Chapter 4

Non-solvable graphs of groups

The non-solvable graph of a finite group G, denoted by $\mathcal{NS}(G)$, is the complement of solvable graph of G considered in Chapter 3. In this chapter, we consider $\mathcal{NS}(G)$ and obtain many results including certain results on graph realization. More precisely, in Section 4.1, we shall study certain properties of degree of a vertex and vertex degree set of $\mathcal{NS}(G)$. We shall also obtain certain bounds for $P_s(G)$, including a better lower bound than the lower bound obtained in Theorem 3.3.3. In Section 4.2, we shall show that $\mathcal{NS}(G)$ is not bipartite, more generally it is not complete multi-partite. We shall also show that $\mathcal{NS}(G)$ is hamiltonian for some classes of finite groups. In Sections 4.3-4.5, we shall obtain several results regarding domination number, vertex connectivity, independence number and clique number of $\mathcal{NS}(G)$. In section 4.6, we shall consider two groups G and H having isomorphic non-solvable graphs and derive some properties of G and H. In the last section, we shall show that the genus of $\mathcal{NS}(G)$ is greater or equal to 4. Hence, $\mathcal{NS}(G)$ is neither planar, toroidal, double-toroidal nor triple-toroidal. We conclude this chapter by showing that $\mathcal{NS}(G)$ is not projective. This chapter is based on our paper [23] published in the *Bulletin of the Malaysian Mathematical Sciences Society*.

4.1 Vertex degree and cardinality of vertex degree set

It is easy to see that $\deg_{\mathcal{NS}(G)}(x) = |G| - |\operatorname{Sol}_G(x)|$ for any vertex x in the non-solvable graph $\mathcal{NS}(G)$ of the group G. In [59], Hai-Reuven have shown that

$$6 \leq \deg_{\mathcal{NS}(G)}(x) \leq |G| - |\operatorname{Sol}(G)| - 2 \qquad (4.1.a)$$

for any $x \in G \setminus \mathrm{Sol}(G)$. In this section, we first obtain some bounds for $P_s(G)$ using $(4.1.a)$. The following result gives a connection between $P_s(G)$ and the number of edges in $\mathcal{NS}(G)$.

Lemma 4.1.1. *If G is a finite non-solvable group then*

$$2|e(\mathcal{NS}(G))| = \sum_{x \in G \setminus \mathrm{Sol}(G)} \deg_{\mathcal{NS}(G)}(x) = |G|^2(1 - P_s(G)).$$

Proof. Let $U = \{(x, y) \in G \times G : \langle x, y \rangle \text{ is not solvable}\}$. Then

$$|U| = |G \times G| - |\{(x, y) \in G \times G : \langle x, y \rangle \text{ is solvable}\}| = |G|^2 - P_s(G)|G|^2.$$

Note that

$$|U| = 2|e(\mathcal{NS}(G))| = \sum_{x \in G \setminus \mathrm{Sol}(G)} \deg_{\mathcal{NS}(G)}(x).$$

Hence the result follows. □

Now we obtain the following bounds for $P_s(G)$.

Theorem 4.1.2. *If G is a finite non-solvable group then*

$$\frac{2(|G| - |\mathrm{Sol}(G)|)}{|G|^2} + \frac{2|\mathrm{Sol}(G)|}{|G|} - \frac{|\mathrm{Sol}(G)|^2}{|G|^2} \leq P_s(G) \leq 1 - \frac{6(|G| - |\mathrm{Sol}(G)|)}{|G|^2}.$$

Proof. By Lemma 4.1.1 and $(4.1.a)$, we have

$$6(|G| - |\mathrm{Sol}(G)|) \leq |G|^2(1 - P_s(G)) \leq (|G| - |\mathrm{Sol}(G)|)(|G| - |\mathrm{Sol}(G)| - 2)$$

and hence the result follows on simplification. □

Note that $\frac{|\mathrm{Sol}(G)|}{|G|} - \frac{|\mathrm{Sol}(G)|^2}{|G|^2} > 0$ for any finite non-solvable group G. Hence, the lower bound obtained in Theorem 4.1.2 for $P_s(G)$ is better than the bound obtained in Theorem 3.3.3.

