Chapter 7

Solvable conjugacy class graph of groups

Extending the notion of CCC-graph, in 2017 Mohammadian and Erfanian [73] introduced NCC-graph. In this chapter, we further extend the notions of CCC-graph and NCC-graph and introduce the *solvable conjugacy class graph* (abbreviated as SCC-graph) of G. The SCC-graph of a group G is a simple undirected graph, denoted by $\mathcal{SCC}(G)$, with vertex set $\{x^G : 1 \neq x \in G\}$ and two distinct vertices x^G and y^G are adjacent if there exist two elements $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is solvable. It is clear that the NCC-graph is a spanning subgraph of the SCC-graph of G.

In Section 7.1, We shall discuss certain properties regarding connectedness, diameter, domination number and girth of SCC-graph. In Section 7.2, we shall obtain some results on distance between two vertices of SCC-graph for some locally finite groups. In Section 7.3, we shall discuss properties of genus and crosscap of SCC-graph of S_n and A_n and determine all positive integer n such that $\mathcal{SCC}(S_n)$ and $\mathcal{SCC}(A_n)$ are planar or toroidal. We shall conclude this chapter by obtaining a relation between $\gamma(\mathcal{SCC}(G))$ and $\Pr(G)$. This chapter is based on our paper [17].

7.1 Certain properties of SCC-graph

We begin with a simple observation. Let a and b be two elements of G such that a^G and b^G are joined in the SCC-graph of G. This means that there exist $a' \in a^G$ and $b' \in b^G$ such that $\langle a', b' \rangle$ is solvable. Without loss of generality, we can assume that $a' = a$. For suppose that $(a')^h = a$. Then $\langle a', b'\rangle^h = \langle a, (b')^h\rangle$ is solvable, since it is a conjugate of (and hence isomorphic to) $\langle a', b'\rangle$.

Theorem 7.1.1. *Let G be a finite group. Then the SCC-graph of G is complete if and only if G is solvable.*

Proof. If G is solvable, $\langle x, y\rangle$ is also solvable for all $x, y \in G$. In particular, if a^G, b^G are two vertices of $\mathcal{SCC}(G)$ and $x \in a^G$, $y \in b^G$ then $\langle x, y\rangle$ is solvable. Therefore, a^G and b^G are adjacent. Hence, $\mathcal{SCC}(G)$ is a complete graph.

Conversely, suppose that $\mathcal{SCC}(G)$ is complete. Then, by the observation at the end of the last section, for every $a, b \in G$, there is a conjugate b' of b such that $\langle a, b'\rangle$ is solvable. By Result 1.2.12, we conclude that G is solvable. □

Next we turn to the questions of connectedness and diameter. The girth will be discussed in the next section, but we begin with a simple observation.

Proposition 7.1.2. *Let G be a non-solvable group such that it has an element of order pq, where p, q are primes. If $p \neq q$ then $\text{girth}(\mathcal{SCC}(G)) = 3$ and hence $\mathcal{SCC}(G)$ is not a tree.*

Proof. Let $a \in G$ be an element of order pq. If $p \neq q$ then $o(a^q) = p$ and $o(a^p) = q$. Also, $\langle a, a^q\rangle, \langle a^q, a^p\rangle$ and $\langle a^p, a\rangle$ are abelian groups. Since a^G, $(a^q)^G$ and $(a^p)^G$ are distinct, we have the following triangle

$$a \sim a^q \sim a^p \sim a$$

in $\mathcal{SCC}(G)$. Therefore, $\text{girth}(\mathcal{SCC}(G)) = 3$ and hence $\mathcal{SCC}(G)$ is not a tree. □

Proposition 7.1.3. *Let $x \in G \setminus \{1\}$ and $a, b \in \text{Sol}_G(x) \setminus \{1\}$. Then a^G and b^G are connected and $d(a^G, b^G) \leq 2$. In particular, if $\text{Sol}(G) \neq \{1\}$ then $\mathcal{SCC}(G)$ is connected and $\text{diam}(\mathcal{SCC}(G)) \leq 2$.*

Proof. Since $a, b \in \text{Sol}_G(x) \setminus \{1\}$, $\langle a, x\rangle$ and $\langle x, b\rangle$ are solvable. Therefore, $d(a^G, x^G) \leq 1$ and $d(x^G, b^G) \leq 1$. Hence, the result follows.

If $\text{Sol}(G) \neq \{1\}$ then there exists an element $z \in G$ such that $z \neq 1$ and $z \in \text{Sol}(G)$. Therefore, $z \in \text{Sol}_G(w)$ for all $w \in G \setminus \{1\}$. Let u^G and v^G be any two vertices of $\mathcal{SCC}(G)$. Then $u, v \in \text{Sol}_G(z) \setminus \{1\}$. Therefore, by the first part it follows that $d(u^G, v^G) \leq 2$. Hence, $\text{diam}(\mathcal{SCC}(G)) \leq 2$. □

Remark 7.1.4. For any two distinct vertices $x^G, y^G \in V(\mathcal{SCC}(G))$, $x^G \sim y^G$ if and only if $\text{Sol}_G(gxg^{-1}) \cap y^G \neq \emptyset$ for all $g \in G$. Also, x^G is an isolated vertex if and only if $\text{Sol}_G(gxg^{-1}) \subseteq x^G \cup \{1\}$ for all $g \in G$.

