Chapter 2

Neutrino Oscillation Phenomenology at Terrestrial Neutrino Experiments

2.1 PMNS matrix parametrization

A general $n \times n$ matrix has $2n^2$ real parameters. However, the unitary condition implies

$$\sum_{i=1}^{n} U_{\alpha,i} U^*_{\beta,i} = \delta_{\alpha\beta} \tag{2.1}$$

This condition yields n constraints for $\alpha = \beta$, and $n^2 - n$ constraints for $\alpha \neq \beta$. An unitary $n \times n$ matrix therefore has n^2 independent real parameters, with $\frac{1}{2}n(n-1)$ angles (magnitudes) and $\frac{1}{2}n(n+1)$ phases. However, in the case of fermions not all these phases are physical. In fact, in a theory with n generations of leptons, we have $2n$ fields that can be rephased. This means that $2n-1$ of these phases can be reabsorbed in a redefinition of the lepton fields. The actual number of physical phases is $\frac{1}{2}n(n+1) - (2n-1) = \frac{1}{2}(n-1)(n-2)$ in case of Dirac neutrino and $\frac{1}{2}n(n-1)$ in case of Marojana neutrino.

For example, in two-flavor oscillation, if we consider ν_e and ν_μ, there should be $\frac{1}{2}2(2-1) = 1$ angle and $\frac{1}{2}2(2+1) = 3$ phases, in which there is 0 physical phase.

In order to prove above conclusion, let us assume

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \tag{2.2}$$

where in general

$$U = \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix} = \begin{pmatrix} e^{i\delta_1}\cos\theta & e^{i\delta_2}\sin\theta \\ -e^{i\delta_3}\sin\theta & e^{i\delta_4}\cos\theta \end{pmatrix} \tag{2.3}$$

We now can compute

$$\begin{aligned} UU^\dagger &= \begin{pmatrix} ce^{i\delta_1} & se^{i\delta_2} \\ -se^{i\delta_3} & ce^{i\delta_4} \end{pmatrix} \begin{pmatrix} ce^{-i\delta_1} & -se^{-i\delta_3} \\ se^{-i\delta_2} & ce^{-i\delta_4} \end{pmatrix} \\ &= \begin{pmatrix} 1 & cs\left(e^{i(\delta_2-\delta_4)} - e^{i(\delta_1-\delta_3)}\right) \\ cs\left(e^{i(\delta_4-\delta_2)} - e^{i(\delta_3-\delta_1)}\right) & 1 \end{pmatrix} \end{aligned}$$

In order to satisfy the unitary condition ($UU^\dagger = 1$), we require

$$e^{i(\delta_2-\delta_4)} - e^{i(\delta_1-\delta_3)} = 0$$

and

$$e^{i(\delta_4-\delta_2)} - e^{i(\delta_3-\delta_1)} = 0$$

From these we derive $\delta_4 = \delta_3 + \delta_2 - \delta_1$. It is obviously seen that among 4 imaginary phases, there are only 3 independent phases. We now prove that these 3 phases can be absorbed into the definition of the lepton fields. The general 2×2 unitary matrix can be of the form

$$U = \begin{pmatrix} ce^{i\delta_1} & se^{i\delta_2} \\ -se^{i\delta_3} & ce^{i(\delta_3+\delta_2-\delta_1)} \end{pmatrix} \tag{2.4}$$

Let us now consider the transformation

$$l_\alpha \to l_\alpha e^{i(\theta_e+\theta'_\alpha)}, \quad \nu_k \to \nu_k e^{i(\theta_e+\theta'_k)}, \quad U_{\alpha k} = U_{\alpha k} e^{i(\theta'_\alpha-\theta'_k)}$$

Under this transformation, the matrix U in equation (2.4) is transformed as

$$U \to \begin{pmatrix} ce^{i(\delta_1+\theta'_e-\theta'_1)} & se^{i(\delta_2+\theta'_e-\theta'_2)} \\ -se^{i(\delta_3+\theta'_\mu-\theta'_1)} & ce^{i(\delta_3+\delta_2-\delta_1+\theta'_\mu-\theta'_2)} \end{pmatrix} \tag{2.5}$$

and the two-flavor weak charged current is invariant

$$-i\frac{g_W}{\sqrt{2}}(\bar{e} \quad \bar{\mu})\gamma^\mu\frac{1}{2}(1-\gamma^5)\begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{2.6}$$

To eliminate all complex phases, we require

$$\theta'_1 - \theta'_e = \delta_1, \qquad \theta'_2 - \theta'_e = \delta_2$$

$$\theta'_1 - \theta'_\mu = \delta_3, \qquad \theta'_2 - \theta'_\mu = \delta_3 + \delta_2 - \delta_1$$

By writing all the phases relative to the phase of the electron, means setting $\theta'_e = 0$, we have

$$\theta'_1 = \delta_1, \qquad \theta'_2 = \delta_2, \qquad \theta'_\mu = \delta_1 - \delta_3$$

In general, by redefining the phases of the lepton fields using

$$\begin{aligned} \theta'_e &= \phi \\ \theta'_\mu &= \phi + \delta_1 - \delta_3 \\ \theta'_1 &= \phi + \delta_1 \\ \theta'_2 &= \phi + \delta_2 \end{aligned}$$

all complex phases can be removed from the 2×2 analogue of the PMNS matrix, where the ϕ fixes the overall phase of (for example) the electron field.

In three-flavor neutrino oscillation framework, the flavor definitive eigenstates are related to the mass definitive eigenstates by a 3×3 unitary PMNS matrix, shown in Eq. 2.7,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U_{\text{PMNS}} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{2.7}$$

If the PMNS matrix were real, it could be described by three rotation angles θ_{12}, θ_{13} and θ_{23} via orthogonal rotation matrix R

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.8}$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$. Since PMNS matrix is unitary and not real, it must contain six more additional degrees of freedom in term of complex phase $e^{i\delta}$. Five among these six phases can be absorbed into the definition of the particles and leaves only one single phase δ. This can be seen as follow.

The charged currents for leptonic weak interaction

$$-i\frac{g_W}{\sqrt{2}}(\bar{e}, \bar{\mu}, \bar{\tau})\gamma^\mu \frac{1}{2}(1-\gamma^5) \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

The four-vector currents are unchanged by transformation

$$l_\alpha \to l_\alpha e^{i\theta_\alpha}, \quad \nu_k \to \nu_k e^{i\theta_k} \quad \text{and} \quad U_{\alpha k} \to U_{\alpha k} e^{i(\theta_\alpha - \theta_k)} \tag{2.9}$$

where l_α is the charged lepton of the type $\alpha = e, \mu, \tau$. Since the phases are arbitrary, all other phases can be defined in term of θ_e:

$$\theta_\alpha = \theta_e + \theta'_\alpha, \qquad \theta_k = \theta_e + \theta'_k$$

The transformation (2.9) therefore becomes

$$l_\alpha \to l_\alpha e^{i(\theta_e+\theta'_\alpha)}, \quad \nu_k \to \nu_k e^{i(\theta_e+\theta'_k)} \quad \text{and} \quad U_{\alpha k} \to U_{\alpha k} e^{i(\theta'_\alpha-\theta'_k)}$$

For electron $\theta_e = \theta_e + \theta'_e \quad \Rightarrow \theta'_e = 0$. It is can be seen now that only five phases are independent and can be absorbed into the particle definitions.

2.1.1 Majorana phase in neutrino oscillation

We will start by Majorana mass term as follow

$$\mathcal{L}_M = -\frac{1}{2} M(\overline{\nu}^c_R \nu_R + \overline{\nu}_R \nu^c_R), \tag{2.10}$$

where $\overline{\nu}_R = (\overline{\nu}_{1R}, \overline{\nu}_{2R}, ..., \overline{\nu}_{nR})$ and $\nu^c_R \equiv C\overline{\nu}^T_R = \nu_L$. The Majorana mass eigenstate is

$$N = N_L + N_R = N_L + N^c_L = N^c \tag{2.11}$$

Then the relations between mass eigenstates and flavor eigenstates are

$$|\nu_i\rangle = \sum_j U^*_{ij} |N_j\rangle \tag{2.12}$$

All phases of U_{ij} are fixed completely by requiring that the mass m_j of N_j is real positive definite.

Since Majorana field must satisfy the condition (2.11) $N = N^c$, there is no freedom to do transformation

$$N_j \to e^{i\theta_j} N_j$$

Now we can see for the charged lepton current

$$-i\frac{g_W}{\sqrt{2}}(\bar{e}, \bar{\mu}, \bar{\tau})\gamma^\mu \frac{1}{2}(1-\gamma^5) \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}$$

some phases of mixing matrix U can only be absorbed by the charged leptons l_α, but not Majorana fields N_j. The number of CP violating phases, therefore in n-Majorana generations are

$$n^2 - \frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-1) \tag{2.13}$$

Hence, if the neutrinos are Majorana particles, we should have $\frac{1}{2}3(3-1) = 3$ phases for 3 generations (n=3), in which we have 1 Dirac phase and 2 Majorana phases. If n = 2, there is one Majorana CPV phase and no Dirac CPV phase in the mixing matrix. Correspondingly, in contrast to the Dirac case, there can exist CP violating effects even in the system of two mixed massive Majorana neutrinos (particles).

The mixing matrix now is rewritten as

$$V = UP \tag{2.14}$$

where $U \equiv U_{PMNS}$ is the PMNS matrix containing 1 Dirac CP violating phase δ and $P = diag(1, e^{i\delta_1}, e^{i\delta_2})$.

We now show that the Majorana phases do not enter into the expressions of the probabilities of oscillations involving the flavour neutrinos and antineutrinos. The oscillation probabilities of neutrino mode and antineutrino mode are defined as in (2.28)

$$P(\nu_\alpha \to \nu_\beta) = \left| \sum_{i=1}^{3} V_{\beta i} V^\dagger_{i\alpha} e^{-i\phi_i} \right|^2 . \tag{2.15}$$

$$P(\overline{\nu}_\alpha \to \overline{\nu}_\beta) = \left| \sum_{i=1}^{3} V_{\alpha i} V^\dagger_{i\beta} e^{-i\phi_i} \right|^2 . \tag{2.16}$$

Inserting equation (2.14) into, for example equation (2.15) we have

$$\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &= \left|\sum_{i=1}^{3} V_{\beta i} V_{i\alpha}^\dagger e^{-i\phi_i}\right|^2 \\
&= \left|\sum_{i=1}^{3} (UP)_{\beta i} (UP)_{i\alpha}^\dagger e^{-i\phi_i}\right|^2 \\
&= \left|\sum_{i=1}^{3} U_{\beta i} (P_{\beta i} P_{i\alpha}^\dagger) U_{i\alpha}^\dagger e^{-i\phi_i}\right|^2 \\
&= \left|\sum_{i=1}^{3} U_{\beta i} U_{i\alpha}^\dagger e^{-i\phi_i}\right|^2
\end{aligned}$$

We see that oscillation probability does not depend on P or equivalently does not contain Majorana phases. Therefore the neutrino oscillation experiment can not tell us whether the nature of neutrinos are Dirac or Majorana particles.

