Chapter 1

Introduction and Preliminaries

Over the last four decades, it is observed that graphs are interesting tool for the study of groups. Properties of a group can be described through the properties of various graphs defined on it. The first graph defined on a group is the commuting graph which was originated from a paper of Brauer and Fowler [27] published in the year 1955. In 1976, Neumann considered non-commuting graph of a finite group, which is the complement of commuting graph, to solve a problem posed by Erdös [79]. Mathematicians have defined many graphs on finite groups, following the works in [27, 79]. Some well-studied graphs defined on finite groups are power graph, enhanced power graph, nilpotent graph, non-nilpotent graph, solvable graph, non-solvable graph, commuting conjugacy class graph and nilpotent conjugacy class graph. In Section 1.3 of this chapter we shall recall definitions and useful results of these graphs. In this thesis we consider commuting graphs, solvable graphs, non-solvable graphs and commuting conjugacy class graphs of finite groups and obtain various results including graph realization and characterization of certain finite groups. We shall also introduce solvable conjugacy class graph of finite groups.

In Chapter 2, we compute genus of commuting graphs of some classes of finite non-abelian groups and characterize those groups such that their commuting graphs have genus $4, 5$ and 6. In Chapter 3, we consider solvable graph of finite groups and show that this graph is not a star graph, a tree, an n-partite graph for any positive integer $n \geq 2$ and not a regular graph for any non-solvable finite group. We compute the girth of solvable graph and derive a lower bound for the clique number of solvable graph. We prove the non-existence of finite non-solvable groups whose solvable graphs are planar, toroidal, double-toroidal, triple-toroidal or projective. We also obtain a relation between solvable graph and solvability degree of finite non-solvable groups. In Chapter 4, we consider non-solvable graphs of finite groups and obtain results on vertex degree, cardinality of vertex degree set, graph realization, domination number, vertex connectivity, independence number and clique number of non-solvable graph. We derive certain properties of the groups G and H if their non-solvable graphs are isomorphic. We also show that non-solvable graph is neither planar, toroidal, double-toroidal, triple-toroidal nor projective. In Chapter 5, we compute various spectra and energies of commuting conjugacy class graph (CCC-graph) of the dihedral groups (D_{2n}), the dicyclic group (Q_{4m}), the groups $U_{(n,m)} = \langle x, y : x^{2n} = y^m = 1, x^{-1}yx = y^{-1}\rangle, V_{8n} = \langle a, b : a^{2n} = b^4 = 1, b^{-1}ab^{-1} = bab = a^{-1}\rangle, SD_{8n} = \langle a, b : a^{4n} = b^2 = 1, bab = a^{2n-1}\rangle$ and $G(p, m, n) = \langle x, y : x^{p^m} = y^{p^n} = [x, y]^p = 1, [x, [x, y]] = [y, [x, y]] = 1\rangle$. Our computations show that CCC-graphs for these groups are super integral. We compare various energies and show that CCC-graphs of these groups satisfy E-LE Conjecture (Conjecture 1.1.7) of Gutman et al. [55]. We also provide negative answer to a question (Question 1.1.8) posed by Dutta et al. [43]. We conclude Chapter 5 by characterizing the above mentioned groups such that their CCC-graphs are hyperenergetic, L-hyperenergetic or Q-hyperenergetic. In Chapter 6, we compute the genus of CCC-graphs and determine whether CCC-graphs for the groups considered in Chapter 5 are planar, toroidal, double-toroidal or triple-toroidal. In Chapter 7, we introduce solvable conjugacy class graph (SCC-graph) of a group G. We discuss the connectivity, girth, clique number, and several other properties of SCC-graph. We also discuss the genus of SCC-graph. We conclude Chapter 7 with a relation between commuting probability and genus of the SCC-graph of a finite non-solvable group with non-trivial center. In Chapter 8, we conclude the thesis by suggesting some problems for future research.

In this chapter, we recall certain results from Graph Theory and Group Theory that are useful in the subsequent chapters.

1.1 Notations and Results from Graph Theory

For all the standard notations and basic results we refer to [94]. All the graphs considered in our study are finite, simple and undirected. We write $V(\Gamma)$ and $e(\Gamma)$ to denote the vertex set and the edge set of a graph Γ respectively. The *degree* of a vertex $x \in V(\Gamma)$, denoted by $\deg_\Gamma(x)$, is defined to be the number of vertices adjacent to x and $\deg(\Gamma) = \{\deg_\Gamma(x) : x \in V(\Gamma)\}$ is the *vertex degree set* of Γ. The *neighborhood* of a vertex x in a graph Γ, denoted by $\mathrm{Nbd}_\Gamma(x)$, is defined to be the set of all vertices adjacent to x and so $\deg_\Gamma(x) = |\mathrm{Nbd}_\Gamma(x)|$.