Theorem 7.1.5. *If G, H are arbitrary non-trivial groups then the graph $\mathcal{SCC}(G \times H)$ is connected and* $\operatorname{diam}(\mathcal{SCC}(G \times H)) \leq 3$. *In particular, $\mathcal{SCC}(G \times G)$ is connected and* $\operatorname{diam}(\mathcal{SCC}(G \times G)) \leq 3$. *Further,* $\operatorname{diam}(\mathcal{SCC}(G \times G)) = 3$ *if and only if either $\mathcal{SCC}(G)$ is disconnected or $\mathcal{SCC}(G)$ is connected with* $\operatorname{diam}(\mathcal{SCC}(G)) \geq 3$.

Proof. Let (x, y) and (u, v) be two non-trivial elements of $G \times H$. Without any loss we may assume that $x \neq 1_G$ and $v \neq 1_H$, where 1_G and 1_H are identity elements of G and H respectively, then

$$(x, y)^{G \times H} \sim (x, 1_H)^{G \times H} \sim (1_G, v)^{G \times H} \sim (u, v)^{G \times H}.$$

This shows that $\mathcal{SCC}(G \times H)$ is connected and $\operatorname{diam}(\mathcal{SCC}(G \times H)) \leq 3$. Putting $H = G$, it follows that $\mathcal{SCC}(G \times G)$ is connected and $\operatorname{diam}(\mathcal{SCC}(G \times G)) \leq 3$.

Let $\operatorname{diam}(\mathcal{SCC}(G \times G)) = 3$. Suppose that $\mathcal{SCC}(G)$ is connected and $\operatorname{diam}(\mathcal{SCC}(G)) \leq 2$ (on the contrary). Let $(x, y), (u, v)$ be two vertices in $\mathcal{SCC}(G \times G)$. Without any loss we may assume that $x, u \neq 1_G$. Since $\mathcal{SCC}(G)$ is connected and $\operatorname{diam}(\mathcal{SCC}(G)) \leq 2$, there exist $a \in G \setminus \{1_G\}$ such that $x^G \sim a^G \sim u^G$. Therefore, $\langle x^f, a^g \rangle$ and $\langle a^h, u^w \rangle$ are solvable for some $f, g, h, w \in G$. We have $\langle (x, y)^{(f,c)}, (a, 1_G)^{(g,d)} \rangle = \langle x^f, a^g \rangle \times \langle y^c \rangle$, where $c, d \in G$. Since $\langle x^f, a^g \rangle$ and $\langle y^c \rangle$ are solvable, $(x, y)^{G \times G} \sim (a, 1_G)^{G \times G}$. Similarly, $(u, v)^{G \times G} \sim (a, 1_G)^{G \times G}$. Thus we get the following path

$$(x, y)^{G \times G} \sim (a, 1_G)^{G \times G} \sim (u, v)^{G \times G}.$$

Therefore, $\operatorname{diam}(\mathcal{S}(G \times G)) \leq 2$, which is a contradiction. Hence, $\mathcal{SCC}(G)$ is disconnected or $\mathcal{SCC}(G)$ is connected with $\operatorname{diam}(\mathcal{SCC}(G)) \geq 3$.

Suppose that either $\mathcal{SCC}(G)$ is disconnected or it is connected with $\operatorname{diam}(\mathcal{S}(G)) \geq 3$. Then there exist two distinct elements $x, y \in G \setminus \{1_G\}$ such that either x^G, y^G are not connected or $d(x^G, y^G) \geq 3$. We are to show that $\operatorname{diam}(\mathcal{SCC}(G \times G)) = 3$. Suppose that $\operatorname{diam}(\mathcal{SCC}(G \times G)) \leq 2$. Consider the following two cases.

Case 1. $\mathcal{SCC}(G)$ is disconnected.

Let u^G and v^G be any two distinct vertices in $\mathcal{SCC}(G)$. Then $d((u, 1_G)^{G \times G}, (v, 1_G)^{G \times G}) = 1$ or 2. If $d((u, 1_G)^{G \times G}, (v, 1_G)^{G \times G}) = 1$ then $(u, 1_G)^{G \times G} \sim (v, 1_G)^{G \times G}$. Therefore, $\langle u^f, v^w \rangle$ is solvable for some $f, w \in G$. Therefore, $u^G \sim v^G$ and so $d(u^G, v^G) = 1$; a contradiction. If $d((u, 1_G)^{G \times G}, (v, 1_G)^{G \times G}) = 2$ then there exists a non-identity element $(a, b) \in G \times G$ such that

$$(u, 1_G)^{G \times G} \sim (a, b)^{G \times G} \sim (v, 1_G)^{G \times G}.$$

It follows that $\langle u^f, a^g\rangle$ and $\langle a^h, v^w\rangle$ are solvable for some $f, g, h, w \in G$ and so

$$u^G \sim a^G \sim v^G.$$

Thus u^G, v^G are connected and $d(u^G, v^G) \leq 2$, a contradiction.

Case 2. $\mathcal{SCC}(G)$ is connected with $\operatorname{diam}(\mathcal{SCC}(G)) \geq 3$.

Proceeding as in Case 1, we get $d(u^G, v^G) \leq 2$ for any two distinct vertices u^G and v^G in $\mathcal{SCC}(G)$. Therefore, $\operatorname{diam}(\mathcal{SCC}(G)) = 2$; a contradiction.