2.2 Three Neutrino Flavour Oscillation in Vacuum

The flavor eigenstates are related to the mass eigenstates by the 3×3 unitary PMNS matrix.

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{2.17}$$

The PMNS matrix can be parameterized by three mixing angles $(\theta_{12}, \theta_{13}, \theta_{23})$ and a single Dirac phase δ_{CP} as expressed in equation (2.18).

$$
\begin{aligned}
U_{PMNS} &= \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}
\end{aligned}
\tag{2.18}
$$

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$ and δ_{CP} Dirac phase represents the CP violation in lepton sector [1].

The PMNS matrix is unitary so $U^{-1} = U^\dagger \equiv (U^*)^T$. And hence, the mass eigenstates also can be performed via flavor eigenstates as

$$
\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U^*_{e1} & U^*_{\mu1} & U^*_{\tau1} \\ U^*_{e2} & U^*_{\mu2} & U^*_{\tau2} \\ U^*_{e3} & U^*_{\mu3} & U^*_{\tau3} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{2.19}
$$

The unitary condition $UU^\dagger = I$ implies:

$$
\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} U^*_{e1} & U^*_{\mu1} & U^*_{\tau1} \\ U^*_{e2} & U^*_{\mu2} & U^*_{\tau2} \\ U^*_{e3} & U^*_{\mu3} & U^*_{\tau3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.20}
$$

In compact form

$$
\sum_{i=1}^{3} U_{\alpha,i} U^*_{\beta,i} = \delta_{\alpha\beta} \tag{2.21}
$$

[1] If neutrino is Majorana particle, the mixing matrix includes two additional phases which do not appear in the expression of oscillation probabilities.

From (2.17), we derive wave function at time $t = 0$

$$|\nu_\mu\rangle_0 \equiv |\psi(0)\rangle = U^*_{\mu 1}|\nu_1\rangle + U^*_{\mu 2}|\nu_2\rangle + U^*_{\mu 3}|\nu_3\rangle \tag{2.22}$$

Time-dependent wave function

$$|\nu_\mu\rangle \equiv |\psi(\vec{x}, t)\rangle = U^*_{\mu 1}|\nu_1\rangle e^{-i\phi_1} + U^*_{\mu 2}|\nu_2\rangle e^{-i\phi_2} + U^*_{\mu 3}|\nu_3\rangle e^{-i\phi_3} \tag{2.23}$$

In compact form

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} U^*_{\alpha,i}|\nu_i\rangle e^{-i\phi_i}, \tag{2.24}$$

where $\phi_i = p_i.x_i = E_i t - \vec{p}_i.\vec{x}_i$. From (2.19) we have

$$|\nu_1\rangle = U_{e1}|\nu_e\rangle + U_{\mu 1}|\nu_\mu\rangle + U_{\tau 1}|\nu_\tau\rangle$$

$$|\nu_2\rangle = U_{e2}|\nu_e\rangle + U_{\mu 2}|\nu_\mu\rangle + U_{\tau 2}|\nu_\tau\rangle$$

$$|\nu_3\rangle = U_{e3}|\nu_e\rangle + U_{\mu 3}|\nu_\mu\rangle + U_{\tau 3}|\nu_\tau\rangle$$

The equation (2.23) can be rewritten as

$$\begin{aligned}
|\nu_\mu\rangle &= U^*_{\mu 1}(U_{e1}|\nu_e\rangle + U_{\mu 1}|\nu_\mu\rangle + U_{\tau 1}|\nu_\tau\rangle)e^{-i\phi_1} \\
&+ U^*_{\mu 2}(U_{e2}|\nu_e\rangle + U_{\mu 2}|\nu_\mu\rangle + U_{\tau 2}|\nu_\tau\rangle)e^{-i\phi_2} \\
&+ U^*_{\mu 3}(U_{e3}|\nu_e\rangle + U_{\mu 3}|\nu_\mu\rangle + U_{\tau 3}|\nu_\tau\rangle)e^{-i\phi_3} \\
&= (U^*_{\mu 1}U_{e1}e^{-i\phi_1} + U^*_{\mu 2}U_{e2}e^{-i\phi_2} + U^*_{\mu 3}U_{e3}e^{-i\phi_3})|\nu_e\rangle \\
&+ (U^*_{\mu 1}U_{\mu 1}e^{-i\phi_1} + U^*_{\mu 2}U_{\mu 2}e^{-i\phi_2} + U^*_{\mu 3}U_{\mu 3}e^{-i\phi_3})|\nu_\mu\rangle \\
&+ (U^*_{\mu 1}U_{\tau 1}e^{-i\phi_1} + U^*_{\mu 2}U_{\tau 2}e^{-i\phi_2} + U^*_{\mu 3}U_{\tau 3}e^{-i\phi_3})|\nu_\tau\rangle \\
&= c_e|\nu_e\rangle + c_\mu|\nu_\mu\rangle + c_\tau|\nu_\tau\rangle.
\end{aligned} \tag{2.25}$$

In compact form

$$|\nu_\mu\rangle = \sum_{\beta}^{e,\mu,\tau} \sum_{i=1}^{3} U^*_{\mu i}U_{\beta i}e^{-i\phi_i}|\nu_\beta\rangle. \tag{2.26}$$

The oscillation probability from muon neutrino to electron neutrino is defined as

$$
\begin{aligned}
P(\nu_\mu \to \nu_e) &= |\langle \nu_e | \nu_\mu \rangle|^2 = c_e c_e^* \\
&= |U_{\mu 1}^* U_{e1} e^{-i\phi_1} + U_{\mu 2}^* U_{e2} e^{-i\phi_2} + U_{\mu 3}^* U_{e3} e^{-i\phi_3}|^2. \qquad (2.27)
\end{aligned}
$$

In compact form

$$
P(\nu_\alpha \to \nu_\beta) = \left| \sum_{i=1}^{3} U_{\alpha i}^* U_{\beta i} e^{-i\phi_i} \right|^2. \qquad (2.28)
$$

Since $v_\nu = c$ and in natural unit $c = 1$, we have $t = L$. Also from relativity relation $E^2 = p^2 + m^2 \Rightarrow E - P = \frac{m^2}{E+p} \approx \frac{m^2}{2E}$. Therefore at distance $x = L$ from the neutrino source, we have

$$
\phi_i = p_i.x_i = E_i t - \vec{p}_i.\vec{x}_i = (E_i - p_i)L \approx \frac{m_i^2 L}{2E}
$$

If $\phi_1 = \phi_2 = \phi_3 (\approx \frac{m^2 L}{2E})$, from unitary condition (2.21) we have

$$
P(\nu_\alpha \to \nu_\beta) = \left| \sum_{i=1}^{3} U_{\alpha i}^* U_{\beta i} \right|^2 e^{i\frac{m^2 L}{2E}} e^{-i\frac{m^2 L}{2E}} = \delta_{\alpha\beta}
$$

This means that the oscillations occur if the neutrinos have mass ($m_i \neq 0$) and the masses are not the same ($m_1 \neq m_2 \neq m_3$).

Using the identity properties of complex number:

$$
|z_1 + z_2 + z_3|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2Re[z_1 z_2^* + z_1 z_3^* + z_2 z_3^*] \qquad (2.29)
$$

Then equation (2.27) becomes

$$
\begin{aligned}
P(\nu_\mu \to \nu_e) &= |U_{\mu 1}^* U_{e1} e^{-i\phi_1} + U_{\mu 2}^* U_{e2} e^{-i\phi_2} + U_{\mu 3}^* U_{e3} e^{-i\phi_3}|^2 \\
&= |U_{\mu 1}^* U_{e1}|^2 + |U_{\mu 2}^* U_{e2}|^2 + |U_{\mu 3}^* U_{e3}|^2 \\
&+ 2Re[U_{\mu 1}^* U_{e1} U_{\mu_2} U_{e2}^* e^{i(\phi_2 - \phi_1)}] \\
&+ 2Re[U_{\mu 1}^* U_{e1} U_{\mu_3} U_{e3}^* e^{i(\phi_3 - \phi_1)}] \\
&+ 2Re[U_{\mu 2}^* U_{e2} U_{\mu_3} U_{e3}^* e^{i(\phi_3 - \phi_2)}] \qquad (2.30)
\end{aligned}
$$

In compact form

$$P(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} |U^*_{\alpha i} U_{\beta i}|^2 + 2\sum_{j>i} Re[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} e^{i(\phi_j - \phi_i)}] \tag{2.31}$$

From the unitary condition we derive

$$\begin{aligned}
& |U^*_{\mu 1} U_{e1} + U^*_{\mu 2} U_{e2} + U^*_{\mu 3} U_{e3}|^2 = 0 \\
\Rightarrow\ & |U^*_{\mu 1} U_{e1}|^2 + |U^*_{\mu 2} U_{e2}|^2 + |U^*_{\mu 3} U_{e3}|^2 \\
+\ & 2Re[U^*_{\mu 1} U_{e1} U_{\mu_2} U^*_{e2} + U^*_{\mu 1} U_{e1} U_{\mu_3} U^*_{e3} + U^*_{\mu 2} U_{e2} U_{\mu_3} U^*_{e3}] \\
=\ & 0
\end{aligned} \tag{2.32}$$

In compact form

$$\sum_{i} |U^*_{\alpha i} U_{\beta i}|^2 + 2\sum_{j>i} Re[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}] = \delta_{\alpha\beta} \tag{2.33}$$

It is followed from (2.30) and (2.32):

$$\begin{aligned}
P(\nu_\mu \to \nu_e) &= 2Re\left[U^*_{\mu 1} U_{e1} U_{\mu_2} U^*_{e2} \left(e^{i(\phi_2 - \phi_1)} - 1\right)\right] \\
&+ 2Re\left[U^*_{\mu 1} U_{e1} U_{\mu_3} U^*_{e3} \left(e^{i(\phi_3 - \phi_1)} - 1\right)\right] \\
&+ 2Re\left[U^*_{\mu 2} U_{e2} U_{\mu_3} U^*_{e3} \left(e^{i(\phi_3 - \phi_2)} - 1\right)\right]
\end{aligned} \tag{2.34}$$

In compact form

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} + 2\sum_{j>i} Re[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} (e^{i(\phi_j - \phi_i)}) - 1)] \tag{2.35}$$