For a graph Γ and a subset S of $V(\Gamma)$ we write $N_\Gamma[S] = S \cup (\underset{x\in S}{\cup}\mathrm{Nbd}_\Gamma(x))$. If $N_\Gamma[S] = V(\Gamma)$ then S is said to be a *dominating set* of Γ. The *domination number* of Γ, denoted by $\lambda(\Gamma)$, is the minimum cardinality of dominating sets of Γ. A subset X of $V(\Gamma)$ is called an *independent set* if the induced subgraph on X has no edges. The maximum size of the independent sets in a graph Γ is called the *independence number* of Γ and it is denoted by $\alpha(\Gamma)$. For any subset S of $V(\Gamma)$, we write $\Gamma[S]$ to denote the *induced subgraph* of Γ on S. For any non-empty subset S of $V(\Gamma)$, we also write $\Gamma\setminus S$ to denote $\Gamma[V(\Gamma)\setminus S]$. A subset of $V(\Gamma)$ is called a *clique* of Γ if it consists entirely of pairwise adjacent vertices. The least upper bound of the sizes of all the cliques of Γ is called the *clique number* of Γ, and it is denoted by $\omega(\Gamma)$. Note that $\omega(\tilde{\Gamma}) \leq \omega(\Gamma)$ for any subgraph $\tilde{\Gamma}$ of Γ. The *girth* of Γ is the minimum of the lengths of all cycles in Γ, and it is denoted by $\mathrm{girth}(\Gamma)$. The distance between two vertices u and v of Γ is denoted by $d(u, v)$. The *diameter* of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is the maximum distance between the pair of vertices. The *vertex connectivity* of a connected graph Γ, denoted by $\kappa(\Gamma)$, is the smallest number of vertices whose removal disconnects Γ. A subset S of $V(\Gamma)$ of a connected graph Γ is called a *vertex cut set*, if $\Gamma \setminus S$ is not a connected graph. The *chromatic number* of a graph Γ, denoted by $\chi(\Gamma)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors.

Result 1.1.1. [24, Theorem 4.3] If Γ is a simple graph with $|V(\Gamma)| > 3$ and $\delta > \frac{|V(\Gamma)|}{2}$, where $\delta = \min\{\deg_\Gamma(v) : v \in V(\Gamma)\}$, then Γ is Hamiltonian.

Result 1.1.2. [95, Corollary 6–14] If Γ is connected and $|V(\Gamma)| > 3$ then

$$\gamma(\Gamma) \geq \frac{|e(\Gamma)|}{6} - \frac{|V(\Gamma)|}{2} + 1.$$

The equality holds if and only if a triangular imbedding can be found for Γ.

The smallest non-negative integer k is called the *genus* of a graph Γ if Γ can be embedded on the surface obtained by attaching k handles to a sphere. We write $\gamma(\Gamma)$ to denote the genus of Γ. If Γ_0 is a subgraph of Γ then it can be easily visualized that

$$\gamma(\Gamma) \geq \gamma(\Gamma_0). \tag{1.1.a}$$

Let K_n be the complete graph on n vertices and mK_n the disjoint union of m copies of K_n. The following results are useful in computing genus of certain graphs.

Result 1.1.3. [15, Corollary 2] If Γ is the disjoint union of K_m and K_n then $\gamma(\Gamma) = \gamma(K_m) + \gamma(K_n)$.

By using Result 1.1.3, we have the following result.

Result 1.1.4. If $\mathcal{G} = m_1K_{n_1} \sqcup m_2K_{n_2}$ then $\gamma(\mathcal{G}) = m_1\gamma(K_{n_1}) + m_2\gamma(K_{n_2})$.

It is well-known that $\gamma(K_n) = 0$ if $n = 1, 2$. If $n \geq 3$ then by [95, Theorem 6-38], we have

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil. \tag{1.1.b}$$

Also, if $m, n \geq 2$ then

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil \text{ and } \gamma(K_{m,m,m}) = \frac{(m-2)(m-1)}{2},$$

where $K_{m,n}$ and $K_{m,m,m}$ are complete bipartite and tripartite graphs respectively. A graph Γ is called *planar, toroidal, double-toroidal* and *triple-toroidal* if $\gamma(\Gamma) = 0, 1, 2$ and 3 respectively.

Result 1.1.5. [94, Theorem 6.3.25](Heawood's Formula) If Γ is a simple graph and $\gamma(\Gamma) = m$ then

$$\chi(\gamma) \leq \left\lfloor \frac{7 + \sqrt{1 + 48m}}{2} \right\rfloor.$$

A compact surface N_k is a connected sum of k projective planes. A simple graph which can be embedded in N_k but not in N_{k-1}, is called a graph of *crosscap* k. We write $\bar{\gamma}(\Gamma)$ to denote the crosscap of a graph Γ. It is easy to see that $\bar{\gamma}(\Gamma_0) \leq \bar{\gamma}(\Gamma)$ for any subgraph Γ_0 of Γ. It was shown in [25] that

$$\bar{\gamma}(K_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil, & \text{if } n \geq 3 \text{ and } n \neq 7 \\ 3, & \text{if } n = 7. \end{cases} \tag{1.1.c}$$

A graph Γ is called *projective* if $\bar{\gamma}(\Gamma) = 1$. It is worth mentioning that $2K_5$ is not a projective graph (see [49]).

Recall that the *spectrum* of a graph Γ, denoted by $\mathrm{Spec}(\Gamma)$, is the set $\{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_n^{k_n}\}$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the adjacency matrix $A(\Gamma)$ of Γ and the exponents $k_1, k_2, \dots, k_n$ are the multiplicities of $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively. A graph Γ is called *integral* if $\mathrm{Spec}(\Gamma)$ contains only integers. Harary and Schwenk [60] introduced the concept of integral graphs in 1974. It is well-known that K_n is integral. Moreover, if $\Gamma = mK_n$ then also Γ is integral. Laplacian and signless Laplacian matrices of a graph Γ, denoted by $L(\Gamma)$ and $Q(\Gamma)$, are defined as $L(\Gamma) = D(\Gamma) - A(\Gamma)$ and $Q(\Gamma) = D(\Gamma) + A(\Gamma)$ respectively, where $D(\Gamma)$ is the degree matrix of Γ. The *Laplacian* and *signless Laplacian spectrum* of Γ are defined by L-spec$(\Gamma) := \{\mu_1^{q_1}, \mu_2^{q_2}, \dots, \mu_m^{q_m}\}$ and Q-spec$(\Gamma) :=$

$\{\nu_1^{r_1}, \nu_2^{r_2}, \ldots, \nu_n^{r_n}\}$ respectively, where $\mu_1, \mu_2, \ldots, \mu_m$ are the eigenvalues of $L(\Gamma)$ with multiplicities $q_1, q_2, \ldots, q_m$ and $\nu_1, \nu_2, \ldots, \nu_n$ are the eigenvalues of $Q(\Gamma)$ with multiplicities $r_1, r_2, \ldots, r_n$. A graph Γ is called *L-integral* and *Q-integral* respectively if L-spec(Γ) and Q-spec(Γ) contain only integers. Several results on these graphs can be found in [5, 13, 33, 66, 69, 87]. A graph Γ is called *super integral* if it is integral, L-integral and Q-integral (see [40]). The following result is useful in computing various spectra of disjoint union of complete graphs.