Thus, from Case 1 and Case 2, we get $\operatorname{diam}(\mathcal{SCC}(G \times G)) \geq 3$. Hence, $\mathcal{SCC}(G \times G) = 3$. □

Proposition 7.1.6. *Let G be a non-solvable group. Then the domination number of SCC-graph, $\lambda(\mathcal{SCC}(G)) = 1$ if $|\operatorname{Sol}(G)| \neq 1$.*

Proof. Let x be a non-trivial element in $\operatorname{Sol}(G)$. Then $x^G \in V(\mathcal{SCC}(G))$. Let $y^G \in V(\mathcal{SCC}(G)) \setminus \{x^G\}$ be an arbitrary vertex. Then $\langle x, y\rangle$ is solvable. Therefore, x^G and y^G are adjacent. Hence, $\{x^G\}$ is a dominating set of $\mathcal{SCC}(G)$ and so $\lambda(\mathcal{SCC}(G)) = 1$. □

Theorem 7.1.7. *Let G be a finite group. If G has an element of order $n = \Pi_{i=1}^{m} p_i^{k_i}$, where p_i's are distinct primes. Then $\mathcal{SCC}(G)$ has a clique of size $\Pi_{i=1}^{m}(k_i + 1) - 1$.*

Proof. Let $x \in G$ be an element of order n. Then $(x^r)^G \sim (x^s)^G$ for all proper divisors r, s of n. Since total number of proper divisors of $n = \Pi_{i=1}^{m} p_i^{k_i}$ is $\Pi_{i=1}^{m}(k_i + 1) - 1$, we get a clique in $\mathcal{SCC}(G)$ of size $\Pi_{i=1}^{m}(k_i + 1) - 1$. □

We conclude this section with the following result.

Theorem 7.1.8. *With the exception of the cyclic groups of orders 1, 2 and 3 and the symmetric group of degree 3, every finite group G has the property that $\mathcal{SCC}(G)$ contains a triangle (that is, has girth 3).*

Proof. If G is solvable then $k(G) = \omega(\mathcal{SCC}(G)) + 1$ (the extra 1 coming from the identity of G), so G has at most three conjugacy classes. The groups listed in the theorem are all those having this property.

So we may assume that G is non-solvable. If G has an element whose order is not a prime power then some power (say g) of this element has order pq, where p and q are distinct primes. Then $\mathcal{SCC}(G)$ contains a clique of size 3, by Theorem 7.1.7.

So we may further assume that every element of G has prime power order.

These groups were first studied by Higman [63] in 1957; Suzuki [89] determined the simple groups with this property in 1965. Subsequently all such groups have been classified [26, 61]. The story is somewhat tangled, perhaps due to the lack of a common name for the class. Subsequently two names were proposed; a group with this property is called a *CP group* by some authors, and an *EPPO group* by others. These groups have arisen in connection with other graphs defined on groups, including the Gruenberg–Kegel graph (or prime graph) and the power graph: see [30]. The result we require is that a non-solvable group in which every element has prime power order satisfies one of the following:

(a) G is one of A_6, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,17)$, M_{10} or $\mathrm{PSL}(3,4)$;

(b) G has a nornal subgroup N such that G/N is $\mathrm{PSL}(2,4)$, $\mathrm{PSL}(2,8)$, $\mathrm{Sz}(8)$ or $\mathrm{Sz}(32)$, and N is a direct sum of copies of the natural G/N-module over its field of definition.

Suppose first that we are in case (b). If we can find a triangle in the solvable conjugacy class group of G/N then it lifts to a triangle in $\mathcal{SCC}(G)$. So it is enough to add the four possibilities for G/N to the list of groups in case (a).

In $\mathrm{Sz}(8)$, there are three conjugacy classes of elements of order 13, all represented in a cyclic subgroup of order 13, giving us a triangle. Similar arguments apply to $\mathrm{Sz}(32)$ (using an element of order 41), $\mathrm{PSL}(2,8)$ (order 7), and $\mathrm{PSL}(2,17)$ (order 3 and two classes of order 9). In $\mathrm{PSL}(2,4)$, a dihedral subgroup of order 10 meets two conjugacy classes of elements of order 5 and one class of involutions. A similar argument applies to A_6 (using a dihedral group of order 10), $\mathrm{PSL}(2,7)$ (using a non-abelian group of order 21) $\mathrm{PSL}(3,4)$ (a non-abelian group of order 21) and M_{10} (a quaternion group of order 8 meets two conjugacy classes of elements of order 4 and one class of involutions). All this information is easily obtained from the $\mathbb{ATLAS}$ *of Finite Groups* [32]. □

7.2 Distance in SCC-Graph for locally finite group

A locally finite group is a group for which every finitely generated subgroup is finite. An element of a group is said to be a p-element if the order of the element is a power of p, where p is a prime. In this section we obtain some results on distance between two vertices of $\mathcal{SCC}(G)$ for some locally finite groups, analogous to the Results 1.3.24 - 1.3.29.

