# Chapter 3

# Solvable graphs of finite groups

Let $G$ be a finite non-solvable group with solvable radical $\mathrm{Sol}(G)$. The solvable graph $\mathcal{S}(G)$ of $G$ is a graph with vertex set $G \setminus \mathrm{Sol}(G)$ and two distinct vertices $u$ and $v$ are adjacent if and only if $\langle u, v\rangle$ is solvable. In Section 3.1, we study graph realization properties of $\mathcal{S}(G)$. More precisely, we show that $\mathcal{S}(G)$ is not a star graph, a tree, an $n$-partite graph for any positive integer $n \geq 2$ and not a regular graph for any non-solvable finite group $G$. We also show that the girth of $\mathcal{S}(G)$ is $3$ and the clique number of $\mathcal{S}(G)$ is greater than or equal to $4$. In Section 3.2, we first show that for a given non-negative integer $k$, there are at the most finitely many finite non-solvable groups whose solvable graphs have genus $k$. We also show that there is no finite non-solvable group, whose solvable graph is planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the chapter by obtaining a relation between $\mathcal{S}(G)$ and $P_s(G)$ in Section 3.3, where $P_s(G)$ is the solvability degree of $G$. This chapter is based on our paper [22] published in *Hacettepe Journal of Mathematics and Statistics*.

## 3.1 Graph realization

We begin with the following lemma.

**Lemma 3.1.1.** *For every $u \in G \setminus \mathrm{Sol}(G)$ we have*

$$\deg_{\mathcal{S}(G)}(u) = |\mathrm{Sol}_G(u)| - |\mathrm{Sol}(G)| - 1.$$

*Proof.* Note that $\deg_{\mathcal{S}(G)}(u)$ represents the number of vertices from $G \setminus \mathrm{Sol}(G)$ which are adjacent to $u$. Since $u \in \mathrm{Sol}_G(u)$, therefore $|\mathrm{Sol}_G(u)| - 1$ represents the number of vertices which are adjacent to $u$. Since we are excluding $\mathrm{Sol}(G)$ from the vertex set therefore $\deg_{\mathcal{S}(G)}(u) = |\mathrm{Sol}_G(u)| - |\mathrm{Sol}(G)| - 1$. □

**Proposition 3.1.2.** *$\mathcal{S}(G)$ is not a star.*

*Proof.* Suppose for a contradiction $\mathcal{S}(G)$ is a star. Let $|G| - |\mathrm{Sol}(G)| = n$. Then there exists $u \in G \setminus \mathrm{Sol}(G)$ such that $\deg_{\mathcal{S}(G)}(u) = n - 1$. Therefore, by Lemma 3.1.1, $|\mathrm{Sol}_G(u)| = |G|$. This gives $u \in \mathrm{Sol}(G)$, a contradiction. Hence, the result follows. □

**Proposition 3.1.3.** *$\mathcal{S}(G)$ is not complete bipartite.*

*Proof.* Let $\mathcal{S}(G)$ be complete bipartite. Suppose that $A_1$ and $A_2$ are parts of the bi-partition. Then, by Proposition 3.1.2, $|A_1| \geq 2$ and $|A_2| \geq 2$. Let $u \in A_1, v \in A_2$. If $|\langle u, v\rangle \mathrm{Sol}(G) \setminus \mathrm{Sol}(G)| > 2$ then there exists $y \in \langle u, v\rangle \mathrm{Sol}(G) \setminus \mathrm{Sol}(G)$ with $u \neq y \neq v$ such that $\langle u, y\rangle$ and $\langle v, y\rangle$ are both solvable. But then $y \notin A_1$ and $y \notin A_2$, a contradiction.

It follows that $|\langle u, v\rangle \mathrm{Sol}(G) \setminus \mathrm{Sol}(G)| = 2$. In particular, $\mathrm{Sol}(G) = 1$ and $\langle u, v\rangle$ is cyclic of order 3 or $|\mathrm{Sol}(G)| = 2$ and $v = uz$ for $z$ an involution in $\mathrm{Sol}(G)$. Now the neighbours of $u \in A_1$ is just $u^2 \in A_2$ or $uz$ in the respective cases. Hence $|A_2| = |A_1| = 1$, a contradiction. Hence, the result follows. □

Following similar arguments as in the proof of Proposition 3.1.3 we get the following result.

