Chapter 6

Genus of commuting conjugacy class graphs of groups

Continuing the works of Ashrafi and Salahshour [86], in Chapter 5 we have obtained various spectra and energies of commuting conjugacy class graphs of finite groups. In this chapter we compute genus of commuting conjugacy class graphs of the groups considered in Chapter 5 and determine whether $\mathcal{CCC}(G)$ for those groups are planar, toroidal, double-toroidal or triple-toroidal. This chapter is based on our paper [19] published in *Algebraic Structures and Their Applications*.

6.1 Genus of $\mathcal{CCC}(G)$

In this section we compute genus of commuting conjugacy class graph of the groups D_{2n}, SD_{8n}, Q_{4m}, V_{8n}, $U_{(n,m)}$ and $G(p, m, n)$ one by one and check their planarity, toroidality etc.

Theorem 6.1.1. *Let* $G = D_{2n}$. *Then*

(a) $\mathcal{CCC}(G)$ *is planar if and only if* $3 \leq n \leq 10$.

(b) $\mathcal{CCC}(G)$ *is toroidal if and only if* $11 \leq n \leq 16$.

(c) $\mathcal{CCC}(G)$ *is double-toroidal if and only if* $n = 17, 18$.

(d) $\mathcal{CCC}(G)$ *is triple-toroidal if and only if* $n = 19, 20$.

(e) $\gamma(\mathcal{CCC}(G)) = \begin{cases} \left\lceil \frac{(n-7)(n-9)}{48} \right\rceil, & \text{if } n \text{ is odd and } n \geq 21 \\ \left\lceil \frac{(n-8)(n-10)}{48} \right\rceil, & \text{if } n \text{ is even and } n \geq 22. \end{cases}$

Proof. Consider the following cases.

Case 1. n is odd.

By Result 1.3.18 we have $\mathcal{CCC}(G) = K_1 \sqcup K_{\frac{n-1}{2}}$. Therefore, for $n = 3$ and 5, it follows that $\mathcal{CCC}(G) = 2K_1$, $K_1 \sqcup K_2$ respectively; and hence $\mathcal{CCC}(G)$ is planar. If $n \geq 7$ then, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = \gamma(K_{\frac{n-1}{2}}) = \left\lceil \frac{(n-7)(n-9)}{48} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 0$ if and only if $n = 7$ or 9. Also, $\gamma(\mathcal{CCC}(G)) = 1$ if $n = 11$, 13 or 15; $\gamma(\mathcal{CCC}(G)) = 2$ if $n = 17$; $\gamma(\mathcal{CCC}(G)) = 3$ if $n = 19$. For $n \geq 21$ we have

$$\frac{(n-7)(n-9)}{48} \geq \frac{7}{2} = 3.5,$$

and so

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(n-7)(n-9)}{48} \right\rceil \geq 4.$$

Thus, $\mathcal{CCC}(G)$ is planar if and only if $n = 3, 5, 7, 9$; toroidal if and only if $n = 11, 13, 15$; double-toroidal if and only if $n = 17$ and triple-toroidal if and only if $n = 19$.

Case 2. n is even.

By Result 1.3.18 we have

$$\mathcal{CCC}(G) = \begin{cases} 2K_1 \sqcup K_{\frac{n}{2}-1}, & \text{if } n \text{ and } \frac{n}{2} \text{ are even} \\ K_2 \sqcup K_{\frac{n}{2}-1}, & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd.} \end{cases}$$

Therefore, for $n = 4$ and 6, it follows that $\mathcal{CCC}(G) = 3K_1$, $2K_2$ respectively; and hence $\mathcal{CCC}(G)$ is planar. If $n \geq 8$ then, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = \gamma(K_{\frac{n}{2}-1}) = \left\lceil \frac{(n-8)(n-10)}{48} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 0$ if and only if $n = 8$ or 10. Also, $\gamma(\mathcal{CCC}(G)) = 1$ if $n = 12$, 14 or 16; $\gamma(\mathcal{CCC}(G)) = 2$ if $n = 18$; $\gamma(\mathcal{CCC}(G)) = 3$ if $n = 20$. For $n \geq 22$ we have

$$\frac{(n-8)(n-10)}{48} \geq \frac{7}{2} = 3.5,$$

and so

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(n-8)(n-10)}{48} \right\rceil \geq 4.$$

Thus, $\mathcal{CCC}(G)$ is planar if and only if $n = 4, 6, 8, 10$; toroidal if and only if $n = 12, 14, 16$; double-toroidal if and only if $n = 18$ and triple-toroidal if and only if $n = 20$. Hence the result follows. □

Theorem 6.1.2. *Let $G = SD_{8n}$. Then*

(a) *$\mathcal{CCC}(G)$ is planar if and only if $n = 2$ or 3.*

(b) *$\mathcal{CCC}(G)$ is toroidal if and only if $n = 4$.*

(c) *$\mathcal{CCC}(G)$ is double-toroidal if and only if $n = 5$.*

(d) *$\mathcal{CCC}(G)$ is not triple-toroidal.*

(e) $\gamma(\mathcal{CCC}(G)) = \begin{cases} \left\lceil \frac{(n-3)(2n-5)}{6} \right\rceil, & \text{if } n \text{ is odd and } n \geq 7 \\ \left\lceil \frac{(n-2)(2n-5)}{6} \right\rceil, & \text{if } n \text{ is even and } n \geq 6. \end{cases}$

Proof. Consider the following cases.