It was also shown in Result 1.3.17 that $|\deg(\mathcal{NS}(G))| \neq 2$, where $\deg(\mathcal{NS}(G))$ is the vertex degree set of $\mathcal{NS}(G)$. However, we observe that the cardinality of $\deg(\mathcal{NS}(G))$ may be equal to 3. In this section, we shall obtain a class of groups G such that $|\deg(\mathcal{NS}(G))|$ $= 3$. Note that $\deg(\mathcal{NS}(A_5)) = \{24, 36, 50\}$. More generally, we have the following result.

Proposition 4.1.3. *Let S be any finite solvable group. Then $|\deg(\mathcal{NS}(A_5 \times S))| = 3$.*

The proof of Proposition 4.1.3 follows from the fact that $|\deg(\mathcal{NS}(A_5))| = 3$ and the result given below.

Lemma 4.1.4. *Let G be a finite non-solvable group and S be any finite solvable group. Then $|\deg(\mathcal{NS}(G))| = |\deg(\mathcal{NS}(G \times S))|$.*

Proof. Let $(x, s), (y, t) \in G \times S$ then $\langle (x, s), (y, t) \rangle \subseteq \langle x, y \rangle \times \langle s, t \rangle$. Therefore, $\langle (x, s), (y, t) \rangle$ is solvable if and only if $\langle x, y \rangle$ is solvable. Also, $\mathrm{Sol}_{G\times S}((x, s)) = \mathrm{Sol}_G(x) \times S$ and hence $\mathrm{Nbd}_{\mathcal{NS}(G\times S)}((x, s)) = \mathrm{Nbd}_{\mathcal{NS}(G)}(x) \times S$. That is, $\deg_{\mathcal{NS}(G\times S)}((x, s)) = |S| \deg_{\mathcal{NS}(G)}(x)$. This completes the proof. □

Now we state the main result of this section.

Theorem 4.1.5. *If G is a finite non-solvable group such that $G/\mathrm{Sol}(G) \cong A_5$ then $|\deg(\mathcal{NS}(G))| = 3$.*

To prove this theorem we need the following results.

Lemma 4.1.6. *Let H be a subgroup of a finite group G and $x, y \in G$.*

(a) *If $\langle x, y \rangle$ is solvable then $\langle xu, yv \rangle$ is also solvable for all $u, v \in \mathrm{Sol}(G)$.*

(b) *If $\langle x, y \rangle$ is not solvable then $\langle xu, yv \rangle$ is not solvable for all $u, v \in \mathrm{Sol}(G)$.*

Proof. Part (a) follows from the Lemma 3.2.1. Also, note that parts (a) and (b) are equivalent. □

Lemma 4.1.7. *Let G be a finite group and $x, y \in G$. Then $\langle x\,\mathrm{Sol}(G), y\,\mathrm{Sol}(G) \rangle$ is solvable if and only if $\langle x, y \rangle$ is solvable.*

Proof. Let $H = \langle x, y \rangle$ and $Z = \mathrm{Sol}(G)$. Note that $\langle xZ, yZ \rangle = \frac{HZ}{Z}$. Suppose $\langle xZ, yZ \rangle$ is solvable. Then $\frac{HZ}{Z}$ is solvable. Since $Z \subset \mathrm{Sol}(HZ)$ and Z is a normal subgroup of HZ, by Result 1.2.9, we have

$$\frac{\mathrm{Sol}_{HZ}(x)}{Z} = \mathrm{Sol}_{\frac{HZ}{Z}}(xZ) = \frac{HZ}{Z}.$$

Therefore, $\mathrm{Sol}_{HZ}(x) = HZ$. In particular, $\mathrm{Sol}_H(x) = H$ and so H is solvable.