**Proposition 3.1.4.** *$\mathcal{S}(G)$ is not complete $n$-partite.*

**Proposition 3.1.5.** *For any finite non-solvable group $G$, $\mathcal{S}(G)$ has no isolated vertex.*

*Proof.* Suppose $x$ is an isolated vertex of $\mathcal{S}(G)$. Then $|\mathrm{Sol}(G)| = 1$; otherwise $x$ is adjacent to $xz$ for any $z \in \mathrm{Sol}(G) \setminus \{1\}$. Thus it follows that $o(x) = 2$; otherwise $x$ is adjacent to $x^2$. Let $y \in G$. Then $\langle x, x^y\rangle$ is dihedral and so $x = x^y$ as $x$ is isolated. Hence $x \in Z(G)$ and so $x \in Z(G) \leq \mathrm{Sol}(G)$, a contradiction. Hence, $\mathcal{S}(G)$ has no isolated vertex. □

The following lemma is useful in proving the next two results as well as some results in subsequent sections.

**Lemma 3.1.6.** *Let $G$ be a finite non-solvable group. Then there exist $x \in G$ such that $x, x^2 \notin \mathrm{Sol}(G)$.*

*Proof.* Suppose that for all $x \in G$, we have $x^2 \in \mathrm{Sol}(G)$. Therefore, $G/\mathrm{Sol}(G)$ is elementary abelian and hence solvable. Also, $\mathrm{Sol}(G)$ is solvable. It follows that $G$ is solvable, a contradiction. Hence, the result follows. □

**Theorem 3.1.7.** *Let $G$ be a finite non-solvable group. Then* $\operatorname{girth}(\mathcal{S}(G)) = 3$.

*Proof.* Suppose for a contradiction that $\mathcal{S}(G)$ has no 3-cycle. Let $x \in G$ such that $x, x^2 \notin \operatorname{Sol}(G)$ (by Lemma 3.1.6). Suppose $|\operatorname{Sol}(G)| \geq 2$. Let $z \in \operatorname{Sol}(G), z \neq 1$, then $x, x^2$ and $xz$ form a 3-cycle, which is a contradiction. Thus $|\operatorname{Sol}(G)| = 1$. In this case, every element of $G$ has order 2 or 3; otherwise, $\{x, x^2, x^3\}$ forms a 3-cycle in $\mathcal{S}(G)$ for all $x \in G$ with $o(x) > 3$. Therefore, $|G| = 2^m 3^n$ for some non-negative integers $m$ and $n$. By Result 1.2.13, it follows that $G$ is solvable; a contradiction. Hence, $\operatorname{girth}(\mathcal{S}(G)) = 3$. □

**Theorem 3.1.8.** *Let $G$ be a finite non-solvable group. Then $\omega(\mathcal{S}(G)) \geq 4$.*

*Proof.* Suppose for a contradiction that $G$ is a finite non-solvable group with $\omega(\Gamma_s(G)) \leq 3$. Let $x \in G \setminus \operatorname{Sol}(G)$ such that $x^2 \notin \operatorname{Sol}(G)$ according to Lemma 3.1.6. Suppose $|\operatorname{Sol}(G)| \geq 2$. Let $z \in \operatorname{Sol}(G), z \neq 1$, then $\{x, x^2, xz, x^2z\}$ is a clique which is a contradiction. Thus $|\operatorname{Sol}(G)| = 1$. In this case every element of $G \setminus \operatorname{Sol}(G)$ has order $2, 3$ or 4 otherwise $\{x, x^2, x^3, x^4\}$ is a clique with $o(x) > 4$, which is a contradiction. Therefore $|G| = 2^m 3^n$ where $m, n$ are non-negative integers. Again, by Result 1.2.13, it follows that $G$ is solvable; a contradiction. This completes the proof. □

As a consequence of Theorem 3.1.7 and Theorem 3.1.8 we have the following corollary.

**Corollary 3.1.9.** *The solvable graph of a finite non-solvable group is not a tree.*

We conclude this section with the following result.

**Proposition 3.1.10.** *$\mathcal{S}(G)$ is not regular.*

*Proof.* Follows from Result 1.3.15, noting the fact that a graph is regular if and only if its complement is regular. □

## 3.2 Genus and diameter

We begin this section with the following useful lemma.