We have

$$\begin{aligned}
& Re[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} (e^{i(\phi_j - \phi_i)}) - 1)] \\
=\ & Re[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} (\cos(\phi_j - \phi_i) - 1 + i\sin(\phi_j - \phi_i))] \\
=\ & Re\left\{\left(Re[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}] + iIm[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}]\right)\left(-2\sin^2\left(\frac{\phi_j - \phi_i}{2}\right) + i\sin(\phi_j - \phi_i)\right)\right\} \\
=\ & -2Re[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}]\sin^2\left(\frac{\phi_j - \phi_i}{2}\right) - Im[U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}]\sin(\phi_j - \phi_i)
\end{aligned} \tag{2.36}$$

From (2.36), we can write the oscillation pobability in a normal form

$$
\begin{aligned}
& P(\nu_\mu \rightarrow \nu_e) = \\
- \quad & 4Re\left[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}\right]\sin^2(\frac{\phi_2-\phi_1}{2}) - 2Im\left[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}\right]\sin(\phi_2-\phi_1) \\
- \quad & 4Re\left[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}\right]\sin^2(\frac{\phi_3-\phi_1}{2}) - 2Im\left[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}\right]\sin(\phi_3-\phi_1) \\
- \quad & 4Re\left[U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}\right]\sin^2(\frac{\phi_3-\phi_2}{2}) - 2Im\left[U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}\right]\sin(\phi_3-\phi_2)
\end{aligned}
\tag{2.37}
$$

In compact form

$$
\begin{aligned}
P(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\alpha\beta} \quad & - \quad 4\sum_{j>i} Re\left[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right]\sin^2(\frac{\phi_j-\phi_i}{2}) \\
& - \quad 2\sum_{j>i} Im\left[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right]\sin(\phi_j-\phi_i)
\end{aligned}
\tag{2.38}
$$

If the neutrinos interact at a time T and at a distance L along its direction of flight, the difference in phase of the three mass eigenstates are written as

$$
\phi_j - \phi_i = p_j.x_j - p_i.x_i = (E_j - E_i)T - (p_j - p_i)L
$$

With assuming that $p_j = p_i = p$ for neutrinos of the same source, then

$$
\begin{aligned}
\phi_j - \phi_i &= (E_j - E_i)T \approx \left[p_j\left(1+\frac{m_j^2}{2p_j^2}\right) - p_i\left(1+\frac{m_i^2}{2p_i^2}\right)\right]T \\
&= \frac{m_j^2 - m_i^2}{2p}T = \frac{\Delta m^2_{ji}L}{2E}
\end{aligned}
\tag{2.39}
$$

In the above calculation, we used the approximation $T \approx L$ and $p \approx E$ for $v_\nu \approx c = 1$ and $m_\nu \ll E_\nu$. We finally get the most common form of the oscillation probability:

$$
\begin{aligned}
P(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\alpha\beta} \quad & - \quad 4\sum_{j>i} Re\left[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right]\sin^2\left(\frac{\Delta m^2_{ji}}{4E}L\right) \\
& - \quad 2\sum_{j>i} Im\left[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right]\sin\left(\frac{\Delta m^2_{ji}}{2E}L\right).
\end{aligned}
\tag{2.40}
$$

For antineutrinos, we just take the complex conjugate of the product matrix and get

$$P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = \delta_{\alpha\beta} - 4\sum_{j>i} Re\left[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right]\sin^2\left(\frac{\Delta m^2_{ji}}{4E}L\right) + 2\sum_{j>i} Im\left[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right]\sin\left(\frac{\Delta m^2_{ji}}{2E}L\right). \quad (2.41)$$

The probabilities (2.40) and (2.41) are called *transition probabilities*, and the *survival probability* for a flavor is

$$P(\nu_\alpha \to \nu_\alpha) = P(\bar{\nu}_\alpha \to \bar{\nu}_\alpha) = 1 - 4\sum_{j>i}|U_{\alpha i}|^2|U_{\alpha j}|^2\sin^2\left(\frac{\Delta m^2_{ji}}{4E}L\right). \quad (2.42)$$

From (2.40) and (2.41), the difference between the neutrino and antineutrino oscillation probability indicates CP violation in neutrino sector

$$\begin{aligned}\mathcal{A}_{\text{CP}} &= P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)\\ &= 4\sum_{j>i} Im\left[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right]\sin\left(\frac{\Delta m^2_{ji}}{2E}L\right). \quad (2.43)\end{aligned}$$

If CP is violated, $U^*_{\alpha i}U_{\beta i}U_{\alpha_j}U^*_{\beta_j}$ has to contain an imaginary component. For $\alpha = \mu$ and $\beta = e$, then

$$\begin{aligned}\mathcal{A}_{\text{CP}} &= P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e)\\ &= 4\sum_{j>i} Im\left[U^*_{\mu i}U_{ei}U_{\mu j}U^*_{ej}\right]\sin\left(\frac{\Delta m^2_{ji}}{2E}L\right)\\ &= 4Im\left[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}\right]\sin\left(\frac{\Delta m^2_{21}}{2E}L\right)\\ &+ 4Im\left[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}\right]\sin\left(\frac{\Delta m^2_{31}}{2E}L\right)\\ &+ 4Im\left[U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}\right]\sin\left(\frac{\Delta m^2_{32}}{2E}L\right). \quad (2.44)\end{aligned}$$

From the unitary condition we have

$$U_{\mu 1}U^*_{e1} + U_{\mu 2}U^*_{e2} + U_{\mu 3}U^*_{e3} = 0. \quad (2.45)$$

Multiply two sides of the equation (2.45) with $U^*_{\mu 1}U_{e1}$ and $U^*_{\mu 2}U_{e2}$ respectively and then add them up, we have

$$\begin{aligned}
& U^*_{\mu 1}U_{e1}U_{\mu 1}U^*_{e1} + U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2} + U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3} \\
+ \quad & U^*_{\mu 2}U_{e2}U_{\mu 1}U^*_{e1} + U^*_{\mu 2}U_{e2}U_{\mu 2}U^*_{e2} + U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3} = 0 \qquad (2.46) \\
\Leftrightarrow \quad & 0 = |U_{\mu 1}|^2|U_{e1}|^2 + |U_{\mu 2}|^2|U_{e2}|^2 \\
+ \quad & Re[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}] + Re[U^*_{\mu 2}U_{e2}U_{\mu 1}U^*_{e1}] + Re[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}] + Re[U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}] \\
+ \quad & i\left\{Im[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}] + Im[U^*_{\mu 2}U_{e2}U_{\mu 1}U^*_{e1}] + Im[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}] + Im[U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}]\right\} \\
\Rightarrow \quad & Im[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}] + Im[U^*_{\mu 2}U_{e2}U_{\mu 1}U^*_{e1}] + Im[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}] + Im[U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}] = 0.
\end{aligned}$$

Note that

$$[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}]^* = U^*_{\mu 2}U_{e2}U_{\mu 1}U^*_{e1} \Rightarrow Im[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}] = -Im[U^*_{\mu 2}U_{e2}U_{\mu 1}U^*_{e1}]$$

Therefore, from (2.46) we get

$$Im[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}] = -Im[U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}]. \qquad (2.47)$$

Multiply two sides of the equation (2.45) with $U^*_{\mu 1}U_{e1}$ and $U^*_{\mu 3}U_{e3}$ respectively and then add them up, we have

$$\begin{aligned}
& U^*_{\mu 1}U_{e1}U_{\mu 1}U^*_{e1} + U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2} + U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3} \\
+ \quad & U^*_{\mu 3}U_{e3}U_{\mu 1}U^*_{e1} + U^*_{\mu 3}U_{e3}U_{\mu 2}U^*_{e2} + U^*_{\mu 3}U_{e3}U_{\mu 3}U^*_{e3} = 0 \\
\Leftrightarrow \quad & 0 = |U_{\mu 1}|^2|U_{e1}|^2 + |U_{\mu 3}|^2|U_{e3}|^2 \\
+ \quad & Re[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}] + Re[U^*_{\mu 3}U_{e3}U_{\mu 1}U^*_{e1}] + Re[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}] + Re[U^*_{\mu 3}U_{e3}U_{\mu 2}U^*_{e2}] \\
+ \quad & i\left\{Im[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}] + Im[U^*_{\mu 3}U_{e3}U_{\mu 1}U^*_{e1}] + Im[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}] + Im[U^*_{\mu 3}U_{e3}U_{\mu 2}U^*_{e2}]\right\} \\
\Rightarrow \quad & Im[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}] + Im[U^*_{\mu 3}U_{e3}U_{\mu 1}U^*_{e1}] + Im[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}] + Im[U^*_{\mu 3}U_{e3}U_{\mu 2}U^*_{e2}] = 0.
\end{aligned} \qquad (2.48)$$

Note that

$$[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}]^* = U^*_{\mu 3}U_{e3}U_{\mu 1}U^*_{e1} \Rightarrow Im[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}] = -Im[U^*_{\mu 3}U_{e3}U_{\mu 1}U^*_{e1}]$$

and

$$[U_{\mu 3}^* U_{e3} U_{\mu 2} U_{e2}^*]^* = U_{\mu 2}^* U_{e2} U_{\mu 3} U_{e3}^* \Rightarrow Im[U_{\mu 3}^* U_{e3} U_{\mu 2} U_{e2}^*] = -[U_{\mu 2}^* U_{e2} U_{\mu 3} U_{e3}^*]$$

Therefore, from (2.48) we get

$$Im[U_{\mu 1}^* U_{e1} U_{\mu 2} U_{e2}^*] = Im[U_{\mu 2}^* U_{e2} U_{\mu 3} U_{e3}^*]. \tag{2.49}$$