Case 1. n is odd.

By Result 1.3.22 we have $\mathcal{CCC}(G) = K_4 \sqcup K_{2n-2}$. For $n \geq 3$, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = \gamma(K_4) + \gamma(K_{2n-2}) = \left\lceil \frac{(n-3)(2n-5)}{6} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 0$ if $n = 3$; $\gamma(\mathcal{CCC}(G)) = 2$ if $n = 5$. For $n \geq 7$ we have

$$\frac{(n-3)(2n-5)}{6} \geq 6,$$

and so

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(n-3)(2n-5)}{6} \right\rceil \geq 6.$$

Thus $\mathcal{CCC}(G)$ is planar if and only if $n = 3$; double-toroidal if and only if $n = 5$.

Case 2. n is even.

By Result 1.3.22 we have $\mathcal{CCC}(G) = 2K_1 \sqcup K_{2n-1}$. For $n \geq 2$, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = \gamma(K_{2n-1}) = \left\lceil \frac{(n-2)(2n-5)}{6} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 0$ if $n = 2$; $\gamma(\mathcal{CCC}(G)) = 1$ if $n = 4$. For $n \geq 6$ we have

$$\frac{(n-2)(2n-5)}{6} \geq \frac{14}{3},$$

and so

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(n-2)(2n-5)}{6} \right\rceil \geq 5.$$

Thus $\mathcal{CCC}(G)$ is planar if and only if $n = 2$; toroidal if and only if $n = 4$. Hence the result follows. □

Theorem 6.1.3. *Let $G = Q_{4m}$. Then*

(a) *$\mathcal{CCC}(G)$ is planar if and only if $m = 2, 3, 4$ or 5.*

(b) *$\mathcal{CCC}(G)$ is toroidal if and only if $m = 6, 7$ or 8.*

(c) *$\mathcal{CCC}(G)$ is double-toroidal if and only if $m = 9$.*

(d) *$\mathcal{CCC}(G)$ is triple-toroidal if and only if $m = 10$.*

(e) *$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(m-4)(m-5)}{12} \right\rceil$ for $m \geq 11$.*

Proof. By Result 1.3.19 we have

$$\mathcal{CCC}(G) = \begin{cases} K_2 \sqcup K_{m-1}, & \text{if } m \text{ is odd} \\ 2K_1 \sqcup K_{m-1}, & \text{if } m \text{ is even.} \end{cases}$$

Therefore, for $m = 2, 3$, it follows that $\mathcal{CCC}(G) = 3K_1, 2K_2$ respectively; and hence $\mathcal{CCC}(G)$ is planar. If $m \geq 4$ then, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = \gamma(K_{m-1}) = \left\lceil \frac{(m-4)(m-5)}{12} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 0$ if and only if $m = 4$ or 5. Also, $\gamma(\mathcal{CCC}(G)) = 1$ if $m = 6$, 7 or 8; $\gamma(\mathcal{CCC}(G)) = 2$ if $m = 9$; $\gamma(\mathcal{CCC}(G)) = 3$ if $m = 10$. For $m \geq 11$ we have

$$\frac{(m-4)(m-5)}{12} \geq \frac{7}{2} = 3.5,$$

and so

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(m-4)(m-5)}{12} \right\rceil \geq 4.$$

Thus, $\mathcal{CCC}(G)$ is planar if and only if $m = 2, 3, 4, 5$; toroidal if and only if $m = 6, 7, 8$; double-toroidal if and only if $m = 9$ and triple-toroidal if and only if $m = 10$. Hence the result follows. □

Theorem 6.1.4. *Let $G = V_{8n}$. Then*

(a) *$\mathcal{CCC}(G)$ is planar if and only if $n = 2$.*

(b) $\mathcal{CCC}(G)$ *is toroidal if and only if* $n = 3$ *or* 4.

(c) $\mathcal{CCC}(G)$ *is not double-toroidal.*

(d) $\mathcal{CCC}(G)$ *is triple-toroidal if and only if* $n = 5$.

(e) $$\gamma(\mathcal{CCC}(G)) = \begin{cases} \left\lceil \frac{(n-2)(2n-5)}{6} \right\rceil, & \text{if } n \text{ is odd and } n \geq 7 \\ \left\lceil \frac{(n-3)(2n-5)}{6} \right\rceil, & \text{if } n \text{ is even and } n \geq 6. \end{cases}$$

Proof. Consider the following cases.

Case 1. n is odd.

By Result 1.3.21 we have $\mathcal{CCC}(G) = 2K_1 \sqcup K_{2n-1}$. For $n \geq 3$, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = \gamma(K_{2n-1}) = \left\lceil \frac{(n-2)(2n-5)}{6} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 1$ if $n = 3$; $\gamma(\mathcal{CCC}(G)) = 3$ if $n = 5$. For $n \geq 7$ we have

$$\frac{(n-2)(2n-5)}{6} \geq \frac{15}{2} = 7.5,$$

and so

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(n-2)(2n-5)}{6} \right\rceil \geq 8.$$

Case 2. n is even.