**Lemma 3.2.1.** *Let $G$ be a finite group and $H$ a solvable subgroup of $G$. Then $\langle H, \operatorname{Sol}(G)\rangle$ is a solvable subgroup of $G$.*

**Proposition 3.2.2.** *Let $G$ be a finite non-solvable group such that $\gamma(\mathcal{S}(G)) = m$.*

(a) *If $S$ is a non-empty subset of $G \setminus \operatorname{Sol}(G)$ such that $\langle x, y\rangle$ is solvable for all $x, y \in S$ then $|S| \leq \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$.*

(b) $|\operatorname{Sol}(G)| \leq \frac{1}{t-1}\left\lfloor\frac{7+\sqrt{1+48m}}{2}\right\rfloor$, *where* $t = \max\{o(x\operatorname{Sol}(G)) : x\operatorname{Sol}(G) \in G/\operatorname{Sol}(G)\}$.

(c) *If $H$ is a solvable subgroup of $G$ then* $|H| \leq \left\lfloor\frac{7+\sqrt{1+48m}}{2}\right\rfloor + |H \cap \operatorname{Sol}(G)|$.

*Proof.* We have $\mathcal{S}(G)[S] \cong K_{|S|}$ and $\gamma(K_{|S|}) = \gamma(\mathcal{S}(G)[S]) \leq \gamma(\mathcal{S}(G))$. Therefore, if $m = 0$ then $\gamma(K_{|S|}) = 0$. This gives $|S| \leq 4$, otherwise $K_{|S|}$ will have a subgraph $K_5$ having genus 1. If $m > 0$ then, by Result 1.1.5, we have

$$|S| = \omega(\mathcal{S}(G)[S]) \leq \omega(\mathcal{S}(G)) \leq \chi(\mathcal{S}(G)) \leq \left\lfloor\frac{7+\sqrt{1+48m}}{2}\right\rfloor$$

where $\chi(\mathcal{S}(G))$ is the chromatic number of $\mathcal{S}(G)$. Hence part (a) follows.

Part (b) follows from Lemma 3.2.1 and part (a) considering $S = \bigsqcup_{i=1}^{t-1} y^i\operatorname{Sol}(G)$, where $y \in G \setminus \operatorname{Sol}(G)$ such that $o(y\operatorname{Sol}(G)) = t$.

Part (c) follows from part (a) noting that $H = (H \setminus \operatorname{Sol}(G)) \cup (H \cap \operatorname{Sol}(G))$. □

**Theorem 3.2.3.** *Let $G$ be a finite non-solvable group. Then $|G|$ is bounded above by a function of $\gamma(\mathcal{S}(G))$.*

*Proof.* Let $\gamma(\mathcal{S}(G)) = m$ and $h_m = \left\lfloor\frac{7+\sqrt{1+48m}}{2}\right\rfloor$. By Lemma 3.2.1, we have

$$\mathcal{S}(G)[x\operatorname{Sol}(G)] \cong K_{|\operatorname{Sol}(G)|},$$

where $x \in G \setminus \operatorname{Sol}(G)$. Therefore by Proposition 3.2.2(a), $|\operatorname{Sol}(G)| \leq h_m$.

Let $P$ be a Sylow $p$-subgroup of $G$ for any prime $p$ dividing $|G|$ having order $p^n$ for some positive integer $n$. Then $P$ is a solvable. Therefore, by Proposition 3.2.2(c), we have $|P| \leq h_m + |\operatorname{Sol}(G)| \leq 2h_m$. Hence, $|G| < (2h_m)^{h_m}$ noting that the number of primes less than $2h_m$ is at most $h_m$. This completes the proof. □

As an immediate consequence of Theorem 3.2.3 we have the following corollary.

**Corollary 3.2.4.** *Let $n$ be a non-negative integer. Then there are at the most finitely many finite non-solvable groups $G$ such that $\gamma(\mathcal{S}(G)) = n$.*

The following lemma is essential in proving the main results of this section.

**Lemma 3.2.5.** *If $G$ is a non-solvable group of order not exceeding 120 then $\mathcal{S}(G)$ has a subgraph isomorphic to $K_{11}$ and $\gamma(\mathcal{S}(G)) \geq 5$.*

*Proof.* If $G$ is a non-solvable group and $|G| \leq 120$ then $G$ is isomorphic to $A_5$, $A_5 \times \mathbb{Z}_2$, $S_5$ or $SL(2,5)$. Note that $|\operatorname{Sol}(A_5)| = |\operatorname{Sol}(S_5)| = 1$ and $|\operatorname{Sol}(A_5 \times \mathbb{Z}_2)| = |\operatorname{Sol}(SL(2,5))| = 2$.

