathbf{U}^n \in \mathbf{H}_h$ and $P^n \in L_h$ with $\mathbf{U}(0) = P_h\mathbf{u}_0$ satisfying

$$\left.\begin{aligned} &(\partial_t \mathbf{U}^n, \mathbf{v}_h) + \mu a(\mathbf{U}^n, \mathbf{v}_h) + b(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h) - (P^n, \nabla\cdot\mathbf{v}_h) \\ &\qquad\qquad +\rho(\nabla\cdot\mathbf{U}^n, \nabla\cdot\mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h) - a(q_r^n(\mathbf{U}), \mathbf{v}_h), \quad \forall\ \mathbf{v}_h \in \mathbf{H}_h, \\ &(\nabla\cdot\mathbf{U}^n, \chi_h) = 0, \quad \forall\ \chi_h \in L_h. \end{aligned}\right\} \tag{5.3}$$

If we consider the discrete solution $\mathbf{U}^n \in \mathbf{J}_h$, then (5.3) becomes: For $0 < n < N$, find $\mathbf{U}^n \in \mathbf{J}_h$ with $\mathbf{U}(0) = P_h\mathbf{u}_0$ satisfying

$$\begin{aligned} (\partial_t \mathbf{U}^n, \mathbf{v}_h) + \mu a(\mathbf{U}^n, \mathbf{v}_h) + b(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h) + \rho(\nabla\cdot\mathbf{U}^n, \nabla\cdot\mathbf{v}_h) \\ = (\mathbf{f}^n, \mathbf{v}_h) - a(q_r^n(\mathbf{U}), \mathbf{v}_h), \quad \forall\ \mathbf{v}_h \in \mathbf{J}_h, \end{aligned} \tag{5.4}$$

Grad-div stabilization was first proposed by Franca and Hughes [51] to increase the conservation of mass in finite element method. But, the method comes with several other benefits. For example, the application of grad-div stabilization results in improved convergence of preconditioned iteration when the stabilization parameter is too small [107], the well-poseness of the continuous solution as well as the accuracy and convergence of the numerical approximation for small values of viscosity [108], the local mass balance of the system in numerical experiments [39]. Moreover, it has been observed that in the simulation of turbulent flows, the use of this stabilization is sufficient for performing a stable simulation, see [86, fig. 3] and [120, fig. 7].

These observations lead us to the present chapter: to derive the error bounds that do not depend on $1/\mu$, for a stable inf-sup mixed finite element method with grad-div stabilization applied to our model (1.4)-(1.6). This is not the first time where similar results have been achieved. In fact, in [41, 42], de Frutos *et. al.* have obtained error bounds with constants which are not depend on the inverse powers of viscosity for evolutionary Oseen equations and Navier-Stokes equations, respectively. There are a few more works in this direction for incompressible flow problems but there is no work available for the Oldroyd model of order one to the best our knowledge. In this chapter, we extend the analysis of [42] to the Oldroyd model of order one. As in [42], we have carried out our analysis for sufficiently smooth initial data, that is, $\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^m$ ($m > 2$), as well as for smooth initial data, $\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^2$. But our proofs are shorter and less technically involved than the ones from [42],especially when $\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^2$.

We would like to point out that the analysis in both these cases does not differ by much. However we get a valuable input that less regularity of the initial velocity puts a restriction in the order of finite element approximation, when keeping estimates independent of inverse of viscosity. For example, with only smooth initial data, we may get maximum of second order convergence rate in case of velocity, even if we employ higher order approximations, see Remark 5.5.

We next would like to point out that the assumption of sufficiently smooth data comes at a cost of non-local compatibility conditions of various order, for the given data, at time $t = 0$. Without these conditions, which do not arise naturally, the solutions of the Oldroyd model of order one can not be assumed to have more than second order derivatives bounded in $L^2(\Omega)$ at $t = 0$ (see [63]). The analysis for smooth initial data takes into account this lack of regularity at $t = 0$.

It is well known that the suitable choice of stabilization parameter for any stabilized scheme is important for accuracy in numerical simulations. In case of grad-div stabilization, a suitable choice of grad-div parameter ρ is shown to be $\mathcal{O}(1)$ for the NSEs and for inf-sup stable finite element pairs, in [107, 109]. And in [101], it is shown that error can be minimized for $\rho \approx 10^{-1}$. However, larger values of ρ may be needed in special cases, see [53]. A detailed investigation to find appropriate values of the grad-div stabilization parameter for Stokes problem has been discussed in [82]. They have observed that the values of grad-div parameter depends on the used norm, the mesh size, the type of mesh, the viscosity, the finite element spaces, and the solution. A similar analysis and numerical simulations have been seen in [4] for the steady state Oseen problem and NSEs.

We have briefly looked into this aspect for the Oldroyd model of order one as well. Based on the error estimate from Theorem 5.1 (see Section 5.3), we have observed that $\rho = \mathcal{O}(1)$ is a suitable choice for stable mixed FE spaces. And for MINI element, the choice of ρ can be in the range of h^2 to 1, see Remark 5.4 (from Section 5.3). Next we have carried out numerical experiments in support of our theoretical finding. First, we have shown numerically that the grad-div parameter depends on the mesh size, the viscosity and the FE spaces. Then we have calculated the values of grad-div parameter ρ for which the $\mathbf{L}^2$ and $\mathbf{H}^1$ velocity errors and L^2 pressure error are minimum, for a known solution, see Section 5.5.

The following are the primary outcomes of this chapter:

(i) *A priori* estimate for the semidiscrete solution which helps us to show the local existence of the discrete solution of (5.2).

(ii) Optimal velocity and pressure errors with error bounds independent of the inverse power of viscosity, that is, these results are valid for high Reynolds number.

(iii) Optimal error bounds for the fully discrete solution by applying a first-order BE method for temporal discretization. The order of convergence for the velocity and the pressure in L^2 norm is $\mathcal{O}(h^m + k)$ when the finite element velocity space and the pressure space are approximated by m-th and $(m-1)$-th degree piecewise polynomial, respectively $(m > 1)$, where h and k are the space and time discretization parameter, respectively. These results are valid for high Reynolds number as well.

(iv) Suitable choice of grad-div parameter for stable mixed FE spaces and for stable equal order spaces like MINI element.

The remainder of this chapter is organised in the following manner. In Section 5.2, we consider the assumptions on the domain and on the given data. In Section 5.3, the semidiscrete formulation and error analysis of the stabilized scheme are carried out and in Section 5.4 BE method is applied to the stabilized system. And finally, in Section 5.5, some numerical examples are given which conform with our theoretical results. We also obtain numerically suitable values of grad-div parameter, for Oldroyd model of order one, that minimize velocity and pressure errors.

Throughout this chapter, we will use $C > 0$ as a constant, which depends on the given data and not on spatial and time discretization parameters. We note that C does not depend on $1/\mu$.

5.2 Preliminaries

In this section, we first consider a couple of assumptions on the domain and the given data. Then, we take a few assumptions on the regularity of the continuous solution. Through out this chapter, we make the following assumption:
(**A1m**) For $\mathbf{g} \in \mathbf{H}^{m-1}$ with $m \geq 1$, let the solution pair $\{\mathbf{w} \in \mathbf{J}_1, z \in L^2/\mathbb{R}\}$ satisfy the following stationary Stokes equation

$$-\Delta \mathbf{w} + \nabla z = \mathbf{g}, \quad \nabla \cdot \mathbf{w} = 0 \ \text{ in } \Omega; \quad \mathbf{w}|_{\partial\Omega} = \mathbf{0},$$

and satisfy $\{\mathbf{w} \in \mathbf{H}^{m+1}, z \in H^m/\mathbb{R}\}$ and the following regularity estimate [80]:

$$\|\mathbf{w}\|_{m+1} + \|z\|_{H^m/\mathbb{R}} \leq C\|\mathbf{g}\|_{m-1}.$$

We first note here that (**A1m**) implies (see [79])

$$\|\mathbf{w}\|_2 \leq C\|\tilde{\Delta}\mathbf{w}\|, \quad \forall \mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2,$$

and

$$\|\mathbf{w}\| \leq \lambda_1^{-1/2}\|\mathbf{w}\|_1, \quad \mathbf{w} \in \mathbf{H}_0^1, \qquad \|\mathbf{w}\|_1 \leq \lambda_1^{-1/2}\|\mathbf{w}\|_2, \quad \mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2,$$

where $\tilde{\Delta} = P\Delta \mathbf{J}_1 \cap \mathbf{H}^2 \to \mathbf{J}$ is the Stokes operator and P is the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{J}$. And $\lambda_1 > 0$ is the least positive eigenvalue of $\tilde{\Delta}$.
We will subsequently use the "Gagliardo-Nirenberg inequality" [81]

$$\|\mathbf{v}\|_{L^p} \leq C\|\mathbf{v}\|^{2/p}\|\nabla \mathbf{v}\|^{1-2/p}, \quad \forall \ \mathbf{v} \in \mathbf{H}_0^1,$$

where $2 \leq p < \infty$ and $C = C(p, \Omega)$. Also, we will consider the "Agmon's inequality" [81]

$$\|\mathbf{v}\|_{L^\infty} \leq C\|\mathbf{v}\|^{1/2}\|\Delta \mathbf{v}\|^{1/2}, \quad \forall \ \mathbf{v} \in \mathbf{H}^2,$$

where $C = C(\Omega)$.

Remark 5.1. *Following [81], the discrete version of the above two inequalities with constants uniform in discretizing parameter h, will be used later for our analysis.*

We now take an assumption on the given data as below.
(**A2m**) For some $M_0 > 0$ and for $0 < T \leq \infty$, the external force $\mathbf{f}$ satisfies $D_t^l\mathbf{f} \in L^\infty(0, T; \mathbf{H}^m)$ with $\sup_{0<t<T} \left\{\|D_t^l\mathbf{f}\|_m\right\} \leq M_0$, for some integer $m \geq 0$ and $l \geq 0$.

For the regularity of the solutions $\mathbf{u}$ and p, we make the following assumptions depending on whether the initial velocity $\mathbf{u}_0$ is sufficiently smooth or just smooth:

(A3m) Let us assume that $\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^{\max\{2,m\}}$ is sufficiently smooth and the pair of solution $(\mathbf{u}, p)$ of (1.8) satisfy

$$\mathbf{u} \in L^2(0,T;\mathbf{H}^{m+1}) \cap L^2(0,T;(W^{1,\infty}(\Omega))^2) \cap L^\infty(0,T;\mathbf{H}^m),\ p \in L^2(0,T;\mathbf{H}^m/\mathbb{R})$$

for all $m \geq 1$. Further, there is a positive constant C that may depend on given data but not on the inverse power of μ such that for all $m \geq 1$, the following holds:

$$\max_{0\leq t\leq T}\Big(\|\mathbf{u}(t)\|_m^2 + \|p(t)\|_{H^{m-1}/\mathbb{R}}^2\Big) \leq C, \quad \int_0^t \|\nabla\mathbf{u}(s)\|_\infty^2 ds \leq Ct$$
$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\big(\|\mathbf{u}(s)\|_{m+1}^2 + \|\mathbf{u}_s(s)\|_{m-1}^2 + \|p(s)\|_{H^m/\mathbb{R}}^2\big)ds \leq C.$$

(A3m′) Let us assume that $\mathbf{u}_0 \in \mathbf{H}_0^1 \cap \mathbf{H}^2$ is smooth and the pair of solution $(\mathbf{u}, p)$ of (1.8) satisfy

$$(\tau(t))^{m-2}\mathbf{u} \in L^2(0,T;\mathbf{H}^{m+1}) \cap L^\infty(0,T;\mathbf{H}^m),\ (\tau(t))^{m-2}p \in L^2(0,T;\mathbf{H}^m/\mathbb{R}),$$
$$\mathbf{u} \in L^2(0,T;(W^{1,\infty}(\Omega))^2)$$

for $m \geq 2$ and for $m = 1$,

$$\mathbf{u} \in L^2(0,T;\mathbf{H}^2) \cap L^2(0,T;(W^{1,\infty}(\Omega))^2) \cap L^\infty(0,T;\mathbf{H}^1),\ p \in L^2(0,T;\mathbf{H}^1/\mathbb{R}).$$

Further, there is a positive constant C that may depends on given data but not on the inverse power of μ such that, for $m = 1$, the following holds:

$$\int_0^t \|\nabla\mathbf{u}(s)\|_\infty^2 ds \leq Ct, \quad \max_{0\leq t\leq T}\|\mathbf{u}(t)\|_1^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}(s)\|_2^2 ds \leq C.$$

For $m \geq 2$, and for $\tau(t) = \min\{1, t\}$

$$\max_{0\leq t\leq T}(\tau(t))^{m-2}\big(\|\mathbf{u}(t)\|_m^2 + \|p(t)\|_{H^{m-1}/\mathbb{R}}^2\big) \leq C,$$
$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\tau(s))^{m-2}\big(\|\mathbf{u}(s)\|_{m+1}^2 + \|\mathbf{u}_s(s)\|_{m-1}^2 + \|p(s)\|_{H^m/\mathbb{R}}^2\big)ds \leq C.$$

We consider the following Sobolev's embedding [3] theorem: "Suppose q be such that $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$ for $1 \leq p \leq d/s$, the following inequality holds

$$\|v\|_{L^{q'}(\Omega)} \leq C\|v\|_{W^{s,p}(\Omega)}, \quad \frac{1}{q'} \geq \frac{1}{q}, \quad v \in W^{s,p}(\Omega).$$

If $p > \frac{d}{s}$ the above result is valid for $q' = \infty$. In our case we consider $d = 2$. The similar embedding inequality holds for vector-valued functions."

5.3 Semidiscrete Formulation

In this section, we first consider a few assumptions on the discrete spaces and some tools which are used in our later analysis. Then we see the semidiscrete error analysis. We make the following assumption on the discrete spaces $\mathbf{H}_h$ and L_h:

(**B1m**) For each $\mathbf{v} \in \mathbf{H}_0^1 \cap \mathbf{H}^{m+1}$ and $\psi \in H^m/\mathbb{R}$ with $m \geq 1$, then there exist approximations $i_h\mathbf{v} \in \mathbf{H}_h$ and $j_h\psi \in L_h$ such that

$$\|\mathbf{v} - i_h\mathbf{v}\| + h\|\nabla(\mathbf{v} - i_h\mathbf{v})\| \leq Ch^{j+1}\|\mathbf{v}\|_{j+1}, \quad \|\psi - j_h\psi\| \leq Ch^j\|\psi\|_j, \quad 0 \leq j \leq m.$$

Further, we will assume that the following inverse hypothesis holds for a quasi-uniform mesh and for $\mathbf{w}_h \in \mathbf{H}_h$, see [35, Theorem 3.2.6]

$$\|\mathbf{w}_h\|_{W^{m,p}(K)^d} \leq Ch^{n-m-d(\frac{1}{q}-\frac{1}{p})}\|\mathbf{w}_h\|_{W^{n,q}(K)^d}, \tag{5.5}$$

where h is the diameter of the mesh cell $K \in \mathcal{T}_h$ and $0 \leq n \leq m < \infty$, $1 \leq q \leq p \leq \infty$, and $\|\cdot\|_{W^{m,p}(K)^d}$ is the norm in Sobolev space $W^{m,p}(K)^d$.