By Result 1.3.21 we have $\mathcal{CCC}(G) = 2K_2 \sqcup K_{2n-2}$. Therefore, for $n = 2$ it follows that $\mathcal{CCC}(G) = 3K_2$; and hence $\mathcal{CCC}(G)$ is planar. If $n \geq 4$ then, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = \gamma(K_{2n-2}) = \left\lceil \frac{(n-3)(2n-5)}{6} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 1$ if $n = 4$. For $n \geq 6$ we have

$$\frac{(n-3)(2n-5)}{6} \geq \frac{7}{2} = 3.5,$$

and so

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(n-3)(2n-5)}{6} \right\rceil \geq 4.$$

Thus $\mathcal{CCC}(G)$ is planar if and only if $n = 2$; toroidal if and only if $n = 4$. Hence the result follows. □

Theorem 6.1.5. *Let* $G = U_{(n,m)}$. *Then*

(a) $\mathcal{CCC}(G)$ *is planar if and only if* $n = 2$ *and* $m = 3, 4, 5, 6$; $n = 3$ *and* $m = 3, 4$; *or* $n = 4$ *and* $m = 3, 4$.

(b) $\mathcal{CCC}(G)$ *is toroidal if and only if* $n = 2$ *and* $m = 7, 8$*; or* $n = 3$ *and* $m = 5, 6$*.*

(c) $\mathcal{CCC}(G)$ *is double-toroidal if and only if* $n = 2$ *and* $m = 9, 10$*;* $n = 4$ *and* $m = 5, 6$*;* $n = 5$ *and* $m = 3$*;* $n = 6$ *and* $m = 3$*; or* $n = 7$ *and* $m = 3$*.*

(d) $\mathcal{CCC}(G)$ *is triple-toroidal if and only if* $n = 3$ *and* $m = 7, 8$*;* $n = 5$ *and* $m = 4$*;* $n = 6$ *and* $m = 4$*; or* $n = 7$ *and* $m = 4$*.*

(e) $\gamma(\mathcal{CCC}(G)) =$

$$\begin{cases} \left\lceil \frac{(mn-n-6)(mn-n-8)}{48} \right\rceil, & \text{if } n = 2,\ m \text{ is odd and } m \geq 11 \\ \left\lceil \frac{(mn-2n-6)(mn-2n-8)}{48} \right\rceil, & \text{if } n = 2,\ m \text{ is even and } m \geq 12 \\ \left\lceil \frac{(mn-n-6)(mn-n-8)}{48} \right\rceil + \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, & \text{if } n = 3,\ m \text{ is odd and } m \geq 9; \\ & n = 4, m \geq 7; n = 5, m \geq 5; \\ & n = 6, m \geq 5; n = 7, m \geq 5; \\ & \text{or } n \geq 8, m \geq 3 \\ \left\lceil \frac{(mn-2n-6)(mn-2n-8)}{48} \right\rceil + 2\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, & \text{if } n = 3,\ m \text{ is even and } m \geq 10; \\ & n = 4, m \geq 8; n = 5, m \geq 6; \\ & n = 6, m \geq 6; n = 7, m \geq 6; \\ & \text{or } n \geq 8, m \geq 4 \end{cases}$$

Proof. Consider the following cases.

Case 1. m is odd.

By Result 1.3.20 we have $\mathcal{CCC}(G) = K_{\frac{n(m-1)}{2}} \sqcup K_n$.

Subcase 1.1 $n = 2$.

If $n = 2$ then we have $\mathcal{CCC}(G) = K_{m-1} \sqcup K_2$. Therefore, for $m = 3$ it follows that $\mathcal{CCC}(G) = 2K_2$; and hence $\mathcal{CCC}(G)$ is planar. For $m \geq 5$, by Result 1.1.4, we have

$$\gamma(\mathcal{CCC}(G)) = \gamma(K_{m-1}) = \left\lceil \frac{(m-4)(m-5)}{12} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 0$ if $m = 5$; $\gamma(\mathcal{CCC}(G)) = 1$ if $m = 7$; $\gamma(\mathcal{CCC}(G)) = 2$ if $m = 9$. For $m \geq 11$ we have

$$\frac{(m-4)(m-5)}{12} \geq \frac{7}{2} = 3.5,$$

and so

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(m-4)(m-5)}{12} \right\rceil \geq 4.$$

Thus $\mathcal{CCC}(G)$ is planar if and only if $m = 3, 5$; toroidal if and only if $m = 7$; double-toroidal if and only if $m = 9$.

Subcase 1.2 $n \geq 3$.