Also, $A_5$ has a solvable subgroup of order 12 and $S_5$, $A_5 \times \mathbb{Z}_2$, $SL(2,5)$ have solvable subgroups of order 24. It follows that $\mathcal{S}(G)$ has a subgraph isomorphic to $K_{11}$. Therefore, by (1.1.$b$), $\gamma(\mathcal{S}(G)) \geq \gamma(K_{11}) = 5$. □

**Theorem 3.2.6.** *The solvable graph of a finite non-solvable group is neither planar, toroidal, double-toroidal nor triple-toroidal.*

*Proof.* Let $G$ be a finite non-solvable group. Note that it is enough to show $\gamma(\mathcal{S}(G)) \geq 4$ to complete the proof. Suppose that $\gamma(\mathcal{S}(G)) \leq 3$. Let $x \in G \setminus \mathrm{Sol}(G)$ such that $x^2 \notin \mathrm{Sol}(G)$. Such element exists by Lemma 3.1.6. Since any two elements of the set $A = x\,\mathrm{Sol}(G) \cup x^2\,\mathrm{Sol}(G)$ generate a solvable group, by Proposition 3.2.2(a), we have $2|\mathrm{Sol}(G)| = |A| \leq \left\lfloor \frac{7+\sqrt{1+48\cdot 3}}{2} \right\rfloor = 9$. Thus $|\mathrm{Sol}(G)| \leq 4$. Let $p$ be a prime divisor of $|G|$ and $P$ is a Sylow $p$-subgroup of $G$. Since $P$ is solvable, by Proposition 3.2.2(c), we get $|P| \leq 9 + |P \cap \mathrm{Sol}(G)| \leq 13$. If $|P| = 11$ or $13$ then $|P \cap \mathrm{Sol}(G)| = 1$. Therefore, $\mathcal{S}(G)[P \setminus \mathrm{Sol}(G)] \cong K_{10}$ or $K_{12}$. Using (1.1.$b$), we get $\gamma(\mathcal{S}(G)[P \setminus \mathrm{Sol}(G)]) = 4$ or $6$. Therefore, $\gamma(\mathcal{S}(G)) \geq \gamma(\mathcal{S}(G)[P \setminus \mathrm{Sol}(G)]) \geq 4$, a contradiction. Thus $|P| \leq 9$ and hence $p \leq 7$. This shows that $|G|$ divides $2^3.3^2.5.7$.

We consider the following cases.

**Case 1.** $|\mathrm{Sol}(G)| = 4$.

If $H$ is a Sylow $p$-subgroup of $G$ where $p = 5$ or $7$ then $\langle H, \mathrm{Sol}(G)\rangle$ is solvable since $H$ is solvable (by Lemma 3.2.1) We have $|H \cap \mathrm{Sol}(G)| = 1$ and $|\langle H, \mathrm{Sol}(G)\rangle| = 20, 28$ according as $p = 5, 7$ respectively. Therefore $\mathcal{S}(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{16}$ or $K_{24}$. By (1.1.$b$) we get $\gamma(\mathcal{S}(G)) \geq \gamma(\mathcal{S}(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) \geq 13$, which is a contradiction.

Thus $|G|$ is a divisor of 72. Therefore, by Lemma 3.2.5 we have $\gamma(\mathcal{S}(G)) \geq 5$, a contradiction.

**Case 2.** $|\mathrm{Sol}(G)| = 3$.

If $H$ is a Sylow $p$-subgroup of $G$ where $p = 5$ or $7$ then $\langle H, \mathrm{Sol}(G)\rangle$ is solvable. We have $|H \cap \mathrm{Sol}(G)| = 1$ and $|\langle H, \mathrm{Sol}(G)\rangle| = 15, 21$ according as $p = 5, 7$ respectively. Therefore

$$\mathcal{S}(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{12} \text{ or } K_{18}.$$

By (1.1.$b$) we get $\gamma(\mathcal{S}(G)) \geq \gamma(\mathcal{S}(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) \geq 6$, which is a contradiction.

Thus $|G|$ is a divisor of 72. Therefore, by Lemma 3.2.5 we have $\gamma(\mathcal{S}(G)) \geq 5$, a contradiction.