Result 1.1.6. If $\mathcal{G} = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup l_3K_{m_3}$ then

$$\begin{aligned}
\text{Spec}(\mathcal{G}) &= \left\{(-1)^{\sum_{i=1}^{3} l_i(m_i-1)}, (m_1-1)^{l_1}, (m_2-1)^{l_2}, (m_3-1)^{l_3}\right\},\\
\text{L-spec}(\mathcal{G}) &= \left\{0^{l_1+l_2++l_3}, m_1^{l_1(m_1-1)}, m_2^{l_2(m_2-1)}, m_3^{l_3(m_3-1)}\right\} \text{ and}\\
\text{Q-spec}(\mathcal{G}) &= \Big\{(2m_1-2)^{l_1}, (m_1-2)^{l_1(m_1-1)}, (2m_2-2)^{l_2}, (m_2-2)^{l_2(m_2-1)},\\
&\qquad (2m_3-2)^{l_3}, (m_3-2)^{l_3(m_3-1)}\Big\}.
\end{aligned}$$

Depending on various spectra of a graph, there are various energies called *energy*, *Laplacian energy* and *signless Laplacian energy* denoted by $E(\Gamma), LE(\Gamma)$ and $LE^+(\Gamma)$ respectively. These energies are defined as follows:

$$E(\Gamma) = \sum_{\lambda \in \text{Spec}(\Gamma)} |\lambda|, \tag{1.1.d}$$

$$LE(\Gamma) = \sum_{\mu \in \text{L-spec}(\Gamma)} \left|\mu - \frac{2|e(\Gamma)|}{|V(\Gamma)|}\right|, \tag{1.1.e}$$

$$LE^+(\Gamma) = \sum_{\nu \in \text{Q-spec}(\Gamma)} \left|\nu - \frac{2|e(\Gamma)|}{|V(\Gamma)|}\right|. \tag{1.1.f}$$

The concept of energy of a graph was introduced by Gutman [58] in the year 1978. Later on, Gutman with his collaborators introduced Laplacian and signless Laplacian energies of a graph in [56] and [4] respectively. In 2008, Gutman et al. [55] posed the following conjecture comparing $E(\Gamma)$ and $LE(\Gamma)$.

Conjecture 1.1.7. (E-LE Conjecture) $E(\Gamma) \leq LE(\Gamma)$ for any graph Γ.

However, in the same year, Stevanović et al. [88] disproved Conjecture 1.1.7. In 2009, Liu and Lin [67] also disproved Conjecture 1.1.7 by providing some counter examples. Following Gutman et al. [55], recently Dutta et al. [43] have posed the following question comparing Laplacian and singless Laplacian energies of graphs.

Question 1.1.8. Is $LE(\Gamma) \leq LE^+(\Gamma)$ for all graphs Γ?

It is well-known that

$$E(K_n) = LE(K_n) = LE^+(K_n) = 2(n-1). \tag{1.1.g}$$

A graph Γ with n vertices is called *hyperenergetic*, *L-hyperenergetic* or *Q-hyperenergetic* according as $E(K_n) < E(\Gamma), LE(K_n) < LE(\Gamma)$ or $LE^+(K_n) < LE^+(\Gamma)$. Also, Γ is called *borderenergetic*, *L-borderenergetic* and *Q-borderenergetic* if $E(K_n) = E(\Gamma), LE(K_n) = LE(\Gamma)$ and $LE^+(K_n) = LE^+(\Gamma)$ respectively. These graphs were considered in [45, 50, 57, 93, 97].

1.2 Notations and Results from Group Theory

In this section, we fix some notations and recall certain results from Group Theory which will be referred in the subsequent chapters. However, for all the standard notations and basic results we refer to [83, 84].

The *centralizer* of an element x in a group G, denoted by $C_G(x)$, is defined as the set $\{y \in G : \langle x, y\rangle$ is abelian$\}$ which is clearly a subgroup of G. For any subset S of G, we write $C_G(S) = \underset{x\in S}{\cap} C_G(x)$. If $S = G$ then $C_G(G) = Z(G)$, the *centre* of the group G. In other words, $Z(G) = \{x \in G : \langle x, y\rangle$ is abelian for all $y \in G\}$. Note that $Z(G) = G$ if and only if G is abelian. A group G is called an *n-centralizer group* if the number of distinct centralizers of G is n. The following characterization of finite n-centralizer groups for $n = 4, 5$ are due to Belcastro and Sherman [16].

Result 1.2.1. [16, Theorem 2] A finite group G is 4-centralizer if and only if $\frac{G}{Z(G)} \cong \mathbb{Z}_2\times\mathbb{Z}_2$.

Result 1.2.2. [16, Theorem 4] A finite group G is 5-centralizer if and only if $\frac{G}{Z(G)} \cong \mathbb{Z}_3\times\mathbb{Z}_3$ or $\frac{G}{Z(G)} \cong S_3$, where S_3 is the symmetric group on three symbols.

In [12], Ashrafi obtained the following characterization of $(p+2)$-centralizer finite p-groups, where p is any prime number.