If H is solvable then, by Lemma 4.1.6(a), $\mathrm{Sol}_{HZ}(x) = HZ$ for all $x \in HZ$. Thus HZ is solvable and so $\frac{HZ}{Z}$ is solvable. Hence, $\langle x_iZ, x_jZ \rangle$ is solvable for $x_i, x_j \in HZ$ and so $\langle xZ, yZ \rangle$ is solvable. □

Proposition 4.1.8. *Let G be a finite non-solvable group. Then for all $x \in G \setminus \mathrm{Sol}(G)$ we have*

$$\deg_{\mathcal{NS}(G)}(x) = \deg_{\mathcal{NS}(G/\mathrm{Sol}(G))}(x\,\mathrm{Sol}(G))|\mathrm{Sol}(G)|.$$

Proof. Let $y \in \mathrm{Nbd}_{\mathcal{NS}(G)}(x)$. By Lemma 4.1.6(b), we have $yz \in \mathrm{Nbd}_{\mathcal{NS}(G)}(x)$ for all $z \in \mathrm{Sol}(G)$. Thus $\mathrm{Nbd}_{\mathcal{NS}(G)}(x)$ is a union of distinct cosets of $\mathrm{Sol}(G)$. Let $\mathrm{Nbd}_{\mathcal{NS}(G)}(x) = y_1 \mathrm{Sol}(G) \cup y_2 \mathrm{Sol}(G) \cup \cdots \cup y_n \mathrm{Sol}(G)$. Then $\deg_{\mathcal{NS}(G)}(x) = n|\mathrm{Sol}(G)|$. By Lemma 4.1.7, we have $\langle x \mathrm{Sol}(G), y_i \mathrm{Sol}(G)\rangle$ is not solvable if and only if $\langle x, y_i\rangle$ is not solvable. Therefore, $\mathrm{Nbd}_{\mathcal{NS}(G/\mathrm{Sol}(G))}(x\mathrm{Sol}(G)) = \{y_1 \mathrm{Sol}(G), y_2 \mathrm{Sol}(G), \dots, y_n \mathrm{Sol}(G)\}$ in $\mathcal{NS}(G/\mathrm{Sol}(G))$. Hence, $\deg_{\mathcal{NS}(G/\mathrm{Sol}(G))}(x\mathrm{Sol}(G)) = n$ and the result follows. □

As a consequence of Proposition 4.1.8 we have the following corollary.

Corollary 4.1.9. *Let G be a finite non-solvable group. Then $|\deg(\mathcal{NS}(G/\mathrm{Sol}(G)))| = |\deg(\mathcal{NS}(G))|$.*

Proof of Theorem 4.1.5: Note that $G/\mathrm{Sol}(G) \cong A_5$ implies $\mathcal{NS}(G/\mathrm{Sol}(G)) \cong \mathcal{NS}(A_5)$. Therefore

$$|\deg(\mathcal{NS}(G/\mathrm{Sol}(G)))| = |\deg(\mathcal{NS}(A_5))| = 3.$$

Hence, the result follows from Corollary 4.1.9.

We conclude this section with the following upper bound for $|\deg(\mathcal{NS}(G))|$.

Theorem 4.1.10. *If G is a finite non-solvable group having n distinct solvabilizers then*

$$|\deg(\mathcal{NS}(G))| \leq n - 1.$$

Proof. Let $G, X_1, X_2, \dots, X_{n-1}$ be the distinct solvabilizers of G where $\mathrm{Sol}_G(x_i) = X_i$ for some $x_i \in G \setminus \mathrm{Sol}(G)$ and $i = 1, 2, \dots, n-1$. Then

$$\deg(\mathcal{NS}(G)) = \{|G| - |X_1|, |G| - |X_2|, \dots, |G| - |X_{n-1}|\}.$$

Hence, the result follows. □

4.2 Graph realization

By using Result 1.2.11, it can be shown that $\mathcal{NS}(G)$ is connected with diameter two. It is also shown that $\mathcal{NS}(G)$ is not regular and hence not a complete graph. Recently, Akbari [8] have shown that $\mathcal{NS}(G)$ is not a tree. In this section, we shall show that $\mathcal{NS}(G)$ is not a complete multi-partite graph. We shall also show that $\mathcal{NS}(G)$ is hamiltonian for some groups.