If $n \geq 3$ then we have $\mathcal{CCC}(G) = K_{\frac{n(m-1)}{2}} \sqcup K_n$. By Result 1.1.4, we have

$$\gamma(\mathcal{CCC}(G)) = \gamma(K_{\frac{n(m-1)}{2}}) + \gamma(K_n) = \left\lceil \frac{(mn-n-6)(mn-n-8)}{48} \right\rceil + \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 0$ if $n = 3, m = 3$ or $n = 4, m = 3$. $\gamma(\mathcal{CCC}(G)) = 1$ if $n = 3, m = 5$; $\gamma(\mathcal{CCC}(G)) = 2$ if $n = 4, m = 5$ or $n = 5, m = 3$ or $n = 6, m = 3$ or $n = 7, m = 3$; $\gamma(\mathcal{CCC}(G)) = 3$ if $n = 3, m = 7$. If $n = 3$ and $m \geq 9$ then

$$\frac{(mn-n-6)(mn-n-8)}{48} = \frac{(m-3)(3m-11)}{16} \geq 6.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-n-6)(mn-n-8)}{48} \right\rceil + \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 6.$$

If $n = 4$ and $m \geq 7$ then

$$\frac{(mn-n-6)(mn-n-8)}{48} = \frac{(2m-5)(m-3)}{6} \geq 6.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-n-6)(mn-n-8)}{48} \right\rceil + \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 6.$$

If $n = 5$ and $m \geq 5$ then

$$\frac{(mn-n-6)(mn-n-8)}{48} = \frac{(5m-11)(5m-13)}{48} \geq \frac{7}{2} = 3.5 \quad \text{and} \quad \frac{(n-3)(n-4)}{12} = \frac{1}{6}.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-n-6)(mn-n-8)}{48} \right\rceil + \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 5.$$

If $n = 6$ and $m \geq 5$ then

$$\frac{(mn-n-6)(mn-n-8)}{48} = \frac{(m-2)(3m-7)}{4} \geq 6 \quad \text{and} \quad \frac{(n-3)(n-4)}{12} = \frac{1}{2}.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-n-6)(mn-n-8)}{48} \right\rceil + \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 7.$$

If $n = 7$ and $m \geq 5$ then

$$\frac{(mn-n-6)(mn-n-8)}{48} = \frac{(7m-13)(7m-15)}{48} \geq \frac{55}{6} \quad \text{and} \quad \frac{(n-3)(n-4)}{12} = 1.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-n-6)(mn-n-8)}{48} \right\rceil + \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 11.$$

If $n \geq 8$ and $m \geq 3$ then

$$\frac{(mn-n-6)(mn-n-8)}{48} \geq \frac{(8(m-1)-6)(8(m-1)-7)}{48} \geq \frac{15}{8}$$

and

$$\frac{(n-3)(n-4)}{12} = \frac{5}{3}.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-n-6)(mn-n-8)}{48} \right\rceil + \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 4.$$

Thus $\mathcal{CCC}(G)$ is planar if and only if $n = 3, m = 3$ or $n = 4, m = 3$; toroidal if and only if $n = 3, m = 5$; double-toroidal if and only if $n = 4, m = 5$ or $n = 5, m = 3$ or $n = 6, m = 3$ or $n = 7, m = 3$; triple-toroidal if and only if $n = 3, m = 7$.

Case 2. m is even.

By Result 1.3.20 we have $\mathcal{CCC}(G) = K_{\frac{n(m-2)}{2}} \sqcup 2K_n$.

Subcase 2.1 $n = 2$.

If $n = 2$ then we have $\mathcal{CCC}(G) = K_{m-2} \sqcup 2K_2$. Therefore, for $m = 4$ it follows that $\mathcal{CCC}(G) = 3K_2$; and hence $\mathcal{CCC}(G)$ is planar. For $m \geq 6$, by Result 1.1.4, we have

$$\gamma(\mathcal{CCC}(G)) = \gamma(K_{m-2}) = \left\lceil \frac{(m-5)(m-6)}{12} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 0$ if $m = 6$; $\gamma(\mathcal{CCC}(G)) = 1$ if $m = 8$; $\gamma(\mathcal{CCC}(G)) = 2$ if $m = 10$. For $m \geq 12$ we have

$$\frac{(m-5)(m-6)}{12} \geq \frac{7}{2} = 3.5$$

and so

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(m-4)(m-5)}{12} \right\rceil \geq 4.$$

Thus $\mathcal{CCC}(G)$ is planar if and only if $m = 4, 6$; toroidal if and only if $m = 8$; double-toroidal if and only if $m = 10$.

Subcase 2.2 $n \geq 3$.

If $n \geq 3$ then we have $\mathcal{CCC}(G) = K_{\frac{n(m-2)}{2}} \sqcup 2K_n$. By Result 1.1.4, we have

$$\begin{aligned}\gamma(\mathcal{CCC}(G)) &= \gamma(K_{\frac{n(m-2)}{2}}) + \gamma(2K_n)\\ &= \left\lceil \frac{(mn-2n-6)(mn-2n-8)}{48} \right\rceil + 2\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.\end{aligned}$$

Clearly $\gamma(\mathcal{CCC}(G)) = 0$ if $n = 3, m = 4$ or $n = 4, m = 4$. $\gamma(\mathcal{CCC}(G)) = 1$ if $n = 3, m = 6$; $\gamma(\mathcal{CCC}(G)) = 2$ if $n = 4, m = 6$; $\gamma(\mathcal{CCC}(G)) = 3$ if $n = 3, m = 8$ or $n = 5, m = 4$ or $n = 6, m = 4$ or $n = 7, m = 4$. If $n = 3$ and $m \geq 10$ then

$$\frac{(mn-2n-6)(mn-2n-8)}{48} = \frac{(m-4)(3m-14)}{16} \geq 6.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-2n-6)(mn-2n-8)}{48} \right\rceil + 2\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 6.$$