Result 1.2.3. [12, Lemma 2.7] If G is a finite non-abelian p-group then G is a $(p+2)$-centralizer group if and only if $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

A group G is said to be a *p-group* if the order of every element in G is a power of a prime number p. For any prime number p, a Sylow p-subgroup of G is a maximal p-subgroup of G. A p-subgroup is a subgroup of a group which is also a p-group. We list some results on Sylow p-subgroup which will be used later.

Result 1.2.4. [36, Lemma 3.4] Let G be a finite group.

(a) If $|G| = 7m$ and the Sylow 7-subgroup is normal in G then G has an abelian subgroup of order at least 14 or $|G| \leq 42$.

(b) If $|G| = 9m$, where $3 \nmid m$ and the Sylow 3-subgroup is normal in G then G has an abelian subgroup of order at least 18 or $|G| \leq 72$.

Result 1.2.5. [83, Theorem 14.3.4] Let G be a locally finite group and suppose that P is a finite Sylow p-subgroup of G. Then all Sylow p-subgroups of G are finite and conjugate.

Result 1.2.6. [83, Theorem 5.3.6] A finite p-group has exactly one subgroup of order p if and only if it is cyclic or a generalized quaternion group.

The *nilpotentizer* of an element x in a group G, denoted by $\mathrm{Nil}_G(x)$, is defined as the set $\{y \in G : \langle x, y\rangle \text{ is nilpotent}\}$. The motivation of defining nilpotentizers comes from the definition of centralizers. For any subset S of G, we write $\mathrm{Nil}_G(S) = \bigcap_{x\in S} \mathrm{Nil}_G(x)$. If $S = G$ then we write $\mathrm{Nil}(G)$ to denote $\mathrm{Nil}_G(G)$. In other words, $\mathrm{Nil}(G) = \{x \in G : \langle x, y\rangle \text{ is nilpotent for all } y \in G\}$. Also $\mathrm{Nil}(G) = G$, if G is nilpotent. Note that nilpotentizers of elements of G are not necessarily subgroups. In fact it is not known whether $\mathrm{Nil}(G)$ is a subgroup of G. A group G is said to be an $\mathfrak{n}$-*group* if $\mathrm{Nil}_G(x)$ is a subgroup of G for all $x \in G$ (see [2]).

The *solvabilizer* of $x \in G$, denoted by $\mathrm{Sol}_G(x)$, is the set given by $\{y \in G : \langle x, y\rangle \text{ is solvable}\}$. Note that $\mathrm{Sol}_G(x)$ is not a subgroup of G in general. A group G is called an *S-group* if $\mathrm{Sol}_G(x)$ is a subgroup of G for all $x \in G$. We write $\mathrm{Sol}(G) = \bigcap_{x\in G} \mathrm{Sol}_G(x)$. In other words, $\mathrm{Sol}(G) = \{x \in G : \langle x, y\rangle \text{ is solvable for all } y \in G\}$ which is also known as *solvable radical* of G. It may be mentioned here that $\mathrm{Sol}(G)$ is a subgroup of G if G is finite (see [53]). Following are some useful results on solvabilizers and solvable groups.

Result 1.2.7. [59, Proposition 2.13] Let G be a finite group. Then $|C_G(x)|$ divides $|\mathrm{Sol}_G(x)|$ for all $x \in G$.

Result 1.2.8. [59, Proposition 2.22] Let G be a finite group. Then G is solvable if and only if G is an S-group.

Result 1.2.9. [59, Lemma 2.11] Let N be a normal subgroup of a finite group G such that $N \subseteq \mathrm{Sol}(G)$ and $x, y, g \in G$. Then we have

(a) $\mathrm{Sol}_G(x) = \mathrm{Sol}_G(y)$ if $\langle x\rangle = \langle y\rangle$.

(b) $\mathrm{Sol}_G(gxg^{-1}) = g\,\mathrm{Sol}_G(x)g^{-1}$.

(c) $\mathrm{Sol}_{G/N}(xN) = \mathrm{Sol}_G(x)/N$.

Result 1.2.10. [59, Proposition 2.16] Let G be a finite group. Then $|G|$ divides $\sum\limits_{x\in G} |\,\mathrm{Sol}_G(x)|$.

Result 1.2.11. [53, Theorem 6.4] Let G be a finite non-solvable group and $x, y \in G\setminus\mathrm{Sol}(G)$. Then there exists $s \in G \setminus \mathrm{Sol}(G)$ such that $\langle x, s\rangle$ and $\langle y, s\rangle$ are not solvable.

Result 1.2.12. [37, Theorem A] Let G be a finite group. Then the following statements are equivalent:

(a) G is solvable.

(b) For all $x, y \in G$, there exists an element $g \in G$ for which $\langle x, y^g\rangle$ is solvable.

(c) For all $x, y \in G$ of prime power order, there exists an element $g \in G$ for which $\langle x, y^g\rangle$ is solvable.

Result 1.2.13. [83, Results 8.5.3] If G is a finite group of order $p^a q^b$, where p, q are prime numbers and a, b are non-negative integers, then G is solvable.

We also have the following result on finite solvable groups.

Result 1.2.14. [37, Section 1] A finite group is solvable if and only if every pair of its elements generates a solvable group.

1.2.1 Commuting probability and solvability degree

We write $\mathrm{Pr}(G)$ to denote the *commuting probability* of a finite group G which is defined as the probability that a pair of elements in G, chosen at random, commute with each other. Thus

$$\mathrm{Pr}(G) := \frac{|\{(u,v) \in G \times G : \langle u, v\rangle \text{ is abelian}\}|}{|G|^2}.$$

In terms of centralizer, $\mathrm{Pr}(G)$ can be written as

$$\mathrm{Pr}(G) = \frac{1}{|G|^2}\sum_{u\in G}|C_G(u)|. \tag{1.2.a}$$

The study of commuting probability of a finite group G was initiated by Erdös and Turán [44] in the year 1968. We recall some results on commuting probability of finite groups.

