Chapter 2

Genus of commuting graphs of certain finite groups

In 2015, Afkhami, Farrokhi and Khashyarmanesh [6] and in 2016, Das and Nongsiang [35] have characterized finite non-abelian groups such that their commuting graphs are planar or toroidal. Recently, Nongsiang [81] has characterized finite non-abelian groups whose commuting graphs are double-toroidal or triple-toroidal. In this Chapter, we compute $\gamma(\mathcal{C}(G))$, the genus of commuting graph of G, for the classes of finite groups such that their central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (where p is a prime), $D_{2n} = \langle a, b : a^n = b^2 = 1, bab^{-1} = a^{-1}\rangle$ (where $n \geq 2$) or $Sz(2) = \langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2\rangle$. We also find conditions such that $\gamma(\mathcal{C}(G)) = 4, 5$ or 6 for the above mentioned groups. As a consequence of our results, we characterize groups of order p^3, the meta-abelian groups $M_{2nk} = \langle a, b : a^n = b^{2k} = 1, bab^{-1} = a^{-1}\rangle$, D_{2n}, $Q_{4m} = \langle a, b : a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1}\rangle$ and $U_{6n} = \langle a, b : a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$ such that their commuting graphs have genus $4, 5$ or 6. It is worth mentioning that the spectral aspects of commuting graphs of these classes of groups have been described in [35, 38, 42]. This chapter is based on our paper [20].

2.1 Genus of $\mathcal{C}(G)$

We begin this section by computing genus of $\mathcal{C}(G)$ for the groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

Theorem 2.1.1. *If G is a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) then $\gamma(\mathcal{C}(G)) = 0$ or $\gamma(\mathcal{C}(G)) = (p+1)\left\lceil \frac{1}{12}((p-1)n-3)((p-1)n-4)\right\rceil$ according as $(p-1)n \leq 2$*

or $(p-1)n \geq 3$, *where* $n = |Z(G)|$.

Proof. By Result 1.3.1 we have $\mathcal{C}(G) = (p+1)K_{(p-1)n}$. If $(p-1)n \leq 2$ then $\gamma(\mathcal{C}(G)) = 0$. If $(p-1)n \geq 3$ then, by (1.1.b) and Result 1.1.4, $\gamma(\mathcal{C}(G)) = (p+1)\gamma(K_{(p-1)n}) = (p+1)\left\lceil \frac{1}{12}((p-1)n-3)((p-1)n-4)\right\rceil$. □

Corollary 2.1.2. *If* G *is a non-abelian group of order* p^3 *(for any prime* p*) then* $\gamma(\mathcal{C}(G)) = 0$ *or* $\gamma(\mathcal{C}(G)) = (p+1)\left\lceil \frac{1}{12}((p-1)p-3)((p-1)p-4)\right\rceil$ *according as* $p = 2$ *or* $p \geq 3$.

Proof. We have $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, $p(p-1) = 2$ or $p(p-1) \geq 6$ according as $p = 2$ or $p \geq 3$. Hence, the result follows from Theorem 2.1.1. □

Corollary 2.1.3. *If* G *is a finite* 4*-centralizer group then* $\gamma(\mathcal{C}(G)) = 0$ *or*

$$\gamma(\mathcal{C}(G)) = 3\left\lceil \frac{1}{12}(n-3)(n-4)\right\rceil$$

according as $n \leq 2$ *or* $n \geq 3$, *where* $n = |Z(G)|$.

Proof. If G is a 4-centralizer group then by Result 1.2.1 we have $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, the result follows from Theorem 2.1.1. □

Corollary 2.1.4. *If* G *is a finite* $(p+2)$*-centralizer* p*-group (for any prime* p*) then* $\gamma(\mathcal{C}(G)) = 0$ *or* $\gamma(\mathcal{C}(G)) = (p+1)\left\lceil \frac{1}{12}((p-1)n-3)((p-1)n-4)\right\rceil$ *according as* $(p-1)n \leq 2$ *or* $(p-1)n \geq 3$, *where* $n = |Z(G)|$.

Proof. If G is a finite $(p+2)$-centralizer p-group (for any prime p) then by Result 1.2.3 we have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, the result follows from Theorem 2.1.1. □

Corollary 2.1.5. *If* G *is a finite* 5*-centralizer group then* $\gamma(\mathcal{C}(G)) = 0$ *or*

$$\gamma(\mathcal{C}(G)) = 4\left\lceil \frac{1}{12}(2n-3)(2n-4)\right\rceil$$

according as $n = 1$ *or* $n \geq 2$, *where* $n = |Z(G)|$.

Proof. If G is a finite 5-centralizer group then by Result 1.2.2 we have $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore, $(p-1)n = 2$ or $(p-1)n \geq 4$ according as $n = 1$ or $n \geq 2$. Hence, the result follows from Theorem 2.1.1. □

Corollary 2.1.6. *If* G *is a finite group and* $\Pr(G) = \frac{p^2+p-1}{p^3}$, *where* p *is the smallest prime divisor of the order of* G, *then* $\gamma(\mathcal{C}(G)) = 0$ *or*

$$\gamma(\mathcal{C}(G)) = (p+1)\left\lceil \frac{1}{12}((p-1)n-3)((p-1)n-4)\right\rceil$$

according as $(p-1)n \leq 2$ *or* $(p-1)n \geq 3$, *where* $n = |Z(G)|$.