If $n = 4$ and $m \geq 8$ then

$$\frac{(mn-2n-6)(mn-2n-8)}{48} = \frac{(m-4)(2m-7)}{6} \geq 6.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-2n-6)(mn-2n-8)}{48} \right\rceil + 2\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 6.$$

If $n = 5$ and $m \geq 6$ then

$$\frac{(mn-2n-6)(mn-2n-8)}{48} = \frac{(5m-16)(5m-18)}{48} \geq \frac{7}{2} = 3.5$$

and

$$\frac{(n-3)(n-4)}{12} = \frac{1}{6}.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-2n-6)(mn-2n-8)}{48} \right\rceil + 2\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 6.$$

If $n = 6$ and $m \geq 6$ then

$$\frac{(mn-2n-6)(mn-2n-8)}{48} = \frac{(m-3)(3m-10)}{4} \geq 6 \quad \text{and} \quad \frac{(n-3)(n-4)}{12} = \frac{1}{6}.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-2n-6)(mn-2n-8)}{48} \right\rceil + 2\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 8.$$

If $n = 7$ and $m \geq 6$ then

$$\frac{(mn-2n-6)(mn-2n-8)}{48} = \frac{(7m-20)(7m-22)}{48} \geq \frac{55}{6} \quad \text{and} \quad \frac{(n-3)(n-4)}{12} = 1.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-2n-6)(mn-2n-8)}{48} \right\rceil + 2\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 12.$$

If $n \geq 8$ and $m \geq 4$ then

$$\frac{(n-3)(n-4)}{12} \geq \frac{5}{3} \quad \text{and} \quad \left\lceil \frac{(mn-2n-6)(mn-2n-8)}{48} \right\rceil \geq 0.$$

Therefore

$$\gamma(\mathcal{CCC}(G)) = \left\lceil \frac{(mn-2n-6)(mn-2n-8)}{48} \right\rceil + 2\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq 4.$$

Thus $\mathcal{CCC}(G)$ is planar if and only if $n = 3, m = 4$ or $n = 4, m = 4$; toroidal if and only if $n = 3, m = 6$; double-toroidal if and only if $n = 4, m = 6$; triple-toroidal if and only if $n = 3, m = 8$ or $n = 5, m = 4$ or $n = 6, m = 4$ or $n = 7, m = 4$. Hence the result follows. □

Theorem 6.1.6. *Let $G = G(p, m, n)$. Then*

(a) *$\mathcal{CCC}(G)$ is planar if and only if $n = 1, m = 1, p = 2, 3, 5$; $n = 1, m = 2, p = 2$; $n = 1, m = 3, p = 2$; $n = 2, m = 1, p = 2$; $n = 2, m = 2, p = 2$; or $n = 3, m = 1, p = 2$.*

(b) *$\mathcal{CCC}(G)$ is not toroidal.*

(c) *$\mathcal{CCC}(G)$ is double-toroidal if and only if $n = 2, m = 1, p = 3$.*

(d) *$\mathcal{CCC}(G)$ is not triple-toroidal.*

(e) $\gamma(\mathcal{CCC}(G)) =$

$$\begin{cases}
(p+1)\left\lceil \frac{(p-4)(p-5)}{12} \right\rceil, & \textit{if } n=1, m=1, p\geq 7\\
(p+1)\left\lceil \frac{(p^2-p-3)(p^2-p-4)}{12} \right\rceil, & \textit{if } n=1, m=2, p\geq 3\\
(p+1)\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil, & \textit{if } n=1, m=3, p\geq 3\\
(p+1)\left\lceil \frac{(p^m-p^{m-1}-3)(p^m-p^{m-1}-4)}{12} \right\rceil, & \textit{if } n=1, m\geq 3, p\geq 2\\
(p^2-p)\left\lceil \frac{(p-4)(p-5)}{12} \right\rceil + 2\left\lceil \frac{(p^2-p-3)(p^2-p-4)}{12} \right\rceil, & \textit{if } n=2, m=1, p\geq 5\\
(p^2-p)\left\lceil \frac{(p^2-p-3)(p^2-p-4)}{12} \right\rceil & \\
\qquad +2\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil, & \textit{if } n=2, m=2, p\geq 3\\
(p^2-p)\left\lceil \frac{(p^{m-1}(p-1)-3)(p^{m-1}(p-1)-4)}{12} \right\rceil & \\
\qquad +2\left\lceil \frac{(p^m(p-1)-3)(p^m(p-1)-4)}{12} \right\rceil, & \textit{if } n=2, m\geq 3, p\geq 2\\
36, & \textit{if } n=3, m=1, p=3\\
p^2(p-1)\left\lceil \frac{(p-4)(p-5)}{12} \right\rceil + 2\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil, & \textit{if } n=3, m=1, p\geq 5\\
4, & \textit{if } n=3, m=2, p=2\\
p^2(p-1)\left\lceil \frac{(p^m-p^{m-1}-3)(p^m-p^{m-1}-4)}{12} \right\rceil & \\
\qquad +2\left\lceil \frac{(p^{m+2}-p^{m+1}-3)(p^{m+2}-p^{m+1}-4)}{12} \right\rceil, & \textit{if } n=3, m=2, p\geq 3\\
 & \textit{or } n=3, m\geq 3, p\geq 2\\
2\left\lceil \frac{(p^{n-1}(p^m-p^{m-1})-3)(p^{n-1}(p^m-p^{m-1})-4)}{12} \right\rceil, & \textit{if } n\geq 4, m\geq 1, p\geq 2\\
 & \textit{and } p^m-p^{m-1}\leq 4\\
(p^n-p^{n-1})\left\lceil \frac{(p^m-p^{m-1}-3)(p^m-p^{m-1}-4)}{12} \right\rceil & \\
\qquad +2\left\lceil \frac{(p^{n-1}(p^m-p^{m-1})-3)(p^{n-1}(p^m-p^{m-1})-4)}{12} \right\rceil, & \textit{if } n\geq 4, m\geq 1, p\geq 2\\
 & \textit{and } p^m-p^{m-1}\geq 5.
\end{cases}$$