Result 1.2.15. [54, pp. 1032] Let $k(G)$ be the number of conjugacy classes of a finite group G. Then $\Pr(G) = \frac{k(G)}{|G|}$.

Result 1.2.16. ([85, pp. 246] and [77, pp. 451]) If $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}, \frac{7}{16}\}$ then $G/Z(G)$ is isomorphic to one of the groups in $\{D_{14}, D_{10}, D_8, D_6, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$.

Result 1.2.17. [68, Theorem 3] If G is a non-abelian finite group and p is the least prime number which divides $|G|$ then $\Pr(G) \leq \frac{p^2+p-1}{p^3}$. Moreover, equality holds if and only if $G/Z(G)$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

Result 1.2.18. [76, Theorem 1] If G is a finite group then

$$\Pr(G) \geq \frac{1}{|G'|}\left(1 + \frac{|G'| - 1}{|G : Z(G)|}\right),$$

where G' is the commutator subgroup of G. In particular, if G is non-abelian then

$$\Pr(G) > \frac{1}{|G'|}.$$

The *solvability degree* of a finite group G is defined by the following ratio

$$P_s(G) := \frac{|\{(u, v) \in G \times G : \langle u, v\rangle \text{ is solvable}\}|}{|G|^2}. \tag{1.2.b}$$

In view of Result 1.2.14, it follows that a finite group G is solvable if and only if $P_s(G) = 1$. It was shown in [52, Theorem A] that $P_s(G) \leq \frac{11}{30}$ for any finite non-solvable group G. It is worth mentioning that solvability degree of a finite group was introduced in [52] and several properties of $P_s(G)$, including some bounds, are studied in [52, 96]. We shall also consider solvability degree of a finite group in Chapter 3 and Chapter 4 and obtain more results.

1.3 Graphs defined on Groups

In this section we recall various graphs defined on groups. Commuting graph is the first such graph originated from the works of Brauer and Fowler [27]. Some extensions of commuting graph are nilpotent graph, solvable graph, commuting conjugacy class graph and nilpotent conjugacy class graph.

1.3.1 Commuting and non-commuting graph

Let G be a finite non-abelian group with centre $Z(G)$. The *commuting graph* of G, denoted by $\mathcal{C}(G)$, is a simple undirected graph whose vertex set is $G\setminus Z(G)$ and two distinct vertices x and y are adjacent whenever $xy = yx$. Various graph theoretic properties, including connectivity and diameter, of $\mathcal{C}(G)$ are studied in [10, 65, 75, 82]. Properties of $\mathcal{C}(G)$ defined on symmetric groups and finite simple groups can be found in [10] and [64]. The complement of $\mathcal{C}(G)$, known as *non-commuting graph* of G, is denoted by $\mathcal{NC}(G)$. Properties of $\mathcal{NC}(G)$ defined on various families of finite non-abelian groups can be found in [1, 34, 72, 90, 92]. The genus of $\mathcal{C}(G)$ and $\mathcal{NC}(G)$, for various families of finite non-abelian groups, are computed in [6, 35] and characterized all finite groups (upto isomorphism) such that $\mathcal{C}(G)$ and $\mathcal{NC}(G)$ are planar or toroidal. Spectral aspects of $\mathcal{C}(G)$ and $\mathcal{NC}(G)$ are studied in [3, 38, 39, 40, 41, 42, 43, 47, 48, 78]. It was shown in [43] that Conjecture 1.1.7 holds for commuting graphs of some families of finite non-abelian groups while the inequality in Question 1.1.8 does not hold for commuting graphs of finite non-abelian groups.

In [38], Dutta and Nath have found the commuting graph of the group G when $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ (p is prime) or $D_{2n} = \langle x, y : x^n = y^2 = 1, yxy = x^{-1}\rangle$, the dihedral group. In particular, we have the following results.

Result 1.3.1. [38, Theorem 2.1] If G is a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime, then $\mathcal{C}(G) \cong (p+1)K_{(p-1)n}$, where $|Z(G)| = n$.

Result 1.3.2. [38, Theorem 2.5] If G is a finite group such that $\frac{G}{Z(G)} \cong D_{2n}$ then $\mathcal{C}(G) \cong K_{(n-1)k} \sqcup nK_k$, where $|Z(G)| = k$.

Consequently we have $\mathcal{C}(Q_{4m}) = K_{2m-2} \sqcup mK_2$ and $\mathcal{C}(U_{6n}) = K_{2n} \sqcup 3K_n$, where $Q_{4m} = \langle x, y : x^{2m} = 1, x^m = y^2, y^{-1}xy = x^{-1}\rangle$, the dicyclic group, and $U_{6n} = \langle a, b : a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$. We also have the following result.

Result 1.3.3. [38, Proposition 2.8] If $G = M_{2nk} = \langle a, b : a^n = b^{2k} = 1, bab^{-1} = a^{-1}\rangle$, where $n > 2$ then

$$\mathcal{C}(G) = \begin{cases} K_{(n-1)k} \sqcup nK_k, & \text{if } n \text{ is odd} \\ K_{(\frac{n}{2}-1)2k} \sqcup \frac{n}{2}K_{2k}, & \text{if } n \text{ is even} \end{cases}$$

and hence

$$\mathcal{C}(D_{2n}) = \begin{cases} K_{n-1} \sqcup nK_1, & \text{if } n \text{ is odd} \\ K_{n-2} \sqcup \frac{n}{2}K_2, & \text{if } n \text{ is even.} \end{cases}$$

In [42], Dutta and Nath have found the commuting graph of the group G if $\frac{G}{Z(G)} \cong Sz(2)$, where $Sz(2)$ is the Suzuki group defined as $\langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle$.