Below, we present *a priori* estimate for the semidiscrete velocity.

Lemma 5.1. *The following stability analysis holds for the semidiscrete velocity for all* $0 \leq t \leq T,\ T > 0$

$$\|\mathbf{u}_h(t)\|^2 + 2e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\big(\mu\|\nabla\mathbf{u}_h(\tau)\|^2 + \rho\|\nabla\cdot\mathbf{u}_h(\tau)\|^2\big)d\tau \leq \Big(e^{-2\alpha t}\|\mathbf{u}_{0h}\|^2 + \frac{\|\mathbf{f}\|_\infty^2}{2\alpha}\Big)e^{(1+2\alpha)t}.$$

Proof. Set $\mathbf{v}_h = \mathbf{u}_h$ in (5.2) and use (1.11) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_h\|^2 + \mu\|\nabla\mathbf{u}_h\|^2 + \rho\|\nabla\cdot\mathbf{u}_h\|^2 + \int_0^t \beta(t-\tau)a(\mathbf{u}_h(\tau),\mathbf{u}_h)\ d\tau \leq (\mathbf{f},\mathbf{u}_h).$$

We multiply both side by $e^{2\alpha t}$ and take time integration. Then, apply the "Cauchy-Schwarz inequality" with the "Young's inequality" to deduce that

$$\begin{aligned} e^{2\alpha t}\|\mathbf{u}_h(t)\|^2 &+ 2\int_0^t e^{2\alpha s}\big(\mu\|\nabla\mathbf{u}_h(s)\|^2 + \rho\|\nabla\cdot\mathbf{u}_h(s)\|^2\big)ds \\ &+ 2\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)a(\mathbf{u}_h(\tau),\mathbf{u}_h(s))\ d\tau\ ds \\ &\leq \|\mathbf{u}_h(0)\|^2 + \int_0^t e^{2\alpha s}\|\mathbf{f}\|^2 ds + (1+2\alpha)\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|^2 ds. \end{aligned}$$

The double integration term on the left of inequality is positive due to Lemma 1.5, hence, we drop it. Then, we use the "Gronwall's lemma" to obtain

$$e^{2\alpha t}\|\mathbf{u}_h(t)\|^2 + 2\int_0^t e^{2\alpha s}\big(\mu\|\nabla\mathbf{u}_h(s)\|^2 + \rho\|\nabla\cdot\mathbf{u}_h(s)\|^2\big)ds \\ \leq \Big(\|\mathbf{u}_{0h}\|^2 + \frac{\|\mathbf{f}\|_\infty^2}{2\alpha}(e^{2\alpha t}-1)\Big)e^{(1+2\alpha)t}.$$

We multiply both side by $e^{-2\alpha t}$ to concludes the proof. □

Lemma 5.1 helps us to show the local existence of the discrete solution of (5.2). Once we have the solution of (5.2), then using this we can easily prove the existence of the solutions of (5.1). The proof is quit similar to that of [116], hence we skip it. And the uniqueness of the solution is found on the quotient space L_h/N_h, where

$$N_h = \{z_h \in L_h : (z_h, \nabla\cdot\mathbf{v}_h) = 0 \text{ for } \mathbf{v}_h \in \mathbf{H}_h\},$$

and the associated norm is defined by

$$\|z_h\|_{L^2/N_h} = \inf_{\chi_h\in N_h}\|z_h + \chi_h\|.$$

Note that $\mathbf{J}_h$ is finite dimensional. Then the system (5.2) becomes a system of nonlinear integro-differential equations with a stabilization term. It is noted that the discrete pressure depends on discrete velocity and hence, we need assume that the following discrete inf-sup (LBB) condition:

(**B2m′**) For each $z_h \in L_h$, there is a function $\mathbf{v}_h \in \mathbf{H}_h$ satisfying

$$|(z_h, \nabla\cdot\mathbf{v}_h)| \geq C\|\nabla\mathbf{v}_h\|\|z_h\|_{L^2/N_h},$$

where $C > 0$ is the constant, independent of h.

Further, we assume the following approximation property on $\mathbf{J}_h$.

(**B2m**) There exists an approximation $r_h\mathbf{v} \in \mathbf{J}_h$ of $\mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^{m+1}$ satisfying

$$\|\mathbf{v} - r_h\mathbf{v}\| + h\|\nabla(\mathbf{v} - r_h\mathbf{v})\| \leq Ch^{j+1}\|\mathbf{v}\|_{j+1}, \quad 0 \leq j \leq m.$$

We define L^2-projection $P_h : \mathbf{L}^2 \to \mathbf{J}_h$ satisfying for $0 \leq j \leq m$ [80]

$$\|\mathbf{v} - P_h\mathbf{v}\| + h\|\nabla(\mathbf{v} - P_h\mathbf{v})\| \leq Ch^{j+1}\|\mathbf{v}\|_{j+1}, \quad \forall\ \mathbf{v} \in \mathbf{J}_1(\Omega)\cap\mathbf{H}^{m+1}(\Omega). \tag{5.6}$$

Let us also consider the Lagrange interpolant $I_h\mathbf{w} \in \mathbf{H}_h$ of a continuous function $\mathbf{w}$ satisfying the following bounds (see [19, Theorem 4.4.4])

$$\|\mathbf{w} - I_h\mathbf{w}\|_{W^{m,p}(K)} \leq Ch^{n-m}\|\mathbf{w}\|_{W^{n,p}(K)}, \quad 0 \leq m \leq n \leq m+1,$$

where $n > \frac{2}{p}$ when $1 < p \leq \infty$ and $n \geq 2$ when $p = 1$. We recall the definition of the discrete operator $\Delta_h : \mathbf{H}_h \to \mathbf{H}_h$ as

$$a(\mathbf{w}_h, \mathbf{v}_h) = (-\Delta_h \mathbf{w}_h, \mathbf{v}_h), \quad \forall\ \mathbf{w}_h, \mathbf{v}_h \in \mathbf{H}_h.$$

And the discrete analogue of the Stokes operator $\tilde{\Delta} = P\Delta$ as $\tilde{\Delta}_h = P_h \Delta_h$. The restriction of $\tilde{\Delta}_h$ to $\mathbf{J}_h$ is invertible and its inverse is denoted as $\tilde{\Delta}_h^{-1}$. We recall the discrete Sobolev norms on $\mathbf{J}_h$ (see [80]): For $r \in \mathbb{R}$, we define

$$\|\mathbf{w}_h\|_r := \|(-\tilde{\Delta}_h)^{r/2} \mathbf{w}_h\|, \quad \mathbf{w}_h \in \mathbf{J}_h.$$

We note that $\|\mathbf{w}_h\|_0 = \|\mathbf{w}_h\|$ and $\|\mathbf{w}_h\|_1 = \|\nabla \mathbf{w}_h\|$. Also the norm $\|\tilde{\Delta}_h(\cdot)\|$ is equivalent to the norm $\|\cdot\|_2$ in $\mathbf{J}_h$ with constant independent of h.

We present below the error analysis due to the space discretization (time remains continuous). Our analysis will be divided in two parts, based on the regularity of the given initial data. First, we consider sufficiently smooth initial data, that is, the initial velocity $\mathbf{u}_0 \in \mathbf{J}_1 \cap \mathbf{H}^m$, and then, we take smooth data, that is, $\mathbf{u}_0 \in \mathbf{J}_1 \cap \mathbf{H}^2$.

5.3.1 Semidiscrete Error Estimates for Sufficiently Smooth Data

In this section, we derive error bounds for the semidiscrete solution for the case in which regularity of the exact solution up to time $t = 0$ is assumed, that is, the given data is as much regular as we need.

Error bounds for the velocity

Since $\mathbf{J}_h \not\subset \mathbf{J}_1$, then for all $\mathbf{v}_h \in \mathbf{J}_h$, $\mathbf{u}$ satisfies

$$(\mathbf{u}_t, \mathbf{v}_h) + \mu a(\mathbf{u}, \mathbf{v}_h) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) + \int_0^t \beta(t-s) a(\mathbf{u}(s), \mathbf{v}_h) ds = (\mathbf{f}, \mathbf{v}_h) + (p, \nabla \cdot \mathbf{v}_h). \quad (5.7)$$

Define $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$, then subtract (5.2) from (5.7) and use $\nabla \cdot \mathbf{u} = 0$ to obtain the following error equation

$$\begin{aligned}(\mathbf{e}_t, \mathbf{v}_h) + \mu a(\mathbf{e}, \mathbf{v}_h) + \int_0^t \beta(t-s) a(\mathbf{e}(s), \mathbf{v}_h) ds + \rho(\nabla \cdot \mathbf{e}, \nabla \cdot \mathbf{v}_h) \\ = (p, \nabla \cdot \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{u}, \mathbf{u}, \mathbf{v}_h),\ \forall \mathbf{v}_h \in \mathbf{J}_h. \quad (5.8)\end{aligned}$$

Theorem 5.1. *Suppose the assumptions* (**A1m**)-(**A3m**), (**B1m**) *and* (**B2m**) *be satisfied. Moreover, assume* $\alpha > 0$ *be such that* $\mu - \left(\frac{\gamma}{\delta-\alpha}\right)^2 > 0$*. Then, the following bounds hold for* $t \in [0,T], T > 0$

$$\|\mathbf{e}(t)\|^2 + \beta_1 e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}(s)\|^2 ds + \rho e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\cdot\mathbf{e}(s)\|^2 ds \le Ch^{2m}e^{L(t)},$$

where, $\beta_1 = \mu - \left(\frac{\gamma}{\delta-\alpha}\right)^2 > 0$*, and*

$$L(t) = \int_0^t \Big(2\alpha + 4\|\nabla\mathbf{u}(s)\|_\infty + (1+\frac{4}{\rho})\|\mathbf{u}(s)\|_2^2\Big)ds, \tag{5.9}$$

and C *depends on*

$$\|\mathbf{u}(t)\|_m^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\Big((\mu+4\rho+2)\|\mathbf{u}(s)\|_{m+1}^2 + \frac{4}{\rho}\|p(s)\|_m^2\Big)ds, \tag{5.10}$$

but not depend on the inverse power of μ*.*

Proof. Choose $\mathbf{v}_h = P_h\mathbf{e} = \mathbf{e} - (\mathbf{u} - P_h\mathbf{u})$ in (5.8) to reach at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{e}\|^2 + \mu\|\nabla\mathbf{e}\|^2 + \rho\|\nabla\cdot\mathbf{e}\|^2 + \int_0^t \beta(t-s)a(\mathbf{e}(s),\mathbf{e})ds \\ = (\mathbf{e}_t, \mathbf{u} - P_h\mathbf{u}) + \mu a(\mathbf{e}, \mathbf{u} - P_h\mathbf{u}) + \rho(\nabla\cdot\mathbf{e}, \nabla\cdot(\mathbf{u} - P_h\mathbf{u})) \\ + \int_0^t \beta(t-s)a(\mathbf{e}(s), \mathbf{u} - P_h\mathbf{u})ds + (p, \nabla\cdot P_h\mathbf{e}) - \Lambda(P_h\mathbf{e}),\end{aligned} \tag{5.11}$$

where

$$\Lambda(\mathbf{v}_h) = b(\mathbf{u},\mathbf{u},\mathbf{v}_h) - b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) = b(\mathbf{u},\mathbf{e},\mathbf{v}_h) + b(\mathbf{e},\mathbf{u},\mathbf{v}_h) - b(\mathbf{e},\mathbf{e},\mathbf{v}_h).$$

Using the properties of P_h we rewrite the following as

$$(\mathbf{e}_t, \mathbf{u} - P_h\mathbf{u}) = (\mathbf{u}_t - P_h\mathbf{u}_t, \mathbf{u} - P_h\mathbf{u}) = \frac{1}{2}\frac{d}{dt}\|\mathbf{u} - P_h\mathbf{u}\|^2. \tag{5.12}$$

We use the "Cauchy-Schwarz inequality" and the "Young's inequality" with (5.6) and $\|\nabla\cdot\mathbf{v}\| \le C\|\nabla\mathbf{v}\|$ to obtain

$$\mu|a(\mathbf{e}, \mathbf{u} - P_h\mathbf{u})| \le \mu\|\nabla\mathbf{e}\|\|\nabla(\mathbf{u} - P_h\mathbf{u})\| \le \frac{C\mu}{2}h^{2m}\|\mathbf{u}\|_{m+1}^2 + \frac{\mu}{2}\|\nabla\mathbf{e}\|^2. \tag{5.13}$$

and

$$\rho|(\nabla\cdot\mathbf{e}, \nabla\cdot(\mathbf{u} - P_h\mathbf{u}))| \le C\rho\|\nabla\cdot\mathbf{e}\|\|\nabla(\mathbf{u} - P_h\mathbf{u})\| \le 2C\rho h^{2m}\|\mathbf{u}\|_{m+1}^2 + \frac{\rho}{8}\|\nabla\cdot\mathbf{e}\|^2.$$

We estimate the integral term as below:

$$\int_0^t \beta(t-s)a(\mathbf{e}(s),\mathbf{u}-P_h\mathbf{u})ds \le Ch^m\Big(\int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|ds\Big)\|\mathbf{u}\|_{m+1} \qquad (5.14)$$
$$\le \frac{C}{2}h^{2m}\|\mathbf{u}\|_{m+1}^2 + \frac{1}{2}(\int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|ds)^2.$$

A use of $(j_h p, \nabla\cdot P_h\mathbf{e}=0)$, with the approximation property (**B1m**) and the "Cauchy-Schwarz inequality" yields

$$|(p,\nabla\cdot P_h\mathbf{e})| = |(p-j_hp,\nabla\cdot P_h\mathbf{e})| \le Ch^m\|p\|_m\|\nabla\cdot\mathbf{e}\| \le \frac{2C}{\rho}h^{2m}\|p\|_m^2 + \frac{\rho}{8}\|\nabla\cdot\mathbf{e}\|^2. \quad (5.15)$$