**Case 3.** $|\mathrm{Sol}(G)| = 2$.

If $H$ is a Sylow 7-subgroup of $G$ then $\langle H, \mathrm{Sol}(G)\rangle$ is solvable. We have $|H \cap \mathrm{Sol}(G)| = 1$ and $|\langle H, \mathrm{Sol}(G)\rangle| = 14$. So, $\mathcal{S}(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{12}$. By (1.1.$b$) we get $\gamma(\mathcal{S}(G)) \geq \gamma(\mathcal{S}(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) \geq 6$, which is a contradiction. Let $K$ be a Sylow 3-subgroup of $G$. If $|K| = 9$ then $\langle K, \mathrm{Sol}(G)\rangle$ is solvable since $K$ is solvable (by Lemma 3.2.1). We have $|K \cap \mathrm{Sol}(G)| = 1$ and $|\langle K, \mathrm{Sol}(G)\rangle| = 18$. So, $\mathcal{S}(G)[\langle K, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{16}$. By (1.1.$b$) we get $\gamma(\mathcal{S}(G)) \geq \gamma(\mathcal{S}(G)[\langle K, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) = 13$, which is a contradiction.

Thus $|G|$ is a divisor of 120. Therefore, by Lemma 3.2.5 we have $\gamma(\mathcal{S}(G)) \geq 5$, a contradiction.

**Case 4.** $|\mathrm{Sol}(G)| = 1$.

In this case, first we shall show that $7 \nmid |G|$. On the contrary, assume that $7 \mid |G|$. Let $n$ be the number of Sylow 7-subgroups of $G$. Then $n \mid 2^3.3^2.5$ and $n \equiv 1 \mod 7$. If $n \neq 1$ then $n \geq 8$. Let $H_1, \ldots, H_8$ be the eight distinct Sylow 7-subgroup of $G$. Then the subgraph induced $\mathcal{S}(G)[H_i \setminus \mathrm{Sol}(G)]$ for each $1 \leq i \leq 8$ will contribute $\gamma(\mathcal{S}(G)[H_i \setminus \mathrm{Sol}(G)]) = 1$ to the genus of $\mathcal{S}(G)$. Thus

$$\gamma(\mathcal{S}(G)) \geq \sum_{i=1}^{8} \gamma(\mathcal{S}(G)[H_i \setminus \mathrm{Sol}(G)]) = 8,$$

a contradiction. Therefore, Sylow 7-subgroup of $G$ is unique and hence normal. Since we have started with a non-solvable group, by Result 1.2.4, it follows that $G$ has an abelian subgroup of order at least 14. Therefore, by (1.1.$b$) we have $\gamma(\mathcal{S}(G)) \geq \gamma(K_{13}) = 8$, a contradiction. Hence, $|G|$ is a divisor of $2^3.3^2.5$.

Now, we shall show that $9 \nmid |G|$. Assume that, on the contrary, $9 \mid |G|$. If Sylow 3-subgroup of $G$ is not normal in $G$ then the number of Sylow 3-subgroup is greater than or equal to 4. Let $H_1, H_2, H_3$ be the three Sylow 3-subgroup of $G$. Then the induced subgraph $\mathcal{S}(G)[H_1 \setminus \mathrm{Sol}(G)] \cong K_8$ and so it contributes $\gamma(\mathcal{S}(G)[H_1 \setminus \mathrm{Sol}(G)]) = 2$ to the genus of $\mathcal{S}(G)$. If $|H_1 \cap H_2| = 1$ then the induced subgraph $\mathcal{S}(G)[H_2 \setminus \mathrm{Sol}(G)] \cong K_8$ and so it contributes $+2$ to the genus $\mathcal{S}(G)$. Thus

$$\gamma(\mathcal{S}(G)) \geq \gamma(\mathcal{S}(G)[(H_1 \cup H_2) \setminus \mathrm{Sol}(G)]) = 4$$

which is a contradiction. So assume that $|H_1 \cap H_2| = 3$. Similarly $|H_1 \cap H_3| = 3$ and $|H_2 \cap H_3| = 3$. Let $M = H_2 \setminus H_1$. Then $|M| = 6$. Also note that if $L = H_1 \cup H_2$ and $K = H_3 \setminus L$ then $|K| \geq 4$. Also $H_1 \cap M = H_1 \cap K = M \cap K = \emptyset$.