Theorem 4.2.1. *Let G be a finite non-solvable group. Then $\mathcal{NS}(G)$ is not a complete multi-partite graph. In particular, $\mathcal{NS}(G)$ is not a complete bipartite graph.*

Proof. Suppose $\mathcal{NS}(G)$ is a complete multi-partite graph. Let $X_1, X_2, \ldots, X_n$ be the partite sets. Let $x \in G \setminus \mathrm{Sol}(G)$ then $x \in X_i$ for some i and $\mathrm{Sol}_G(x) = \mathrm{Sol}(G) \cup X_i$. Let $y, z \in \mathrm{Sol}_G(x)$. Then $\langle y, z\rangle$ is solvable and $yz \in \mathrm{Sol}_G(y) = \mathrm{Sol}_G(x)$. Thus $\mathrm{Sol}_G(x)$ is a subgroup of G. By Result 1.2.8, G is solvable, a contradiction. Hence, the result follows. □

Theorem 4.2.2. *Let G be a finite non-solvable group. Then $\mathcal{NS}(G)$ is not a bipartite graph.*

Proof. Suppose $\mathcal{NS}(G)$ is a bipartite graph. Let X, Y be the partite sets. Let $x \in X$ and $y \in Y$. Then, by Result 1.2.11, there exists $z \in G \setminus \mathrm{Sol}(G)$ such that $\langle x, z\rangle$ and $\langle y, z\rangle$ are not solvable. Therefore, $z \notin X \cup Y$, a contradiction. Hence the result follows. □

Theorem 4.2.3. *Let G be a finite non-solvable group such that $|\mathrm{Sol}_G(x)| \leq \frac{|G|}{2}$ for all $x \in G \setminus \mathrm{Sol}(G)$. Then $\mathcal{NS}(G)$ is hamiltonian.*

Proof. Note that $\deg_{\mathcal{NS}(G)}(x) = |G| - |\mathrm{Sol}_G(x)|$ for all $x \in G \setminus \mathrm{Sol}(G)$. Since $|\mathrm{Sol}_G(x)| \leq \frac{|G|}{2}$ for all $x \in G \setminus \mathrm{Sol}(G)$ we have $|G| \geq 2|\mathrm{Sol}_G(x)|$. Thus, it follows that $\deg_{\mathcal{NS}(G)}(x) > (|G| - |\mathrm{Sol}(G)|)/2$. Therefore by Result 1.1.1, $\mathcal{NS}(G)$ is hamiltonian. □

Corollary 4.2.4. *The non-solvable graph of the group $PSL(3,2) \rtimes \mathbb{Z}_2$, A_6 and $PSL(2,8)$ are hamiltonian.*

Proof. The result follows from Theorem 4.2.3 using the fact that $|\mathrm{Sol}_G(x)| \leq \frac{|G|}{2}$ for all $x \in G \setminus \mathrm{Sol}(G)$ where $G = PSL(3,2) \rtimes \mathbb{Z}_2, A_6$ and $PSL(2,8)$. □

The following result shows that there is a group G with $|\mathrm{Sol}_G(x)| > |G|/2$ for some $x \in G \setminus \mathrm{Sol}(G)$ such that $\mathcal{NS}(G)$ is hamiltonian.

Proposition 4.2.5. *The non-solvable graph of A_5 is Hamiltonian.*

Proof. For any two vertex a and b we write $a \sim b$ if a is adjacent to b. It can be verified that

$(1,5,4,3,2) \sim (1,3)(2,5) \sim (2,3,4) \sim (1,4)(3,5) \sim (2,5,4) \sim (1,2)(3,4) \sim (1,5,4) \sim (2,5)(3,4) \sim (1,3,5) \sim (1,4)(2,5) \sim (2,4,3) \sim (1,3)(4,5) \sim (1,2,5) \sim (1,4)(2,3) \sim (3,5,4) \sim (1,5)(2,4) \sim (1,2,3) \sim (1,5)(3,4) \sim (2,3,5) \sim (1,4,2) \sim (2,3)(4,5) \sim (1,5,2) \sim (2,4)(3,5) \sim (1,4,5) \sim (1,2)(3,5) \sim (1,3,4) \sim (1,2)(4,5) \sim (1,5,3) \sim (1,4,2,5,3) \sim (1,3,2) \sim (3,4,5) \sim (1,3)(2,4) \sim (2,5,3) \sim (1,2,4) \sim (1,5)(2,3) \sim (2,4,5) \sim (1,4,3) \sim (1,3,5,2,4) \sim (1,4,5,3,2) \sim (1,2,3,4,5) \sim (1,2,4,3,5) \sim (1,5,3,2,4) \sim (1,4,5,2,3) \sim (1,5,4,2,3) \sim (1,3,4,5,2) \sim (1,5,3,4,2) \sim (1,3,2,4,5) \sim (1,3,2,5,4) \sim (1,2,4,5,3) \sim$