Proposition 7.2.1. *Let G be a locally finite group. If $x, y \in G \setminus \{1\}$ are p-elements, where p is a prime, then $d(x^G, y^G) \leq 1$.*

Proof. Since G is a locally finite group and $x, y \in G \setminus \{1\}$ are p-elements, the subgroup $\langle x, y\rangle$ is finite. Let P be a Sylow p-subgroup of $\langle x, y\rangle$ containing x. Then $y^g = gyg^{-1} \in P$ for some $g \in G$ since all the Sylow p-subgroups are conjugate. Therefore, $\langle x, y^g\rangle$ is solvable and so $d(x^G, y^G) \leq 1$. □

Proposition 7.2.2. *Let G be a locally finite group. If $x, y \in G$ are of non-coprime orders then $d(x^G, y^G) \leq 3$. If either x or y is of prime order then $d(x^G, y^G) \leq 2$.*

Proof. Let $o(x) = pm$ and $o(y) = pn$, where p is a prime and m, n are positive integers. Then x^m and y^n are non-trivial p-elements of G. Therefore, by Proposition 7.2.1, we have

$$d((x^m)^G, (y^n)^G) \leq 1.$$

Clearly, $d(x^G, (x^m)^G) \leq 1$ and $d((y^n)^G, y^G) \leq 1$. Therefore, if $x^G \neq y^G$ then $x^G \sim (x^m)^G \sim (y^n)^G \sim y^G$ is a path from x^G to y^G. Hence, $d(x^G, y^G) \leq 3$.

Suppose that $o(x) = pm$ and $o(y) = p$. Then x^m and y are non-trivial p-elements of G. Therefore, by Proposition 7.2.1, we have

$$d((x^m)^G, y^G) \leq 1.$$

Thus $x^G \sim (x^m)^G \sim y^G$ is a path from x^G to y^G. Hence, $d(x^G, y^G) \leq 2$. □

Proposition 7.2.3. *Let G be a locally finite group and $x, y \in G$. Suppose p and q are prime divisors of $o(x)$ and $o(y)$, respectively, and that G has an element of order pq. Then*

(a) *$d(x^G, y^G) \leq 5$, and moreover $d(x^G, y^G) \leq 4$ if either x or y is of prime power order.*

(b) *If either a Sylow p-subgroup or a Sylow q-subgroup of G is a cyclic or generalized quaternion finite group then $d(x^G, y^G) \leq 4$. Moreover, $d(x^G, y^G) \leq 3$ if either x or y is of prime order.*

(c) *If both Sylow p-subgroup and Sylow q-subgroup of G are either cyclic or generalized quaternion finite groups then $d(x^G, y^G) \leq 3$. Moreover, $d(x^G, y^G) \leq 2$ if either x or y is of prime order.*

Proof. Let $o(x) = pm$ and $o(y) = qn$ for some positive integers m, n. Let $a \in G$ be an element of order pq. Then $o(a^q) = p$ and $o(a^p) = q$. Also, a^p commutes with a^q.

(a) We have

$$d(x^G, (x^m)^G) \leq 1, \; d((a^q)^G, (a^p)^G) = 1, \text{ and } d((y^n)^G, y^G) \leq 1.$$

Since $o(x^m) = o(a^q) = o(y^n) = p$, by Proposition 7.2.1, we have

$$d((x^m)^G, (a^q)^G) \leq 1 \text{ and } d((a^p)^G, (y^n)^G) \leq 1.$$

Therefore, $d(x^G, y^G) \leq 5$.

If $o(x) = p^s$ for some positive integer s then, by Proposition 7.2.1, we have $d(x^G, (a^q)^G) \leq 1$. Similarly, if $o(y) = q^t$ for some positive integer t then $d(y^G, (a^p)^G) \leq 1$. Therefore, $d(x^G, y^G) \leq 4$.

(b) Without any loss of generality assume that Sylow p-subgroup of G is either a cyclic group or a generalized quaternion finite group. Let P and Q be two Sylow p-subgroups of G containing x^m and a^q respectively. Since P is finite, by Result 1.2.5, Q is also finite and $P = gQg^{-1}$ for some $g \in G$ and so $ga^qg^{-1} \in P$. Therefore, $\langle x^m \rangle$ and $\langle ga^qg^{-1} \rangle$ are subgroups of P having order p. Since P is cyclic or a generalized quaternion group, by Result 1.2.6, we have $\langle x^m \rangle = \langle ga^qg^{-1} \rangle$. Therefore, $ga^qg^{-1} = (x^m)^i$ for some integer i and so $\langle x, ga^qg^{-1} \rangle = \langle x, (x^m)^i \rangle = \langle x \rangle$. Hence $d(x^G, (a^q)^G) \leq 1$. We also have

$$d((a^q)^G, (a^p)^G) = 1, \; d((a^p)^G, (y^n)^G) \leq 1, \text{ and } d((y^n)^G, y^G) \leq 1.$$

Thus $d(x^G, y^G) \leq 4$.

If $o(x) = p$ then $\langle x \rangle = \langle ga^qg^{-1} \rangle$. Therefore, $x = ga^{qt}g^{-1}$ for some integer t. We have $x^G = (a^{qt})^G$ and so $\langle a^{qt}, a^p \rangle$ is abelian. Hence, $d(x^G, (a^p)^G) \leq 1$ and so $d(x^G, y^G) \leq 3$.

(c) If both Sylow p-subgroup and Sylow q-subgroup of G are either cyclic or generalized quaternion finite groups then proceeding as part (b) we get

$$d(x^G, (a^q)^G) \leq 1, \; d((a^q)^G, (a^p)^G) = 1, \text{ and } d((a^p)^G, y^G) \leq 1.$$

Therefore, $d(x^G, y^G) \leq 3$.

If $o(x) = p$ then proceeding as in part (b), we have $d(x^G, (a^p)^G) \leq 1$ and so $d(x^G, y^G) \leq 2$.

□

We conclude this section with the following consequence.