Proof. If $\Pr(G) = \frac{p^2+p-1}{p^3}$ then by Result 1.2.17, we have $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence, the result follows from Theorem 2.1.1. □

Theorem 2.1.7. *Let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2n}$ $(n \geq 2)$. Then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{if } k = 1, n = 2, 3 \text{ and } k = n = 2 \\ \lceil \frac{1}{12}((n-1)k-3)((n-1)k-4) \rceil, & \text{if } k = 1, n \geq 4 \text{ and } k = 2, n \geq 3 \\ \lceil \frac{1}{12}((n-1)k-3)((n-1)k-4) \rceil + n \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil, & \text{if } k \geq 3, n \geq 2 \end{cases},$$

where $k = |Z(G)|$.

Proof. By Result 1.3.2 we have $\mathcal{C}(G) = K_{(n-1)k} \sqcup nK_k$. Therefore,

$$\mathcal{C}(G) = \begin{cases} K_1 \sqcup 2K_1, & \text{if } k = 1 \text{ and } n = 2 \\ K_2 \sqcup 3K_1, & \text{if } k = 1 \text{ and } n = 3 \\ K_2 \sqcup 2K_2, & \text{if } k = n = 2 \end{cases}$$

and so $\gamma(\mathcal{C}(G)) = 0$ in these cases. We also have

$$\mathcal{C}(G) = \begin{cases} K_{n-1} \sqcup nK_1, & \text{if } k = 1 \text{ and } n \geq 4 \\ K_{2(n-1)} \sqcup nK_2, & \text{if } k = 2 \text{ and } n \geq 3. \end{cases}$$

In these cases, $(n-1)k \geq 3$ and so by Result 1.1.4 and (1.1.b) we get

$$\gamma(\mathcal{C}(G)) = \lceil \frac{1}{12}((n-1)k-3)((n-1)k-4) \rceil.$$

If $k \geq 3$ and $n \geq 2$ then $(n-1)k \geq 3$. Therefore, by Result 1.1.4 and (1.1.b) we get the required expression for $\gamma(\mathcal{C}(G))$. □

Corollary 2.1.8. *Let $G = M_{2nk}$, where $n > 2$ and $k \geq 1$. If n is odd then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{if } k = 1, n = 3 \\ \left\lceil \frac{1}{12}((n-1)k-3)((n-1)k-4) \right\rceil, & \text{if } k = 1, n \geq 5 \\ & \text{or } k = 2, n \geq 3 \\ \left\lceil \frac{1}{12}((n-1)k-3)((n-1)k-4) \right\rceil + n \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil, & \text{if } k \geq 3, n \geq 3. \end{cases}$$

If n is even then

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{if } k=1, n=4 \\ \left\lceil \frac{1}{12}((n-2)k-3)((n-2)k-4) \right\rceil, & \text{if } k=1, n\geq 6 \\ \left\lceil \frac{1}{12}((n-2)k-3)((n-2)k-4) \right\rceil + \frac{n}{2}\left\lceil \frac{1}{12}(2k-3)(2k-4) \right\rceil, & \text{if } k\geq 2, n\geq 4 \end{cases}$$

Proof. We have $\frac{M_{2nk}}{Z(M_{2nk})} \cong D_{2n}$ or D_n depending on n is odd or even respectively. Also, $|Z(M_{2nk})| = k$ or $2k$ for n odd or even respectively. Therefore, if n is odd then the result follows from Theorem 2.1.7. If n is even then replacing n by $\frac{n}{2}$ and k by $2k$ in Theorem 2.1.7 we get the required result. □

Corollary 2.1.9. *Let $G = D_{2n}$ $(n \geq 3)$. Then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{if } n=3,4 \\ \left\lceil \frac{1}{12}(n-4)(n-5) \right\rceil, & \text{if } n \text{ is odd and } n\geq 5 \\ \left\lceil \frac{1}{12}(n-5)(n-6) \right\rceil, & \text{if } n \text{ is even and } n\geq 6. \end{cases}$$

Proof. We have $M_{2nk} = D_{2n}$ if $k = 1$. Hence, the result follows from Corollary 2.1.8. □

Corollary 2.1.10. *Let $G = Q_{4m}$ $(m \geq 3)$. Then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{if } m=2 \\ \left\lceil \frac{1}{12}(2m-5)(2m-6) \right\rceil, & \text{if } m\geq 3. \end{cases}$$

Proof. We have $|Z(Q_{4m})| = 2$ and $\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}$. Hence, the result follows from Theorem 2.1.7. □

Corollary 2.1.11. *Let $G = U_{6n}$. Then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{if } n=1,2 \\ 3\left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil + \left\lceil \frac{1}{12}(2n-3)(2n-4) \right\rceil, & \text{if } n\geq 3. \end{cases}$$

Proof. We have $Z(U_{6n}) = \langle a^2 \rangle$ and $\frac{U_{6n}}{Z(U_{6n})} \cong D_6$. Hence, the result follows from Theorem 2.1.7 considering $m = 3$. □

