Abstract

Let G be a finite non-abelian group with centre $Z(G)$. The commuting graph of G, denoted by $\mathcal{C}(G)$, is a simple undirected graph whose vertex set is $G \setminus Z(G)$ and two distinct vertices x and y are adjacent whenever $xy = yx$. The solvable graph $\mathcal{S}(G)$ of a finite non-solvable group G is a simple undirected graph whose vertex set is $G \setminus \mathrm{Sol}(G)$, where $\mathrm{Sol}(G) = \{x \in G : \langle x, y\rangle \text{ is solvable for all } y \in G\}$, and two distinct vertices x and y are adjacent if $\langle x, y\rangle$ is solvable. The complement of $\mathcal{S}(G)$ is known as the non-solvable graph of G denoted by $\mathcal{NS}(G)$. Another way of defining graphs on G is given by considering the conjugacy classes as the vertices and adjacency is defined using various group properties. The commuting conjugacy class graph (or CCC-graph) of a group G, denoted by $\mathcal{CCC}(G)$, is a simple undirected graph whose vertex set is the set $\{x^G : x \in G \setminus Z(G)\}$, where x^G is the conjugacy class of x, and two distinct vertices x^G and y^G are adjacent if $\langle x', y'\rangle$ is abelian for some $x' \in x^G$ and $y' \in y^G$. Following the notion of CCC-graph, we introduce the notion of solvable conjugacy class graph (or SCC-graph) of a group. The solvable conjugacy class graph of a group G, denoted by $\mathcal{SCC}(G)$, is a simple undirected graph whose vertex set is the set $\{x^G : x \in G \setminus \{1\}\}$, and two distinct vertices x^G and y^G are adjacent if $\langle x', y'\rangle$ is solvable for some $x' \in x^G$ and $y' \in y^G$.

In Chapter 1 of this thesis, we recall several definitions, notations and results from Group Theory and Graph Theory. We also review literature on commuting graphs of groups and its various extensions.

In Chapter 2, we compute genus of commuting graphs for the classes of finite groups such that their central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (where p is a prime), $D_{2n} = \langle x, y : x^n = y^2 = 1, yxy = x^{-1}\rangle$ (where $n \geq 2$) or $Sz(2) = \langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2\rangle$ and find conditions such that $\gamma(\mathcal{C}(G)) = 4, 5$ or 6. We also characterize groups of order p^3, the

meta-abelian groups $M_{2nk} = \langle a, b : a^n = b^{2k} = 1, bab^{-1} = a^{-1}\rangle$, D_{2n}, $Q_{4m} = \langle x, y : x^{2m} = 1, x^m = y^2, y^{-1}xy = x^{-1}\rangle$ and $U_{6n} = \langle a, b : a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$ such that their commuting graphs have genus $4, 5$ or 6.

In Chapter 3, We consider solvable graphs of groups (denoted by $\mathcal{S}(G)$) and show that it is not a star graph, a tree, an n-partite graph for any positive integer $n \geq 2$ and a regular graph for any non-solvable finite group. We show that the girth of $\mathcal{S}(G)$ is 3 and the clique number of $\mathcal{S}(G)$ is greater than or equal to 4. Further, it is shown that there is no finite non-solvable group whose solvable graph is planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude Chapter 3 by obtaining a relation between the number of edges in $\mathcal{S}(G)$ and the solvability degree of G (denoted by $P_s(G)$).

In Chapter 4, we consider non-solvable graphs of groups (denoted by $\mathcal{NS}(G)$), which is the complement of $\mathcal{S}(G)$, and obtain a relation between the number of edges in $\mathcal{NS}(G)$ and $P_s(G)$. Consequently, we obtain better bounds for $P_s(G)$. Further, we show that $|\deg(\mathcal{NS}(G))| = 3$ if $G/\operatorname{Sol}(G) \cong A_5$, where $\deg(\mathcal{NS}(G)) = \{\deg_{\mathcal{NS}(G)}(x) : x \in G \setminus \operatorname{Sol}(G)\}$. It is shown that $\mathcal{NS}(G)$ is not complete multi-partite. In general, it is not known whether $\mathcal{NS}(G)$ is Hamiltonian. However, we show that $\mathcal{NS}(A_5)$ is Hamiltonian. We show that the domination number of $\mathcal{NS}(G)$ is not equal to 1 and the clique number of $\mathcal{NS}(G)$ is greater than or equal to 6 if G is any finite non-solvable groups. Finally, we prove that the non-solvable graph is neither planar, toroidal, double-toroidal, triple-toroidal nor projective.

In Chapter 5, we compute spectrum, Laplacian spectrum, signless Laplacian spectrum and their corresponding energies for the commuting conjugacy classes of the groups D_{2n}, Q_{4m}, $U_{(n,m)}$, V_{8n}, SD_{8n}, $G(p, m, n)$ and show that the commuting conjugacy class graphs of these groups are super integral and they satisfy E-LE Conjecture. We also characterize these groups such that their commuting conjugacy class graphs are hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic and Q-borderenergetic.

In Chapter 6, we compute genus of commuting conjugacy class graphs of the groups D_{2n}, Q_{4m}, $U_{(n,m)}$, V_{8n}, SD_{8n}, $G(p, m, n)$ and characterize these groups such that their commuting conjugacy class graphs are planar, toroidal, double-toroidal or triple-toroidal.

In Chapter 7, we introduce solvable conjugacy class graph of groups (denoted by $\mathcal{SCC}(G)$)

extending the notions of commuting conjugacy class graphs and nilpotent conjugacy class graphs of groups. Among other results we prove that $\mathcal{SCC}(G)$ is not triangle-free if G is not isomorphic to the cyclic groups of order $1, 2$ and 3 and the symmetric group of degree 3. We also characterize the symmetric groups of degree n and the alternating groups of degree n such that their solvable conjugacy class graphs are planar, toroidal, double-toroidal or triple-toroidal.

In Chapter 8, we conclude the thesis by suggesting some problems for future research.