By using (2.47) and (2.49), we can rewrite (2.44) as

$$\begin{aligned} \mathcal{A}_{\text{CP}} &= P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e) \\ &= 4Im\left[U_{\mu 1}^* U_{e1} U_{\mu 2} U_{e2}^*\right] (\sin\Delta_{21} - \sin\Delta_{31} + \sin\Delta_{32})\,, \end{aligned} \tag{2.50}$$

where $\Delta_{31} = \frac{\Delta m_{31}^2}{2E}L$, $\Delta_{21} = \frac{\Delta m_{21}^2}{2E}L$ and $\Delta_{32} = \frac{\Delta m_{32}^2}{2E}L = \Delta_{31} - \Delta_{21}$. By a simple trigonometry calculation, we have

$$\begin{aligned} & \sin\Delta_{21} - \sin\Delta_{31} + \sin(\Delta_{31} - \Delta_{21}) \\ = \;& \sin\Delta_{21} - \sin\Delta_{31} + (\sin\Delta_{31}\cos\Delta_{21} - \cos\Delta_{31}\sin\Delta_{21}) \\ = \;& \sin\Delta_{21}(1-\cos\Delta_{31}) - \sin\Delta_{31}(1-\cos\Delta_{21}) \\ = \;& 2\frac{\sin\Delta_{21}}{2}\frac{\cos\Delta_{21}}{2}\frac{2(1-\cos 2\frac{\Delta_{31}}{2})}{2} - 2\sin\frac{\Delta_{31}}{2}\cos\frac{\Delta_{31}}{2}\frac{2(1-\cos 2\frac{\Delta_{21}}{2})}{2} \\ = \;& 4\frac{\sin\Delta_{21}}{2}\frac{\cos\Delta_{21}}{2}\sin^2\frac{\Delta_{31}}{2} - 4\sin\frac{\Delta_{31}}{2}\cos\frac{\Delta_{31}}{2}\sin^2\frac{\Delta_{21}}{2} \\ = \;& 4\sin\frac{\Delta_{21}}{2}\sin\frac{\Delta_{31}}{2}\left(\cos\frac{\Delta_{21}}{2}\sin\frac{\Delta_{31}}{2} - \sin\frac{\Delta_{21}}{2}\cos\frac{\Delta_{31}}{2}\right) \\ = \;& 4\sin\frac{\Delta_{21}}{2}\sin\frac{\Delta_{31}}{2}\sin\frac{\Delta_{32}}{2} \end{aligned}$$

Then we can rewrite (2.50) as

$$\begin{aligned} & \mathcal{A}_{\text{CP}} = P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e) \\ & = 16Im\left[U_{\mu 1}^* U_{e1} U_{\mu 2} U_{e2}^*\right]\left(\sin\frac{\Delta_{21}}{2}\sin\frac{\Delta_{31}}{2}\sin\frac{\Delta_{32}}{2}\right) \\ & = 16Im\left[U_{\mu 1}^* U_{e1} U_{\mu 2} U_{e2}^*\right]\sin\frac{\Delta m_{21}^2 L}{4E}\sin\frac{\Delta m_{31}^2 L}{4E}\sin\frac{\Delta m_{32}^2 L}{4E} \\ & = 2\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\sin\delta_{CP}\sin\frac{\Delta m_{21}^2 L}{4E}\sin\frac{\Delta m_{31}^2 L}{4E}\sin\frac{\Delta m_{32}^2 L}{4E}. \end{aligned} \tag{2.51}$$

In the last line, we already used

$$U_{e1} = c_{12}c_{13}, \quad U^*_{e2} = s_{12}c_{13}$$

$$U^*_{\mu 1} = -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta}, \quad U_{\mu 2} = c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta}$$

In practical, CP violation can be measured by comparing the rate of electron neutrinos appearance from muon neutrinos, $P(\nu_\mu \rightarrow \nu_e)$, with its of electron antineutrinos appearance from muon anti-neutrinos, $P(\overline{\nu}_\mu \rightarrow \overline{\nu}_e)$ in accelerator-based experiments or comparing the first with electron antineutrino disappearance in the reactor-based experiments.[2]

2.3 Three Neutrino Flavour Oscillation in Matter

In matter, unlike oscillation probabilities in vacuum, oscillation patterns are altered due to the MSW effect [1–3]. This occurs as a result of neutrinos experiencing potentials due to charged current(CC) and neutral current (NC) scattering on electrons, neutrons, and protons as they propagate through matter on Earth. The explicit expressions for these matter potentials can be found in much literature, and a brief sketch for how these quantities are derived is given in [4–6]. Neglecting Majorana phases that are irrelevant for oscillation experiments, the evolution of the mass eigenstates in vacuum is described by

$$i\frac{d}{dt}\Psi(x) = H_0^M \Psi(x) \tag{2.52}$$

where $H_0^M = diag(E_1, E_2, E_3)$ with $E_k = \sqrt{\vec{p}_k^2 + m_k^2} \simeq p_k + \frac{m_k^2}{2p_k}$, $k = 1, 2, 3$ being the energy eigenvalues of the mass eigen states. The Hamiltonian in the flavour space is obtained as $H_0 = UH_M^0 U^\dagger$, where U is the lepton mixing matrix from Equation 2.18.

[2] Accelerator-based measurements lead to an intrinsic $\delta_{CP} - \theta_{13}$ degeneracy while reactor-based measurement can precisely measure θ_{13}. Their combined information thus can provide constraint on δ_{CP}.

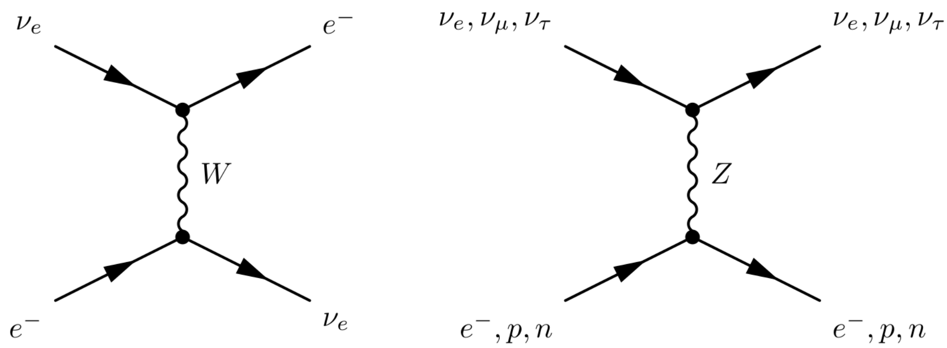

Figure 2-1: Feynman diagrams of the coherent forward elastic scattering processes experienced by neutrinos on the particles in matter.

When the neutrinos propagate through matter, the hamiltonian is modified as

$$H = H_0 + H_1 \tag{2.53}$$

where, in the interaction Hamiltonian H_1, we add up the interaction terms from scattering on the particles present. Thus, we can write

$$H_1 = H_Z^n + H_Z^p + H_Z^e + H_W^e \tag{2.54}$$

In Equation 2.54, $H_Z^i = diag(V_Z^i, V_Z^i, V_Z^i)$, $i = n, p, e$ and $H_W^i = diag(V_W^e, 0, 0)$ such that V_W^e represents the effective matter potential due to CC scattering on electrons, V_Z^e represents the effective matter potential due to NC scattering on electrons, and so on. The reason we consider matter potentials from scattering with electrons, protons and neutrons, is because the compositions of μ and τ leptons is zero on Earth. The Feynman diagrams for coherent forward scattering process that the CC potential via W-boson exchange (*left*) and the NC potential through the Z-boson exchange (*right*) is given in Figure 2-1. The effective weak CC and NC interaction Hamiltonians are:

$$\mathcal{H}_W^{eff} = \frac{G_F}{\sqrt{2}} J_{W\mu} J_W^{\mu\dagger}, \tag{2.55}$$

$$\mathcal{H}_Z^{eff} = \frac{4G_F}{\sqrt{2}} J_Z^{\mu} J_{Z\mu}. \tag{2.56}$$

G_F is the weak Fermi constant and the currents are:

$$J_W^\mu = \sum_l \Big[\bar{l}\gamma^\mu(1-\gamma_5)\nu_l + \bar{d}\gamma^\mu(1-\gamma_5)u\Big], \tag{2.57}$$

$$J_Z^\mu = \frac{1}{2}\sum_i \bar{\psi}_i\gamma^\mu\Big[I_i^3(1-\gamma_5) - 2Q_i\sin^2\theta_W\Big]\psi_i, \tag{2.58}$$

where $i = (l, \nu_l, u, d)$, I_i^3 is the associated particle isospin, and Q_i is the particle charge. The derivation of neutrino matter potentials has been studied alos in Refs. [7, 8].

Now, the effective CC potential due to scattering on electrons is given by

$$V_W^e = < \nu_e(p_1, s_1)e(p_2, s_2)|H_W|\nu_e(p_1, s_1)e(p_2, s_2) >, \tag{2.59}$$

The electron states correspond to the electrons in the left diagram in Figure 2-1. For the interaction to leave the medium unchanged in order to contribute coherently to the neutrino potential, it is fair to assume that the neutrinos and electrons conserve their momentum. The low-energy effective Hamiltonian density relevant for CC $\nu_e e$ scattering from Equation 2.55 is,

$$\mathcal{H}_W(x) = \frac{G_F}{\sqrt{2}}\Big[\bar{e}(x)\gamma^\beta(1-\gamma_5)\nu_e(x)\Big]\Big[\bar{\nu}_e(x)\gamma_\beta(1-\gamma_5)e(x)\Big] \tag{2.60}$$

The presence of electrons in a medium results in two observations. One is the statistical energy distribution of the electrons in the medium is accounted for by integration over the Fermi function $f(E_e, T)$ which is normalized to $\int f(E_e, T)dp_e = 1$, T being the temperature T of the electron background. The other is an averaging over spins $\frac{1}{2}\sum_s$, since we do not know the polarization of the electrons. Thus, Equation 2.60 is transformed as,

$$\mathcal{H}_W(x) = \int f(E_e, T)\frac{G_F}{2\sqrt{2}}\sum_s \Big[\bar{e}(x)\gamma^\beta(1-\gamma_5)\nu_e(x)\Big]\Big[\bar{\nu}_e(x)\gamma_\beta(1-\gamma_5)e(x)\Big]dp_2 \tag{2.61}$$

Since, only electrons contribute to CC potential in Equation 2.59, we can write

$$\begin{aligned}&\Big[\bar{e}(x)\gamma^{\beta}(1-\gamma_5)\nu_e(x)\Big]\Big[\bar{\nu}_e(x)\gamma_{\beta}(1-\gamma_5)e(x)\Big]|\nu_e(p_1,s_1)e(p_2,s_2)>\\ =&\frac{1}{2VE_2(p_2)}\Big[a^{\dagger}_{s_2}(p_2)a_{s_2}(p_2)\bar{u}_{s_2}(p_2)\gamma^{\beta}(1-\gamma_5)\nu_e(x)\Big]\Big[\bar{\nu}_e(x)\gamma_{\beta}(1-\gamma_5)u_{s_2}(p_2)\Big]|\nu_e e>\end{aligned} \tag{2.62}$$

Now, considering $H_W = \int_V \mathcal{H}_W(x)dx$, substituting Equations 2.62 and 2.61 in 2.59, we have

$$\begin{aligned}V_W^e =&< \nu_e(p_1,s_1)e(p_2,s_2)|\frac{G_F}{4\sqrt{2}V}\int\int f(E_e,T)\sum_{s_2}\frac{a^{\dagger}_{s_2}(p_2)a_{s_2}(p_2)}{E_e(p_2)}\\ &\Big[\bar{u}_{s_2}(p_2)\gamma^{\beta}(1-\gamma_5)\nu_e(x)\Big]\Big[\bar{\nu}_e(x)\gamma_{\beta}(1-\gamma_5)u_{s_2}(p_2)\Big]dxdp_2|\nu_e(p_1,s_1)e(p_2,s_2)>\end{aligned} \tag{2.63}$$