Corollary 2.1.12. *If G is a finite group such that $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}, \frac{7}{16}\}$ then $\gamma(\mathcal{C}(G)) \in \{0, 1, 2, 6, \lceil \frac{1}{2}(2n-1)(3n-2) \rceil + 7\lceil \frac{1}{12}(n-3)(n-4) \rceil, \lceil \frac{1}{3}(n-1)(4n-3) \rceil + 5\lceil \frac{1}{12}(n-3)(n-4) \rceil, \lceil \frac{1}{4}(n-1)(3n-4) \rceil + 4\lceil \frac{1}{12}(n-3)(n-4) \rceil, \lceil \frac{1}{6}(n-2)(2n-3) \rceil + 3\lceil \frac{1}{12}(n-3)(n-4) \rceil, 3\lceil \frac{1}{12}(n-3)(n-4) \rceil, 4\lceil \frac{1}{6}(n-2)(2n-3) \rceil\}$, where $n = |Z(G)| \geq 3$.*

Proof. If $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}, \frac{7}{16}\}$ then as given in Result 1.2.16, we have $\frac{G}{Z(G)}$ is isomorphic to one of the groups in $\{D_{14}, D_{10}, D_8, D_6, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3\}$. Let $n = |Z(G)|$.

If $\frac{G}{Z(G)} \cong D_{14}$ then considering $m = 7$ in Theorem 2.1.7, we get

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(6n-3)(6n-4) \right\rceil$$

if $n = 1, 2$. Therefore, $\gamma(\mathcal{C}(G)) = 1$ or 6 according as $n = 1$ or 2. If $n \geq 3$ then we get

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{2}(2n-1)(3n-2) \right\rceil + 7\left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil.$$

If $\frac{G}{Z(G)} \cong D_{10}$ then considering $m = 5$ in Theorem 2.1.7, we get

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(4n-3)(4n-4) \right\rceil$$

if $n = 1, 2$. Therefore, $\gamma(\mathcal{C}(G)) = 0$ or 2 according as $n = 1$ or 2. If $n \geq 3$ then we get

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{3}(4n-3)(n-1) \right\rceil + 5\left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil.$$

If $\frac{G}{Z(G)} \cong D_8$ then considering $m = 4$ in Theorem 2.1.7, we get

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(3n-3)(3n-4) \right\rceil$$

if $n = 1, 2$. Therefore, $\gamma(\mathcal{C}(G)) = 0$ or 1 according as $n = 1$ or 2. If $n \geq 3$ then we get

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{4}(n-1)(3n-4) \right\rceil + 4\left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil.$$

If $\frac{G}{Z(G)} \cong D_6$ then considering $m = 3$ in Theorem 2.1.7, we get $\gamma(\mathcal{C}(G)) = 0$ if $n = 1$. If $n = 2$ then $\gamma(\mathcal{C}(G)) = \lceil \frac{1}{6}(2n-3)(n-2) \rceil = 0$. If $n \geq 3$ then we get

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{6}(2n-3)(n-2) \right\rceil + 3\left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil.$$

If $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ then considering $p = 2$ in Theorem 2.1.1 we get $\gamma(\mathcal{C}(G)) = 0$ if $n = 1, 2$. If $n \geq 3$ we get

$$\gamma(\mathcal{C}(G)) = 3\left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil.$$

If $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ then considering $p = 3$ in Theorem 2.1.1 we get $\gamma(\mathcal{C}(G)) = 0$ if $n = 1$ or 2. If $n \geq 3$ we get

$$\gamma(\mathcal{C}(G)) = 4\left\lceil \frac{1}{6}(2n-3)(n-2) \right\rceil.$$

□

Theorem 2.1.13. *If G is a finite group such that $\frac{G}{Z(G)} \cong Sz(2)$ then*

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{3}(n-1)(4n-3) \right\rceil + 5\left\lceil \frac{1}{4}(n-1)(3n-4) \right\rceil,$$

where $n = |Z(G)|$.

Proof. By Result 1.3.4 we have $\mathcal{C}(G) = K_{4n} \sqcup 5K_{3n}$. Therefore by (1.1.$b$) and Result 1.1.4,

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \gamma(K_{4n}) + 5\gamma(K_{3n})\\ &= \left\lceil \frac{1}{12}(4n-3)(4n-4) \right\rceil + 5\left\lceil \frac{1}{12}(3n-3)(3n-4) \right\rceil\\ &= \left\lceil \frac{1}{3}(n-1)(4n-3) \right\rceil + 5\left\lceil \frac{1}{4}(n-1)(3n-4) \right\rceil.\end{aligned}$$

□

Corollary 2.1.14. *If $G = Sz(2)$ then $\gamma(\mathcal{C}(G)) = 0$.*

Proof. If $G = Sz(2)$ then we have $|Z(G)| = 1$. Hence, the result follows from Theorem 2.1.13. □

Theorem 2.1.15. *If $G = V_{8n} = \langle a, b : a^{2n} = b^4 = 1, b^{-1}ab^{-1} = bab = a^{-1}\rangle$ then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{if } n = 1, 2\\ \left\lceil \frac{1}{6}(2n-3)(4n-5) \right\rceil, & \text{if } n \geq 3 \text{ and } n \text{ is odd}\\ \left\lceil \frac{1}{3}(n-2)(4n-7) \right\rceil, & \text{if } n \geq 4 \text{ and } n \text{ is even.}\end{cases}$$

Proof. If n is odd then by Result 1.3.5 we have $\mathcal{C}(G) = K_{2(2n-1)} \sqcup 2nK_2$. If $n = 1$ then $2(2n-1) = 2$ and so $\gamma(\mathcal{C}(G)) = 0$. If $n \geq 3$ then by (1.1.b) and Result 1.1.4, we get

$$\gamma(\mathcal{C}(G)) = \gamma(K_{2(2n-1)}) + 2n\gamma(K_2) = \left\lceil \frac{1}{6}(2n-3)(4n-5) \right\rceil.$$