If $|K| \geq 5$ then $H_1$ contribute $+2$ to genus of $\mathcal{S}(G)$, $M$ and $K$ each contribute $+1$ to genus of $\mathcal{S}(G)$. Hence genus of $\mathcal{S}(G)$ is greater than or equal to 4, a contradiction.

Assume that $|K| = 4$. In this case $|M \cap H_3| = 2$. Let $x \in M \cap H_3$. Then $H_1$ contribute $+2$ to genus of $\mathcal{S}(G)$, $M \setminus \{x\}$ and $K \cup \{x\}$ each contribute $+1$ to genus of $\mathcal{S}(G)$. Hence genus of $\mathcal{S}(G)$ is greater than or equal to 4, a contradiction.

These show that the Sylow 3-subgroup of $G$ is unique and hence normal in $G$. Therefore, by Result 1.2.4 and Lemma 3.2.5, $G$ has an abelian subgroup $A$ of order at least 18. Hence,

$$\gamma(\mathcal{S}(G)) \geq \gamma(\mathcal{S}(G)[A \setminus \mathrm{Sol}(G)]) \geq \gamma(K_{17}) = 16$$

which is a contradiction.

It follows that $9 \nmid |G|$ and $|G|$ is a divisor of 120. Therefore, by Lemma 3.2.5 we get $\gamma(\mathcal{S}(G)) \geq 5$, a contradiction. Hence, $\gamma(\mathcal{S}(G)) \geq 4$ and the result follows. □

The above theorem gives that $\gamma(\mathcal{S}(G)) \geq 4$. Usually, genera of solvable graphs of finite non-solvable groups are very large. For example, if $G$ is the smallest non-solvable group $A_5$ then $\mathcal{S}(G)$ has 59 vertices and 571 edges. Also $\gamma(\mathcal{S}(G)) \geq 571/6 - 59/2 + 1 = 68$ (follows from Result 1.1.2).

The following theorem shows that the crosscap number of the solvable graph of a finite non-solvable group is greater than 1.

**Proposition 3.2.7.** *The solvable graph of a finite non-solvable group is not projective.*

*Proof.* Suppose $G$ is a finite non-solvable group whose solvable graph is projective. Note that if $\mathcal{S}(G)$ has a subgraph isomorphic to $K_n$ then, by (1.1.$c$), we must have $n \leq 6$. Let $x \in G$, such that $x, x^2 \notin \mathrm{Sol}(G)$. Then

$$\mathcal{S}(G)[x\,\mathrm{Sol}(G) \cup x^2\,\mathrm{Sol}(G)] \cong K_{2|\mathrm{Sol}(G)|}.$$

Therefore, $2|\mathrm{Sol}(G)| \leq 6$ and hence $|\mathrm{Sol}(G)| \leq 3$.

Let $p \mid |G|$ be a prime and $P$ be a Sylow $p$-subgroup of $G$. Then $\mathcal{S}(G)[P \setminus \mathrm{Sol}(G)] \cong K_{|P \setminus \mathrm{Sol}(G)|}$ since $P$ is solvable. Therefore, $|P \setminus \mathrm{Sol}(G)| = |P| - |P \cap \mathrm{Sol}(G)| \leq 6$ and hence $|P| \leq 9$. This shows that $|G|$ is a divisor of $2^3.3^2.5.7$.

If $7 \mid |G|$ then the Sylow 7-subgroup of $G$ is unique and hence normal in $G$; otherwise, let $H$ and $K$ be two Sylow 7-subgroup of $G$. Then $|H \cap K| = |H \cap \mathrm{Sol}(G)| = |K \cap \mathrm{Sol}(G)| = 1$. Therefore, $\mathcal{S}(G)[(H \cup K) \setminus \mathrm{Sol}(G)]$ has a subgraph isomorphic to $2K_6$. Hence, $\mathcal{S}(G)$ has a subgraph isomorphic to $2K_5$, which is a contradiction. Similarly, if $9 \mid |G|$ then the Sylow 3-subgroup of $G$ is normal in $G$. Therefore, by Result 1.2.4, it follows that $|G| \leq 72$ or $|G|$ is a divisor of $2^3.3.5$. In the both cases, by Lemma 3.2.5, $\mathcal{S}(G)$ has complete subgraphs isomorphic to $K_{11}$, which is a contradiction. This completes the proof. □