$(1,2,5,4,3) \sim (1,5,2,3,4) \sim (1,2,3,5,4 \sim (1,4,3,2,5) \sim (1,4,3,5,2) \sim (1,3,4,2,5) \sim (1,4,2,3,5) \sim (1,5,2,4,3) \sim (1,3,5,4,2) \sim (1,2,5,3,4) \sim (1,5,4,3,2)$
is a hamiltonian cycle of $\mathcal{NS}(A_5)$. Hence, $\mathcal{NS}(A_5)$ is hamiltonian. □

We conclude this section with the following question.

Question 4.2.6. Is $\mathcal{NS}(G)$ Hamiltonian for any finite non-solvable group G?

4.3 Domination number and vertex connectivity

In this section, we shall obtain a few results regarding $\lambda(\mathcal{NS}(G))$, the domination number of $\mathcal{NS}(G)$.

Proposition 4.3.1. *Let G be a finite non-solvable group. Then $\lambda(\mathcal{NS}(G)) \neq 1$.*

Proof. Let $\{x\}$ be a dominating set for $\mathcal{NS}(G)$. If $\mathrm{Sol}(G)$ contains a non-trivial element z then xz is adjacent to x, a contradiction. Hence, $|\mathrm{Sol}(G)| = 1$.

If $o(x) \neq 2$ then x is adjacent to x^{-1}, which is a contradiction. Hence, $o(x) = 2$ and so $x \in P_2$, for some Sylow 2-subgroup P_2 of G. Since $|\mathrm{Sol}(G)| = 1$ and x is adjacent to all vertices of $\mathcal{NS}(G)$ we have $\mathrm{Sol}_G(x) = \langle x \rangle$. Also, $P_2 \subseteq \mathrm{Sol}_G(x)$ and so $P_2 = \langle x \rangle$. If Q_2 is another Sylow 2-subgroup of G then $|Q_2| = 2$ and so $\langle P_2, Q_2 \rangle$ is a dihedral group and hence solvable. That is, x is not adjacent to $y \in Q_2$, $y \neq 1$, which is a contradiction. Thus it follows that P_2 is normal in G. Let $g \in G \setminus P_2$. Then $gxg^{-1} = x$, that is $xg = gx$ and so $x \in Z(G)$, which is a contradiction. Hence, the result follows. □

Using GAP [91], it can be seen that $\lambda(\mathcal{NS}(A_5)) = \lambda(\mathcal{NS}(S_5)) = 4$. In fact, $\{(3,4,5), (1,2,3,4,5), (1,2,4,5,3), (1,5)(2,4)\}$ and $\{(4,5), (1,2)(3,4,5), (1,3)(2,4,5), (1,5)(2,4)\}$ are dominating sets for A_5 and S_5 respectively. At this point we would like to ask the following question.

Question 4.3.2. Is there any finite non-solvable group G such that $\lambda(\mathcal{NS}(G)) = 2, 3$?

Proposition 4.3.3. *Let G be a non-solvable group. Then a subset S of $V(\mathcal{NS}(G))$ is a dominating set if and only if* $\mathrm{Sol}_G(S) \subset \mathrm{Sol}(G) \cup S$.

Proof. Suppose S is a dominating set. If $a \notin \mathrm{Sol}(G) \cup S$ then, by definition of dominating set, there exists $x \in S$ such that $\langle x, a \rangle$ is not solvable. Thus $a \notin \mathrm{Sol}_G(S)$. It follows that $\mathrm{Sol}_G(S) \subset S \cup \mathrm{Sol}(G)$.

Now assume that $\mathrm{Sol}_G(S) \subset \mathrm{Sol}(G) \cup S$. If $a \notin \mathrm{Sol}(G) \cup S$ then by hypothesis, $a \notin \mathrm{Sol}_G(S)$. Therefore, a is adjacent to at least one element of S. This completes the proof. □

We conclude this section with the following result on vertex cut set and vertex connectivity of $\mathcal{NS}(G)$.