We assume the medium to be isotropic and non-magnetic and, then, apply the Fierz transformation (page 64, [4]) to re-arrange the ν_e and e spinors. Using $\int p_2 f(E_e,T)dp_2 = 0$ for isotropy, and the total electron number density of the medium $\int f(E_e,T)N_e(p_2)dp_2 = N_e$, only integration over x remains. Hence, we have,

$$V_W^e =< \nu_e(p_1,s_1)e(p_2,s_2)|\frac{G_F N_e}{\sqrt{2}}\times\frac{1}{2VE_{\nu_e}}\int 4E_{\nu_e}dx|\nu_e(p_1,s_1)e(p_2,s_2)> . \tag{2.64}$$

Assuming normalized state vectors, $|\nu_e(p_1,s_1)e(p_2,s_2)>$, Equation 2.64 reduces to

$$\begin{aligned}V_W^e =&\frac{G_F N_e}{\sqrt{2}}\times\frac{2}{V}\int dx < \nu_e(p_1,s_1)e(p_2,s_2)||\nu_e(p_1,s_1)e(p_2,s_2)>\\ V_W^e =&V_{CC} = \sqrt{2}G_F N_e\end{aligned} \tag{2.65}$$

For anti-neutrinos, $V_{CC} \rightarrow -V_{CC}$, because of the anti-commutation relation between the creation and annihilation operators.

The NC scattering is mediated by Z^0 boson and we use the NC hamiltonian from Equation 2.56 to obtain V_Z^n due to ν_α, $\alpha = e,\mu,\tau$ scattering. The effective hamiltonian is given by,

$$\mathcal{H}_Z(x) = -\frac{G_F}{4\sqrt{2}} \int f(E_n, T) \sum_s \left[\bar{\psi}_n(x)\gamma^\mu(1-\gamma_5)\psi_n(x)\right]\left[\bar{\nu}_\alpha(x)\gamma_\mu(1-\gamma_5)\nu_\alpha(x)\right] dp_n \tag{2.66}$$

Here, we have introduced the statistical Fermi distribution for neutrons $f(E_n, T)$ and summation over the neutron spins due to the assumption of unpolarized medium, just as for the electrons. Equation 2.66 is of the form as in Equation 2.61. Applying the Fierz transformation and performing similar derivation of V_W^e, we find

$$V_Z^n = -\frac{G_F N_n}{\sqrt{2}} \tag{2.67}$$

The potentials V_Z^e and V_Z^p are equal in magnitude and opposite in sign, and therefore cancels out each other. Thus, the potential induced by NC scattering is,

$$V_{NC} = V_Z^n = -\frac{G_F N_n}{\sqrt{2}} \tag{2.68}$$

where, N_n is the neutron no. density of the medium.

2.3.1 Neutrino Oscillation Probability in Matter

The relation between mass eigenstates and flavor eigenstates

$$|\nu_\alpha\,\rangle = \sum_k U^*_{\alpha k}|\nu_k\,\rangle$$

The total Hamiltonian in matter is

$$H = H_0 + H_1,$$

where H_0 is the hamiltonian in vaccum and H_1 is the perturbed hamiltonian.

$$\begin{aligned} H_0|\nu_k\,\rangle &= E_k|\nu_k\,\rangle; \quad \text{with} \quad E_k = \sqrt{\vec{p_k}^{\,2} + m_k^2} \approx p_k + \frac{m_k^2}{2p_k} \\ H_1|\nu_\alpha\,\rangle &= V_\alpha|\nu_\alpha\,\rangle = (V_{CC} + V_{NC})|\nu_\alpha\,\rangle\,. \end{aligned}$$

In the literature, many authors have considered perturbation/expansion H_1 around naturally appearing small parameters such as the matter potential $\frac{a}{\Delta m^2_{31}}$ [9], $\sin\theta_{13}$, $\sin^2\theta_{13}$ [10, 11], the ratio of mass of mass-squared differences $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ and $\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}$ [9, 12–17], where $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$. In context of ongoing accelerator-based long baseline and reactor-based neutrino based medium baseline experiments, the oscillation channels $P_{\nu_\mu\to\nu_e}$ $(P_{\bar{\nu}_\mu\to\bar{\nu}_e})$, $P_{\nu_\mu\to\nu_\mu}$ $(P_{\bar{\nu}_\mu\to\bar{\nu}_\mu})$ and $P_{\bar{\nu}_e\to\bar{\nu}_e}$ are of great interest. We will use the S-matrix method to show a derivation, presented as in Ref.[9] for propagation of neutrinos in matter of constant density. The Schrodinger equation for neutrino in matter is

$$\begin{aligned}
i\frac{d}{dt}|\nu_\alpha(t)\,\rangle &= H|\nu_\alpha(t)\,\rangle \\
&= (H_0 + H_1)|\nu_\alpha(t)\,\rangle \\
&= (E_k + V_\alpha)|\nu_\alpha(t)\,\rangle \\
&= \left[\left(p_k + \frac{m_k^2}{2p_k}\right) + V_\alpha\right]|\nu_\alpha(t)\,\rangle
\end{aligned}$$

For $v \approx c = 1$ (means $t \approx x$) we have $p_k \approx E$. We can see that $E + V_{NC}$ is the same for all neutrinos. They generate a phase common to all flavors and will cancel out in transition. Hence we can ignore them here for simplicity. So we rewrite the above equation as

$$\begin{aligned}
i\frac{d}{dt}|\nu_\alpha(t)\,\rangle &= \left[\left(p_k + \frac{m_k^2}{2p_k}\right) + V_\alpha\right]|\nu_\alpha(t)\,\rangle \\
&= \left[\left(\frac{m_k^2}{2E} + V_{CC}\delta_{\alpha e}\right) + (E + V_{NC})\right]|\nu_\alpha(t)\,\rangle \\
&\Longrightarrow \left(\frac{m_k^2}{2E} + V_{CC}\delta_{\alpha e}\right)|\nu_\alpha(t)\,\rangle
\end{aligned}$$

Or in explicit form

$$i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \left[\frac{1}{2E}U\begin{pmatrix}m_1^2 & 0 & 0\\ 0 & m_2^2 & 0\\ 0 & 0 & m_3^2\end{pmatrix}U^\dagger + \begin{pmatrix}V_{CC} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right]\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}, \tag{2.69}$$

where U is an unitary matrix. We can rewrite the Schrodinger equation in matter as

$$i\frac{d\nu}{dx} = H\nu,$$

where

$$\begin{aligned} H &= H_0 + H_1 \\ &= \frac{1}{2E}U\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix}U^\dagger + \frac{1}{2E}\left[U\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}U^\dagger + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \end{aligned}$$

and $a = 2EV_{CC} = 2\sqrt{2}G_F E N_e$. Since Δm_{21}^2 and $a \ll \Delta m_{31}^2$, we can treat H_1 as a pertubation. The Schrodinger equation has a solution of Dyson series form

$$\nu(x) = S(x)\nu(0), \tag{2.70}$$

with

$$S(x) \equiv Te^{\int_0^x H(s)ds},$$

T is the symbol of time ordering. The oscillation probability at distance L then can be calculate through $S(x)$

$$P(\nu_\alpha \to \nu_\beta) = |S_{\beta\alpha}(L)|^2. \tag{2.71}$$

We can calculate the pertubation to the first order in a and Δm_{21}^2. We have

$$S_0(x) = e^{-iH_0x} \tag{2.72}$$

and

$$S_1(x) = e^{-iH_0x}(-i)\int_0^x dsH_1(s) = e^{-iH_0x}(-i)\int_0^x ds e^{iH_0s}H_1e^{-iH_0s} \tag{2.73}$$

Substituting H_o and H_1 and calculating $(S(x))_{\beta\alpha} = (S_0(x))_{\beta\alpha} + (S_1(x))_{\beta\alpha}$, the general form of oscillation probability is given by,

$$\begin{aligned}
P(\nu_\alpha \to \nu_\beta) \;=\;& |(S(x))_{\beta\alpha}|^2 = \delta_{\alpha\beta}\left[1 - 4|U_{\alpha 3}|^2 \sin^2\Delta_{31}\left(1 - \frac{2a}{\Delta m^2_{31}}(|U_{13}|^2 - \delta_{\alpha 1})\right)\right] \\
& -\delta_{\alpha\beta}\left[\frac{ax}{E}|U_{\alpha 3}|^2|U_{13}|^2 \sin 2\Delta_{31}\right] \\
& +4\sin^2\Delta_{31}|U_{\beta 3}|^2|U_{\alpha 3}|^2\left[1 - 2\frac{a}{\Delta m^2_{31}}(2|U_{13}|^2 - \delta_{\alpha 1} - \delta_{\beta 1})\right] \\
& -8\Delta_{21}\sin^2\Delta_{31} Im(U^*_{\beta 3}U_{\alpha 3}U_{\beta 2}U^*_{\alpha 2}) \\
& +4\sin 2\Delta_{31}\left[\Delta_{21} Re(U^*_{\beta 3}U_{\alpha 3}U_{\beta 2}U^*_{\alpha 2})\right. \\
& \left.+\frac{ax}{4E}\left(|U_{13}|^2\delta_{\alpha 1}\delta_{\beta 1} + |U_{\beta 3}|^2|U_{\alpha 3}|^2(2|U_{13}|^2 - \delta_{\alpha 1} - \delta_{\beta 1})\right)\right] \\
& +4\Delta^2_{21}|U_{\beta 2}|^2|U_{\alpha 2}|^2
\end{aligned} \tag{2.74}$$

For $\alpha = \mu$ and $\beta = e$ we get the electron neutrino appearance probability from muon neutrino, given by

$$\begin{aligned}
P(\nu_\mu \to \nu_e) \;=\;& 4\sin^2\Delta_{31}|U_{e3}|^2|U_{\mu 3}|^2 \\
& -8\sin^2\Delta_{31}|U_{e3}|^2|U_{\mu 3}|^2\frac{a}{\Delta m^2_{31}}(2|U_{e3}|^2 - 1) \\
& +4\sin 2\Delta_{31}\frac{ax}{4E}|U_{e3}|^2|U_{\mu 3}|^2(2|U_{e3}|^2 - 1) \\
& -8\Delta_{21}\sin^2\Delta_{31} Im(U^*_{e3}U_{\mu 3}U_{e2}U^*_{\mu 2}) \\
& +4\Delta_{21}\sin 2\Delta_{31} Re(U^*_{e3}U_{\mu 3}U_{e2}U^*_{\mu 2}) \\
& +4\Delta^2_{21}|U_{e2}|^2|U_{\mu 2}|^2
\end{aligned} \tag{2.75}$$