Result 1.3.4. [42, Theorem 2.2] If G is a finite group such that $\frac{G}{Z(G)} \cong Sz(2)$ then $\mathcal{C}(G) = K_{4|Z(G)|} \sqcup 5K_{3|Z(G)|}$ and hence $\mathcal{C}(Sz(2)) = K_4 \sqcup 5K_3$.

In [70], Mirzargar et al. and in [46], Fasfous et al. have derived the commuting graphs of the groups V_{8n} and SD_{8n} respectively.

Result 1.3.5. [70, Example 2.4] If $G = V_{8n}$ then

$$\mathcal{C}(G) = \begin{cases} K_{2(2n-1)} \sqcup 2nK_2, & \text{if } n \text{ is odd} \\ K_{4(n-1)} \sqcup nK_4, & \text{if } n \text{ is even.} \end{cases}$$

Result 1.3.6. [46, Theorem 4.2] If $G = SD_{8n}$ then

$$\mathcal{C}(G) = \begin{cases} K_{4(n-1)} \sqcup nK_4, & \text{if } n \text{ is odd} \\ K_{2(2n-1)} \sqcup 2nK_2, & \text{if } n \text{ is even.} \end{cases}$$

In [35], Das and Nongsiang have given the structure of the commuting graph of the group $QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1} \rangle$, where $n \geq 4$.

Result 1.3.7. [35, Proposition 4.3] If $G = QD_{2^n}$, where $n \geq 4$, then $\mathcal{C}(G) = K_{2^{n-1}-2} \sqcup 2^{n-2}K_2$.

We have the following characterizations of finite non-abelian groups such that $\mathcal{C}(G)$ is either planar or toroidal.

Result 1.3.8. Let G be a finite non-abelian group. Then

(a) [35, Theorem 5.7] $\mathcal{C}(G)$ is planar if and only if G is isomorphic to either S_3, D_{10}, A_4, $Sz(2)$, S_4, A_5, D_8, Q_8, D_{12}, Q_{12}, $SL(2,3)$, $\mathbb{Z}_2 \times D_8$, $\mathbb{Z}_2 \times Q_8$, $SG(16,3)$, $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$, $D_8 * \mathbb{Z}_4$ or $\mathcal{M}_{16}$.

(b) [42, Theorem 3.3] $\mathcal{C}(G)$ is toroidal if and only if $\mathcal{C}(G)$ is projective if and only if G is isomorphic to either D_{14}, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_2 \times A_4$, $\mathbb{Z}_3 \times S_3$, D_{16}, Q_{16} or SD_{16}.

In 2006, Abdollahi, Akbari and Maimani [1] asked the following question.

Question 1.3.9. [1, Question 2.10] Let G be a group such that $\mathcal{NC}(G)$ has no infinite independent sets. Is it true that the independence number, $\alpha(\mathcal{NC}(G))$ is finite?

Abdollahi et al. [1] also answered Question 1.3.9 affirmatively for some classes of groups.

Result 1.3.10. [1, Theorem 2.11] Let G be a group such that $\mathcal{NC}(G)$ has no infinite independent sets. If G is an Engel group, a locally finite group, a locally solvable group, a linear group or a 2-group then G is a finite. In particular, $\alpha(\mathcal{NC}(G))$ is finite.

We also have the following characterizations of finite non-abelian groups such that $\mathcal{NC}(G)$ is either planar, toroidal or double-toroidal.

Result 1.3.11. Let G be a finite non-abelian group. Then

(a) [1, Proposition 2.3] $\mathcal{NC}(G)$ is planar if and only if $G \cong S_3$, D_8 or Q_8.

(b) [6, Theorem 3.2] $\mathcal{NC}(G)$ is not toroidal.

(c) [80, Proposition 5.5] $\mathcal{NC}(G)$ is double-toroidal if and only if $G \cong D_{10}, Q_{12}$ or D_{12}.

(d) [6, Theorem 3.3] $\mathcal{NC}(G)$ is not projective.

1.3.2 Nilpotent and non-nilpotent graph

The *nilpotent graph* of a finite non-nilpotent group G, denoted by $\mathcal{N}(G)$, is a simple undirected graph whose vertex set is $G \setminus \mathrm{Nil}(G)$ and two distinct vertices x and y are adjacent if $\langle x, y\rangle$ is nilpotent. The complement of $\mathcal{N}(G)$ is known as *non-nilpotent graph* of G and it is denoted by $\mathcal{NN}(G)$. In 2010, Abdollahi and Zarrin [2] have introduced and studied this graph. Several results on $\mathcal{N}(G)$ and $\mathcal{NN}(G)$ can be found in [2, 36, 80]. Nongsiang and Saikia [80] asked the following question similar to Question 1.3.9 for $\mathcal{NN}(G)$.

Question 1.3.12. [80, Question 3.17] Let G be a group such that $\mathcal{NN}(G)$ has no infinite independent sets. Is it true that the independence number, $\alpha(\mathcal{NN}(G))$ is finite?

Nongsiang and Saikia [80] also answered Question 1.3.12 affirmatively for some classes of finite groups. However, in general the answer is no (see [80, page 86]).

Result 1.3.13. [80, Theorem 3.18] Let G be a non-weakly nilpotent group such that $\mathcal{NN}(G)$ has no infinite independent sets. If $\mathrm{Nil}(G)$ is a subgroup and G is an Engel group, a locally finite group, a locally solvable group, a linear group or a 2-group then G is finite. In particular, $\alpha(\mathcal{NN}(G))$ is finite.

We have the following characterizations of finite non-nilpotent groups such that $\mathcal{N}(G)$ is either planar or toroidal.