Theorem 7.2.4. *Let G be a finite group. Let H and K be two subgroups of G such that H is normal in G, $G = HK$ and $\mathcal{SCC}(H), \mathcal{SCC}(K)$ are connected. If there exist two elements $h \in H \setminus \{1\}$ and $x \in G \setminus H$ such that h^G and x^G are connected in $\mathcal{SCC}(G)$ then $\mathcal{SCC}(G)$ is connected.*

Proof. Let $a, b \in G$ such that a^G and b^G are two distinct vertices in $\mathcal{SCC}(G)$.

If $a, b \in H$ then there exists a path from a^H to b^H, since $\mathcal{SCC}(H)$ is connected. Hence, a^G and b^G are connected. Let $a \notin H$ and $o(a) = n$. Let $f : G/H \to K/(H \cap K)$ be an isomorphism and $f(aH) = x(H \cap K)$, where $x \in K$. Then $x^n(H \cap K) = f(a^nH) = H \cap K$ and so $x^n \in H \cap K$. Let $d = \gcd(o(a), |K|)$. Then there exist integers r, s such that

$$x^d = x^{nr+|K|s} = (x^n)^r.(x^{|K|})^s \in H \cap K.$$

Therefore, $d > 1$. Let p be a prime divisor of d. Then there exists an element $k_1 \in K$ such that $\gcd(o(a), o(k_1)) \neq 1$. Therefore, by Proposition 7.2.2, there is a path from a^G to k_1^G. Similarly, if $b \notin H$ then there exists an element $k_2 \in K$ such that there is a path from k_2^G to b^G. We have $k_1^G = k_2^G$ or there is a path from k_1^K to k_2^K, since $\mathcal{SCC}(K)$ is connected. Therefore, $k_1^G = k_2^G$ or there is a path from k_1^G to k_2^G. Thus a^G and b^G are connected. If $b \in H$ then, by given conditions, there exist two elements $h \in H \setminus \{1\}$ and $x \in G \setminus H$ such that there is a path from x^G to h^G and a path from h^G to b^G (since $\mathcal{SCC}(H)$ is connected). Since $x \notin H$, proceeding as above we get a path from x^G to k_3^G for some $k_3 \in K$ and hence a path from a^G to x^G. Thus we get a path from a^G to b^G. Hence, $\mathcal{SCC}(G)$ is connected. □

7.3 Genus of SCC-graph

In this section, we discuss certain properties of genus and crosscap of $\mathcal{SCC}(G)$ for the groups S_n and A_n. In particular, we determine all positive integers n such that $\mathcal{SCC}(S_n)$ and $\mathcal{SCC}(A_n)$ are planar or toroidal. We shall also obtain a lower bound for $\gamma(\mathcal{SCC}(G))$ in terms of order of the center and number of conjugacy classes for certain groups. As a consequence, we shall derive a relation between $\gamma(\mathcal{SCC}(G))$ and commuting probability of certain finite non-solvable group.

The groups S_3, S_4, A_3 and A_4 are solvable, with respectively $3, 5, 3$ and 4 conjugacy classes; so their SCC-graphs are complete graphs on $2, 4, 2$ and 3 vertices respectively. All these graphs are planar. The SCC-graphs of other small symmetric and alternating groups are shown in the following figures, where a vertex is labelled with a representative of its conjugacy class.