Substituting the elements from the PMNS matrix, we find

$$\begin{aligned}
P(\nu_\mu \to \nu_e) \;=\;& 4s^2_{13}s^2_{23}c^2_{13}\sin^2\Delta_{31} \\
& -8s^2_{13}s^2_{23}c^2_{13}\frac{a}{\Delta m^2_{31}}(2s^2_{13} - 1)\sin^2\Delta_{31} \\
& +4s^2_{13}s^2_{23}c^2_{13}\frac{ax}{4E}(2s^2_{13} - 1)\sin 2\Delta_{31} \\
& -8s_{12}s_{13}s_{23}c_{12}c^2_{13}c_{23}\sin\delta\Delta_{21}\sin^2\Delta_{31} \\
& +4s_{12}s_{13}s_{23}c^2_{13}(c_{12}c_{23}\cos\delta - s_{12}s_{13}s_{23})\Delta_{21}\sin 2\Delta_{31} \\
& +4s^2_{12}c^2_{13}(c^2_{12}c^2_{23} + s^2_{12}s^2_{13}s^2_{23} - 2s_{12}s_{13}s_{23}c_{12}c_{23}\cos\delta)\Delta^2_{21}
\end{aligned} \tag{2.76}$$

For $\frac{\Delta m_{21}^2 x}{4E} \ll 1$ and $\Delta m_{31}^2 \approx \Delta m_{32}^2$, we can make a replacement with: $\Delta_{21} = \sin\Delta_{21}$; $\cos\Delta_{31} = \cos\Delta_{32}$; $\sin\Delta_{31} = \sin\Delta_{32}$ and the probability of $\nu_\mu \to \nu_e$ oscillation can be written as follows

$$\begin{aligned} P(\nu_\mu \to \nu_e) \approx\ & 4s_{13}^2 s_{23}^2 c_{13}^2 \sin^2\Delta_{31} \\ & -8s_{13}^2 s_{23}^2 c_{13}^2 \frac{a}{\Delta m_{31}^2}(2s_{13}^2 - 1)\sin^2\Delta_{31} \\ & +8s_{13}^2 s_{23}^2 c_{13}^2 \frac{aL}{4E}(2s_{13}^2 - 1)\sin\Delta_{31}\cos\Delta_{32} \\ & -8s_{12}s_{13}s_{23}c_{12}c_{13}^2 c_{23}\sin\delta_{CP}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32} \\ & +8s_{12}s_{13}s_{23}c_{13}^2(c_{12}c_{23}\cos\delta_{CP} - s_{12}s_{13}s_{23})\sin\Delta_{21}\sin\Delta_{31}\cos\Delta_{32} \\ & +4s_{12}^2 c_{13}^2(c_{12}^2 c_{23}^2 + s_{12}^2 s_{13}^2 s_{23}^2 - 2s_{12}s_{13}s_{23}c_{12}c_{23}\cos\delta_{CP})\sin^2\Delta_{21}, \end{aligned} \tag{2.77}$$

where $\Delta_{ji} = \frac{\Delta m_{ji}^2}{4E}L$, and $a = 2\sqrt{2}G_F n_e E = 7.56 \times 10^{-5}[eV^2](\frac{\rho}{g/cm^3})(\frac{E}{GeV})$, n_e is the electron density of the matter and ρ is the density of the Earth.

- The appearances of a in the equation (2.77) is due to the matter effect which is rooted from the fact that electron neutrino when passing through ordinary matter will interact weakly with electrons.
- For anti-neutrino counterpart,$P(\overline{\nu}_\mu \to \overline{\nu}_e)$ can be obtained from Eq.(2.77) by replacing $\delta \to -\delta$ and $a \to -a$.
- The matter effect, represented by a constant, involves to the second and third terms.
- While the term proportional to $\sin\delta_{CP}$ is called *CP-violating* since their contribution for total probability are opposite for neutrino and antineutrino, the fifth ,which contains $\cos\delta_{CP}$, is called *CP-conserving term* since their contributions are the same for neutrino and antineutrino.
- The last one depends on Δm_{21}^2 and can be ignored in the case of long baseline experiments. At present landscape of neutrino oscillations, this channels is the only hope to provide information about δ_{CP}.

Fig. 2-2 and Fig. 2-3 show the oscillation probabilities of $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ as a function of neutrino energy at different true value of δ_{CP} for T2K baseline

Table 2.1: Global constraint of oscillation parameters with *normal* mass hierarchy assumed [18].

Parameter	Best fit$\pm 1\sigma$
$\sin^2\theta_{12}$	$0.310^{+0.013}_{-0.012}$
$\sin^2\theta_{13}(\times 10^{-2})$	$2.241^{+0.067}_{-0.066}$
$\sin^2\theta_{23}$	$0.558^{+0.020}_{-0.033}$
$\delta_{CP}(^\circ)$	222^{+38}_{-28}
$\Delta m^2_{21}(10^{-5}\text{eV}^2/c^4)$	$7.39^{+0.21}_{-0.20}$
$\Delta m^2_{31}(10^{-3}\text{eV}^2/c^4)$	$2.523^{+0.032}_{-0.030}$

$L = 295$ km (with peak of neutrino flux at 0.6 GeV) and NOνA baseline $L = 810$ km (with peak of neutrino flux at 2 GeV), respectively. In the figure 2-4, the difference between solid and dashed blue lines indicates the matter effect, and the difference between solid and dashed red lines shows the combined effect of both matter and CP-violation. In the case of T2K experiment, the matter effect is much smaller than the CP-violation effect. For NOνA, due to its longer baseline the matter effect is larger. The plots are made with assumed values of oscillation parameters as listed Table 2.1.

However, challenges for this channel measurement are the smallness of oscillation amplitude and its degeneracy with other oscillation parameters. Along with the appearance channels, the accelerator-based long-baseline neutrino experiments typically can measure precisely the probability of $\nu_\mu \to \nu_\mu$ and $\overline{\nu}_\mu \to \overline{\nu}_\mu$.

Substituting $\alpha = \beta = \mu$ in Equation 2.74, we obtain the survival/disappearance probability for muon-neutrinos, given by

$$\begin{aligned} P(\nu_\mu \to \nu_\mu) \;=\; & 1 + 4\sin^2\Delta_{31}|U_{\mu 3}|^2 \left[(|U_{\mu 3}|^2 - 1) - \frac{2a}{\Delta m^2_{31}}|U_{e3}|^2 \left(2|U_{\mu 3}|^2 - 1\right) \right] \\ & + 4\Delta_{31}\sin 2\Delta_{31}|U_{\mu 3}|^2 \left[\frac{\Delta m^2_{21}}{\Delta m^2_{31}}|U_{\mu 2}|^2 + \frac{a}{\Delta m^2_{31}}|U_{e3}|^2 \left(2|U_{\mu 3}|^2 - 1\right) \right] \\ & + 4\Delta^2_{21}|U_{\mu 2}|^4 \end{aligned} \tag{2.78}$$

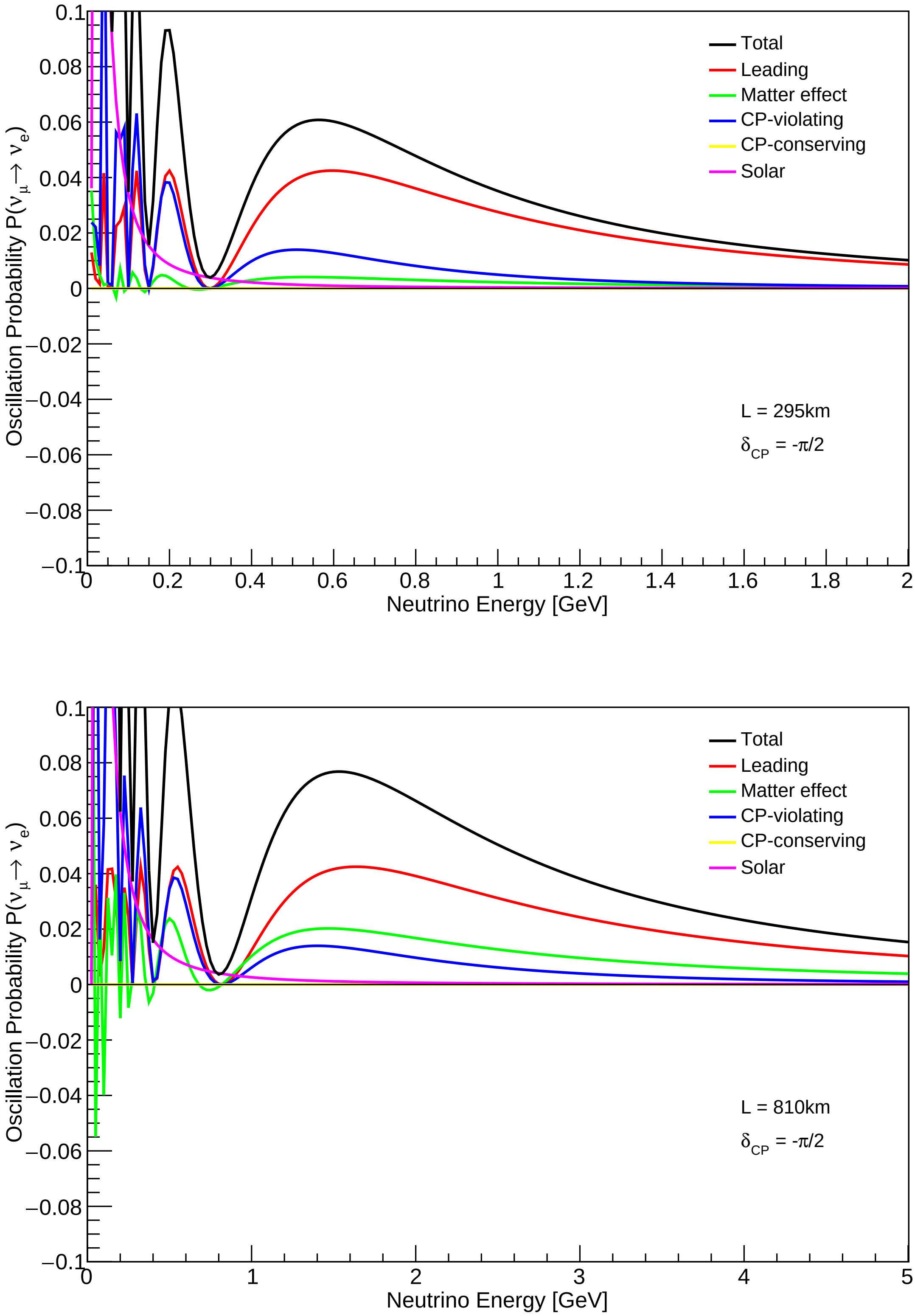

Figure 2-2: $P(\nu_\mu \to \nu_e)$ for L =295 (*top*) and L=810 km (*bottom*) for $\delta_{CP} = -\pi/2$.