Result 1.3.14. [36, Proposition 5.1] Let G be a finite non-nilpotent group. Then the following assertions hold:

(a) $\mathcal{N}(G)$ is planar if and only if G is isomorphic to S_3, D_{10}, D_{12}, Q_{12}, A_4, A_5, or $S_z(2)$.

(b) $\mathcal{N}(G)$ is toroidal if and only if G is isomorphic to $SL(2,3)$, D_{14}, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_2 \times A_4$, or $\mathbb{Z}_3 \times S_3$.

1.3.3 Solvable and non-solvable graph

The *solvable graph* of a finite non-solvable group G, denoted by $\mathcal{S}(G)$, is a simple undirected graph whose vertex set is $G \setminus \mathrm{Sol}(G)$ and two distinct vertices x and y are adjacent if $\langle x, y \rangle$ is solvable. The complement of $\mathcal{S}(G)$ is known as *non-solvable graph* of G and it is denoted by $\mathcal{NS}(G)$. This graph was introduced by Hai-Reuvan [59], in 2013. Results on $\mathcal{NS}(G)$ can be found in [8, 59]. Some graph realization results on $\mathcal{NS}(G)$ are as given below.

Result 1.3.15. [59, Corollary 3.17] Let G be a finite non-solvable group. Then $\mathcal{NS}(G)$ is irregular.

Result 1.3.16. [59, Corollary 3.14] $\mathcal{NS}(G)$ is not planar for any finite non-solvable group G.

Result 1.3.17. [59, Proposition 3.16] Let G be a finite non-solvable group. Then

$$|\deg(\mathcal{NS}(G))| \neq 2.$$

In Chapter 3 and Chapter 4, we shall consider solvable and non-solvable graph and obtain more results.

1.3.4 Conjugacy class graphs

The *conjugacy class* of an element $x \in G$, denoted by x^G, is the set given by $\{yxy^{-1} : y \in G\}$. Extending the notion of commuting graph, Herzog, Longobardi and Maj [62] introduced commuting conjugacy class graphs of groups in the year 2009. The *commuting conjugacy class graph* (or CCC-graph) of a group G, denoted by $\mathcal{CCC}(G)$, is a simple undirected graph whose vertex set is $\{x^G : x \in G \setminus Z(G)\}$ and two distinct vertices x^G and y^G are adjacent if $\langle x', y' \rangle$ is abelian for some $x' \in x^G$ and $y' \in y^G$. In 2020, Salahshour and Ashrafi [86] have obtained the following results on the structures of CCC-graph of the groups $D_{2n}\,(n \geq 3)$, $Q_{4m}\,(m \geq 2)$, $U_{(n,m)} = \langle x, y : x^{2n} = y^m = 1, x^{-1}yx = y^{-1}\rangle\,(m \geq 3$ and $n \geq 2)$, $V_{8n}, SD_{8n}, G(p,m,n) = \langle x, y : x^{p^m} = y^{p^n} = [x,y]^p = 1, [x,[x,y]] = [y,[x,y]] = 1\rangle$ (where p is any prime, $m \geq 1$ and $n \geq 1$).

Result 1.3.18. [86, Proposition 2.1] The commuting conjugacy class graph of dihedral groups are given by

$$\mathcal{CCC}(D_{2n}) = \begin{cases} K_{\frac{n-1}{2}} \sqcup K_1, & \text{if } n \text{ is odd} \\ K_{\frac{n}{2}-1} \sqcup 2K_1, & \text{if } n \text{ and } \frac{n}{2} \text{ are even} \\ K_{\frac{n}{2}-1} \sqcup K_2, & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd.} \end{cases}$$

Result 1.3.19. [86, Proposition 2.2] The commuting conjugacy class graph of the dicyclic group Q_{4m} is given by

$$\mathcal{CCC}(Q_{4m}) = \begin{cases} K_{m-1} \sqcup 2K_1, & \text{if } m \text{ is even} \\ K_{m-1} \sqcup K_2, & \text{if } m \text{ is odd.} \end{cases}$$

Result 1.3.20. [86, Proposition 2.3] The commuting conjugacy class graph of the group $U_{(n,m)}$ is given by

$$\mathcal{CCC}(U_{(n,m)}) = \begin{cases} 2K_n \sqcup K_{n(\frac{m}{2}-1)}, & \text{if } m \text{ is even} \\ K_n \sqcup K_{n(\frac{m-1}{2})}, & \text{if } m \text{ is odd.} \end{cases}$$

Result 1.3.21. [86, Proposition 2.4] The commuting conjugacy class graph of the group V_{8n} is given by

$$\mathcal{CCC}(V_{8n}) = \begin{cases} K_{2n-2} \sqcup 2K_2, & \text{if } n \text{ is even} \\ K_{2n-1} \sqcup 2K_1, & \text{if } n \text{ is odd.} \end{cases}$$

Result 1.3.22. [86, Proposition 2.5] The commuting conjugacy class graph of the semi-dihedral group SD_{8n} is given by

$$\mathcal{CCC}(SD_{8n}) = \begin{cases} K_{2n-1} \sqcup 2K_1, & \text{if } n \text{ is even} \\ K_{2n-2} \sqcup K_4, & \text{if } n \text{ is odd.} \end{cases}$$

Result 1.3.23. [86, Proposition 2.6] The commuting conjugacy class graph of the group $G(p,m,n)$ is given by

$$\mathcal{CCC}(G(p,m,n)) = (p^n - p^{n-1})K_{p^{m-n}(p^n-p^{n-1})} \sqcup K_{p^{n-1}(p^m-p^{m-1})} \sqcup K_{p^{m-1}(p^n-p^{n-1})}.$$

We write $\mathcal{CCC}(G \setminus \{1\})$ to denote the graph whose vertex set is $\{x^G : x \in G \setminus \{1\}\}$ and two distinct vertices x^G and y^G are adjacent if $\langle x', y' \rangle$ is abelian for some $x' \in x^G$ and $y' \in y^G$. Herzog, Longobardi and Maj [62] have obtained the following results on the distance between two vertices in $\mathcal{CCC}(G \setminus \{1\})$.

