Chapter 8

Conclusion and problems for future research

The thesis began with an introductory chapter, where we introduced some notations and recalled certain useful results from Graph Theory and Group Theory. In that chapter we also reviewed literature on commuting graphs of groups and its various extensions. In Chapter 2, we had computed genus of commuting graphs for the classes of finite groups such that their central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (where p is a prime), $D_{2n} = \langle x, y : x^n = y^2 = 1, yxy = x^{-1}\rangle$ (where $n \geq 2$) or $Sz(2) = \langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2\rangle$ and found conditions such that $\gamma(\mathcal{C}(G)) = 4, 5$ or 6. We also characterized groups p^3, the meta-abelian groups $M_{2nk} = \langle a, b : a^n = b^{2k} = 1, bab^{-1} = a^{-1}\rangle$, D_{2n}, $Q_{4m} = \langle x, y : x^{2m} = 1, x^m = y^2, y^{-1}xy = x^{-1}\rangle$ and $U_{6n} = \langle a, b : a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$ such that their commuting graphs have genus $4, 5$ or 6.

In Chapter 3, we had considered solvable graphs of groups (denoted by $\mathcal{S}(G)$) and showed that it is not a star graph, a tree, an n-partite graph for any positive integer $n \geq 2$ and a regular graph for any non-solvable finite group. We showed that the girth of $\mathcal{S}(G)$ is 3 and the clique number of $\mathcal{S}(G)$ is greater than or equal to 4. Further, it was shown that there is no finite non-solvable group whose solvable graph is planar, toroidal, double-toroidal, triple-toroidal or projective. We had concluded Chapter 3 by obtaining a relation between the number of edges in $\mathcal{S}(G)$ and the solvability degree of G (denoted by $P_s(G)$).

In Chapter 4, we had considered non-solvable graphs of groups (denoted by $\mathcal{NS}(G)$), which is the complement of $\mathcal{S}(G)$, and obtained a relation between the number of edges in $\mathcal{NS}(G)$ and $P_s(G)$. Consequently, we obtained better bounds for $P_s(G)$. Further, we had shown that $|\deg(\mathcal{NS}(G))| = 3$ if $G/\operatorname{Sol}(G) \cong A_5$, where $\deg(\mathcal{NS}(G)) = \{\deg_{\mathcal{NS}(G)}(x) : x \in G \setminus \operatorname{Sol}(G)\}$. It was shown that $\mathcal{NS}(G)$ is not complete multi-partite. In general, it is not known whether $\mathcal{NS}(G)$ is Hamiltonian. However, we had shown that $\mathcal{NS}(A_5)$ is Hamiltonian. We had shown that the domination number of $\mathcal{NS}(G)$ is not equal to 1 and the clique number of $\mathcal{NS}(G)$ is greater than or equal to 6 if G is any finite non-solvable groups. Finally, we had proved that the non-solvable graph is neither planar, toroidal, double-toroidal, triple-toroidal nor projective.

In Chapter 5, we had computed spectrum, Laplacian spectrum, signless Laplacian spectrum and their corresponding energies for the commuting conjugacy classes of the groups D_{2n}, Q_{4m}, $U_{(n,m)}$, V_{8n}, SD_{8n}, $G(p,m,n)$ and showed that the commuting conjugacy class graphs of these groups are super integral and they satisfy E-LE Conjecture. We had also characterized these groups such that their commuting conjugacy class graphs are hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic and Q-borderenergetic.

In Chapter 6, we had computed genus of commuting conjugacy class graphs of the groups D_{2n}, Q_{4m}, $U_{(n,m)}$, V_{8n}, SD_{8n}, $G(p,m,n)$ and characterized these groups such that their commuting conjugacy class graphs are planar, toroidal, double-toroidal or triple-toroidal.

In Chapter 7, we had introduced solvable conjugacy class graph of groups (denoted by $\mathcal{SCC}(G)$) extending the notions of commuting conjugacy class graphs and nilpotent conjugacy class graphs of groups. Among other results we had proved that $\mathcal{SCC}(G)$ is not triangle-free if G is not isomorphic to the cyclic groups of order $1, 2$ and 3 and the symmetric group of degree 3. We had also characterized the symmetric groups of degree n and the alternating groups of degree n such that their solvable conjugacy class graphs are planar, toroidal, double-toroidal or triple-toroidal.

8.1 Problems for future research

During our study on various properties of commuting graphs of finite groups and its extensions we came up with certain problems for future research. In this section we list all those problems.

In Chapter 2, it was observed that $\gamma(\mathcal{C}(G)) \neq 5$ for all the groups considered in our study. It may be interesting to give examples of groups G such that $\gamma(\mathcal{C}(G)) = 5$. In this regard, we pose the following general problem.

Problem 8.1.1. To determine all the positive integers that can be realized as genus of commuting graphs of some finite non-abelian groups. Also, characterize all finite non-abelian groups such that $\gamma(\mathcal{C}(G)) = n$, if $n \geq 4$ is any positive integer that can be realized as the genus of $\mathcal{C}(G)$ for some finite non-abelian groups.

It is worth mentioning here that finite non-abelian groups whose commuting graphs have genus $\gamma(\mathcal{C}(G)) \leq 3$ were characterized in [35, 81].

Problems similar to Problem 8.1.1 can also be posed for solvable and non-solvable groups of finite groups. Spectral aspects of commuting and non-commuting graphs of finite groups are well-studied. However, the same is not carried for solvable and non-solvable graphs of finite groups. Therefore, it is worth studying spectral aspects of solvable and non-solvable graphs of finite groups. Regarding non-solvable groups of finite groups we also pose the following problems.

Problem 8.1.2. To determine whether $\mathcal{NS}(G)$ is Hamiltonian for any finite non-solvable group G.

Problem 8.1.3. To determine whether the domination number, $\lambda(\mathcal{NS}(G)) = 2, 3$ for any finite non-solvable group G.

Problem 8.1.4. To prove/disprove Conjecture 4.5.5 regarding the clique number of non-solvable graphs of groups.

In Chapter 5, it was shown that $\mathcal{CCC}(G)$ is super integral if $G = D_{2n}, Q_{4m}, U_{(n,m)}, V_{8n}$, SD_{8n} or $G(p, m, n)$. In this regard the following problem arises naturally.

Problem 8.1.5. To determine all finite groups such that $\mathcal{CCC}(G)$ is super integral.

Regarding genus of commuting conjugacy class graphs of finite groups we pose the following problem.

Problem 8.1.6. To characterize all finite groups whose commuting conjugacy class graphs are planar, toroidal, double-toroidal or triple-toroidal.

Regarding genus of solvable conjugacy class graphs of finite groups we pose the following problem.

Problem 8.1.7. To characterize all finite groups whose solvable conjugacy class graphs are planar, toroidal, double-toroidal or triple-toroidal.

We conclude this section noting that problems similar to Problem 8.1.1 can also be posed for commuting/solvable conjugacy class graphs of finite groups.