If n is even then by Result 1.3.5 we have $\mathcal{C}(G) = K_{4(n-1)} \sqcup nK_4$. If $n = 2$ then $4(n-1) = 4$ and so $\gamma(\mathcal{C}(G)) = 0$. If $n \geq 4$ then by (1.1.b) and Result 1.1.4, we get

$$\gamma(\mathcal{C}(G)) = \gamma(K_{4(n-1)}) + n\gamma(K_4) = \left\lceil \frac{1}{3}(n-2)(4n-7) \right\rceil.$$

□

Theorem 2.1.16. *If $G = QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1}\rangle$, where $n \geq 4$, then $\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(2^{n-1}-5)(2^{n-1}-6) \right\rceil$.*

Proof. By Result 1.3.7 we have $\mathcal{C}(G) = K_{2^{n-1}-2} \sqcup 2^{n-2}K_2$. Therefore by (1.1.$b$) and Result 1.1.4,

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \gamma(K_{2^{n-1}-2}) + 2^{n-2}\gamma(K_2)\\ &= \left\lceil \frac{1}{12}(2^{n-1}-5)(2^{n-1}-6)\right\rceil.\end{aligned}$$

□

Theorem 2.1.17. *If $G = SD_{8n}$ then*

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{if } n = 1\\ \left\lceil \frac{1}{3}(n-2)(4n-7)\right\rceil, & \text{if } n \text{ is odd and } n \geq 3\\ \left\lceil \frac{1}{6}(2n-3)(4n-5)\right\rceil, & \text{if } n \text{ is even and } n \geq 2.\end{cases}$$

Proof. If n is odd then by Result 1.3.6, we have $\mathcal{C}(G) = K_{4(n-1)} \sqcup nK_4$. If $n = 1$ then $4(n-1) = 0$. Therefore $\gamma(\mathcal{C}(G)) = 0$. If $n \geq 3$ then by (1.1.b) and Result 1.1.4,

$$\gamma(\mathcal{C}(G)) = \gamma(K_{4(n-1)}) + n\gamma(K_4) = \left\lceil \frac{1}{3}(n-2)(4n-7)\right\rceil.$$

If n is even then by Result 1.3.6, we have $\mathcal{C}(G) = K_{2(2n-1)} \sqcup 2nK_2$. Therefore by (1.1.$b$) and Result 1.1.4,

$$\gamma(\mathcal{C}(G)) = \gamma(K_{2(2n-1)}) + 2n\gamma(K_2) = \left\lceil \frac{1}{6}(2n-3)(4n-5)\right\rceil.$$

□

2.2 Some consequences

Using the results on $\gamma(\mathcal{C}(G))$ obtained in Section 2.1, in this section we derive necessary and sufficient conditions such that $\gamma(\mathcal{C}(G)) = 4, 5$ and 6 respectively.

Theorem 2.2.1. *If G is a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) then*

(a) $\gamma(\mathcal{C}(G)) = 4$ *if and only if* $p = 3$ *and* $|Z(G)| = 3$.

(b) $\gamma(\mathcal{C}(G)) \neq 5$.

(c) $\gamma(\mathcal{C}(G)) = 6$ *if and only if* $p = 2$ *and* $|Z(G)| = 8$.

(d) $\gamma(\mathcal{C}(G)) \geq 7$ *for* $p = 2, |Z(G)| \geq 9; p = 3, |Z(G)| \geq 4;$ *or* $p \geq 5, |Z(G)| \geq 1$.

Proof. By Theorem 2.1.1, we have

$$\gamma(\mathcal{C}(G)) = (p+1)\left\lceil \frac{1}{12}((p-1)n-3)((p-1)n-4)\right\rceil,$$

for $(p-1)n \geq 3$, where $|Z(G)| = n$.

If $p = 2$ and $n \geq 3$ then

$$\gamma(\mathcal{C}(G)) = 3\left\lceil \frac{1}{12}(n-3)(n-4)\right\rceil.$$

For $3 \leq n \leq 7$, it can be seen that $\gamma(\mathcal{C}(G)) \leq 3$. For $n = 8$, we have $\gamma(\mathcal{C}(G)) = 6$. If $n \geq 9$ then

$$\frac{1}{12}(n-3)(n-4) = \frac{1}{12}(n(n-9)+2n+12) > 2.$$

Hence $\gamma(\mathcal{C}(G)) = 3\left\lceil \frac{1}{12}(n-3)(n-4)\right\rceil > 6$.

If $p = 3$ and $n \geq 2$ then

$$\gamma(\mathcal{C}(G)) = 4\left\lceil \frac{1}{12}(2n-3)(2n-4)\right\rceil = 4\left\lceil \frac{1}{6}(n-2)(2n-3)\right\rceil.$$

If $n = 1$ then $(p-1)n = 2$. Therefore by Theorem 2.1.1, $\gamma(\mathcal{C}(G)) = 0$. For $n = 2$, we have $\gamma(\mathcal{C}(G)) = 0$. For $n = 3$, we have $\gamma(\mathcal{C}(G)) = 4$. If $n \geq 4$ then

$$\frac{1}{6}(2n-3)(n-2) = \frac{1}{6}(2n(n-4)+n+6) > 1.$$

Hence $\gamma(\mathcal{C}(G)) = 4\left\lceil \frac{1}{6}(2n-3)(n-2)\right\rceil \geq 8$.