We conclude this section, by an observation and a couple of problems regarding the diameter and connectedness of $\mathcal{S}(G)$. Using the following programme in GAP[91], we see that the solvable graph of the groups $A_5$, $S_5$, $A_5 \times \mathbb{Z}_2$, $SL(2,5)$, $PSL(3,2)$ and $GL(2,4)$ are connected with diameter 2. The solvable graphs of $S_6$ and $A_6$ are connected with diameters greater than 2.

```
g:=PSL(3,2);
sol:=RadicalGroup(g);
L:=[];
gsol:=Difference(g,sol);
for x in gsol do
 AddSet(L,[x]);
 for y in Difference(gsol,L) do
  if IsSolvable(Subgroup(g,[x,y]))=true then
   break;
  fi;
  i:=0;
  for z in gsol do
   if IsSolvable(Subgroup(g,[x,z]))=true and
   IsSolvable(Subgroup(g,[z,y]))=true
   then
    i:=1;
    break;
   fi;
  od;
  if i=0 then
   Print("Diameter>2");
   Print(x,"   ",y);
  fi;
 od;
od;
```

In this connection, the following problems were posed in [22].

**Problem 3.2.8.** Is $\mathcal{S}(G)$ connected for any finite non-solvable group $G$?

**Problem 3.2.9.** Is there any finite bound for the diameter of $\mathcal{S}(G)$ when $\mathcal{S}(G)$ is connected?

It is worth mentioning that Akbari et al. [9] have answered these problems by proving that $\mathcal{S}(G)$ is connected and diameter of $\mathcal{S}(G)$ is at the most 11. Akbari et al. [9] also remarked that the actual bound for the diameter of $\mathcal{S}(G)$ is much smaller than 11. Recently, Burness, Lucchini and Nemmi [28] have shown that $\mathcal{S}(G)$ is connected and its diameter is less than or equal to 5.

## 3.3 Relations with solvability degree

In this section, we study a few properties of $P_s(G)$, the solvability degree of $G$, and derive a connection between $P_s(G)$ and $\mathcal{S}(G)$ for finite non-solvable groups $G$. We begin with the following lemma.

**Lemma 3.3.1.** *Let $G$ be a finite group. Then $P_s(G) = \frac{1}{|G|^2}\sum\limits_{u\in G}|\operatorname{Sol}_G(u)|$.*

*Proof.* Let $\mathcal{S} = \{(u, v) \in G \times G : \langle u, v\rangle \text{ is solvable}\}$. Then

$$\mathcal{S} = \bigcup_{u\in G}(\{u\} \times \{v \in G : \langle u, v\rangle \text{ is solvable}\}) = \bigcup_{u\in G}(\{u\} \times \operatorname{Sol}_G(u)).$$

Therefore, $|\mathcal{S}| = \sum\limits_{u\in G}|\operatorname{Sol}_G(u)|$. Hence, the result follows from (1.2.$b$). □

**Corollary 3.3.2.** *$|G|P_s(G)$ is an integer for any finite group $G$.*

*Proof.* By Result 1.2.10 we have that $|G|$ divides $\sum\limits_{u\in G}|\operatorname{Sol}_G(u)|$. Hence, the result follows from Lemma 3.3.1. □

We have the following lower bound for $P_s(G)$.

**Theorem 3.3.3.** *For any finite group $G$,*

$$P_s(G) \geq \frac{|\operatorname{Sol}(G)|}{|G|} + \frac{2(|G| - |\operatorname{Sol}(G)|)}{|G|^2}.$$

*Proof.* By Lemma 3.3.1, we have

$$\begin{aligned}|G|^2P_s(G) &= \sum_{u\in\operatorname{Sol}(G)}|\operatorname{Sol}_G(u)| + \sum_{u\in G\setminus\operatorname{Sol}(G)}|\operatorname{Sol}_G(u)|\\ &= |G||\operatorname{Sol}(G)| + \sum_{u\in G\setminus\operatorname{Sol}(G)}|\operatorname{Sol}_G(u)|. \qquad (3.3.a)\end{aligned}$$

By Result 1.2.7, $|C_G(u)|$ is a divisor of $|\operatorname{Sol}_G(u)|$ for all $u \in G$ where $C_G(u) = \{v \in G : uv = vu\}$, the centralizer of $u \in G$. Since $|C_G(u)| \geq 2$ for all $u \in G$ we have $|\operatorname{Sol}_G(u)| \geq 2$ for all $u \in G$. Therefore

$$\sum_{u \in G \setminus \operatorname{Sol}(G)} |\operatorname{Sol}_G(u)| \geq 2(|G| - |\operatorname{Sol}(G)|).$$

Hence, the result follows from (3.3.$a$). □

The following theorem shows that $P_s(G) > \Pr(G)$ for any finite non-solvable group.