Using (1.11), we can rewrite the nonlinear terms as

$$|\Lambda(P_h\mathbf{e})| = |b(\mathbf{e},\mathbf{u},\mathbf{e}) - b(\mathbf{u},\mathbf{e},\mathbf{u}-P_h\mathbf{u}) - b(\mathbf{e},\mathbf{u},\mathbf{u}-P_h\mathbf{u}) + b(\mathbf{e},\mathbf{e},\mathbf{u}-P_h\mathbf{u})| \quad (5.16)$$

We use (1.10) and (1.11) with the "Hölder's inequality", the "Garliardo-Nirenberg inequality", the "Agmon's inequality" and the "Young's inequality" and (5.6) to bound the nonlinear terms as

$$\begin{aligned}
|b(\mathbf{u},\mathbf{e},\mathbf{u}-P_h\mathbf{u})| &= |b(\mathbf{u},\mathbf{u}-P_h\mathbf{u},\mathbf{e})| \\
&= ((\mathbf{u}\cdot\nabla)(\mathbf{u}-P_h\mathbf{u}),\mathbf{e}) + \frac{1}{2}((\nabla\cdot\mathbf{u})(\mathbf{u}-P_h\mathbf{u}),\mathbf{e})| \\
&\le \|\mathbf{u}\|_\infty\|\nabla(\mathbf{u}-P_h\mathbf{u})\|\|\mathbf{e}\| + \frac{1}{2}\|\nabla\cdot\mathbf{u}\|_{L^4}\|\mathbf{u}-P_h\mathbf{u}\|_{L^4}\|\mathbf{e}\| \\
&\le Ch^m\|\mathbf{e}\|\|\mathbf{u}\|_2\|\mathbf{u}\|_{m+1} \\
&\le \frac{C}{2}h^{2m}\|\mathbf{u}\|_{m+1}^2 + \frac{C}{2}\|\mathbf{u}\|_2^2\|\mathbf{e}\|^2. \qquad (5.17)
\end{aligned}$$

And

$$\begin{aligned}
|b(\mathbf{e},\mathbf{u},\mathbf{u}-P_h\mathbf{u})| &= |\frac{1}{2}((\mathbf{e}\cdot\nabla)\mathbf{u},\mathbf{u}-P_h\mathbf{u}) - \frac{1}{2}((\mathbf{e}\cdot\nabla)(\mathbf{u}-P_h\mathbf{u}),\mathbf{u})| \\
&\le \frac{1}{2}(\|\mathbf{e}\|\|\nabla\mathbf{u}\|_{L^4}\|\mathbf{u}-P_h\mathbf{u}\|_{L^4} + \|\mathbf{e}\|\|\nabla(\mathbf{u}-P_h\mathbf{u})\|\|\mathbf{u}\|_\infty) \\
&\le Ch^m\|\mathbf{e}\|\|\mathbf{u}\|_2\|\mathbf{u}\|_{m+1} \\
&\le \frac{C}{2}h^{2m}\|\mathbf{u}\|_{m+1}^2 + \frac{C}{2}\|\mathbf{u}\|_2^2\|\mathbf{e}\|^2. \qquad (5.18)
\end{aligned}$$

Also use (1.10) with the "Agmon's inequality" and the "Cauchy-Schwarz inequality" to bound another nonlinear terms

$$\begin{aligned}
|b(\mathbf{e},\mathbf{u},\mathbf{e})| &\le ((\mathbf{e}\cdot\nabla)\mathbf{u},\mathbf{e}) + \frac{1}{2}((\nabla\cdot\mathbf{e})\mathbf{u},\mathbf{e}) \\
&\le \|\nabla\mathbf{u}\|_\infty\|\mathbf{e}\|^2 + \frac{1}{2}\|\nabla\cdot\mathbf{e}\|\|\mathbf{u}\|_\infty\|\mathbf{e}\|
\end{aligned}$$

$$\leq C(\|\nabla \mathbf{u}\|_\infty + \frac{1}{\rho}\|\mathbf{u}\|_\infty^2)\|\mathbf{e}\|^2 + \frac{\rho}{8}\|\nabla\cdot\mathbf{e}\|^2$$
$$\leq C(\|\nabla \mathbf{u}\|_\infty + \frac{1}{\rho}\|\mathbf{u}\|_2^2)\|\mathbf{e}\|^2 + \frac{\rho}{8}\|\nabla\cdot\mathbf{e}\|^2. \tag{5.19}$$

We use (1.11) and then as in (5.19) we bound remaining nonlinear terms

$$\begin{aligned} |b(\mathbf{e},\mathbf{e},\mathbf{u}-P_h\mathbf{u})| &= |b(\mathbf{e},\mathbf{u}-P_h\mathbf{u},\mathbf{e})| \\ &\leq C(\|\nabla(\mathbf{u}-P_h\mathbf{u})\|_\infty + \frac{1}{\rho}\|\mathbf{u}-P_h\mathbf{u}\|_\infty^2)\|\mathbf{e}\|^2 + \frac{\rho}{8}\|\nabla\cdot\mathbf{e}\|^2 \\ &\leq C(\|\nabla \mathbf{u}\|_\infty + \frac{1}{\rho}\|\mathbf{u}\|_2^2)\|\mathbf{e}\|^2 + \frac{\rho}{8}\|\nabla\cdot\mathbf{e}\|^2. \end{aligned} \tag{5.20}$$

Inserting (5.12)-(5.20) in (5.11) and then multiplying both side by $e^{2\alpha t}$, we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}e^{2\alpha t}\|\mathbf{e}\|^2 + \frac{\mu}{2}e^{2\alpha t}\|\nabla\mathbf{e}\|^2 + \frac{\rho}{2}e^{2\alpha t}\|\nabla\cdot\mathbf{e}\|^2 + e^{2\alpha t}\int_0^t \beta(t-\tau)a(\mathbf{e}(\tau),\mathbf{e})d\tau \\ &\leq \frac{1}{2}\frac{d}{dt}e^{2\alpha t}\|\mathbf{u}-P_h\mathbf{u}\|^2 - \alpha e^{2\alpha t}\|\mathbf{u}-P_h\mathbf{u}\|^2 + Ch^{2m}e^{2\alpha t}\Big((\frac{\mu}{2}+2\rho+1)\|\mathbf{u}\|_{m+1}^2 + \frac{2}{\rho}\|p\|_m^2\Big) \\ &\quad + e^{2\alpha t}\Big(2\|\nabla\mathbf{u}\|_\infty + (\frac{1}{2}+\frac{2}{\rho})\|\mathbf{u}\|_2^2 + \alpha\Big)\|\mathbf{e}\|^2 + \frac{1}{2}e^{2\alpha t}\Big(\int_0^t \beta(t-\tau)\|\nabla\mathbf{e}(\tau)\|d\tau\Big)^2. \end{aligned} \tag{5.21}$$

Take time integration and use $\|\mathbf{e}(0)\| = \|\mathbf{u}(0) - P_h\mathbf{u}(0)\|$ to obtain

$$\begin{aligned} &e^{2\alpha t}\|\mathbf{e}(t)\|^2 + \int_0^t e^{2\alpha s}(\mu\|\nabla\mathbf{e}(s)\|^2 + \rho\|\nabla\cdot\mathbf{e}(s)\|^2)ds + 2\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)a(\mathbf{e}(\tau),\mathbf{e}(s))d\tau ds \\ &\leq e^{2\alpha t}\|(\mathbf{u}-P_h\mathbf{u})(t)\|^2 + Ch^{2m}\int_0^t e^{2\alpha s}((\mu+4\rho+2)\|\mathbf{u}(s)\|_{m+1}^2 + \frac{4}{\rho}\|p(s)\|_m^2)ds \\ &+ \int_0^t e^{2\alpha s}(2\alpha + 4\|\nabla\mathbf{u}\|_\infty + (1+\frac{4}{\rho})\|\mathbf{u}\|_2^2)\|\mathbf{e}\|^2 ds + \int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\nabla\mathbf{e}(\tau)\|d\tau\Big)^2 ds. \end{aligned} \tag{5.22}$$

From Lemma 1.5, the double integration term on left of inequality is positive. We drop it. And the another double integration term can be bounded as similar to (2.15) of Chapter 2 as

$$\int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\nabla\mathbf{e}(\tau)\|d\tau\Big)^2 ds \leq \Big(\frac{\gamma}{\delta-\alpha}\Big)^2\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}(s)\|^2 ds. \tag{5.23}$$

Use (5.23) in (5.22) with $\beta_1 = \mu - \big(\frac{\gamma}{\delta-\alpha}\big)^2 > 0$ and use the "Gronwall's lemma" to conclude

$$\begin{aligned} &e^{2\alpha t}\|\mathbf{e}(t)\|^2 + \beta_1\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}(s)\|^2 ds + \rho\int_0^t e^{2\alpha s}\|\nabla\cdot\mathbf{e}(s)\|^2 ds \\ &\quad \leq Ch^{2m}e^{L(t)}\Big[e^{2\alpha t}\|\mathbf{u}(t)\|_m^2 + \int_0^t e^{2\alpha s}\Big((\mu+4\rho+2)\|\mathbf{u}(s)\|_{m+1}^2 + \frac{4}{\rho}\|p(s)\|_m^2\Big)ds\Big]. \end{aligned}$$

Multiply both side by $e^{-2\alpha t}$, which concludes the remaining of the proof. □

Remark 5.2. *In Theorem 5.1, such a choice of* $\alpha > 0$ *is possible by choosing* $\alpha < \delta - \frac{\gamma}{\sqrt{\mu}}$.

Remark 5.3. *From the assumption **(A3m)**, it is clear that* $L(t)$ *defined on (5.9) is bounded by* Ct *and the quantity in (5.10) is also bounded by* C*, where* C *does not depend on* μ^{-1}.

Remark 5.4. *For stable mixed finite element spaces* (P_m, P_{m-1})*,* $m > 1$*, the constant* C *of Theorem 5.1 does not depends on the inverse power of* μ*, but it depends on* ρ *and* ρ^{-1}*. This justifies the standard values of grad-div stabilization parameter to be* $\rho \approx 1$ *(as for NSEs, see [107, 109]). But for a pair of equal degree inf-sup stable FE spaces like MINI element* (P_1b, P_1)*, the constant* C *depends on*

$$\|\mathbf{u}(t)\|_1^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\Big((\mu + 4\rho + 2)\|\mathbf{u}(s)\|_2^2 + \frac{4h^2}{\rho}\|p(s)\|_2^2\Big)ds,$$

Then we can choose $\rho \approx h^2$ *or* h*, which give us the optimal result. In other words, we can choose the stabilization parameter* $\rho \approx h^2$ *to* 1.

However, it is important to note that ρ *depends on the mesh size, the viscosity and the finite element spaces which we have numerically verified in Section 5.5. (Detailed discussion for Stokes and NSEs can be found in [4, 82].)*

Error bounds for the pressure

Theorem 5.2. *Let us assume that the hypothesis of the Lemma* 5.1 *holds true. Additionally, let* $\mathbf{u}_t \in L^2(0, T; \mathbf{H}^{m-1})$*, then, for all* $t > 0$*, the following holds*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|(p - p_h)(s)\|^2_{L^2/N_h}ds \le Ch^{2m}e^{L(t)},$$

where, $L(t)$ *is defined in (5.9) and* C *depends on the following*

$$\|\mathbf{u}(t)\|_m^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\big(\|\mathbf{u}_s(s)\|^2_{m-1} + (\mu + 4\rho + 2)\|\mathbf{u}(s)\|^2_{m+1} + \frac{4}{\rho}\|p(s)\|_m^2\big)ds, \qquad (5.24)$$

but independent of μ^{-1}.

To achieve a proof, we need some intermediate results. We start with splitting the pressure error $p - p_h$ as

$$\|p - p_h\| \le \|p - j_hp\| + \|j_hp - p_h\|. \qquad (5.25)$$

We just need to estimate the second term which we rewrite as

$$\|j_hp - p_h\|_{L^2/N_h} \leq C \sup_{\mathbf{v}_h\in\mathbf{H}_h/\{0\}} \left\{ \frac{|(j_hp - p_h, \nabla\cdot\mathbf{v}_h)|}{\|\nabla\mathbf{v}_h\|} \right\}$$
$$\leq C\Big(\|j_hp - p\| + \sup_{\mathbf{v}_h\in\mathbf{H}_h/\{0\}} \left\{ \frac{|(p - p_h, \nabla\cdot\mathbf{v}_h)|}{\|\nabla\mathbf{v}_h\|} \right\}\Big). \tag{5.26}$$

The first term of the above inequality (5.26) can be estimated by using (**B2m**). And for the second term, we first look at the error equation in pressure obtained by subtracting (5.1) from (1.8):

$$(p - p_h, \nabla\cdot\mathbf{v}_h) = (\mathbf{e}_t, \mathbf{v}_h) + \mu a(\mathbf{e}, \mathbf{v}_h) + \int_0^t \beta(t-s)a(\mathbf{e}(s), \mathbf{v}_h)ds$$
$$+\rho(\nabla\cdot\mathbf{e}, \nabla\cdot\mathbf{v}_h) + \Lambda(\mathbf{v}_h), \tag{5.27}$$

where

$$\Lambda(\mathbf{v}_h) = -b(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = -b(\mathbf{u}_h, \mathbf{e}, \mathbf{v}_h) - b(\mathbf{e}, \mathbf{u}, \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{H}_h.$$

Similar to (5.17) and (5.18), we bound the nonlinear terms as

$$|\Lambda(\mathbf{v}_h)| = C(\|\mathbf{u}\|_2 + \|\mathbf{u}_h\|_2)\|\mathbf{e}\|\|\nabla\mathbf{v}\|.$$

Since $\mathbf{u}$ is regular enough, $\mathbf{u}$ is continuous and hence, $\|I_h\mathbf{u}\|_2 \leq C\|\mathbf{u}\|_2$, for some $C > 0$. Then, using (5.5), (5.6) and Lemma 5.1, one can find

$$\|\mathbf{u}_h\|_2 \leq \|\mathbf{u}_h - I_h\mathbf{u}\|_2 + \|I_h\mathbf{u}\|_2 \leq Ch^{-2}\|\mathbf{u}_h - I_h\mathbf{u}\| + C\|\mathbf{u}\|_2 \leq C\|\mathbf{u}\|_3. \tag{5.28}$$