If $p = 5$ then by Result 1.3.1 we have $\mathcal{C}(G) = 6K_{4n}$. For $n = 1$, we have $\gamma(\mathcal{C}(G)) = 0$. If $n \geq 2$ then $6K_{4n}$ has a subgraph $6K_8$. Since $\gamma(6K_8) \geq 7$, by (1.1.a), $\gamma(\mathcal{C}(G)) \geq 7$.

If $p \geq 7$ then by Result 1.3.1 we have $\mathcal{C}(G) = (p+1)K_{6n}$ which has a subgraph $8K_6$ for $n \geq 1$. Since $\gamma(8K_6) \geq 7$, by (1.1.a), $\gamma(\mathcal{C}(G)) \geq 7$. □

Corollary 2.2.2. *If G is a non-abelian group of order p^3 (for any prime p) then*

(a) $\gamma(\mathcal{C}(G)) = 4$ *if and only if* $p = 3$.

(b) $\gamma(\mathcal{C}(G)) \geq 7$ *for any prime* $p \geq 5$.

Corollary 2.2.2 can be proved by using Theorem 2.2.1 noting the fact that if G is a non-abelian group of order p^3 then $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Theorem 2.2.3. *If G is a finite group such that $\frac{G}{Z(G)} \cong D_{2n}$, where $n \geq 2$, then*

(a) $\gamma(\mathcal{C}(G)) = 4$ *if and only if* $n = 6, |Z(G)| = 2;$ *or* $n = 11, |Z(G)| = 1$.

(b) $\gamma(\mathcal{C}(G)) \neq 5$.

(c) $\gamma(\mathcal{C}(G)) = 6$ *if and only if* $n = 2, |Z(G)| = 8; n = 4, |Z(G)| = 4; n = 5, |Z(G)| = 3; n = 7, |Z(G)| = 2;$ *or* $n = 13, |Z(G)| = 1$.

(d) $\gamma(\mathcal{C}(G)) \geq 7$ *for* $n = 2, |Z(G)| \geq 9; n = 3, |Z(G)| \geq 5; n = 4, |Z(G)| \geq 5; n = 5, |Z(G)| \geq 4; n = 6, |Z(G)| \geq 3; n = 7, |Z(G)| \geq 3; n = 8, |Z(G)| \geq 2; n = 9, |Z(G)| \geq 2; n = 10, |Z(G)| \geq 2; n = 11, |Z(G)| \geq 2; n = 12, |Z(G)| \geq 2; n = 13, |Z(G)| \geq 2;$ *or* $n \geq 14, |Z(G)| \geq 1$.

Proof. By Theorem 2.1.7 we have

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{if } k = 1, n = 2, 3 \text{ and } k = n = 2 \\ \lceil \frac{1}{12}((n-1)k-3)((n-1)k-4) \rceil, & \text{if } k = 1, n \geq 4 \text{ and } k = 2, n \geq 3 \\ \lceil \frac{1}{12}((n-1)k-3)((n-1)k-4) \rceil + n \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil, & \text{if } k \geq 3, n \geq 2 , \end{cases}$$

where $k = |Z(G)|$. We consider the following cases.

Case 1. If $n = 2$ then we have $\gamma(\mathcal{C}(G)) = 0$ for $k = 1$ and $k = 2$. For $k \geq 3$

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil + 2 \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil = 3 \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil .$$

For $k \leq 7$, it can be seen that $\gamma(\mathcal{C}(G)) \leq 3$. For $k = 8$, we have $\gamma(\mathcal{C}(G)) = 6$. If $k \geq 9$ then

$$\frac{1}{12}(k-3)(k-4) = \frac{1}{12}(k^2 - 7k + 12) = \frac{1}{12}(k(k-9) + 2k + 12) > 2.$$

Hence $\gamma(\mathcal{C}(G)) = 3 \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil > 6$.

Case 2. If $n = 3$ then we have $\gamma(\mathcal{C}(G)) = 0$ for $k = 1$. For $k = 2$, we have

$$\gamma(\mathcal{C}(G)) = \lceil \frac{1}{12}((n-1)k-3)((n-1)k-4) \rceil = 0.$$

For $k \geq 3$, we have

$$\begin{aligned} \gamma(\mathcal{C}(G)) &= \left\lceil \frac{1}{12}(2k-3)(2k-4) \right\rceil + 3 \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil \\ &= \left\lceil \frac{1}{6}(k-2)(2k-3) \right\rceil + 3 \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil . \end{aligned}$$

For $k = 3, 4$ we have $\gamma(\mathcal{C}(G)) = 1, 2$ respectively. If $k \geq 5$ then

$$\frac{1}{6}(k-2)(2k-3) = \frac{1}{6}(2k^2 - 7k + 6) = \frac{2k(k-5)}{6} + \frac{k+2}{2} > 3,$$

also $k-3>0$ and $k-4>0$, which gives $\frac{1}{12}(k-3)(k-4)>0$. Therefore,

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{6}(k-2)(2k-3) \right\rceil + 3\left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil > 7$$

.