**Theorem 3.3.4.** *Let $G$ be a finite group. Then $P_s(G) \geq \Pr(G)$ with equality if and only if $G$ is abelian.*

*Proof.* For all $u \in G$ we have $C_G(u) \subseteq \operatorname{Sol}_G(u)$ and so $|\operatorname{Sol}_G(u)| \geq |C_G(u)|$. Therefore, by Lemma 3.3.1 and (1.2.$a$) we get

$$\begin{aligned} P_s(G) &= \frac{1}{|G|^2} \sum_{x \in G} |\operatorname{Sol}_G(x)| \\ &\geq \frac{1}{|G|^2} \sum_{u \in G} |C_G(u)| = \Pr(G). \end{aligned}$$

Clearly, $P_s(G) = \Pr(G)$ if and only if $\operatorname{Sol}_G(u) = C_G(u)$ for all $u \in G$. Therefore, if $G$ is abelian then $\operatorname{Sol}_G(u) = G = C_G(u)$ for all $u \in G$ and so

$$P_s(G) = 1 = \Pr(G).$$

Suppose that $\operatorname{Sol}_G(u) = C_G(u)$ for all $u \in G$. Then $G$ is an $S$-group. Let $a, b \in G$. Then $\langle a, b\rangle$ is solvable. Therefore

$$b \in \operatorname{Sol}_G(a) = C_G(a)$$

and so $ab = ba$. Hence, $G$ is abelian. This completes the proof. □

Let $|e(\mathcal{S}(G))|$ be the number of edges of the solvable graph $\mathcal{S}(G)$ of $G$. The following theorem gives a relation between $P_s(G)$ and $|e(\mathcal{S}(G))|$.

**Theorem 3.3.5.** *Let $G$ be a finite non-solvable group. Then*

$$2|e(\mathcal{S}(G))| = |G|^2 P_s(G) + |\operatorname{Sol}(G)|^2 + |\operatorname{Sol}(G)| - |G|(2|\operatorname{Sol}(G)| + 1).$$

*Proof.* We have

$$2|e(\mathcal{S}(G))| = |\{(x, y) \in (G \setminus \operatorname{Sol}(G)) \times (G \setminus \operatorname{Sol}(G)) : \langle x, y\rangle \text{ is solvable}\}| - |G| + |\operatorname{Sol}(G)|.$$

Also

$$\begin{aligned}\mathcal{S} &= \{(x,y) \in G \times G : \langle x,y\rangle \text{ is solvable}\}\\ &= \operatorname{Sol}(G) \times \operatorname{Sol}(G) \quad \sqcup \quad \operatorname{Sol}(G) \times (G \setminus \operatorname{Sol}(G)) \quad \sqcup \quad (G \setminus \operatorname{Sol}(G)) \times \operatorname{Sol}(G)\\ &\quad \sqcup \quad \{(x,y) \in (G \setminus \operatorname{Sol}(G)) \times (G \setminus \operatorname{Sol}(G)) : \langle x,y\rangle \text{ is solvable}\}.\end{aligned}$$

Therefore

$$|\mathcal{S}| = |\operatorname{Sol}(G)|^2 + 2|\operatorname{Sol}(G)|(|G| - |\operatorname{Sol}(G)|) + 2|e(\mathcal{S}(G))| + |G| - |\operatorname{Sol}(G)|,$$

so by Lemma 3.3.1,

$$|G|^2 P_s(G) = |G|(2|\operatorname{Sol}(G)| + 1) - |\operatorname{Sol}(G)|^2 - |\operatorname{Sol}(G)| + 2|e(\mathcal{S}(G))|.$$

Hence, the result follows. □

We conclude this chapter noting that lower bounds for $|e(\mathcal{S}(G))|$ can be obtained from Theorem 3.3.5 using the lower bounds given in Theorem 3.3.3, Theorem 3.3.4 and the lower bounds for $\Pr(G)$ obtained in Result 1.2.18.