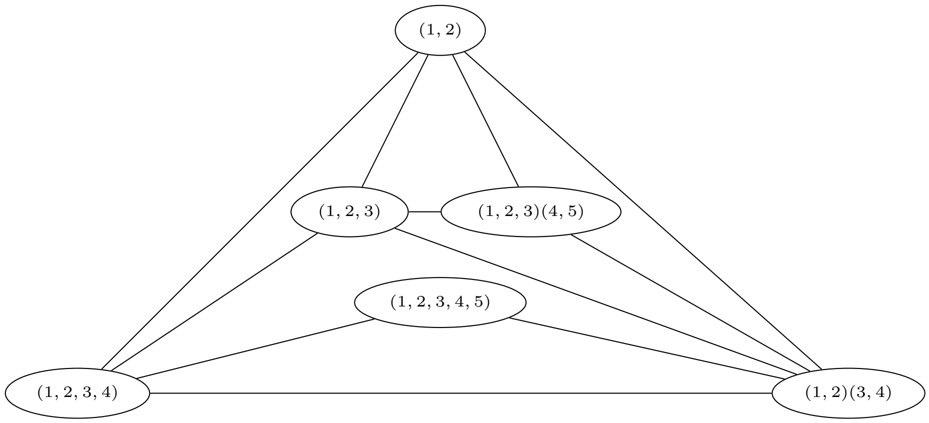

Figure 7.1: $\mathcal{SCC}(S_5)$

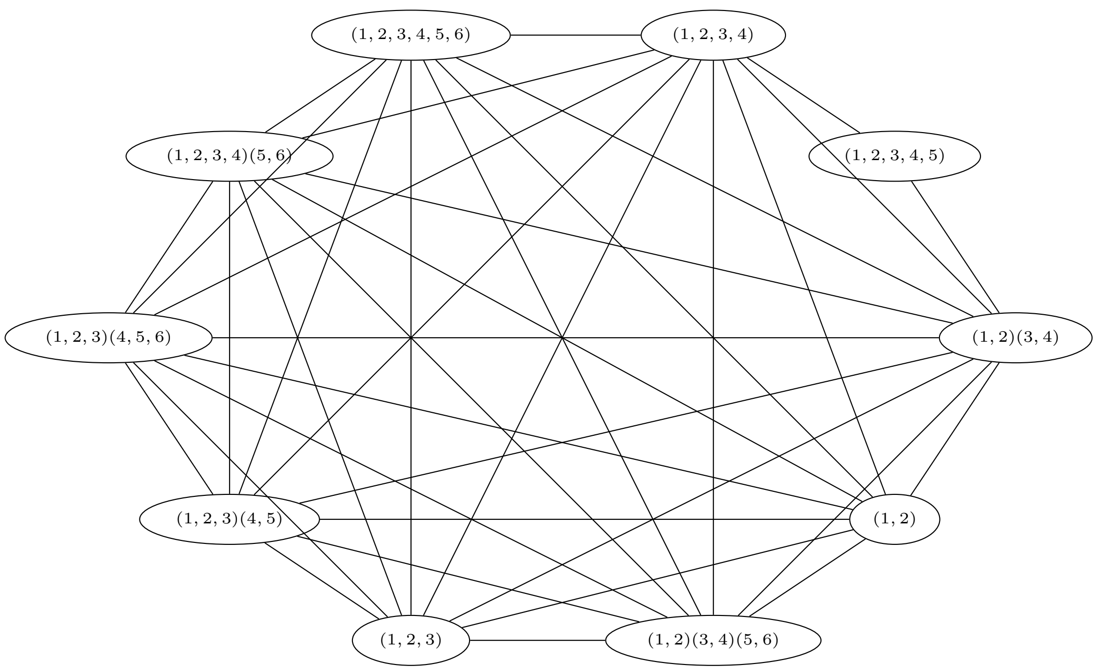

Figure 7.2: $\mathcal{SCC}(S_6)$

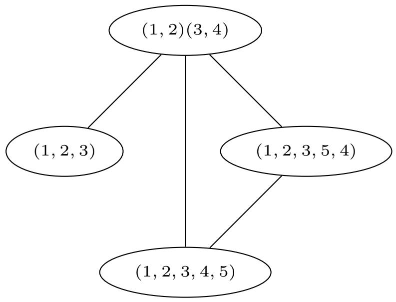

Figure 7.3: $\mathcal{SCC}(A_5)$

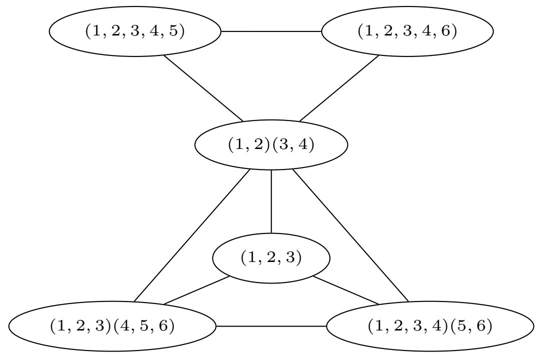

Figure 7.4: $\mathcal{SCC}(A_6)$

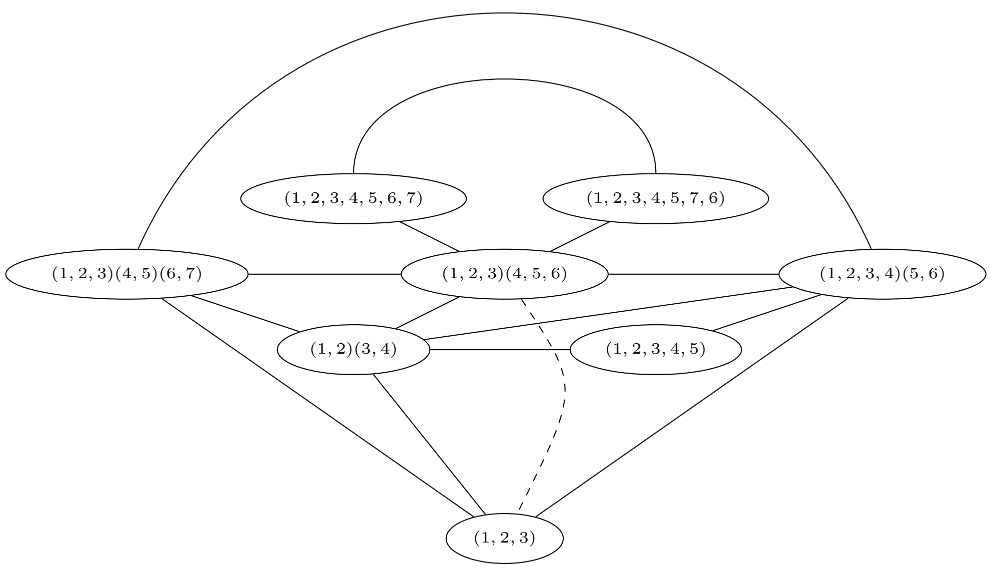

Figure 7.5: $\mathcal{SCC}(A_7)$

The symmetric and alternating groups whose SCC-graphs have small genus or are projective are given in the following results.