Case 3. If $n=4$ then we have $\gamma(\mathcal{C}(G)) = 0, 1$ for $k = 1, 2$ respectively. For $k \geq 3$ we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(3k-3)(3k-4) \right\rceil + 4\left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil.$$

For $k = 3, 4$ we have $\gamma(\mathcal{C}(G)) = 3, 6$ respectively. If $k \geq 5$ then

$$\gamma(\mathcal{C}(G)) > \left\lceil \frac{1}{12}(3k-3)(3k-4) \right\rceil = \left\lceil \frac{1}{4}(3k^2-7k+4) \right\rceil = \left\lceil \frac{3k(k-5)}{4} + (2k+1) \right\rceil \geq 11.$$

Case 4. If $n=5$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(4k-3)(4k-4) \right\rceil$$

for $k \leq 2$. Therefore $\gamma(\mathcal{C}(G)) = 0, 2$ for $k = 1, 2$ respectively. If $k \geq 3$ we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(4k-3)(4k-4) \right\rceil + 5\left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil.$$

For $k=3$, we have $\gamma(\mathcal{C}(G)) = 6$. If $k \geq 4$ then

$$\gamma(\mathcal{C}(G)) \geq \left\lceil \frac{1}{12}(4k-3)(4k-4) \right\rceil = \left\lceil \frac{1}{3}(4k^2-7k+3) \right\rceil = \left\lceil \frac{4k(k-4)}{3} + \frac{9k+3}{3} \right\rceil \geq 13.$$

Case 5. If $n=6$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(5k-3)(5k-4) \right\rceil$$

for $k \leq 2$. Therefore $\gamma(\mathcal{C}(G)) = 1, 4$ for $k = 1, 2$ respectively. If $k \geq 3$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(5k-3)(5k-4) \right\rceil + 6\left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil.$$

Now,

$$\left\lceil \frac{1}{12}(5k-3)(5k-4) \right\rceil = \left\lceil \frac{1}{12}(25k^2-35k+12) \right\rceil = \left\lceil \frac{25k(k-3)}{12} + \frac{40k+12}{12} \right\rceil \geq 11,$$

for $k \geq 3$. Therefore $\gamma(\mathcal{C}(G)) \geq 11$.

Case 6. If $n=7$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(6k-3)(6k-4) \right\rceil$$

for $k \leq 2$. Therefore $\gamma(\mathcal{C}(G)) = 1, 6$ for $k = 1, 2$ respectively. If $k \geq 3$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(6k-3)(6k-4) \right\rceil + 7\left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil.$$

Now,

$$\left\lceil \frac{1}{12}(6k-3)(6k-4) \right\rceil = \left\lceil \frac{6k^2-7k+2}{2} \right\rceil = \left\lceil \frac{6k(k-3)+11k+2}{2} \right\rceil \geq 18,$$

for $k \geq 3$. Therefore $\gamma(\mathcal{C}(G)) \geq 18$.

Case 7. If $n = 8$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(7k-3)(7k-4) \right\rceil$$

for $k \leq 2$. Therefore $\gamma(\mathcal{C}(G)) = 1, 10$ for $k = 1, 2$ respectively. If $k \geq 3$ we have

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \left\lceil \frac{1}{12}(7k-3)(7k-4) \right\rceil + 8\left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil \\ &\geq \left\lceil \frac{1}{12}(7k-3)(7k-4) \right\rceil \\ &= \left\lceil \frac{1}{12}(49k^2-49k+12) \right\rceil = \left\lceil \frac{1}{12}(49k(k-3)+(98k+12) \right\rceil \geq 26.\end{aligned}$$

Case 8. If $n = 9$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(8k-3)(8k-4) \right\rceil$$

for $k \leq 2$. Therefore $\gamma(\mathcal{C}(G)) = 2, 13$ for $k = 1, 2$ respectively. For $k \geq 3$ then we have

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \left\lceil \frac{1}{12}(8k-3)(8k-4) \right\rceil + 9\left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil \\ &\geq \left\lceil \frac{1}{12}(8k-3)(8k-4) \right\rceil \\ &= \left\lceil \frac{1}{12}(64k^2-56k+12) \right\rceil = \left\lceil \frac{1}{12}(64k(k-3)+(136k+12) \right\rceil = 35,\end{aligned}$$

therefore $\gamma(\mathcal{C}(G)) \geq 35$.

Case 9. If $n = 10$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(9k-3)(9k-4) \right\rceil$$

for $k \leq 2$. Therefore $\gamma(\mathcal{C}(G)) = 3, 18$ for $k = 1, 2$ respectively. For $k \geq 3$ we have

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \left\lceil \frac{1}{12}(9k-3)(9k-4) \right\rceil + 10\left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil \\ &\geq \left\lceil \frac{1}{12}(9k-3)(9k-4) \right\rceil \\ &= \left\lceil \frac{1}{12}(81k^2-63k+12) \right\rceil = \left\lceil \frac{1}{12}(81k(k-3)+(180k+12)) \right\rceil \geq 46,\end{aligned}$$

therefore $\gamma(\mathcal{C}(G)) \geq 46$.