Proof. By Result 1.3.23 we have

$$\mathcal{CCC}(G) = (p^n - p^{n-1})K_{p^{m-n}(p^n - p^{n-1})} \sqcup K_{p^{n-1}(p^m - p^{m-1})} \sqcup K_{p^{m-1}(p^n - p^{n-1})}.$$

Consider the following cases.

Case 1. $n = 1$.

We have $\mathcal{CCC}(G) = (p+1)K_{p^{m-1}(p-1)}$. For $m = 1$ and $p = 2, 3$, it follows that $\mathcal{CCC}(G) = 2K_1$ or $3K_2$ which is planar. If $m = 1$ and $p \geq 5$, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = (p+1)\gamma(K_{p-1}) = (p+1)\left\lceil \frac{(p-4)(p-5)}{12} \right\rceil.$$

Clearly $\gamma(\mathcal{CCC}(G)) = 0$ for $p = 5$. If $p \geq 7$ then

$$\frac{(p-4)(p-5)}{12} \geq \frac{1}{2}$$

and so

$$\gamma(\mathcal{CCC}(G)) = (p+1)\left\lceil \frac{(p-4)(p-5)}{12} \right\rceil \geq 8.$$

If $m = 2$ and $p = 2$ then $\gamma(\mathcal{CCC}(G)) = 3\gamma(K_2) = 0$. For $m = 2$ and $p \geq 3$, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = (p+1)\gamma(K_{p(p-1)}) = (p+1)\left\lceil \frac{(p^2-p-3)(p^2-p-4)}{12} \right\rceil.$$

If $p \geq 3$ then

$$\frac{(p^2-p-3)(p^2-p-4)}{12} \geq \frac{1}{2}$$

and so

$$\gamma(\mathcal{CCC}(G)) = (p+1)\left\lceil \frac{(p^2-p-3)(p^2-p-4)}{12} \right\rceil \geq 4.$$

If $m = 3$ then $\gamma(\mathcal{CCC}(G)) = (p+1)\gamma(K_{p^2(p-1)})$. Therefore, if $m = 3$ and $p \geq 2$ then by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = (p+1)\gamma(K_{p^2(p-1)}) = (p+1)\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil.$$

Clearly if $m = 3$ and $p = 2$ then $\gamma(\mathcal{CCC}(G)) = 0$. If $p \geq 3$ then

$$\frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \geq \frac{35}{2}$$

and so

$$\gamma(\mathcal{CCC}(G)) = (p+1)\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil \geq 72.$$

If $m \geq 4$ and $p \geq 2$ then $\gamma(\mathcal{CCC}(G)) = (p+1)\gamma(K_{p^{m-1}(p-1)})$. Therefore, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = (p+1)\gamma(K_{p^{m-1}(p-1)}) = (p+1)\left\lceil \frac{(p^m - p^{m-1} - 3)(p^m - p^{m-1} - 4)}{12} \right\rceil.$$

We have

$$\frac{(p^m - p^{m-1} - 3)(p^m - p^{m-1} - 4)}{12} \geq \frac{20}{12}$$

and so

$$\gamma(\mathcal{CCC}(G)) = (p+1)\left\lceil \frac{(p^m - p^{m-1} - 3)(p^m - p^{m-1} - 4)}{12} \right\rceil \geq 6.$$

Therefore, $\mathcal{CCC}(G)$ is planar if and only if $n = 1, m = 1, p = 2, 3, 5$; $n = 1, m = 2, p = 2$; or $n = 1, m = 3, p = 2$. Also, in this case, $\mathcal{CCC}(G)$ is neither toroidal, double-torodal nor triple-toroidal.

Case 2. $n = 2$.