Incorporating (5.28) in (5.27), we arrive at

$$(p - p_h, \nabla\cdot\mathbf{v}_h) \leq C\Big(\|\mathbf{e}_t\|_{-1;h} + \mu\|\nabla\mathbf{e}\| + \rho\|\nabla\cdot\mathbf{e}\| + \|\mathbf{u}\|_3\|\mathbf{e}\|$$
$$+ \int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|ds\Big)\|\nabla\mathbf{v}_h\| \tag{5.29}$$

where,

$$\|\mathbf{e}_t\|_{-1;h} = \sup\Big\{ \frac{< \mathbf{e}_t, \mathbf{v}_h >}{\|\nabla\mathbf{v}_h\|} : \mathbf{v}_h \in \mathbf{H}_h, \mathbf{v}_h \neq 0 \Big\}.$$

Since all the estimate on the right of inequality in (5.29) are known except $\|\mathbf{e}_t\|_{-1;h}$, we now derive $\|\mathbf{e}_t\|_{-1;h}$. Since $\mathbf{H}_h \subset \mathbf{H}_0^1$, then we have

$$\|\mathbf{e}_t\|_{-1;h} = \sup\Big\{ \frac{\langle\mathbf{e}_t, \mathbf{v}_h\rangle}{\|\nabla\mathbf{v}_h\|} : \mathbf{v}_h \in \mathbf{H}_h, \mathbf{v}_h \neq 0 \Big\}$$
$$\leq \sup\Big\{ \frac{\langle\mathbf{e}_t, \mathbf{v}\rangle}{\|\nabla\mathbf{v}\|} : \mathbf{v} \in \mathbf{H}_0^1, \mathbf{v} \neq 0 \Big\} = \|\mathbf{e}_t\|_{-1}.$$

Lemma 5.2. *Under the hypothesis of previous lemma, the following negative norm error estimate holds for* $0 < t < T$*:*

$$\|\mathbf{e}_t\|_{-1} \leq C\Big(h^m(\|\mathbf{u}_t\|_{m-1}+\|p\|_m)+\mu\|\nabla\mathbf{e}\|+\rho\|\nabla\cdot\mathbf{e}\|+\|\mathbf{u}\|_3\|\mathbf{e}\|+\int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|ds\Big).$$

Proof. For any $\boldsymbol{\psi} \in \mathbf{H}_0^1$, we use the orthogonal projection $P_h : \mathbf{L}^2 \to \mathbf{J}_h$ and(5.8) with $\mathbf{v}_h = P_h\boldsymbol{\psi}$ to obtain

$$\begin{aligned}(\mathbf{e}_t, \boldsymbol{\psi}) &= (\mathbf{e}_t, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) + (\mathbf{e}_t, P_h\boldsymbol{\psi})\\ &= (\mathbf{e}_t, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) - \mu a(\mathbf{e}, P_h\boldsymbol{\psi}) - \int_0^t \beta(t-s)a(\mathbf{e}(s), P_h\boldsymbol{\psi})ds\\ &\quad + (p, \nabla\cdot P_h\boldsymbol{\psi}) - \rho(\nabla\cdot\mathbf{e}, \nabla\cdot P_h\boldsymbol{\psi}) - \Lambda(P_h\boldsymbol{\psi}). \qquad (5.30)\end{aligned}$$

An application of the approximation property of P_h helps to bound the followings

$$(\mathbf{e}_t, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) = (\mathbf{u}_t - P_h\mathbf{u}_t, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) \leq Ch^m\|\mathbf{u}_t\|_{m-1}\|\nabla\boldsymbol{\psi}\|. \qquad (5.31)$$

And

$$(p, \nabla\cdot P_h\boldsymbol{\psi}) \leq (p - j_hp, \nabla\cdot P_h\boldsymbol{\psi}) \leq Ch^m\|p\|_m\|\nabla\boldsymbol{\psi}\|. \qquad (5.32)$$

Now, substitute (5.31)-(5.32) in (5.30) and use (5.28) with $\mathbf{v}_h = P_h\boldsymbol{\psi}$ to obtain

$$\begin{aligned}(\mathbf{e}_t, \boldsymbol{\psi}) \leq C\Big(&h^m(\|\mathbf{u}_t\|_{m-1} + \|p\|_m) + \mu\|\nabla\mathbf{e}\| + \rho\|\nabla\cdot\mathbf{e}\| + \|\mathbf{u}\|_3\|\mathbf{e}\|\\ &+ \int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|ds\Big)\|\nabla\boldsymbol{\psi}\|.\end{aligned}$$

and therefore,

$$\begin{aligned}\|\mathbf{e}_t\|_{-1} &\leq \sup\Big\{\frac{<\mathbf{e}_t, \mathbf{v}>}{\|\nabla\mathbf{v}\|} : \mathbf{v} \in \mathbf{H}_0^1, \mathbf{v} \neq 0\Big\}\\ &\leq C\Big(h^m(\|\mathbf{u}_t\|_{m-1} + \|p\|_m) + \mu\|\nabla\mathbf{e}\| + \rho\|\nabla\cdot\mathbf{e}\| + \|\mathbf{u}\|_3\|\mathbf{e}\|\\ &\qquad + \int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|ds\Big),\end{aligned}$$

This concludes the proof. □

Proof of the Theorem 5.2: From (5.25), (5.26), (5.29) and Lemma 5.2, we obtain

$$\begin{aligned}\|(p - p_h)\|^2_{L^2/N_h} \leq &C\Big(h^{2m}(\|\mathbf{u}_t\|^2_{m-1} + \|p\|^2_m) + \mu\|\nabla\mathbf{e}\|^2 + \rho\|\nabla\cdot\mathbf{e}\|^2 + \|\mathbf{u}\|^2_3\|\mathbf{e}\|^2\\ &+ \big(\int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|ds\big)^2\Big)\end{aligned}$$

We multiply both side by $e^{2\alpha t}$ and take time integration. Then, the resulting double integration term can be written as a single integration similar to (5.23) and we finally reach at

$$\int_0^t e^{2\alpha s}\|(p-p_h)(s)\|^2_{L^2/N_h}ds \le C\Big(h^{2m}\int_0^t e^{2\alpha s}(\|\mathbf{u}_s(s)\|^2_{m-1}+\|p(s)\|^2_m)ds$$
$$+\beta_1\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}(s)\|^2ds+\rho\int_0^t e^{2\alpha s}\|\nabla\cdot\mathbf{e}(s)\|^2ds+\|\mathbf{e}(t)\|^2_{L^\infty}\int_0^t e^{2\alpha s}\|\mathbf{u}(s)\|^2_3ds\Big).$$

We use Theorem 5.1 and multiply both side by $e^{-2\alpha t}$ to conclude the proof. □

5.3.2 Semidiscrete Error Estimates for Smooth Data

As discussed in the introduction, the assumption of sufficiently smooth initial data is not realistic. And so we restrict the initial velocity $\mathbf{u}_0$ to be just smooth, that is, $\mathbf{u}_0 \in \mathbf{J}_1 \cap \mathbf{H}^2$. The loss of regularity at $t=0$ is taken into consideration in this section.

Theorem 5.3. *Suppose the conditions* (**A1m**),(**A2m**), (**A3m′**), (**B1m**) *and* (**B2m**) *be satisfied. Moreover, assume* $\alpha > 0$ *be such that* $\mu - \left(\frac{\gamma}{\delta-\alpha}\right)^2 > 0$. *Then, for* $t \in [0,T], T > 0$ *and* $m \in \{1,2\}$, *the following bounds hold:*

$$\|\mathbf{e}(t)\|^2+\beta_1 e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{e}(s)\|^2ds+\rho e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\cdot\mathbf{e}(s)\|^2ds \le Ch^{2m}e^{L(t)},$$

and

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|(p-p_h)(s)\|^2_{L^2/N_h}ds \le Ch^{2m}e^{L(t)},$$

where, C, β_1, *and* $L(t)$ *are defined on Theorem 5.1.*

We skip the proof since it follows the proofs of Theorems 5.1 and 5.2.

Remark 5.5. *Unlike in the case of sufficiently smooth data, where the estimates of Theorems 5.1 and 5.2 are valid for all* $m \ge 1$, *here, in the case of smooth only data, these estimates remain valid only for* $m \in \{1,2\}$. *That is, for* $m \ge 3$, *for higher order approximations of velocity and pressure, we do not obtain higher order rate of convergence, but is restricted to second order convergence for velocity and pressure, in case of smooth only data, and in case the estimates do not depend on inverse power of* μ.

In view of the above remark, we look into the case $m \geq 3$ for smooth initial data. We set $\mathbf{v}_h = P_h\mathbf{e} = \mathbf{e} - (\mathbf{u} - P_h\mathbf{u})$ in (5.8) and following the steps (5.11)-(5.20), we obtain (5.21), that is,

$$\frac{1}{2}\frac{d}{dt}e^{2\alpha t}\|\mathbf{e}\|^2 + \frac{\mu}{2}e^{2\alpha t}\|\nabla\mathbf{e}\|^2 + \frac{\rho}{2}e^{2\alpha t}\|\nabla\cdot\mathbf{e}\|^2 + e^{2\alpha t}\int_0^t \beta(t-s)a(\mathbf{e}(s),\mathbf{e})ds$$
$$\leq \frac{1}{2}\frac{d}{dt}e^{2\alpha t}\|\mathbf{u}-P_h\mathbf{u}\|^2 - \alpha e^{2\alpha t}\|\mathbf{u}-P_h\mathbf{u}\|^2 + Ch^{2m}e^{2\alpha t}\Big((\frac{\mu}{2}+2\rho+1)\|\mathbf{u}\|^2_{m+1} + \frac{2}{\rho}\|p\|^2_m\Big)$$
$$+ e^{2\alpha t}\Big(2\|\nabla\mathbf{u}\|_\infty + (\frac{1}{2}+\frac{2}{\rho})\|\mathbf{u}\|_2^2 + \alpha\Big)\|\mathbf{e}\|^2 + \frac{1}{2}e^{2\alpha t}\Big(\int_0^t \beta(t-\tau)\|\nabla\mathbf{e}(\tau)\|d\tau\Big)^2. \quad (5.33)$$

Here we can not integrate from 0 to t directly, since the third term on the right of inequality (5.33) is no longer integrable near $t = 0$ for $m \geq 3$. For example, from **(A3m′)**, and for $m = 3$, we have

$$\int_0^t e^{2\alpha s}\tau(s)(\|\mathbf{u}\|_4^2 + \|p\|_3^2)\ ds \leq C.$$

Here the kernel $\tau(t)$ compensates for the singularity at $t = 0$ of the higher order estimates of the solutions.

Keeping this in mind we multiply (5.33) by $\tau^{m-2}(t)$ and use the fact $\sigma_t^{m-2}(t) \leq 2\alpha\sigma^{m-2}(t) + (m-2)\sigma^{m-3}(t)$, where $\sigma^m(t) = (\tau(t))^m e^{2\alpha t}$. Then we take time integration the resulting inequality to find

$$\sigma^{m-2}(t)\|\mathbf{e}(t)\|^2 + \int_0^t \sigma^{m-2}(s)\big(\mu\|\nabla\mathbf{e}\|^2 ds + \rho\|\nabla\cdot\mathbf{e}\|^2\big)ds$$
$$+ 2\int_0^t \sigma^{m-2}(s)\int_0^s \beta(s-\tau)a(\mathbf{e}(\tau),\mathbf{e}(s))d\tau ds$$
$$\leq \sigma^{m-2}(t)\|\mathbf{u}(t) - P_h\mathbf{u}(t)\|^2 + (m-2)\int_0^t \sigma^{m-3}(s)\|\mathbf{e}(s)\|^2 ds$$
$$- \int_0^t (2\alpha\sigma^{m-2}(s) + (m-2)\sigma^{m-3}(s))\|\mathbf{u}(t) - P_h\mathbf{u}(t)\|^2 ds$$
$$+ Ch^{2m}\int_0^t \sigma^{m-2}(s)\Big((\mu+4\rho+2)\|\mathbf{u}(s)\|^2_{m+1} + \frac{4}{\rho}\|p(s)\|^2_m\Big)ds$$
$$+ \int_0^t \sigma^{m-2}(s)\Big(2\alpha + 4\|\nabla\mathbf{u}\|_\infty + (1+\frac{4}{\rho})\|\mathbf{u}\|_2^2\Big)\|\mathbf{e}\|^2 ds$$
$$+ \int_0^t \sigma^{m-2}(s)\Big(\int_0^s \beta(s-\tau)\|\nabla\mathbf{e}(\tau)\|d\tau\Big)^2 ds. \quad (5.34)$$

However the bound for the second term on the right of inequality (5.34), that is,

$$(m-2)\int_0^t \sigma^{m-3}(s)\|\mathbf{e}(s)\|^2 ds \quad (5.35)$$

is no longer independent of inverse power of μ. To see this for the case $m = 3$, we first write the error $\mathbf{e}$ in two parts, as $\mathbf{e} = (\mathbf{u} - \mathbf{w}_h) + (\mathbf{w}_h - \mathbf{u}_h)$, where $\mathbf{w}_h : [0, T) \to \mathbf{J}_h$

is the auxiliary function satisfying

$$(\mathbf{u}_t - \mathbf{w}_{ht}, \mathbf{v}_h) + \mu a(\mathbf{u} - \mathbf{w}_h, \mathbf{v}_h) + \int_0^t \beta(t-\tau) a(\mathbf{u}(\tau) - \mathbf{w}_h(\tau), \mathbf{v}_h) d\tau = 0. \quad (5.36)$$

Let $\boldsymbol{\xi} = \mathbf{u} - \mathbf{w}_h$. Then choose $\mathbf{v}_h = P_h(-\Delta_h)^{-1}\boldsymbol{\xi} = (-\Delta_h)^{-1}\boldsymbol{\xi} - (-\Delta_h)^{-1}(\mathbf{u} - P_h\mathbf{u})$ in (5.36) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\xi}\|_{-1}^2 + \mu\|\boldsymbol{\xi}\|^2 + \int_0^t \beta(t-\tau)(\boldsymbol{\xi}(\tau), \boldsymbol{\xi}) d\tau = (\boldsymbol{\xi}_t, (-\Delta_h)^{-1}(\mathbf{u} - P_h\mathbf{u}))$$
$$+\mu(\boldsymbol{\xi}, \mathbf{u} - P_h\mathbf{u}) + \int_0^t \beta(t-\tau)(\boldsymbol{\xi}(\tau), \mathbf{u} - P_h\mathbf{u}) d\tau.$$