Case 10. If $n = 11$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(10k-3)(10k-4) \right\rceil$$

for $k \leq 2$. Therefore $\gamma(\mathcal{C}(G)) = 4, 23$ for $k = 1, 2$ respectively. For $k \geq 3$

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \left\lceil \frac{1}{12}(10k-3)(10k-4) \right\rceil + 11 \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil \\ &\geq \left\lceil \frac{1}{12}(10k-3)(10k-4) \right\rceil \\ &= \left\lceil \frac{1}{12}(100k^2 - 70k + 12) \right\rceil = \left\lceil \frac{1}{12}(100k(k-3) + (230k + 12)) \right\rceil \geq 59,\end{aligned}$$

therefore $\gamma(\mathcal{C}(G)) \geq 59$.

Case 11. If $n = 12$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(11k-3)(11k-4) \right\rceil$$

for $k \leq 2$. Note that $k \neq 1$. Otherwise $G \cong D_{24}$ and so $k = |Z(G)| = 2$, a contradiction. If $k = 2$ then $\gamma(\mathcal{C}(G)) = 29$. For $k \geq 3$

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \left\lceil \frac{1}{12}(11k-3)(11k-4) \right\rceil + 12 \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil \\ &\geq \left\lceil \frac{1}{12}(11k-3)(11k-4) \right\rceil \\ &= \left\lceil \frac{1}{12}(121k^2 - 77k + 12) \right\rceil = \left\lceil \frac{1}{12}(121k(k-3) + (286k + 12)) \right\rceil \geq 73,\end{aligned}$$

therefore $\gamma(\mathcal{C}(G)) \geq 73$.

Case 12. If $n = 13$ then we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(12k-3)(12k-4) \right\rceil$$

for $k \leq 2$. Therefore $\gamma(\mathcal{C}(G)) = 6, 35$ for $k = 1, 2$ respectively. For $k \geq 3$

$$\begin{aligned}\gamma(\mathcal{C}(G)) &= \left\lceil \frac{1}{12}(12k-3)(12k-4) \right\rceil + 13 \left\lceil \frac{1}{12}(k-3)(k-4) \right\rceil \\ &\geq \left\lceil \frac{1}{12}(12k-3)(12k-4) \right\rceil \\ &= \lceil 12k^2 - 7k + 1 \rceil = \lceil 12k(k-3) + (29k + 1) \rceil \geq 88\end{aligned}$$

therefore $\gamma(\mathcal{C}(G)) \geq 88$.

Case 13. If $n \geq 14$ then by Result 1.3.2 we have

$$\mathcal{C}(G) = K_{(n-1)k} \sqcup nK_k.$$

Therefore $K_{13} \sqcup 14K_1$ is a subgraph of $K_{(n-1)k} \sqcup nK_k$ for every $k \geq 1$. We know the genus of $K_{13} \sqcup 14K_1$ is equal to 15. Hence by (1.1.a), $\gamma(\mathcal{C}(G)) \geq 15$. □

Corollary 2.2.4. *If $G = M_{2nk}$, where $n > 2$, then*

(a) $\gamma(\mathcal{C}(G)) = 4$ *if and only if* $n = 11, k = 1;$ *or* $n = 12, k = 1.$

(b) $\gamma(\mathcal{C}(G)) \neq 5.$

(c) $\gamma(\mathcal{C}(G)) = 6$ *if and only if* $n = 4, k = 4; n = 5, k = 3; n = 7, k = 2; n = 8, k = 2; n = 13, k = 1;$ *or* $n = 14, k = 1.$

(d) $\gamma(\mathcal{C}(G)) \geq 7$ *for* $n = 3, k \geq 5; n = 4, k \geq 5; n = 5, k \geq 4; n = 6, k \geq 3; n = 7, k \geq 3; n = 8, k \geq 3; n = 9, k \geq 2; n = 10, k \geq 2; n = 11, k \geq 2; n = 12, k \geq 2; n = 13, k \geq 2; n = 14, k \geq 2;$ *or* $n \geq 15, k \geq 1.$

Corollary 2.2.4 can be proved by using Theorem 2.2.3 noting the fact that if $G = M_{2nk}$ then $\frac{M_{2nk}}{Z(M_{2nk})} \cong D_{2n}$ or D_n depending on n is odd or even respectively also $|Z(M_{2nk})| = k$ or $2k$ for n is odd or even respectively.

Corollary 2.2.5. *If $G = D_{2n}$ then*

(a) $\gamma(\mathcal{C}(G)) = 4$ *if and only if* $n = 11$ *or* $12.$

(b) $\gamma(\mathcal{C}(G)) \neq 5.$

(c) $\gamma(\mathcal{C}(G)) = 6$ *if and only if* $n = 13$ *or* $14.$

(d) $\gamma(\mathcal{C}(G)) \geq 7$ *for* $n \geq 15.$

Corollary 2.2.5 can be proved by using Corollary 2.2.4 noting the fact that $M_{2nk} = D_{2n}$ if $k = 1$.

Corollary 2.2.6. *If $G = Q_{4m}$ then*

(a) $\gamma(\mathcal{C}(G)) = 4$ *if and only if* $m = 6.$

(b) $\gamma(\mathcal{C}(G)) \neq 5.$

(c) $\gamma(\mathcal{C}(G)) = 6$ *if and only if* $m = 7.$

(d) $\gamma(\mathcal{C}(G)) \geq 7$ *for* $m \geq 8$.

Corollary 2.2.6 can be proved by using Theorem 2.2.3 noting the fact that if $G = Q_{4m}$ then $|Z(Q_{4m})| = 2$ and $\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}$.