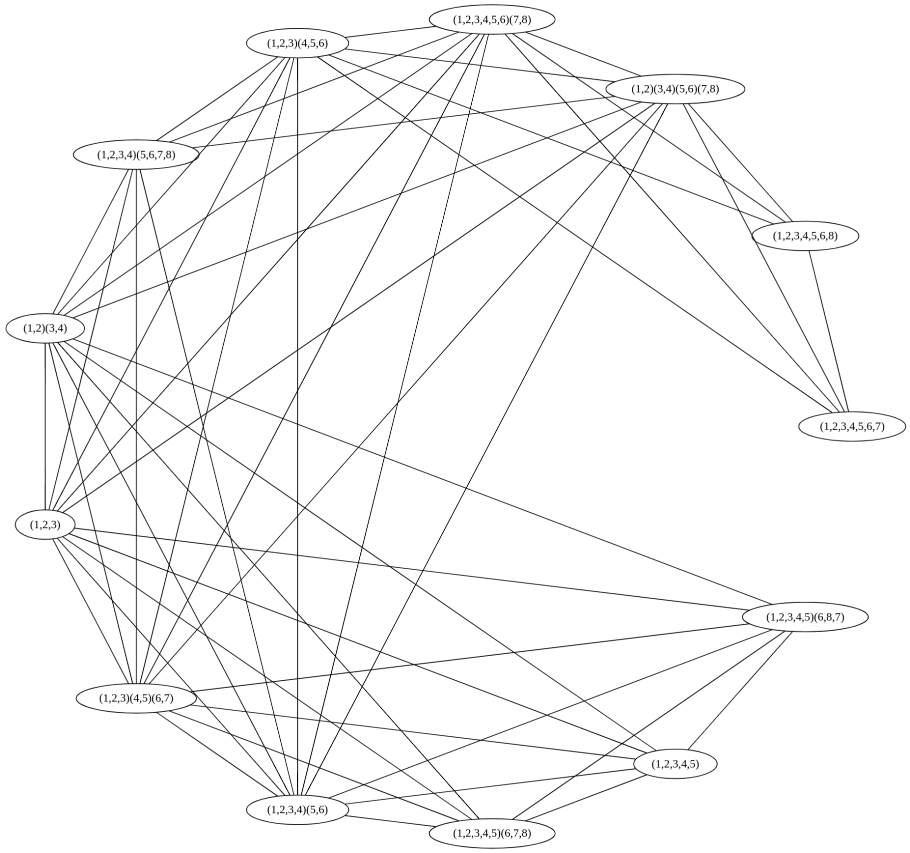

Figure 7.6: $\mathcal{SCC}(A_8)$

Theorem 7.3.1. (a) *$\mathcal{SCC}(S_n)$ is planar if and only if $n \leq 5$.*

(b) *If $n \geq 7$ then $\mathcal{SCC}(S_n)$ is neither planar, toroidal, double-toroidal nor triple-toroidal.*

(c) *$\mathcal{SCC}(S_6)$ is neither toroidal nor double-toroidal.*

(d) *If $n \geq 6$ then $\mathcal{SCC}(S_n)$ is not projective.*

Proof. (a) If $n \leq 5$ then, from our earlier remarks and Figure 7.1, it follows that $\mathcal{SCC}(S_n)$ is planar. If $n \geq 6$ then it is easy to show that the elements $(1,2)$, $(1,2,3)$, $(1,2)(3,4)$, $(1,2,3,4)$, $(1,2,3)(4,5)$ induce a clique in $\mathcal{SCC}(S_n)$. Hence,

$$\gamma(\mathcal{SCC}(S_n)) \geq \gamma(K_5) = 1$$

and so $\mathcal{SCC}(S_n)$ is not planar.

(b) One can show that the ten elements

$$(1,2),(1,2,3),(1,2)(3,4),(1,2,3,4),(1,2,3)(4,5),(1,2)(3,4)(5,6),$$
$$(1,2,3,4)(5,6),(1,2,3)(4,5,6),(1,2,3)(4,5)(6,7),(1,2,3,4)(5,6,7)$$

induce a clique in $\mathcal{SCC}(S_n)$. Hence,

$$\gamma(\mathcal{SCC}(S_n)) \geq \gamma(K_{10}) = 4$$

and so $\mathcal{SCC}(S_n)$ is neither planar, toroidal, double-toroidal nor triple-toroidal.

(c) From Figure 7.2, it follows that $\mathcal{SCC}(S_6)$ contains a subgraph isomorphic to K_9 (which is induced by $V(\mathcal{SCC}(S_6)) \setminus \{(1,2,3,4,5)^{S_6}\}$). Therefore,

$$\gamma(\mathcal{SCC}(S_6)) \geq \gamma(K_9) = 3.$$

Hence, the result follows from (a) and (b).

(d) In addition to the five permutations listed in the proof of (a), also the elements

$$(1,2),(1,2)(3,4)(5,6),(1,2,3,4)(5,6),(1,2,3)(4,5,6),(1,2,3,4,5,6)$$

induce a clique. Consequently, $\mathcal{SCC}(S_n)$ contains two copies of K_5 which share a single vertex. This subgraph is isomorphic to the graph denoted by A_1 in [49]. Therefore, $\mathcal{SCC}(S_n)$ is not projective. □

Here is the analogous results for alternating groups.

Theorem 7.3.2. (a) *$\mathcal{SCC}(A_n)$ is planar if and only if $n \leq 6$.*

(b) *If $n \geq 9$ then $\mathcal{SCC}(A_n)$ is neither planar, toroidal, double-toroidal nor triple-toroidal.*

(c) *$\mathcal{SCC}(A_n)$ is toroidal if and only if $n = 7$.*

(d) *If $n \geq 8$ then $\mathcal{SCC}(A_n)$ is not projective.*

Proof. (a) If $n \leq 6$ then, as shown in Figures 7.3 and 7.4, it follows that $\mathcal{SCC}(A_n)$ is planar.