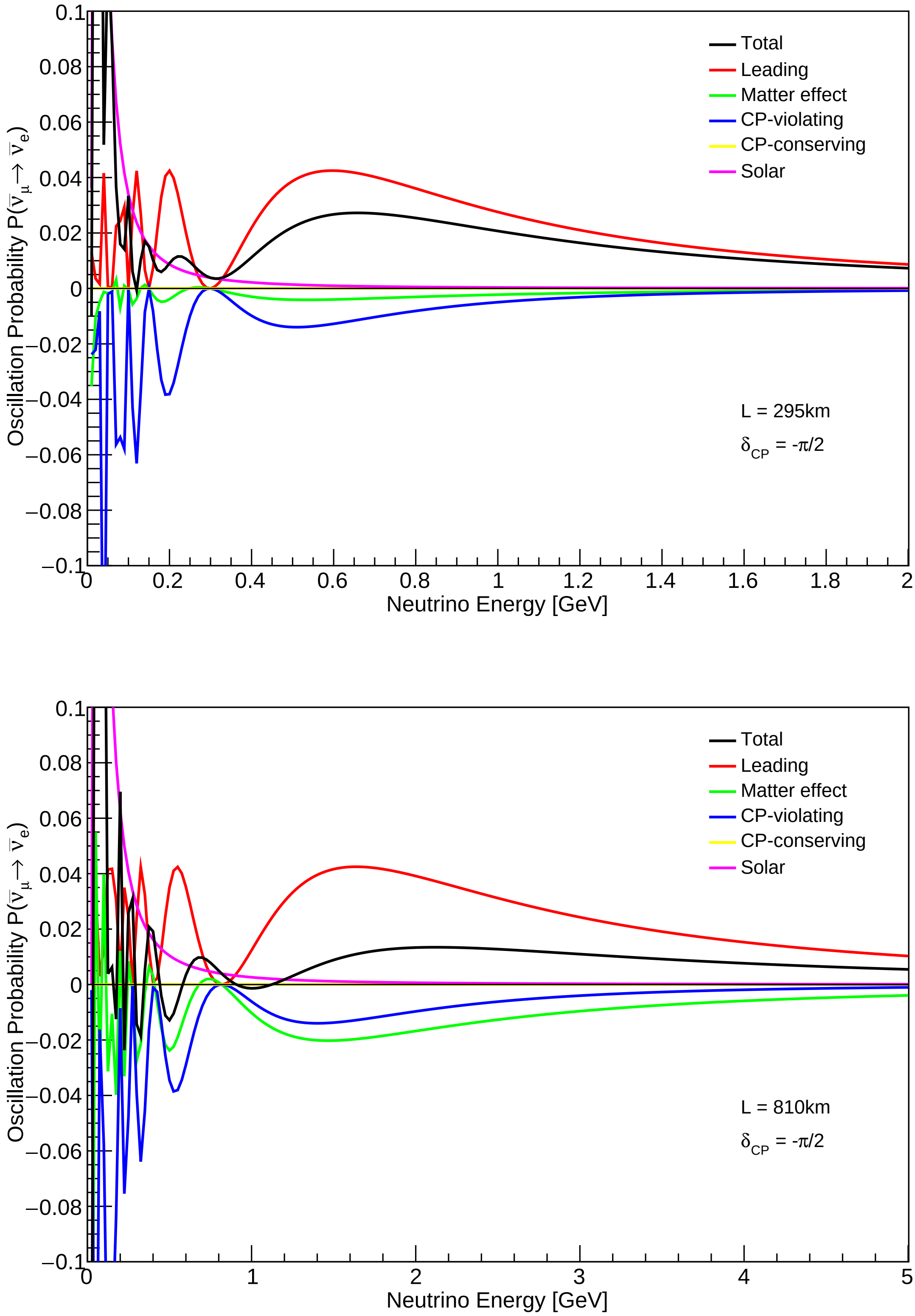

Figure 2-3: $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$ for L =295 (*top*) and L=810 km (*bottom*) for $\delta_{CP} = -\pi/2$.

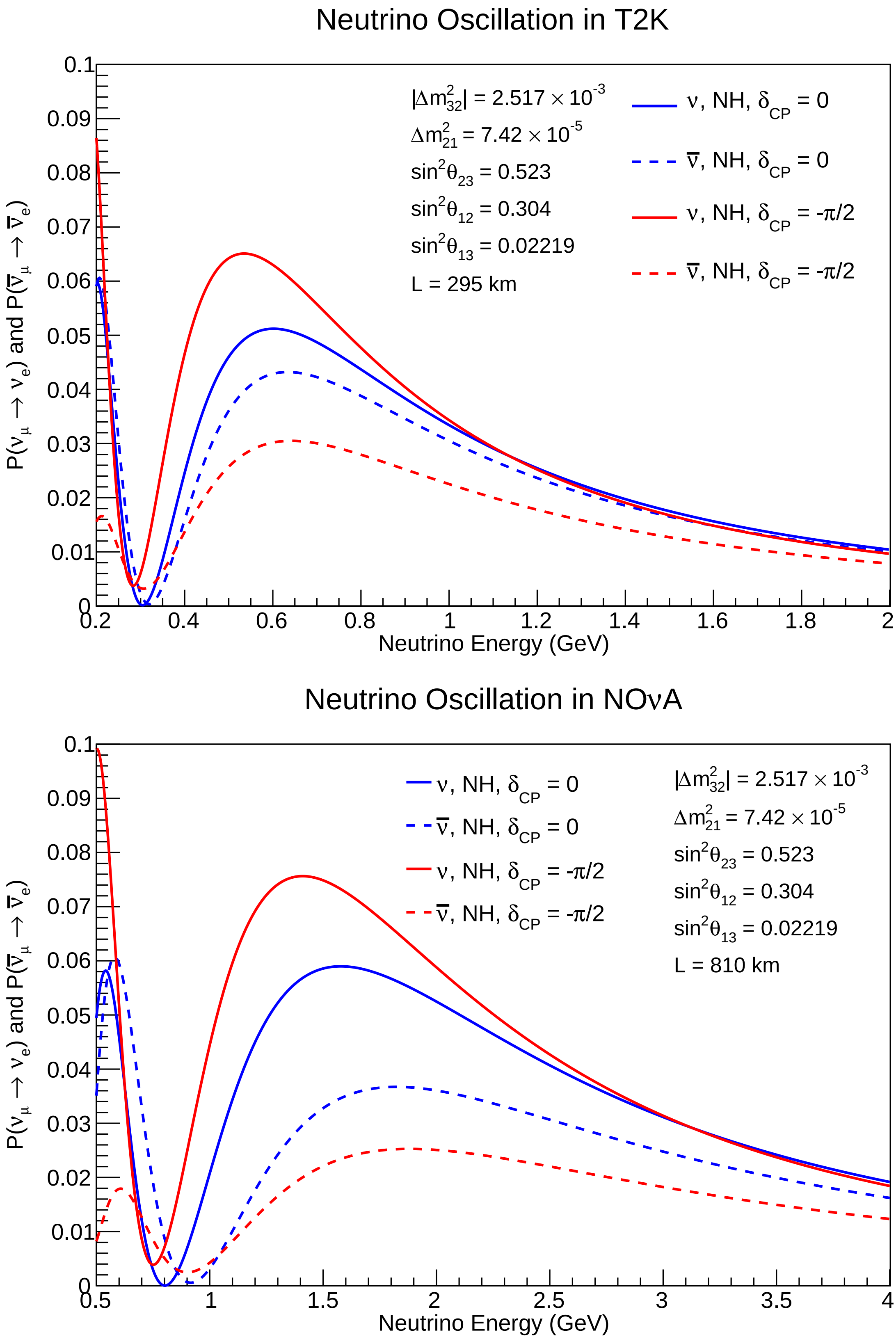

Figure 2-4: $P(\nu_\mu \to \nu_e$ and $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ for L =295 (*top*) and L=810 km (*bottom*) for $\delta_{CP} = 0, -\pi/2$.

Using the elements from the PMNS matrix, we obtain

$$\begin{aligned} P(\nu_\mu \to \nu_\mu) \;=\; & 1 + 4s_{23}^2c_{13}^2(s_{23}^2c_{13}^2 - 1)\sin^2\Delta_{31} \\ & \pm 4s_{23}^2c_{13}^2s_{13}^2\left(2s_{23}^2c_{13}^2 - 1\right)\frac{2a}{\Delta m_{31}^2}\sin^2\Delta_{31} \\ & \pm 4s_{23}^2c_{13}^2s_{13}^2\left(2s_{23}^2c_{13}^2 - 1\right)\frac{a}{\Delta m_{31}^2}\Delta_{31}\sin 2\Delta_{31} \\ & + 4s_{23}^2c_{13}^2(c_{12}^2c_{23}^2 + s_{12}^2s_{13}^2s_{23}^2 - 2s_{12}s_{13}s_{23}c_{12}c_{23}\cos\delta)\Delta_{21}\sin 2\Delta_{31} \\ & + 4(c_{12}^2c_{23}^2 + s_{12}^2s_{13}^2s_{23}^2 - 2s_{12}s_{13}s_{23}c_{12}c_{23}\cos\delta)^2\Delta_{21}^2 \end{aligned} \tag{2.79}$$

where positive (negative) signs are taken for neutrino (antineutrino) oscillations respectively.

- As can be seen in the equation (2.79), the second term dominates.
- The third and the forth are related to matter effect.
- Due to relative smallness of θ_{13} the first term is dominated in the accelerator-based long-baseline neutrino experiment and measurement with this channel is essentially sensitive to mixing angle θ_{23} and Δm_{31}^2.

In practice, neutrino oscillation analyses take advance of combining both appearance channel and disappearance channel in order to provide the most precise measurements of oscillation parameters and explore CP violation from constraints on δ_{CP}.