A use of the properties of P_h, the "Cauchy-Schwarz inequality" and the "Young's inequality" yields

$$\frac{d}{dt}\|\boldsymbol{\xi}\|_{-1}^2 + \mu\|\boldsymbol{\xi}\|^2 + 2\int_0^t \beta(t-\tau)(\boldsymbol{\xi}(\tau), \boldsymbol{\xi}) d\tau \leq \frac{d}{dt}\|\mathbf{u} - P_h\mathbf{u}\|_{-1}^2 + C(\mu+1)h^6\|\mathbf{u}\|_3^2$$
$$+\Big(\int_0^t \beta(t-\tau)\|\boldsymbol{\xi}(\tau)\| d\tau\Big)^2.$$

We multiply both side by $e^{2\alpha t}$ and take time integration to arrive

$$e^{2\alpha t}\|\boldsymbol{\xi}(t)\|_{-1}^2 + \mu\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|^2 ds + 2\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)(\boldsymbol{\xi}(\tau), \boldsymbol{\xi}(s)) d\tau ds$$
$$\leq 2\alpha\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|_{-1}^2 ds + e^{2\alpha t}\|\mathbf{u} - P_h\mathbf{u}\|_{-1}^2 + C(\mu+1)h^6\int_0^t e^{2\alpha s}\|\mathbf{u}(s)\|_3^2 ds$$
$$+\int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\boldsymbol{\xi}(\tau)\| d\tau\Big)^2 ds. \quad (5.37)$$

We drop the double integration term on the left of inequality (5.37) and as in (5.23) we bound the last term on the right of inequality. Now for the first term on the right of inequality, an application of the orthogonal property of P_h and the "Cauchy-Schwarz inequality" yield

$$\|\mathbf{u} - P_h\mathbf{u}\|_{-1}^2 = (\mathbf{u} - \mathbf{w}_h, (-\Delta_h)^{-1}(\mathbf{u} - P_h\mathbf{u})) + (\mathbf{w}_h - P_h\mathbf{u}, (-\Delta_h)^{-1}(\mathbf{u} - P_h\mathbf{u}))$$
$$= (\boldsymbol{\xi}, (-\Delta_h)^{-1}(\mathbf{u} - P_h\mathbf{u})) \leq \|\boldsymbol{\xi}\|_{-1}\|\mathbf{u} - P_h\mathbf{u}\|_{-1}.$$

Above we have used the fact that $\boldsymbol{\xi} = \mathbf{u} - \mathbf{w}_h$. On simplifying, we find

$$\|\mathbf{u} - P_h\mathbf{u}\|_{-1} \leq \|\boldsymbol{\xi}\|_{-1}. \quad (5.38)$$

And finally, a use of the "Gronwall's lemma" and (5.38) in (5.37) give

$$\Big(\mu - \frac{\gamma^2}{(\delta-\alpha)^2}\Big)\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|^2 ds \leq Ce^{2\alpha t}(\mu+1)h^6\int_0^t e^{2\alpha s}\|\mathbf{u}(s)\|_3^2 ds.$$

By our assumption we have $\mu - \frac{\gamma^2}{(\delta-\alpha)^2} > 0$. Now the resulting estimate will depend on the inverse power of μ. Using similar arguments we can show that the estimate of (5.35) will depend on the inverse power of μ for $m > 3$. And as a result, so will the estimate of (5.34).

In order to show that only second order convergence is possible, in case estimates are independent of inverse power of μ, and in case $m \geq 3$, we obtain (5.33) as earlier. But now we restrict ourselves to lower order projection properties, that is, $\|\nabla(\mathbf{u}-P_h\mathbf{u})\| \leq Ch^2\|\mathbf{u}\|_3$ etc., which no longer demands a time weight $\tau(t)$. Following the lines of argument for (5.21), we can obtained the desired result.

5.4 Fully Discrete Formulation

This section deals with the fully discrete error estimates for sufficient smooth initial data, that is, $\mathbf{u}_0 \in \mathbf{H}^{\max\{3,m\}}$. Then we carry out the analysis for smooth initial data, that is, $\mathbf{u}_0 \in \mathbf{H}^2 \cap \mathbf{J}_1$.

5.4.1 Fully Discrete Error Estimates for Sufficiently Smooth Data

We define $\mathbf{u}(t_n) = \mathbf{u}^n$, $p(t_n) = p^n$ and set $\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}^n$. For the error equation, we consider (5.7) at $t = t_n$ and subtract from (5.4): For all $\mathbf{v}_h \in \mathbf{J}_h$

$$
\begin{aligned}
(\partial_t \mathbf{e}^n, \mathbf{v}_h) + \mu a(\mathbf{e}^n, \mathbf{v}_h) + \rho(\nabla\cdot\mathbf{e}^n, \nabla\cdot\mathbf{v}_h) + a(q_r^n(\mathbf{e}), \mathbf{v}_h)& \\
= (p^n, \nabla\cdot\mathbf{v}_h) + R^n(\mathbf{v}_h) + \Lambda^n(\mathbf{v}_h) + E^n(\mathbf{v}_h),& \qquad (5.39)
\end{aligned}
$$

where

$$
\begin{aligned}
R^n(\mathbf{v}_h) &= (\mathbf{u}_t^n, \mathbf{v}_h) - (\partial_t \mathbf{u}^n, \mathbf{v}_h) = (\mathbf{u}_t^n, \mathbf{v}_h) - \frac{1}{k}\int_{t_{n-1}}^{t_n} (\mathbf{u}_s, \mathbf{v}_h)\, ds \\
&= \frac{1}{k}\int_{t_{n-1}}^{t_n} (s - t_{n-1})(\mathbf{u}_{ss}, \mathbf{v}_h)\, ds, \qquad (5.40) \\
E^n(\mathbf{v}_h) &= \int_0^{t_n} \beta(t-s) a(\mathbf{u}(s), \mathbf{v}_h)\, ds - k\sum_{i=1}^{n} \beta(t_n - t_i) a(\mathbf{u}^i, \mathbf{v}_h) \qquad (5.41) \\
&\leq C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} (s - t_{i-1})\big(\beta_s(t_n - s)a(\mathbf{u}(s), \mathbf{v}_h) + \beta(t_n - s)a(\mathbf{u}_s(s), \mathbf{v}_h)\big)\, ds. \\
\Lambda^n(\mathbf{v}_h) &= b(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}_h) - b(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h)
\end{aligned}
$$

$$= b(\mathbf{e}^n, \mathbf{e}^n, \mathbf{v}_h) - b(\mathbf{u}^n, \mathbf{e}^n, \mathbf{v}_h) - b(\mathbf{e}^n, \mathbf{u}^n, \mathbf{v}_h). \tag{5.42}$$

Fully discrete error bounds for velocity

In this section, we consider the exact solution to be sufficiently smooth. Our main result of this section is as follows:

Theorem 5.4. *Let the initial velocity satisfy $\mathbf{u}_0 \in \mathbf{H}^{\max\{3,m\}}$ and let all other assumptions of Theorem 5.1 hold true. Further, let $\mathbf{u}_t \in L^2(0,T;\mathbf{H}^2) \cap L^2(0,T;\mathbf{H}^m)$ and $\mathbf{u}_{tt} \in L^2(0,T;\mathbf{L}^2)$. Then, the following bounds hold for $1 \leq n \leq N$*

$$\|\mathbf{e}^n\|^2 + ke^{-2\alpha t_n}\sum_{i=1}^{n} e^{2\alpha t_i}\big(\beta_1\|\nabla \mathbf{e}^i\|^2 + \rho\|\nabla\cdot\mathbf{e}^i\|^2\big) \leq Ce^{\hat{L}^n}\big(K_1(t_n)h^{2m} + K_2(t_n)k^2\big),$$

where $\beta_1 = \mu - \big(\frac{\gamma}{\delta-\alpha}\big)^2 > 0$, and

$$\hat{L}^n = \sum_{i=1}^{n}\big(C(\alpha) + 4\|\nabla\mathbf{u}^i\|_\infty + (1+\frac{4}{\rho})\|\mathbf{u}^i\|_2^2\big), \tag{5.43}$$

$$K_1(t_n) = \|\mathbf{u}(t_n)\|_m^2 + e^{-2\alpha t_n}\int_0^{t_n} e^{2\alpha s}\big(\|\mathbf{u}_s(s)\|_m^2 + (\mu+4\rho+2)\|\mathbf{u}(s)\|_{m+1}^2 + \frac{4}{\rho}\|p(s)\|_m^2\big)ds$$

$$K_2(t_n) = e^{-2\alpha t_n}\int_0^{t_n} e^{2\alpha s}\big(\|\mathbf{u}_{ss}(s)\|^2 + \|\mathbf{u}(s)\|_2^2 + \|\mathbf{u}_s(s)\|_2^2\big)ds,$$

and $C > 0$ is a constant may depends on given data but not on the inverse power of μ as well as h and k.

Proof. We take $n = i$ and $\mathbf{v}_h = P_h\mathbf{e}^i = \mathbf{e}^i - (\mathbf{u}^i - P_h\mathbf{u}^i)$ in (5.39) to arrive at

$$\begin{aligned}(\partial_t\mathbf{e}^i, \mathbf{e}^i) + \mu a(\mathbf{e}^i, \mathbf{e}^i) + \rho(\nabla\cdot\mathbf{e}^i, \nabla\cdot\mathbf{e}^i) + a(q_r^i(\mathbf{e}), \mathbf{e}^i) &= (\partial_t\mathbf{e}^i, \mathbf{u}^i - P_h\mathbf{u}^i)\\ + \mu a(\mathbf{e}^i, \mathbf{u}^i - P_h\mathbf{u}^i) + \rho(\nabla\cdot\mathbf{e}^i, \nabla\cdot(\mathbf{u}^i - P_h\mathbf{u}^i)) &+ a(q_r^i(\mathbf{e}), \mathbf{u}^i - P_h\mathbf{u}^i)\\ + (p^n, \nabla\cdot P_h\mathbf{e}^i) + R_h^i(P_h\mathbf{e}^i) + \Lambda_h^i(P_h\mathbf{e}^i) &+ E_h^i(P_h\mathbf{e}^i).\end{aligned}$$

We note that

$$(\partial_t\mathbf{v}^i, \mathbf{v}^i) = \frac{1}{k}(\mathbf{v}^i - \mathbf{v}^{i-1}, \mathbf{v}^i) = \frac{1}{2}\partial_t\|\mathbf{v}^i\|^2 + \frac{k}{2}\|\partial_t\mathbf{v}^i\|^2 \geq \frac{1}{2}\partial_t\|\mathbf{v}^i\|^2. \tag{5.44}$$

And a use of the property of P_h yields

$$(\partial_t\mathbf{e}^i, \mathbf{u}^i - P_h\mathbf{u}^i) = (\partial_t(\mathbf{u}^i - P_h\mathbf{u}^i), \mathbf{u}^i - P_h\mathbf{u}^i) \leq \frac{Ch^{2m}}{2}(\partial_t\|\mathbf{u}^i\|_m^2 + k\|\partial_t\mathbf{u}^i\|_m^2). \tag{5.45}$$

Now, we use the "Cauchy-Schwarz inequality" and the "Young's inequality" along with (5.6), (5.44) and (5.45) to obtain

$$\partial_t\|\mathbf{e}^i\|^2 + \mu\|\nabla\mathbf{e}^i\|^2 + \frac{3\rho}{2}\|\nabla\cdot\mathbf{e}^i\|^2 + 2a(q_r^i(\mathbf{e}), \mathbf{e}^i)$$

$$\le Ch^{2m}\Big(\partial_t\|\mathbf{u}^i\|_m^2 + k\|\partial_t\mathbf{u}^i\|_m^2 + (\mu+4\rho+1)\|\mathbf{u}^i\|_{m+1}^2 + \frac{4}{\rho}\|p^i\|_m^2\Big)$$
$$+ (q_r^i(\|\nabla\mathbf{e}\|))^2 + 2R_h^i(P_h\mathbf{e}^i) + 2\Lambda_h^i(P_h\mathbf{e}^i) + 2E_h^i(P_h\mathbf{e}^i).$$

We multiply both side by $ke^{2\alpha t_i}$ then take summation over $1 \le i \le n$ to find that,

$$k\sum_{i=1}^n e^{2\alpha t_i}\partial_t\|\mathbf{e}^i\|^2 + \mu k\sum_{i=1}^n e^{2\alpha t_i}\|\nabla\mathbf{e}^i\|^2 + \frac{3\rho}{2}k\sum_{i=1}^n e^{2\alpha t_i}\|\nabla\cdot\mathbf{e}^i\|^2 + 2k\sum_{i=1}^n e^{2\alpha t_i}a(q_r^i(\mathbf{e}),\mathbf{e}^i)$$
$$\le Ch^{2m}k\sum_{i=1}^n e^{2\alpha t_i}\Big(\partial_t\|\mathbf{u}^i\|_m^2 + k\|\partial_t\mathbf{u}^i\|_m^2 + (\mu+4\rho+1)\|\mathbf{u}^i\|_{m+1}^2 + \frac{4}{\rho}\|p^i\|_m^2\Big)$$
$$+ k\sum_{i=1}^n e^{2\alpha t_i}(q_r^i(\|\nabla\mathbf{e}\|))^2 + 2k\sum_{i=1}^n e^{2\alpha t_i}(R^i(P_h\mathbf{e}^i) + \Lambda^i(P_h\mathbf{e}^i) + E^i(P_h\mathbf{e}^i)). \quad (5.46)$$

The last term on the left of inequality is vanished due to positivity property (1.18). Similar to (5.23), with a use of the "Cauchy-Schwarz inequality" and the change of order of summation, we can write the quadrature term as

$$k\sum_{i=1}^n e^{2\alpha t_i}(q_r^i(\|\nabla\mathbf{e}\|))^2 = k\sum_{i=1}^n e^{2\alpha t_i}k\sum_{j=0}^i \beta(t_n-t_j)\|\nabla\mathbf{e}^j\|^2 \le \Big(\frac{\gamma}{\delta-\alpha}\Big)^2 k\sum_{i=1}^n e^{2\alpha t_i}\|\nabla\mathbf{e}^i\|^2.$$