Corollary 2.2.7. *If* $G = U_{6n}$ *then* $\gamma(\mathcal{C}(G)) \neq 4, 5, 6$ *also* $\gamma(\mathcal{C}(G)) \geq 7$ *for* $n \geq 5$.

Corollary 2.2.7 can be proved by using Theorem 2.2.3 noting the fact that if $G = U_{6n}$ then $|Z(U_{6n})| = n$ and $\frac{U_{6n}}{Z(U_{6n})} \cong D_6$.

Theorem 2.2.8. *If G is a finite group such that* $\frac{G}{Z(G)} \cong Sz(2)$ *then* $\gamma(\mathcal{C}(G)) \neq 4, 5, 6$ *also* $\gamma(\mathcal{C}(G)) \geq 7$ *for* $n \geq 2$.

Proof. By Theorem 2.1.13 we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{3}(n-1)(4n-3) \right\rceil + 5 \left\lceil \frac{1}{4}(n-1)(3n-4) \right\rceil$$

where $|Z(G)| = n$. It can be seen that $\gamma(\mathcal{C}(G)) = 0$ for $n = 1$. If $n \geq 2$ then

$$\frac{1}{3}(n-1)(4n-3) = \frac{4n(n-2)}{3} + \frac{n+3}{3} > 1,$$

also $n - 1 > 0$ and $3n - 4 > 0$, so $\frac{1}{2}(n-1)(3n-4) > 0$. Therefore

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{3}(n-1)(4n-3) \right\rceil + 5 \left\lceil \frac{1}{4}(n-1)(3n-4) \right\rceil > 7.$$

□

Theorem 2.2.9. *If* $G = V_{8n}$ *then*

(a) $\gamma(\mathcal{C}(G)) = 4$ *if and only if* $n = 3$.

(b) $\gamma(\mathcal{C}(G)) \neq 5$.

(c) $\gamma(\mathcal{C}(G)) = 6$ *if and only if* $n = 4$.

(d) $\gamma(\mathcal{C}(G)) > 18$ *for* $n \geq 5$.

Proof. By Theorem 2.1.15 we have, $\gamma(\mathcal{C}(G)) = 0$ for $n = 1, 2$.
Case 1. n is odd. If $n \geq 3$ then by Theorem 2.1.15 we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{6}(4n-5)(2n-3) \right\rceil.$$

Clearly, $\gamma(\mathcal{C}(G)) = 4$ for $n = 3$. If $n \geq 5$ then

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{6}(4n-5)(2n-3) \right\rceil = \left\lceil \frac{1}{3}(8n(n-5) + 18n + 15) \right\rceil > 18.$$

Case 2. n is even. If $n \geq 4$ then by Theorem 2.1.15 we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{3}(4n-7)(n-2) \right\rceil.$$

Clearly, $\gamma(\mathcal{C}(G)) = 6$ for $n = 4$. If $n \geq 6$ then

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{3}(4n-7)(n-2) \right\rceil = \left\lceil \frac{1}{3}(4n(n-6)+9n+14) \right\rceil > 22.$$

□

Theorem 2.2.10. *If $G = QD_{2^n}$ or SD_{8n} then $\gamma(\mathcal{C}(G)) \neq 4, 5, 6$ also $\gamma(\mathcal{C}(G)) \geq 7$ for $n \geq 5$ or $n \geq 4$ respectively.*

Proof. If $G = QD_{2^n}$ then by Theorem 2.1.16 we have

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{12}(2^{n-1}-5)(2^{n-1}-6) \right\rceil.$$

If $n = 4$ then $\gamma(\mathcal{C}(G)) = 1$. If $n \geq 5$ then $(2^{n-1}-5) \geq 11$ and $(2^{n-1}-6) \geq 10$. So $\frac{1}{12}(2^{n-1}-5)(2^{n-1}-6) \geq \frac{110}{12}$. Therefore $\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{6}(2^{n-1}-5)(2^{n-1}-6) \right\rceil \geq 10$. Hence the result follows.

If $G = SD_{8n}$ then by Theorem 2.1.17 we have

$$\gamma(\mathcal{C}(G)) = \begin{cases} 0, & \text{if } n = 1 \\ \left\lceil \frac{1}{3}(4n-7)(n-2) \right\rceil, & \text{if } n \text{ is odd and } n \geq 3 \\ \left\lceil \frac{1}{6}(4n-5)(2n-3) \right\rceil, & \text{if } n \text{ is even and } n \geq 2. \end{cases}$$

For $n = 3$ we have $\gamma(\mathcal{C}(G)) = 2$. If $n \geq 5$ and n is odd then

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{3}(4n-7)(n-2) \right\rceil = \left\lceil \frac{1}{3}(4n(n-5)+5n+14) \right\rceil \geq 13.$$

If $n = 2$ then $\gamma(\mathcal{C}(G)) = 1$. If n is even and $n \geq 4$ then

$$\gamma(\mathcal{C}(G)) = \left\lceil \frac{1}{6}(4n-5)(2n-3) \right\rceil = \left\lceil \frac{1}{6}(8n(n-4)+10n+15) \right\rceil > 10.$$

Hence the result follows. □

It is observed that $\gamma(\mathcal{C}(G)) \neq 5$ for all the groups considered in our study. It may be interesting to give examples of groups G such that $\gamma(\mathcal{C}(G)) = 5$. In general we pose the following question:

"Which positive integers can be realized as genus of commuting graphs of some finite non-abelian groups?"