2.4 Oscillation Parameter Degeneracy

We revisit on the degeneracies between the various oscillation parameters in this section. In the context of neutrino oscillations, parameter degeneracy refers to the probability of obtaining the same value for various sets of oscillation parameters. The number of neutrino and antineutrino events that are functions of neutrino oscillation probabilities determines how sensitive an experiment is. This suggests that different sets of parameters can provide an equally good fit to the data in the presence of degeneracies, making it challenging to determine the precise values of the parameters. Determining the unknown parameters, therefore, requires a thorough understanding of various degeneracies, their dependence on various oscillation parameters, and their resolution. When θ_{13} was unknown, three types of degeneracies have been discussed widely in the literature [19–21].

1. The intrinsic degeneracy of the $P_{\mu e}$ channel refers to the same value of probability coming from a different θ_{13} and δ_{CP} value and can be expressed as
$$P_{\mu e(\bar{\mu}\bar{e})}(\theta_{13}, \delta_{CP}) = P_{\mu e(\bar{\mu}\bar{e})}(\theta_{13}^{'}, \delta_{CP}^{'}) \tag{2.80}$$

2. The hierarchy-δ_{CP} degeneracy of the $P_{\mu e}$ channel leads to wrong hierarchy solutions arising due to a different value of δ_{CP} other than the true value. This degeneracy can be expressed mathematically as
$$P_{\mu e(\bar{\mu}\bar{e})}(NH, \delta_{CP}) = P_{\mu e(\bar{\mu}\bar{e})}(IH, \delta_{CP}^{'}) \tag{2.81}$$

3. The intrinsic octant degeneracy of the $P_{\mu\mu}$ channel refers to the clone solutions occurring for θ_{23} and $\pi/2 - \theta_{23}$ and expressed as
$$P_{\mu\mu(\bar{\mu}\bar{\mu})}(\theta_{23}) = P_{\mu\mu(\bar{\mu}\bar{\mu})}(\pi/2 - \theta_{23}) \tag{2.82}$$

Summing up the above degeneracies, a system of solutions, illustrated in Figure 2-5, is given by

$$P_{\alpha\to\beta(\bar{\alpha}\to\bar{\beta})}(\theta_{13}, \delta_{CP}, NH, \theta_{23}) = P_{\alpha\to\beta(\bar{\alpha}\to\bar{\beta})}(\theta_{13}^{'}, \delta_{CP}^{'}, IH, \theta_{23}^{'}) \tag{2.83}$$

where, $\alpha = \nu_\mu$ and $\beta = \nu_\mu, \nu_e$. Solving these equation gives us a true solution and additional clone solutions to form the eight fold degeneracy [22]. However, there is no intrinsic octant degeneracy in the $P_{\mu e}$ channel as the dependence of θ_{23} in the leading order term of $P_{\mu e}$ channel goes as $\sin^2\theta_{13}\sin^2\theta_{23}$.

At present, the measurement of the non-zero precise value of θ_{13} from the reactor experiments resolves the degeneracies associated with θ_{13}. The intrinsic

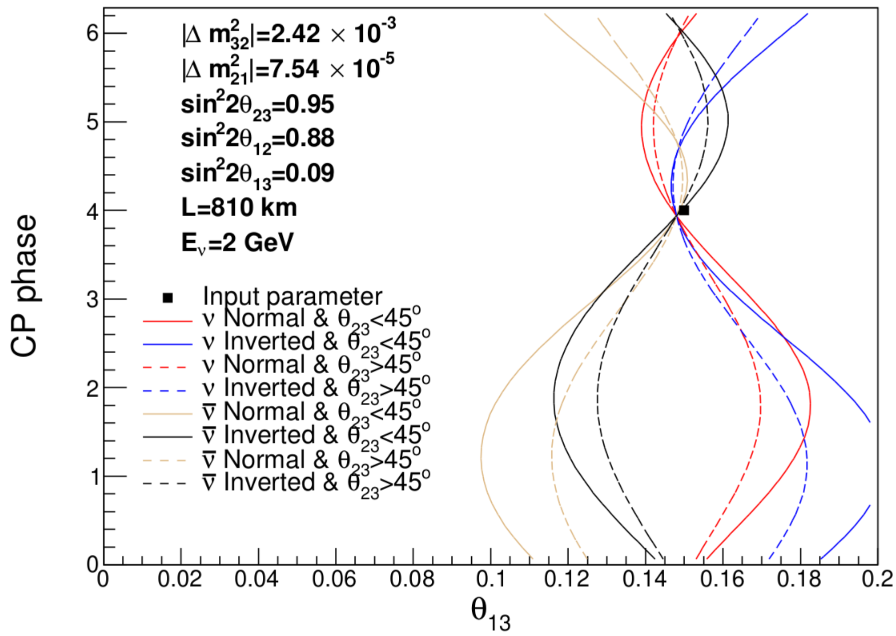

Figure 2-5: The Eight-fold degeneracy in $\nu_\mu \to \nu_e$ appearance channel for NOνA baseline. Courtesy: Son Cao [23].

degeneracy is largely resolved and the octant sensitivity of the appearance channel has greatly improved. But due to the completely unknown value of δ_{CP}, the hierarchy-δ_{CP} degeneracy still persists and there are also degenerate solutions arising due to different values of θ_{23} and δ_{CP}. This degeneracy is referred as octant-δ_{CP} degeneracy. The degeneracy illustration are shown in Figures 2-6, 2-7 and 2-8. There are several methods discussed in the literature to break

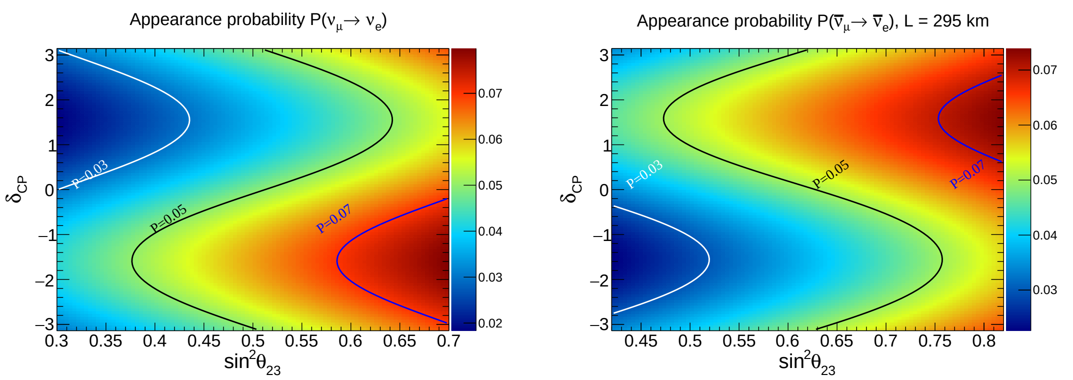

Figure 2-6: $P_{\mu e}(\theta_{23}, \delta_{CP}) = P_{\mu e}(\theta'_{23}, \delta'_{CP})$ Degeneracy at E_ν =0.6 GeV (T2K).

these degeneracies and to have a clean measurement of the neutrino oscillation parameters. They include combination of experiments at various baselines and L/E values [20, 22, 24], use of spectral information [25, 26], combination of different oscillation channels [27], and combination of experiments with varying neutrino

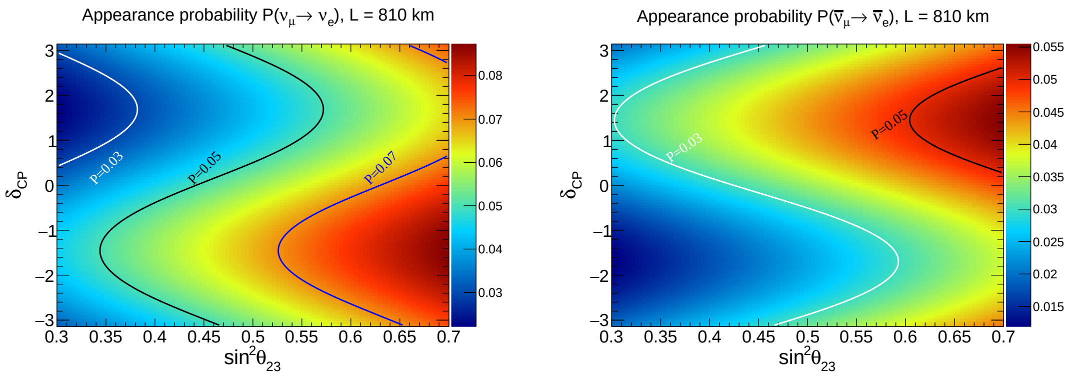

Figure 2-7: $P_{\mu e}(\theta_{23}, \delta_{CP}) = P_{\mu e}(\theta_{23}', \delta_{CP}')$ Degeneracy at E_ν=2.0 GeV (NOνA).

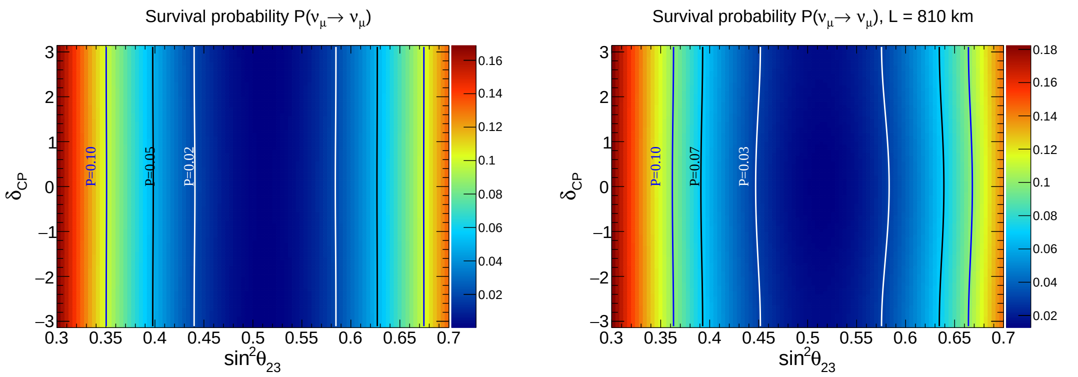

Figure 2-8: $P_{\mu\mu}(\theta_{23}) = P_{\mu\mu}(\pi/2 - \theta_{23})$ Degeneracy at E_ν =0.6 GeV for T2K(*left*) and 2.0 GeV for NOνA(*right*).

sources, such as A-LBL and reactor-based neutrino experiments [28–32], or A-LBL with atmospheric neutrino experiments [33, 34].

2.5 Summary

This chapter forms the motivation of the thesis to adopt a framework to address the objectives. We describe the neutrino oscillation phenomena in vacuum and matter and revisited the oscillation paramater degeneracies associated with physics studies in A-LBL experiments. In the upcoming chapters, we will show how these parameter degeneracies affect the CP measurement, MH determination, and θ_{23} octant resolution capabilities of the long-baseline experiments and how adding a reactor-based oscillation experiment can be crucial to determining the present unknowns.

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