Now using the fact that

$$k\sum_{i=1}^n e^{2\alpha t_i}\partial_t\|\mathbf{e}^i\|^2 \ge e^{2\alpha t_n}\|\mathbf{e}^n\|^2 - k\sum_{i=1}^{n-1}\frac{(e^{2\alpha k}-1)}{k\lambda_1}e^{2\alpha t_i}\|\mathbf{e}^i\|^2$$

in (5.46) with $\beta_1 = \mu - (\frac{\gamma}{\delta-\alpha})^2 > 0$, we arrive at

$$e^{2\alpha t_n}\|\mathbf{e}^n\|^2 + \beta_1 k\sum_{i=1}^n e^{2\alpha t_i}\|\nabla\mathbf{e}^i\|^2 + \frac{3\rho}{2}k\sum_{i=1}^n e^{2\alpha t_i}\|\nabla\cdot\mathbf{e}^i\|^2$$
$$\le \sum_{i=1}^{n-1} e^{2\alpha t_i}(e^{2\alpha k}-1)\|\mathbf{e}^i\|^2 + Ch^{2m}\Big[e^{2\alpha t_n}\|\mathbf{u}^n\|_m^2 - \|\mathbf{u}_0\|^2$$
$$- \sum_{i=1}^{n-1} e^{2\alpha t_i}(e^{2\alpha k}-1)\|\mathbf{u}^i\|^2 + k^2\sum_{i=1}^n e^{2\alpha t_i}\|\partial_t\mathbf{u}^i\|_m^2$$
$$+ k\sum_{i=1}^n e^{2\alpha t_i}\Big((\mu+4\rho+1)\|\mathbf{u}^i\|_{m+1}^2 + \frac{4}{\rho}\|p^i\|_m^2\Big)\Big]$$
$$+ 2k\sum_{i=1}^n e^{2\alpha t_i}(R^i(P_h\mathbf{e}^i) + \Lambda^i(P_h\mathbf{e}^i) + E^i(P_h\mathbf{e}^i)). \quad (5.47)$$

The second and third terms within the bracket are positive, so we drop them and the third and fourth terms can be written as

$$k^2\sum_{i=1}^n e^{2\alpha t_i}\|\partial_t\mathbf{u}^i\|_m^2 \le k^2\sum_{i=1}^n e^{2\alpha t_i}\Big(\frac{1}{k}\int_{t_{i-1}}^{t_i}\|\mathbf{u}_s(s)\|_m ds\Big)^2 \le C\int_0^{t_n} e^{2\alpha s}\|\mathbf{u}_s(s)\|_m^2 ds. \quad (5.48)$$

And

$$k\sum_{i=1}^{n} e^{2\alpha t_i}\Big((\mu+4\rho+1)\|\mathbf{u}^i\|_{m+1}^2+\frac{4}{\rho}\|p^i\|_m^2\Big)$$
$$\leq \int_0^{t_n} e^{2\alpha s}\Big((\mu+4\rho+1)\|\mathbf{u}(s)\|_{m+1}^2+\frac{4}{\rho}\|p(s)\|_m^2\Big)ds. \quad (5.49)$$

We use the "Cauchy-Schwarz inequality" and the "Young's inequality" with $t_{i-1}\leq t,\ t\in[t_{i-1},t_i]$ in (5.40) to find

$$\begin{aligned}
k\sum_{i=1}^{n} e^{2\alpha t_i}R^i(P_h\mathbf{e}^i) &\leq k\sum_{i=1}^{n} e^{2\alpha t_i}\Big(\frac{1}{k}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|\mathbf{u}_{ss}(s)\|ds\Big)^2+\frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2\\
&\leq k\sum_{i=1}^{n} e^{2\alpha t_i}\frac{1}{k^2}\Big(\int_{t_{i-1}}^{t_i}(s-t_{i-1})^2ds\Big)\Big(\int_{t_{i-1}}^{t_i}\|\mathbf{u}_{ss}(s)\|^2ds\Big)+\frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2\\
&\leq k^2\sum_{i=1}^{n} e^{2\alpha t_i}\int_{t_{i-1}}^{t_i}\|\mathbf{u}_{ss}(s)\|^2ds+\frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2\\
&\leq Ce^{2\alpha k}k^2\int_0^{t_n} e^{2\alpha s}\|\mathbf{u}_{ss}(s)\|^2ds+\frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2. \quad (5.50)
\end{aligned}$$

Again from (5.41)

$$\begin{aligned}
k\sum_{i=1}^{n} e^{2\alpha t_i}|E^i(P_h\mathbf{e}^i)| &\leq \frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2\\
&\quad + Ck\sum_{i=1}^{n} e^{2\alpha t_i}\Big(\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\beta(t_i-s)\big(\delta\|\mathbf{u}(s)\|_2+\|\mathbf{u}_s(s)\|_2\big)ds\Big)^2\\
&\leq Ck^2\int_0^{t_n} e^{2\alpha s}(\|\mathbf{u}(s)\|_2^2+\|\mathbf{u}_s(s)\|_2^2)ds+\frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2. \quad (5.51)
\end{aligned}$$

Also, similar to (5.16)-(5.20), we can bound the nonlinear terms of (5.42) as

$$\begin{aligned}
|\Lambda^i(P_h\mathbf{e}^i)| &= |b(\mathbf{u}^i,\mathbf{e}^i,\mathbf{u}^i-P_h\mathbf{u}^i)+b(\mathbf{e}^i,\mathbf{u}^i,\mathbf{e}^i)-b(\mathbf{e}^i,\mathbf{u}^i,\mathbf{u}^i-P_h\mathbf{u}^i)+b(\mathbf{e}^i,\mathbf{e}^i,\mathbf{u}^i-P_h\mathbf{u}^i)|\\
&\leq Ch^{2m}\|\mathbf{u}^i\|_{m+1}^2+(2\|\nabla\mathbf{u}^i\|_\infty+(\frac{1}{2}+\frac{2}{\rho})\|\mathbf{u}^i\|_2^2)\|\mathbf{e}^i\|^2+\frac{\rho}{4}\|\nabla\cdot\mathbf{e}^i\|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
k\sum_{i=1}^{n} e^{2\alpha t_i}|\Lambda^i(P_h\mathbf{e}^i)| &\leq Ch^{2m}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{u}^i\|_{m+1}^2+\frac{\rho}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\cdot\mathbf{e}^i\|^2\\
&\quad + Ck\sum_{i=1}^{n} e^{2\alpha t_i}(2\|\nabla\mathbf{u}^i\|_\infty+(1+\frac{1}{2\rho})\|\mathbf{u}^i\|_2^2)\|\mathbf{e}^i\|^2. \quad (5.52)
\end{aligned}$$

Now, we use (5.48)-(5.52) in (5.47) to arrive at

$$e^{2\alpha t_n}\|\mathbf{e}^n\|^2+\beta_1 k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\mathbf{e}^i\|^2+\rho k\sum_{i=1}^{n} e^{2\alpha t_i}\|\nabla\cdot\mathbf{e}^i\|^2$$

$$\leq Ch^{2m}\Big[e^{2\alpha t_n}\|\mathbf{u}(t_n)\|_m^2 + \int_0^{t_n} e^{2\alpha s}\big(\|\mathbf{u}_s(s)\|_m^2 + (\mu+4\rho+2)\|\mathbf{u}(s)\|_{m+1}^2$$
$$+ \frac{4}{\rho}\|p(s)\|_m^2\big)ds\Big] + Ck^2\int_0^{t_n} e^{2\alpha s}\big(\|\mathbf{u}_{ss}\|^2 + \|\mathbf{u}(s)\|_2^2 + \|\mathbf{u}_s(s)\|_2^2\big)ds$$
$$+ k\sum_{i=1}^{n} e^{2\alpha t_i}\big(\frac{e^{2\alpha k}-1}{k} + 4\|\nabla\mathbf{u}^i\|_\infty + (1+\frac{4}{\rho})\|\mathbf{u}^i\|_2^2\big)\|\mathbf{e}^i\|^2.$$

Note that $e^{2\alpha k} - 1 \leq C(\alpha)k$. We now use the "discrete Gronwall's Lemma" and then multiply the resulting equation by $e^{-2\alpha t_n}$ to conclude the proof. □

Remark 5.6. *From (**A3m**), $\hat{L}^n$ defined in (5.43) is bounded by Ct_n and $K_1(t_n)$ and $K_2(t_n)$ defined in (5.10) and (5.24) respectively, both are bounded by C, where C is not dependent of inverse power of μ.*

Fully discrete error bounds for pressure

To obtain the fully discrete pressure error estimate, first we consider (5.7) with $t = t_n$ and subtract (5.3) from the resulting equation to arrive at

$$(p^n - P^n, \nabla\cdot\mathbf{v}_h) = (\partial_t\mathbf{e}^n, \mathbf{v}_h) + \mu a(\mathbf{e}^n, \mathbf{v}_h) + \rho(\nabla\cdot\mathbf{e}^n, \nabla\cdot\mathbf{v}_h) + a(q_r^n(\mathbf{e}), \mathbf{v}_h)$$
$$+ R^n(\mathbf{v}_h) + \Lambda^n(\mathbf{v}_h) + E^n(\mathbf{v}_h).$$

A use of (5.40), (5.41), (5.42) with the "Cauchy-Schwarz inequality" yields

$$(p^n - P^n, \nabla\cdot\mathbf{v}_h) \leq C\Big[\|\partial_t\mathbf{e}^n\|_{-1} + \mu\|\nabla\mathbf{e}^n\| + \rho\|\nabla\cdot\mathbf{e}^n\| + \|q_r^n(\nabla\mathbf{e})\|$$
$$+ \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\big(\beta_s(t_n-s)\|\nabla\mathbf{u}(s)\| + \beta(t_n-s)\|\nabla\mathbf{u}_s(s)\|\big)\,ds$$
$$+ \frac{1}{k}\int_{t_{n-1}}^{t_n}(s-t_{n-1})\|\mathbf{u}_{ss}(s)\|_{-1}\,ds + (\|\mathbf{u}^n\|_2 + \|\mathbf{U}^n\|_2)\|\mathbf{e}^n\|\Big]\|\nabla\mathbf{v}_h\|. \tag{5.53}$$

Arguing as in the proof of Lemma 5.2, we can find a bound for the first term on the right of inequality (5.53) as

$$\|\partial_t\mathbf{e}^n\|_{-1} \leq C\Big[h^{2m}\|\partial_t\mathbf{u}^n\|_{m-1} + \mu\|\nabla\mathbf{e}^n\| + \rho\|\nabla\cdot\mathbf{e}^n\| + \|q_r^n(\nabla\mathbf{e})\|$$
$$+ \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\big(\beta_s(t_n-s)\|\nabla\mathbf{u}(s)\| + \beta(t_n-s)\|\nabla\mathbf{u}_s(s)\|\big)\,ds$$
$$+ \frac{1}{k}\int_{t_{n-1}}^{t_n}(s-t_{n-1})\|\mathbf{u}_{ss}(s)\|_{-1}\,ds + (\|\mathbf{u}^n\|_2 + \|\mathbf{U}^n\|_2)\|\mathbf{e}^n\|\Big]. \tag{5.54}$$

Incorporate (5.54) in (5.53) and divide the resulting inequality by $\|\nabla \mathbf{v}_h\|,\ \mathbf{v}_h \neq 0$. Similar to (5.26), we then have

$$\|p^n - P^n\|_{L^2/N_h} \le C\Big[h^m(\|\partial_t \mathbf{u}^n\|_{m-1} + \|p^n\|_m) + \mu\|\nabla \mathbf{e}^n\| + \rho\|\nabla\cdot\mathbf{e}^n\| + \|q_r^n(\nabla \mathbf{e})\| + \frac{1}{k}\int_{t_{n-1}}^{t_n}(s-t_{n-1})\|\mathbf{u}_{ss}(s)\|_{-1}\,ds + (\|\mathbf{u}^n\|_2 + \|\mathbf{U}^n\|_2)\|\mathbf{e}^n\| + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\big(\beta_s(t_n-s)\|\nabla \mathbf{u}(s)\| + \beta(t_n-s)\|\nabla \mathbf{u}_s(s)\|\big)\,ds\Big].$$

Squaring and multiplying both side by $ke^{2\alpha t_n}$ with $n=i$ and taking sum over $1 \le i \le n$ to obtain

$$k\sum_{i=1}^{n}\|p^i - P^i\|^2_{L^2/N_h} \le Ck\sum_{i=1}^{n}\Big[h^{2m}(\|\partial_t \mathbf{u}^i\|^2_{m-1} + \|p^i\|^2_m) + \mu\|\nabla \mathbf{e}^i\|^2 + \rho\|\nabla\cdot\mathbf{e}^i\|^2 + \|q_r^i(\nabla \mathbf{e})\|^2 + \Big(\frac{1}{k}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|\mathbf{u}_{ss}(s)\|_{-1}\,ds\Big)^2 + (\|\mathbf{u}^i\|_2^2 + \|\mathbf{U}^i\|_2^2)\|\mathbf{e}^i\|^2 + \Big(\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\big(\beta_s(t_i-s)\|\nabla \mathbf{u}(s)\| + \beta(t_i-s)\|\nabla \mathbf{u}_s(s)\|\big)\,ds\Big)^2\Big].$$

Incorporating with Theorem 5.4 and the "Young's inequality" we reach at

$$k\sum_{i=1}^{n}\|p^i - P^i\|^2_{L^2/N_h} \le Ce^{\hat{L}t_n}h^{2m}\Big(\|\mathbf{u}(t_n)\|_m^2 + e^{-2\alpha t_n}\int_0^{t_n}e^{2\alpha s}\|\mathbf{u}_s(s)\|^2_m ds + e^{-2\alpha t_n}\int_0^{t_n}e^{2\alpha s}\big((\mu+4\rho+2)\|\mathbf{u}(s)\|^2_{m+1} + \frac{4}{\rho}\|p(s)\|^2_m\big)ds\Big) + Ce^{\hat{L}t_n}k^2e^{-2\alpha t_n}\int_0^{t_n}e^{2\alpha s}\big(\|\mathbf{u}_{ss}(s)\|^2 + \|\mathbf{u}(s)\|_2^2 + \|\mathbf{u}_s(s)\|_2^2\big)ds.$$

Multiply both sides by $e^{-2\alpha t}$. We summarize our result in the following Theorem.

Theorem 5.5. *Suppose the hypothesis of Theorem 5.4 be satisfied. Then, the following holds true:*

$$ke^{-2\alpha t_n}\sum_{i=0}^{n}e^{2\alpha t_i}\|p^n - P^n\|^2_{L^2/N_h} \le Ce^{\hat{L}^n}\big(K_1(t_n)h^{2m} + K_2(t_n)k^2\big).$$

5.4.2 Fully Discrete Error Estimates for Smooth Data

We now consider, the initial data $\mathbf{u}_0 \in \mathbf{J}_1 \cap \mathbf{H}^2$. Then, the following result holds.