We have $\mathcal{CCC}(G) = (p^2 - p)K_{p^{m-1}(p-1)} \sqcup 2K_{p^m(p-1)}$. For $m = 1$ and $p = 2$, it follows that $\mathcal{CCC}(G) = 2K_1 \sqcup 2K_2$ which is planar. If $m = 1$ and $p = 3$ then, by Result 1.1.4 and (1.1.b), we have

$$\gamma(\mathcal{CCC}(G)) = 2\gamma(K_6) = 2.$$

If $m = 1$ and $p \geq 5$, by Result 1.1.4 and (1.1.b), we have

$$\begin{aligned}\gamma(\mathcal{CCC}(G)) &= (p^2 - p)\gamma(K_{p-1}) + 2\gamma(K_{p(p-1)})\\ &= (p^2 - p)\left\lceil \frac{(p-4)(p-5)}{12} \right\rceil + 2\left\lceil \frac{(p^2 - p - 3)(p^2 - p - 4)}{12} \right\rceil.\end{aligned}$$

Since $p \geq 5$ then

$$\frac{(p^2 - p - 3)(p^2 - p - 4)}{12} \geq \frac{68}{3}$$

and so

$$\gamma(\mathcal{CCC}(G)) \geq 2\left\lceil \frac{(p^2 - p - 3)(p^2 - p - 4)}{12} \right\rceil \geq 46.$$

If $m = 2$ and $p \geq 2$ then $\mathcal{CCC}(G) = (p^2 - p)K_{p(p-1)} \sqcup 2K_{p^2(p-1)}$. Therefore, if $p = 2$ then $\mathcal{CCC}(G) = 2K_2 \sqcup 2K_4$ hence by (1.1.b) we have $\gamma(\mathcal{CCC}(G)) = 2\gamma(K_4) = 0$. If $p \geq 3$, by Result 1.1.4 and (1.1.b), we have

$$\begin{aligned}\gamma(\mathcal{CCC}(G)) &= (p^2 - p)\gamma(K_{p(p-1)}) + 2\gamma(K_{p^2(p-1)})\\ &= (p^2 - p)\left\lceil \frac{(p^2 - p - 3)(p^2 - p - 4)}{12} \right\rceil + 2\left\lceil \frac{(p^3 - p^2 - 3)(p^3 - p^2 - 4)}{12} \right\rceil.\end{aligned}$$

Also, $\frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \geq \frac{35}{2}$ and so

$$\gamma(\mathcal{CCC}(G)) \geq 2\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil \geq 36.$$

If $m \geq 3$ and $p \geq 2$ then

$$\begin{aligned}\gamma(\mathcal{CCC}(G)) &= (p^2-p)\gamma(K_{p^{m-1}(p-1)}) + 2\gamma(K_{p^m(p-1)}) \\ &= (p^2-p)\left\lceil \frac{(p^{m-1}(p-1)-3)(p^{m-1}(p-1)-4)}{12} \right\rceil \\ &\quad + 2\left\lceil \frac{(p^m(p-1)-3)(p^m(p-1)-4)}{12} \right\rceil \geq 4.\end{aligned}$$

Therefore, $\mathcal{CCC}(G)$ is planar if and only if $n = 2, m = 1, p = 2$; $n = 2, m = 2, p = 2$; or $n = 3; m = 1; p = 2$ and double-toroidal if and only if $n = 2, m = 1, p = 3$. In this case, $\mathcal{CCC}(G)$ is neither toroidal nor triple-toroidal.

Case 3. $n = 3$.

We have $\mathcal{CCC}(G) = p^2(p-1)K_{p^{m-1}(p-1)} \sqcup 2K_{p^{m+1}(p-1)}$. If $m = 1$ and $p = 2$ then $\mathcal{CCC}(G) = 4K_1 \sqcup 2K_4$, and so by Result 1.1.4 and (1.1.b) $\gamma(\mathcal{CCC}(G)) = 2\gamma(K_4) = 0$. For $p = 3$ we have $\mathcal{CCC}(G) = 18K_2 \sqcup 2K_{18}$. Therefore, by Result 1.1.4 and (1.1.b) we have $\gamma(\mathcal{CCC}(G)) = 2\gamma(K_{18}) = 36$. For $p \geq 5$, by Result 1.1.4 and (1.1.b) we have

$$\begin{aligned}\gamma(\mathcal{CCC}(G)) &= p^2(p-1)\gamma(K_{p-1}) + 2\gamma(K_{p^2(p-1)}) \\ &= p^2(p-1)\left\lceil \frac{(p-4)(p-5)}{12} \right\rceil + 2\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil > 36.\end{aligned}$$

If $m = 2$ and $p = 2$ then we have $\mathcal{CCC}(G) = 4K_2 \sqcup 2K_8$. By Result 1.1.4 and (1.1.b) we have

$$\gamma(\mathcal{CCC}(G)) = 2\gamma(K_8) = 4.$$

If $m = 2$ and $p \geq 3$ or $m \geq 3$ and $p \geq 2$ then we have $\mathcal{CCC}(G) = p^2(p-1)K_{p^{m-1}(p-1)} \sqcup 2K_{p^{m+1}(p-1)}$. By Result 1.1.4 and (1.1.b) we have

$$\begin{aligned}\gamma(\mathcal{CCC}(G)) &= p^2(p-1)\gamma(K_{p^{m-1}(p-1)}) + 2\gamma(K_{p^{m+1}(p-1)}) \\ &= p^2(p-1)\left\lceil \frac{(p^{m-1}(p-1)-3)(p^{m-1}(p-1)-4)}{12} \right\rceil \\ &\quad + 2\left\lceil \frac{(p^{m+1}(p-1)-3)(p^{m+1}(p-1)-4)}{12} \right\rceil.\end{aligned}$$

We have

$$\frac{(p^{m+1}(p-1)-3)(p^{m+1}(p-1)-4)}{12} \geq \frac{5}{3}$$

and so

$$\gamma(\mathcal{CCC}(G)) \geq 2\left\lceil \frac{(p^{m+1}(p-1)-3)(p^{m+1}(p-1)-4)}{12} \right\rceil \geq 4.$$

Therefore, $\mathcal{CCC}(G)$ is planar if and only if $n = 3, m = 1, p = 2$. Also, in this case, $\mathcal{CCC}(G)$ is neither toroidal, double-toridal nor triple-toroidal.