Result 1.3.24. Let G be a locally finite group.

(a) If $x, y \in G \setminus \{1\}$ are p-elements, where p is a prime, then x^G and y^G are connected in $\mathcal{CCC}(G \setminus \{1\})$ and $d(x^G, y^G) \leq 2$ ([62, Lemma 3]).

(b) If $x, y \in G\setminus\{1\}$ are of non-coprime orders then x^G and y^G are connected in $\mathcal{CCC}(G\setminus\{1\})$ and $d(x^G, y^G) \leq 4$. If either x or y is of prime power order then $d(x^G, y^G) \leq 3$ in $\mathcal{CCC}(G \setminus \{1\})$ ([62, Lemma 4]).

Result 1.3.25. [62, Lemma 5] Let G be a locally finite group and let $x, y \in G \setminus \{1\}$. Suppose that $p \mid o(x)$ and $q \mid o(y)$, where p, q are distinct primes, and that G contains an element z of order pq. Then the following statements hold:

(a) The classes x^G and y^G are connected in $\mathcal{CCC}(G \setminus \{1\})$ and $d(x^G, y^G) \leq 7$; moreover, $d(x^G, y^G) \leq 6$ if either x or y is of prime order.

(b) If G is locally solvable then $d(x^G, y^G) \leq 6$ in $\mathcal{CCC}(G \setminus \{1\})$; moreover, $d(x^G, y^G) \leq 5$ if either x or y is of prime order.

(c) If either a Sylow p-subgroup or a Sylow q-subgroup of G is a cyclic or generalized quaternion finite group then $d(x^G, y^G) \leq 5$ in $\mathcal{CCC}(G \setminus \{1\})$; moreover, $d(x^G, y^G) \leq 4$ if either x or y is of prime order.

(d) If both a Sylow p-subgroup and a Sylow q-subgroup of G are either cyclic or generalized quaternion finite groups then $d(x^G, y^G) \leq 3$ in $\mathcal{CCC}(G \setminus \{1\})$; moreover, $d(x^G, y^G) \leq 2$ if either x or y is of prime order.

Result 1.3.26. [62, Proposition 6] Let G be a finite group with a normal subgroup H and a subgroup S. Suppose that for each $h_1, h_2 \in H \setminus \{1\}$ and for each $s_1, s_2 \in S \setminus \{1\}$ the classes h_1^G and h_2^G are connected in $\mathcal{CCC}(G \setminus \{1\})$ and so are the classes s_1^G and s_2^G. Moreover, suppose that there exist $h \in H \setminus \{1\}$ and $k \in HS \setminus H$ such that h^G and k^G are connected in $\mathcal{CCC}(G \setminus \{1\})$. Then for each $x, y \in HS \setminus \{1\}$, x^G and y^G are connected in $\mathcal{CCC}(G \setminus \{1\})$.

In 2017, Mohammadian and Erfanian [73] have extended the notion of CCC-graph and introduced nilpotent conjugacy class graph of a group. The *nilpotent conjugacy class graph* (or NCC-graph) of a group G, denoted by $\mathcal{NCC}(G)$, is a simple undirected graph whose vertex set is $\{x^G : x \in G \setminus \{1\}\}$ and two distinct vertices x^G and y^G are adjacent if $\langle x', y' \rangle$ is nilpotent for some $x' \in x^G$ and $y' \in y^G$. We have the following results on the distance between two vertices in $\mathcal{NCC}(G)$.

Result 1.3.27. [73, Lemma 2.1] Let G be a locally finite group and p be a prime number. Then the following statements hold:

(a) If $x, y \in G \setminus \{1\}$ are p-elements then $d(x^G, y^G) \leq 1$ in $\mathcal{NCC}(G)$.

(b) If $x, y \in G \setminus \{1\}$ are of non-coprime orders then $d(x^G, y^G) \leq 3$ in $\mathcal{NCC}(G)$. Moreover $d(x^G, y^G) \leq 2$, whenever either x or y is of prime power order.

Result 1.3.28. [73, Lemma 2.2] Let G be a locally finite group and $x, y \in G \setminus \{1\}$. Suppose p and q are prime divisors of $o(x)$ and $o(y)$, respectively, and that G has an element of order pq. Then

(a) $d(x^G, y^G) \leq 5$ in $\mathcal{NCC}(G)$, and moreover $d(x^G, y^G) \leq 4$ if either x or y is of prime power order.

(b) If either a Sylow p-subgroup or a Sylow q-subgroup of G is a cyclic or generalized quaternion finite group then $d(x^G, y^G) \leq 4$ in $\mathcal{NCC}(G)$. Moreover, $d(x^G, y^G) \leq 3$ if either x or y is of prime order.

(c) If both a Sylow p-subgroup and a Sylow q-subgroup of G are either cyclic or generalized quaternion finite groups then $d(x^G, y^G) \leq 3$ in $\mathcal{NCC}(G)$. Moreover, $d(x^G, y^G) \leq 2$ if either x or y is of prime order.

Result 1.3.29. [73, Lemma 2.3] Let $G = HK$ be a finite group with a normal subgroup H and a subgroup K such that $\mathcal{NCC}(H)$ and $\mathcal{NCC}(K)$ are connected. If there exist two elements $h \in H \setminus \{1\}$ and $x \in G \setminus H$ such that h^G and x^G are connected in $\mathcal{NCC}(G)$ then $\mathcal{NCC}(G)$ is connected.

Following the notions of CCC-graph and NCC-graph, in Chapter 7, we shall introduce the notion of solvable conjugacy class graph (or SCC-graph) of a group.