Theorem 5.6. *Suppose the hypothesis of Theorem 5.3 be satisfied. Further, assume $\tau(t)\mathbf{u}_t \in L^2(0,T;\mathbf{H}^2)$ and $\tau(t)\mathbf{u}_{tt} \in L^2(0,T;\mathbf{L}^2)$. Then, the following bounds hold for*

$1 \leq n \leq N$ and $m \in \{1,2\}$

$$\|\mathbf{e}^n\|^2 + ke^{-2\alpha t_n} \sum_{i=1}^{n} e^{2\alpha t_i}\big(\beta_1 \|\nabla \mathbf{e}^i\|^2 + \rho \|\nabla\cdot \mathbf{e}^i\|^2\big) \leq Ce^{\hat{L}^n}\big(K_3(t_n)h^{2m} + K_4(t_n)k\big),$$

and

$$ke^{-2\alpha t_n} \sum_{i=0}^{n} e^{2\alpha t_i} \|p^n - P^n\|^2_{L^2/N_h} \leq Ce^{\hat{L}^n}\big(K_3(t_n)h^{2m} + K_4(t_n)k\big),$$

where $\beta_1 = \mu - \big(\frac{\gamma}{\delta-\alpha}\big)^2 > 0$, and $\hat{L}^n$ is defined in (5.43), and

$$\begin{aligned} K_3(t_n) =& \|\mathbf{u}(t_n)\|^2_m + e^{-2\alpha t_n}\int_0^{t_n} e^{2\alpha s}\tau^{m-1}(s)\|\mathbf{u}_s(s)\|^2_m ds \\ &+ e^{-2\alpha t_n}\int_0^{t_n} e^{2\alpha s}\big((\mu + 4\rho + 2)\|\mathbf{u}(s)\|^2_{m+1} + \frac{4}{\rho}\|p(s)\|^2_m\big)ds, \\ K_4(t_n) =& e^{-2\alpha t_n}\int_0^{t_n} e^{2\alpha s}\tau(s)\big(\|\mathbf{u}_{ss}(s)\|^2 + \|\mathbf{u}(s)\|^2_2 + \|\mathbf{u}_s(s)\|^2_2\big)ds. \end{aligned}$$

and $C > 0$ is a constant that may depends on given data but not on the inverse power of μ as well as h and k.

Proof. The proof goes in the similar way of the proof of Theorem 5.4 except the estimates (5.48),(5.50) and (5.51) since $\|\mathbf{u}_{tt}(t)\|$ and $\|\mathbf{u}_t(t)\|_2$ are not integrable at $t = 0$, when $\mathbf{u}_0 \in \mathbf{J}_1 \cap \mathbf{H}^2$.

For $m = 1$, (5.48) will go through as it is but for $m = 2$, we modify it as follows, keeping in mind $\tau(t_n) \leq \tau(t_{n-1}) + k \leq C\tau(t)$ for $t \in [t_{n-1}, t_n]$

$$\begin{aligned} k^2\sum_{i=1}^{n} e^{2\alpha t_i}\|\partial_t \mathbf{u}^i\|^2_2 \leq& k^2 \sum_{i=1}^{n} e^{2\alpha t_i}\Big(\frac{1}{k}\int_{t_{i-1}}^{t_i}\|\mathbf{u}_s(s)\|_2 ds\Big)^2 \\ \leq& e^{2\alpha k}\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}\frac{1}{\tau(t_i)}ds\Big)\Big(\int_{t_{i-1}}^{t_i}\tau(t_i)e^{2\alpha t_{i-1}}\|\mathbf{u}_s(s)\|^2_2 ds\Big) \\ \leq& e^{2\alpha k}\sum_{i=1}^{n}\Big(\frac{k}{\tau(t_i)}\Big)\Big(\int_{t_{i-1}}^{t_i}\sigma(s)\|\mathbf{u}_s(s)\|^2_2 ds\Big) \end{aligned}$$

When $0 < t_i < 1$, we have $\tau(t_i) = t_i = ik$. Hence

$$\sum_{i=1}^{n}\Big(\frac{k}{\tau(t_i)}\Big)\Big(\int_{t_{i-1}}^{t_i}\sigma(s)\|\mathbf{u}_s(s)\|^2_2 ds\Big) \leq \sum_{i=1}^{n}\frac{1}{i}\Big(\int_{t_{i-1}}^{t_i}\sigma(s)\|\mathbf{u}_s(s)\|^2_2 ds\Big) \leq \int_0^{t_n}\sigma(s)\|\mathbf{u}_s(s)\|^2_2 ds.$$

And when $t_i \geq 1$, we have $\tau(t_i) = 1$ and then

$$\sum_{i=1}^{n}\Big(\frac{k}{\tau(t_i)}\Big)\Big(\int_{t_{i-1}}^{t_i}\sigma(s)\|\mathbf{u}_s(s)\|^2_2 ds\Big) \leq k\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}\sigma(s)\|\mathbf{u}_s(s)\|^2_2 ds\Big) \leq k\int_0^{t_n}\sigma(s)\|\mathbf{u}_s(s)\|^2_2 ds.$$

We modify (5.50) for both $m = 1, 2$, using the fact $t - t_{i-1} \leq \tau(t)$ for $t \in [t_{i-1}, t_i]$ as

$$
\begin{aligned}
k\sum_{i=1}^{n} e^{2\alpha t_i} R^i(P_h\mathbf{e}^i) &\leq k\sum_{i=1}^{n} e^{2\alpha t_i}\frac{1}{k}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|\mathbf{u}_{ss}(s)\|ds\ \|P_h\mathbf{e}^i\| \\
&\leq k\sum_{i=1}^{n} e^{2\alpha t_i}\left(\frac{1}{k}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|\mathbf{u}_{ss}(s)\|ds\right)^2 + \frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2 \\
&\leq \frac{1}{k}\sum_{i=1}^{n} e^{2\alpha t_i}\left(\int_{t_{i-1}}^{t_i}(s-t_{i-1})ds\right)\left(\int_{t_{i-1}}^{t_i}(s-t_{i-1})\|\mathbf{u}_{ss}(s)\|^2ds\right) + \frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2 \\
&\leq k\sum_{i=1}^{n} e^{2\alpha t_i}\int_{t_{i-1}}^{t_i}\tau(s)\|\mathbf{u}_{ss}(s)\|^2ds + \frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2 \\
&\leq Ck\int_0^{t_n} e^{2\alpha s}\tau(s)\|\mathbf{u}_{ss}(s)\|^2ds + \frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2.
\end{aligned}
$$

Similarly we modify (5.51) and obtain

$$
\begin{aligned}
k\sum_{i=1}^{n} e^{2\alpha t_i}|E^i(P_h\mathbf{e}^i)| &\leq \frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2 \\
&+ Ck\sum_{i=1}^{n} e^{2\alpha t_i}\sum_{j=1}^{i}\Big(\int_{t_{j-1}}^{t_j}(s-t_{j-1})^2ds\Big)\Big(\int_{t_{j-1}}^{t_j}(\|\tilde{\Delta}_h\mathbf{u}(s)\|^2 + \|\tilde{\Delta}_h\mathbf{u}_s(s)\|^2)ds\Big) \\
&\leq Ck\int_0^{t_n} e^{2\alpha s}\tau(s)(\|\mathbf{u}(s)\|_2^2 + \|\mathbf{u}_s(s)\|_2^2)ds + \frac{1}{4}k\sum_{i=1}^{n} e^{2\alpha t_i}\|\mathbf{e}^i\|^2.
\end{aligned}
$$

We use these modified estimates in the proof of Theorem 5.4 to conclude the velocity estimates. And now based on these modified estimates we can easily obtain the pressure estimate, similar to Theorem 5.5 which concludes the proof. □

Remark 5.7. *Similar to semidiscrete case, here also we can not extend the analysis for $m \geq 3$ to obtain better convergence rate.*

5.5 Numerical Experiments

This section is devoted to verifying the theoretical findings by numerical examples. First, we mainly verify the order of convergence of the solution. For simplicity, we use examples with known solution. In all cases, computation are done in FreeFem++ [78].

5.5.1 Known Analytic Solutions

We consider the Oldroyd model of order one subject to the homogeneous Dirichlet boundary condition. We approximate the equation using the Mini-element $(P_1 b, P_1)$

and the Taylor-Hood element (P_2, P_1) over a union jack (criss-cross) triangulation of Ω . We take the domain $\Omega = [0,1] \times [0,1]$, which is partitioned into triangles with size $h = 2^{-i},\ i = 2, 3, \ldots, 6$. To verify the theoretical results, we consider the following examples:

Example 5.1. *In our first example, we consider the forcing term $f(x,t)$ so as to get the following exact solutions*

$$u_1(x,t) = -e^t(\cos(2\pi x) - 1)\sin(2\pi y),$$
$$u_2(x,t) = e^t(\cos(2\pi y) - 1)\sin(2\pi x),$$
$$p(x,t) = 2\pi e^t(\cos(2\pi y) - \cos(2\pi x)).$$

Table 5.1: Numerical results for $\mu = 10^{-5}$ at time $t = 1$.

	(P_1b, P_1)				(P_2, P_1)			
h	$\lVert\mathbf{U}^n - \mathbf{u}(t_n)\rVert$	C.R.	$\lVert P^n - p(t_n)\rVert$	C.R.	$\lVert\mathbf{U}^n - \mathbf{u}(t_n)\rVert$	C.R.	$\lVert P^n - p(t_n)\rVert$	C.R.
1/4	1.48858661		3.93977224		0.63433473		3.23651019	
1/8	0.64373813	1.2094	1.89611585	1.0551	0.18140785	1.8060	0.86467803	1.9042
1/16	0.23477703	1.4552	0.32472939	2.5457	0.04941585	1.8761	0.22448926	1.9455
1/32	0.10624476	1.1439	0.14171058	1.1962	0.01246026	1.9874	0.05667220	1.9859
1/64	0.04821498	1.1398	0.07036613	1.0100	0.00284780	2.1296	0.01429384	1.9872

First, we consider stable equal order finite element pair (MINI-element) and we discretize the domain with mesh size $h = 2^{-i},\ i = 2, 3, \ldots, 6$, in each coordinate directions. We take different values of $\mu = 1, 10^{-2}, 10^{-4}, 10^{-6}$ and 10^{-8} for each values of h with fixed $k = \mathcal{O}(h), \delta = 0.1$ and $\gamma = 0.1\mu$. In Fig 5.1, we represent the velocity and the pressure errors in $\mathbf{L}^2$-norm. For this case, we set the grad-div parameter $\rho = h^2$ for optimal values (see, Remark 5.4).

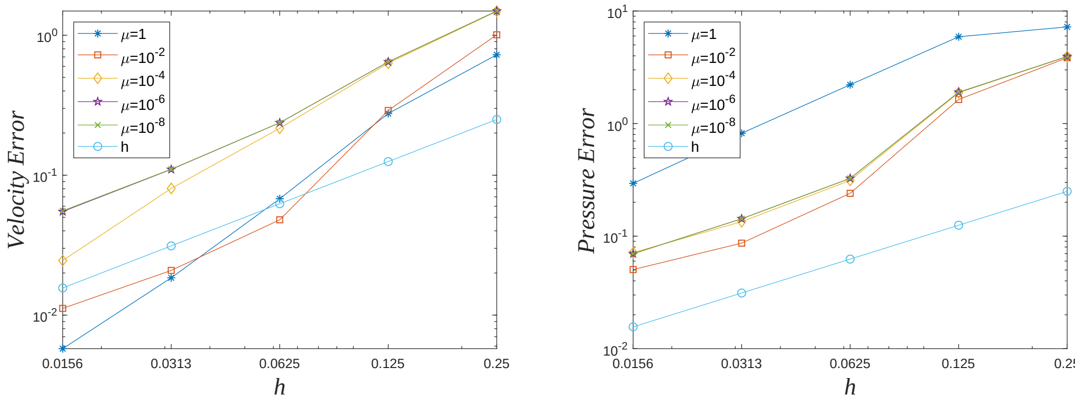

Figure 5.1: Velocity and pressure errors in $\mathbf{L}^2$-norm for Example 5.1 with $\rho = h^2$ for (P_1b, P_1) element.

Next, we consider the well-known Taylor-Hood element. We present the velocity and the pressure errors for different values of μ as above with $k = \mathcal{O}(h^2), \delta = 0.1$ and $\gamma = 0.1\mu$ in Fig 5.2. In this case, we set the grad-div parameter $\rho = 0.25$ for optimal values. In Table 5.1, we present the velocity and pressure errors and the rate of convergence in L^2-norm for $\mu = 10^{-5}$.

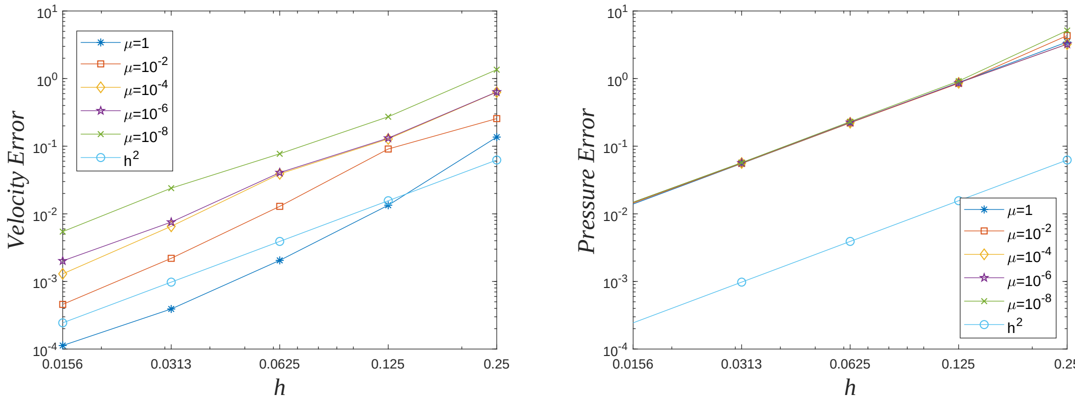

Figure 5.2: Velocity and pressure errors in $\mathbf{L}^2$-norm for Example 5.1 with $\rho = 0.25$ for (P_2, P_1) element.

We next look at a benchmark problem.