Case 4. $n \geq 4$.

We have $\mathcal{CCC}(G) = (p^n - p^{n-1})K_{p^m - p^{m-1}} \sqcup 2K_{p^{n-1}(p^m - p^{m-1})}$. Therefore, by Result 1.1.4, we have

$$\gamma(\mathcal{CCC}(G)) = (p^n - p^{n-1})\gamma(K_{p^m-p^{m-1}}) + 2\gamma(K_{p^{n-1}(p^m-p^{m-1})}) \qquad (6.1.a)$$

For $m \geq 1$ and $p \geq 2$ we have

$$\gamma(K_{p^{n-1}(p^m-p^{m-1})}) \geq \gamma(K_{p^{n-1}}) \geq \gamma(K_8) = 2,$$

noting that K_8 and $K_{p^{n-1}}$ are subgraphs of $K_{p^{n-1}}$ and $K_{p^{n-1}(p^m-p^{m-1})}$ respectively. Therefore

$$\gamma(\mathcal{CCC}(G)) \geq 2\gamma(K_{p^{n-1}(p^m-p^{m-1})}) \geq 4.$$

Further, if $p^m - p^{m-1} \leq 4$ then, by (6.1.a) and (1.1.b), we have

$$\begin{aligned}\gamma(\mathcal{CCC}(G)) &= 2\gamma(K_{p^{n-1}(p^m-p^{m-1})}) \\ &= 2\left\lceil \frac{(p^{n-1}(p^m-p^{m-1})-3)(p^{n-1}(p^m-p^{m-1})-4)}{12} \right\rceil.\end{aligned}$$

If $p^m - p^{m-1} \geq 5$ then, by (6.1.a) and (1.1.b), we have

$$\begin{aligned}\gamma(\mathcal{CCC}(G)) = (p^n - p^{n-1})&\left\lceil \frac{(p^m-p^{m-1}-3)(p^m-p^{m-1}-4)}{12} \right\rceil + \\ 2&\left\lceil \frac{(p^{n-1}(p^m-p^{m-1})-3)(p^{n-1}(p^m-p^{m-1})-4)}{12} \right\rceil.\end{aligned}$$

Hence the result follows. □

We conclude this chapter with the following characterization of $\mathcal{CCC}(G)$ for the groups considered above.

Corollary 6.1.7. *Let $G = D_{2n}$, SD_{8n}, Q_{4m}, V_{8n}, $U_{(n,m)}$ or $G(p,m,n)$. Then*

(a) *$\mathcal{CCC}(G)$ is planar if and only if $G = D_6, D_8, D_{10}, D_{12}, D_{14}, D_{16}$, D_{18}, D_{20}, SD_{16}, SD_{24}, $Q_8, Q_{12}, Q_{16}, Q_{20}, V_{16}, U_{(2,2)}$, $U_{(2,3)}$, $U_{(2,4)}$, $U_{(2,5)}$, $U_{(2,6)}$, $U_{(3,2)}$, $U_{(3,3)}$, $U_{(3,4)}$, $U_{(4,2)}$, $U_{(4,3)}$, $U_{(4,4)}$, $G(2,1,1)$, $G(3,1,1)$, $G(5,1,1)$, $G(2,2,1)$, $G(2,3,1)$, $G(2,1,2)$, $G(2,2,2)$ or $G(2,1,3)$.*

(b) $\mathcal{CCC}(G)$ *is toroidal if and only if* $G = D_{22}$, D_{24}, D_{26}, D_{28}, D_{30}, D_{32}, SD_{32}, Q_{24}, Q_{28}, Q_{32}, V_{24}, V_{32}, $U_{(2,7)}$, $U_{(2,8)}$, $U_{(3,5)}$ *or* $U_{(3,6)}$.

(c) $\mathcal{CCC}(G)$ *is double-toroidal if and only if* $G = D_{34}$, D_{36}, SD_{40}, Q_{36}, $U_{(2,9)}$, $U_{(2,10)}$, $U_{(4,5)}$, $U_{(4,6)}$, $U_{(5,2)}$, $U_{(5,3)}$, $U_{(6,2)}$, $U_{(6,3)}$, $U_{(7,2)}$, $U_{(7,3)}$ *or* $G(3,1,2)$.

(d) $\mathcal{CCC}(G)$ *is triple-toroidal if and only if* $G = D_{38}$, D_{40}, Q_{40}, V_{40}, $U_{(3,7)}$, $U_{(3,8)}$, $U_{(5,4)}$, $U_{(6,4)}$ *or* $U_{(7,4)}$.