If $n \geq 7$ then the permutations

$$(1,2,3),(1,2)(3,4),(1,2,3,4)(5,6),(1,2,3)(4,5,6),(1,2,3)(4,5)(6,7)$$

induce a clique in $\mathcal{SCC}(A_n)$ (note that the elements have pairwise distinct cycle types). Therefore,

$$\gamma(\mathcal{SCC}(A_n)) \geq \gamma(K_5) = 1.$$

and so $\mathcal{SCC}(A_n)$ is not planar.

(b) The ten elements

$$(1,2,3),(1,2)(3,4),(1,2,3,4)(5,6),(1,2,3)(4,5,6),(1,2,3)(4,5)(6,7),(1,2)(3,4)(5,6)(7,8),$$
$$(1,2,3,4,5,6)(7,8),(1,2,3,4)(5,6,7)(8,9),(1,2,3)(4,5,6)(7,8,9),(1,2,3,4,5,6,7,8,9)$$

induce a clique in $\mathcal{SCC}(A_n)$. Thus, the result follows as in Theorem 7.3.1(b).

(c) The fact that $\mathcal{SCC}(A_7)$ is toroidal follows from Figure 7.5 and part (a).

It is easy to see in Figure 7.6, that the subgraph induced by the permutations

$$(1,2,3)(4,5,6),(1,2,3,4,5,6,7),(1,2,3,4,5,6,8),(1,2)(3,4)(5,6)(7,8),(1,2,3,4,5,6)(7,8)$$

and

$$(1,2,3,4,5),(1,2,3,4)(5,6),(1,2,3)(4,5)(6,7),(1,2,3,4,5)(6,7,8),(1,2,3,4,5)(6,8,7)$$

contains a subgraph isomorphic to $K_5 \sqcup K_5$. Therefore,

$$\gamma(\mathcal{SCC}(A_8)) \geq \gamma(K_5 \sqcup K_5) = 2.$$

Hence, the result follows from parts (a) and (b).

(d) There are two 5-cliques induced by

$$(1,2,3),(1,2)(3,4),(1,2,3,4,5),(1,2,3,4)(5,6),(1,2,3)(4,5)(6,7),$$
$$(1,2,3),(1,2,3)(4,5,6),(1,2)(3,4)(5,6)(7,8),(1,2,3,4,5,6)(7,8),(1,2,3,4)(5,6,7,8),$$

which share a single vertex. Thus, the claim follows as in Theorem 7.3.1(d). □

Recall that $k(G)$ denotes the number of conjugacy classes of G. The following lemma is useful in obtaining a lower bound for $\gamma(\mathcal{SCC}(G))$ as mentioned above.

Lemma 7.3.3. *Let G be a finite non-solvable group with non-trivial center $Z(G)$. Then $\mathcal{SCC}(G)$ has a subgraph isomorphic to $K_{|Z(G)|-1,\ k(G)-|Z(G)|}$.*

Proof. Let $S = \{x^G : x \in Z(G) \setminus \{1\}\}$ and $T = \{y^G : y \in G \setminus Z(G)\}$. We consider the subgraph S_Γ of $\mathcal{SCC}(G)$ by removing edges between the vertices of S as well as removing edges between the vertices of T. Then the subgraph thus obtained is isomorphic to $K_{|Z(G)|-1,\ k(G)-|Z(G)|}$. □

Theorem 7.3.4. *Let G be a finite non-solvable group with non-trivial center $Z(G)$. Then*

$$4\gamma(\mathcal{SCC}(G)) \geq (|Z(G)| - 3)(k(G) - |Z(G)| - 2).$$

Proof. By Lemma 7.3.3, it follows that $\mathcal{SCC}(G)$ has a subgraph which is isomorphic to $K_{|Z(G)|-1,\, k(G)-|Z(G)|}$. We have

$$\gamma(\mathcal{SCC}(G)) \geq \gamma(K_{|Z(G)|-1, k(G)-|Z(G)|}).$$

Therefore,

$$\gamma(\mathcal{SCC}(G)) \geq \left\lceil \frac{(|Z(G)| - 3)(k(G) - |Z(G)| - 2)}{4} \right\rceil \geq \frac{(|Z(G)| - 3)(k(G) - |Z(G)| - 2)}{4}.$$

Hence, the result follows on simplification. □

We conclude this chapter with the following relation between commuting probability and genus of SCC-graph of finite non-solvable group with non-trivial center.

Corollary 7.3.5. *Let G be a finite non-solvable group and $|Z(G)| > 3$. If $\Pr(G)$ is the commuting probability of G then*

$$\Pr(G) \leq \frac{4\gamma(\mathcal{SCC}(G)) + (|Z(G)| - 3)(|Z(G)| + 2)}{|G|(|Z(G)| - 3)}.$$

Proof. The result follows from Theorem 7.3.4 and the fact that $\Pr(G) = \frac{k(G)}{|G|}$ as given in Result 1.2.15. □

It is worth mentioning that many bounds for $\Pr(G)$ have been obtained using various group theoretic notions over the years (see [51, 76]). However, the bound for $\Pr(G)$ obtained in Corollary 7.3.5 is the first of its kind involving genus of certain graph defined on groups though it is difficult to compute genus of $\mathcal{SCC}(G)$ in general.