Example 5.2. *"In this example, we consider a benchmark problem related to a 2D lid driven cavity flow on a unit square with zero body force. Also, no slip boundary condition is considered everywhere except the non zero velocity* $\mathbf{u} = (1, 0)^T$ *on upper boundary."*

For numerical simulations, we take two lines $(x, 0.5)$ and $(0.5, y)$ and plot the solutions along these two lines.. In Figure 5.3, we present the comparison between velocity obtained by Ghia et. al. [54], velocity obtained without stabilization and velocity obtained with stabilization of Oldroyd model of order one for final time $t = 150$ for $\mu = 10^{-4}$, with the choice of time step $k = 0.01$, $\delta = 0.1$ and $\gamma = 0.1\mu$. We also choose the stabilization parameter $\rho = 0.1$. We approximate the equation using Taylor-Hood (P_2, P_1) element over a union jack (criss-cross) triangulation of Ω and discretize the domain with mesh size $h = 1/64$. From the graphs, it is observed that the velocity profiles with stabilization coincide with those of Ghia's results quite well in comparison with the velocity profile without stabilization for large time and for small ν, (see Fig 5.3).

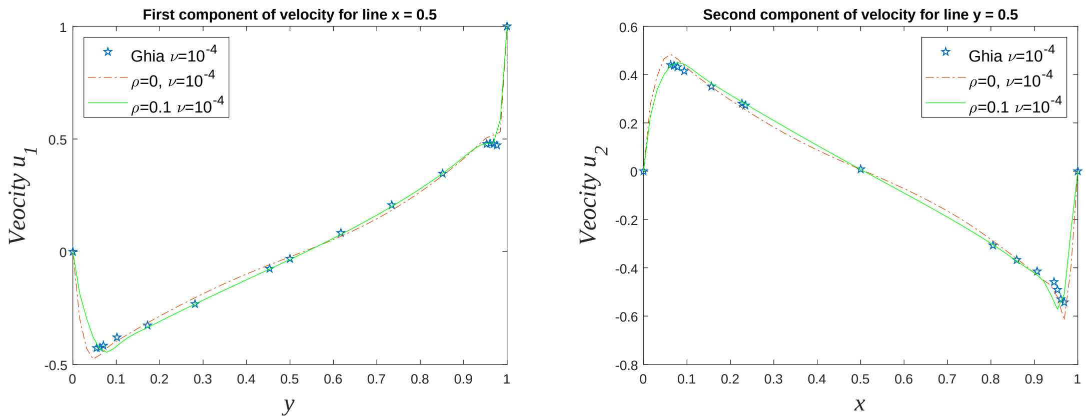

Figure 5.3: Velocity components for Example 5.2.

5.5.2 Choice of Grad-div Parameter

As discussed in the introduction, the choice of grad-div stabilization parameter plays a vital role in numerical simulations. Here we present a few numerical examples to find a suitable choice of grad-div parameter for the Oldroyd model of order one.

The numerical simulations are performed for different values of ρ that lie between 10^{-3} to 10^4, approximating the equation using the MINI element (P_1b, P_1) and the Taylor-Hood element (P_2, P_1). The numerical results were computed for three successively finer meshes with union jack (criss-cross) type triangulation with mesh sizes $h = 2^{-3}, 2^{-4}$ and 2^{-5}. Figs 5.4 and 5.5 represent the velocity and the pressure errors graphs with respect to the grad-div stabilization parameter ρ for the MINI element and the Taylor-Hood element, respectively. For all cases, we take $\mu = 1, 10^{-2}, 10^{-4}, 10^{-6}$

and 10^{-8} and mark the point where the error is minimum. We observe that for $\mathbf{L}^2$ error of velocity, a suitable range of ρ would be from 10^{-1} to 10^1. And for $\mathbf{H}^1$ error of velocity, ρ in the range of 10^{-1} to 10^4 will work. However, for L^2 error of pressure, the suitable range of ρ is 10^{-1} to 10^4. Finally, these figures also give us a rough picture how the grad-div parameter ρ changes with h and μ.

We also present the values of grad-div parameter ρ, that minimize the $\mathbf{L}^2$ and $\mathbf{H}^1$ velocity errors and L^2 pressure error. We have used the known solution of Example 5.1. In the Tables 2 and 3, we present the corresponding minimum errors and the errors for standard choice of grad-div parameter $\rho = 1$, for different values of h. We have used boldface for the minimizing value of ρ in each case.

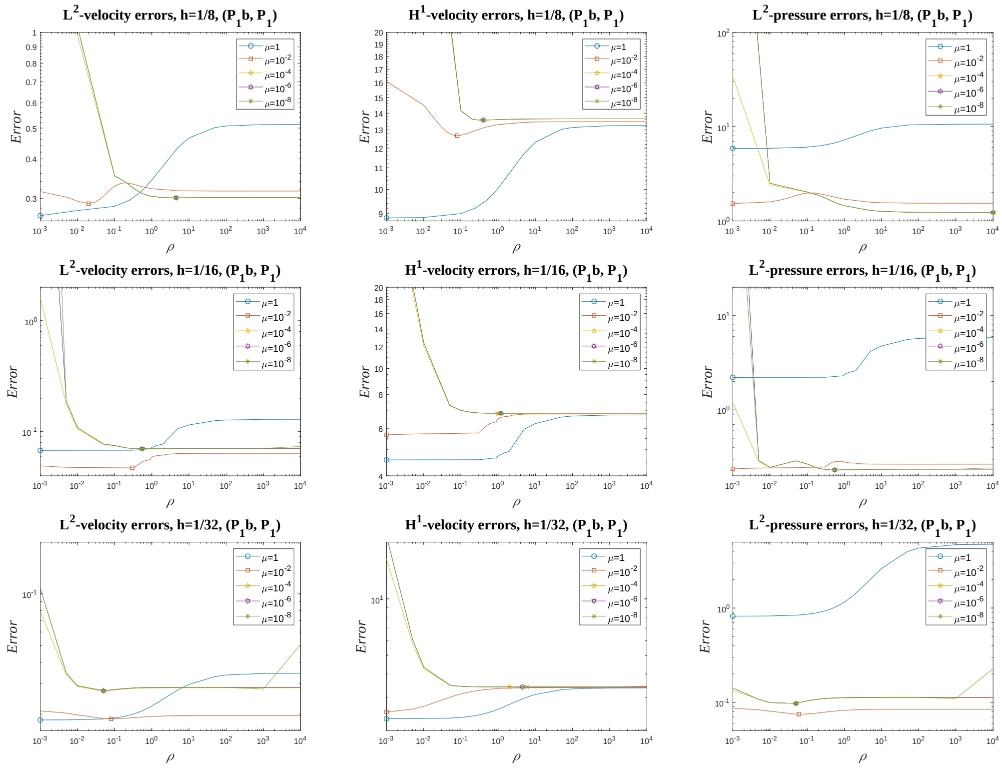

Figure 5.4: Errors Vs stabilization parameter for (P_1b, P_1) element.

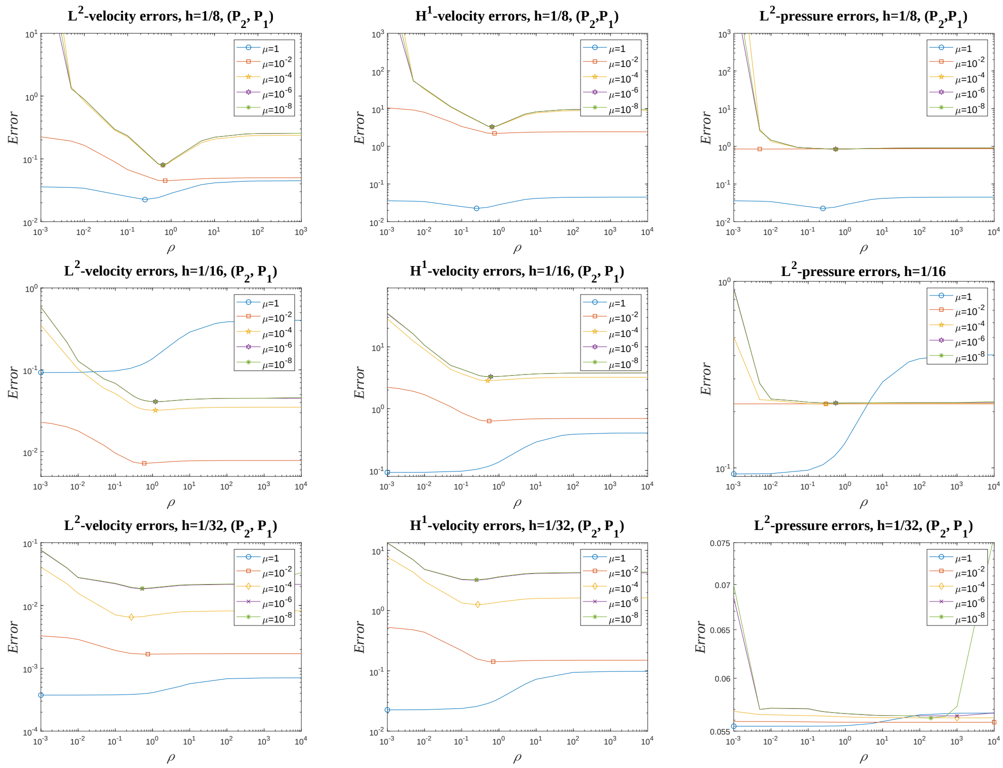

Figure 5.5: Errors Vs stabilization parameter for (P_2, P_1) element.

Table 5.2: Minimum errors and corresponding stabilization parameter ρ for (P_1b, P_1).

		Velocity errors in $\mathbf{L}^2$-norm			Velocity errors in $\mathbf{H}^1$-norm			Pressure errors in L^2-norm		
	μ	ρ	Min	Std.(ρ=1)	ρ	Min	Std.(ρ=1)	ρ	Min	Std.(ρ=1)
h=$\frac{1}{8}$	1	**0.001**	0.27419	0.34335	**0.001**	8.81884	10.05271	**0.001**	5.8799	7.21910
	1e-2	**0.02**	0.28931	0.32214	**0.08**	12.6806	13.31144	**0.001**	1.5288	1.70077
	1e-4	**4.5**	0.30166	0.30490	**0.4**	13.5764	13.60262	**10000**	1.23236	1.45157
	1e-6	**4.5**	0.30166	0.30486	**0.4**	13.5823	13.60660	**10000**	1.22929	1.44892
	1e-8	**4.5**	0.30166	0.30486	**0.4**	13.5823	13.60664	**10000**	1.22746	1.44890
h=$\frac{1}{16}$	1	**0.001**	0.06776	0.08406	**0.001**	4.58129	5.16855	**0.001**	2.21019	2.91551
	1e-2	**0.01**	0.04718	0.06249	**0.001**	5.67357	6.71433	**0.001**	0.23700	0.26881
	1e-4	**0.55**	0.07000	0.07011	**1.0**	6.82155	6.82155	**0.55**	0.23076	0.23147
	1e-6	**0.55**	0.07017	0.07030	**1.2**	6.82284	6.82290	**0.55**	0.23090	0.23174
	1e-8	**0.55**	0.07015	0.07030	**1.2**	6.82285	6.82291	**0.55**	0.23090	0.23175
h=$\frac{1}{32}$	1	**0.001**	0.01848	0.02229	**0.001**	2.31312	2.59893	**0.001**	0.82050	1.15744
	1e-2	**0.08**	0.01875	0.01936	**0.001**	2.51021	3.34415	**0.06**	0.07475	0.08212
	1e-4	**0.05**	0.02717	0.02836	**2.0**	3.41308	3.41329	**0.05**	0.09716	0.11123
	1e-6	**0.05**	0.02735	0.02855	**4.5**	3.41332	3.41436	**0.05**	0.09740	0.11218
	1e-8	**0.05**	0.02735	0.02855	**6.0**	3.41334	3.41437	**0.05**	0.09740	0.11217

Table 5.3: Minimum errors and corresponding stabilization parameter ρ for (P_2, P_1).

		Velocity errors in $\mathbf{L}^2$-norm			Velocity errors in $\mathbf{H}^1$-norm			Pressure errors in L^2-norm		
	μ	ρ	Min	Std.(ρ=1)	ρ	Min	Std.(ρ=1)	ρ	Min	Std.(ρ=1)
h=$\frac{1}{8}$	1	**0.001**	0.01011	0.01365	**0.001**	0.40921	0.54307	**0.001**	0.86810	0.88401
	1e-2	**0.73**	0.04502	0.04525	**0.76**	2.21141	2.21968	**0.005**	0.84583	0.85761
	1e-4	**0.65**	0.07898	0.09129	**0.65**	3.26339	3.57391	**0.55**	0.84359	0.84799
	1e-6	**0.65**	0.08016	0.09427	**0.65**	3.29902	3.66404	**0.55**	0.84148	0.84708
	1e-8	**0.65**	0.08017	0.09430	**0.65**	3.29939	3.66500	**0.55**	0.84146	0.84707
h=$\frac{1}{16}$	1	**0.001**	0.00133	0.00182	**0.001**	0.09276	0.13683	**0.001**	0.22091	0.22190
	1e-2	**0.6**	0.00723	0.00731	**0.55**	0.63038	0.63938	**0.3**	0.22028	0.22029
	1e-4	**1.2**	0.03211	0.03214	**0.5**	2.84648	2.89919	**0.3**	0.22017	0.22060
	1e-6	**1.2**	0.04087	0.04098	**0.6**	3.29495	3.32761	**0.55**	0.22258	0.22268
	1e-8	**1.2**	0.04099	0.04110	**0.6**	3.30041	3.33304	**0.55**	0.22262	0.22271
h=$\frac{1}{32}$	1	**0.001**	0.00037	0.00041	**0.001**	0.02253	0.03440	**0.001**	0.05545	0.05549
	1e-2	**0.73**	0.00168	0.00168	**0.7**	0.14200	0.14244	**10000**	0.05581	0.05581
	1e-4	**0.27**	0.00652	0.00704	**0.27**	1.26200	1.38753	**1000**	0.05621	0.05630
	1e-6	**0.55**	0.01876	0.01888	**0.25**	3.26332	3.59116	**1000**	0.05638	0.05660
	1e-8	**0.55**	0.01876	0.01924	**0.25**	3.30041	3.65619	**200**	0.05619	0.05662

5.6 Conclusion

We have considered here, an inf-sup mixed finite element method for Oldroyd model of order one with grad-div stabilization. We have obtained the error estimates in $L^\infty(\mathbf{L}^2)$-norm for the velocity and $L^2(\mathbf{L}^2)$-norm for the pressure in semidiscrete case as well as in the fully discrete case with the error bounds independent of the inverse power of μ. We have carried out our analysis for both sufficiently smooth and smooth initial data. And finally we have briefly looked at suitable values of the grad-div parameter.