Proposition 4.3.4. *Let G be a finite non-solvable group and let S be a vertex cut set of $\mathcal{NS}(G)$. Then S is a union of cosets of $\operatorname{Sol}(G)$. In particular $\kappa(\mathcal{NS}(G)) = t|\operatorname{Sol}(G)|$, where $t > 1$ is an integer.*

Proof. Let $a \in S$. Then there exist two distinct components G_1, G_2 of $\mathcal{NS}(G) \setminus S$ and two vertices $x \in V(G_1)$, $y \in V(G_2)$ such that a is adjacent to both x and y. By Lemma 4.1.6(b), x and y are also adjacent to az for any $z \in \operatorname{Sol}(G)$, and so $a\operatorname{Sol}(G) \subset S$. Thus S is a union of cosets of $\operatorname{Sol}(G)$. Hence, $\kappa(\mathcal{NS}(G)) = t|\operatorname{Sol}(G)|$, where $t \geq 1$ is an integer.

Suppose that $|S| = \kappa(\mathcal{NS}(G))$. It follows from the first part that $\kappa(\mathcal{NS}(G)) = t|\operatorname{Sol}(G)|$ for some integer $t \geq 1$. If $t = 1$ then $S = b\operatorname{Sol}(G)$ for some element $b \in G\setminus\operatorname{Sol}(G)$. Therefore, there exist two distinct components G_1, G_2 of $\mathcal{NS}(G) \setminus S$ and $r \in V(G_1)$, $s \in V(G_2)$ such that b is adjacent to both r and s. In other words, $\langle b, r\rangle$ and $\langle b, s\rangle$ are not solvable. Suppose that $o(b) \neq 2$. Then the number of integers less than $o(b)$ and relatively prime to it is greater or equal to 2. Let $1 \neq i \in \mathbb{N}$ such that $\gcd(i, o(b)) = 1$. Then

$$\langle b^i, r\rangle = \langle b, r\rangle \text{ and } \langle b^i, s\rangle = \langle b, s\rangle.$$

Therefore, b^i is adjacent to both r and s. This is a contradiction since $b^i \notin b\operatorname{Sol}(G)$. Hence, $o(b) = 2$.

Suppose $x' \in V(G_1)$ and $y' \in V(G_2)$ are adjacent to bz for some $z \in \operatorname{Sol}(G)$. Then, by Lemma 4.1.6(b), b is adjacent to x' and y'. Again, by Lemma 4.1.7, $x'\operatorname{Sol}(G)$ and $y'\operatorname{Sol}(G)$ are adjacent to $b\operatorname{Sol}(G)$ in the graph $\mathcal{NS}(G/\operatorname{Sol}(G))$. That is, $g\operatorname{Sol}(G)$ and $b\operatorname{Sol}(G)$ are adjacent for all $g\operatorname{Sol}(G) \in V(\mathcal{NS}(G/\operatorname{Sol}(G)))$. Therefore, $\{b\operatorname{Sol}(G)\}$ is a dominating set of $\mathcal{NS}(G/\operatorname{Sol}(G))$ and so $\lambda(\mathcal{NS}(G/\operatorname{Sol}(G))) = 1$. Hence, the result follows in view of Proposition 4.3.1. □

4.4 Independence Number

In this section we consider the following question on independence number of $\mathcal{NS}(G)$.

Question 4.4.1. Suppose G is a non-solvable group such that $\mathcal{NS}(G)$ has no infinite independent set. Is it true that $\alpha(\mathcal{NS}(G))$ is finite?

It is worth mentioning that Question 4.4.1 is similar to Question 1.3.9 and Question 1.3.12 where Abdollahi et al. and Nongsiang et al. considered non-commuting and non-nilpotent graphs of finite groups respectively. Note that the group considered in [80, Page

86] in order to answer Question 1.3.12 negatively, also gives negative answer to Question 4.4.1. However, the next theorem gives affirmative answer to Question 4.4.1 for some classes of groups.

Theorem 4.4.2. *Let G be a non-solvable group such that $\mathcal{NS}(G)$ has no infinite independent sets. If* $\mathrm{Sol}(G)$ *is a subgroup and G is an Engel, locally finite, locally solvable, a linear group or a 2-group then G is a finite group. In particular $\alpha(\mathcal{NS}(G))$ is finite.*

Note that Theorem 4.4.2 and its proof are similar to Result 1.3.10 and Result 1.3.13.

Proposition 4.4.3. *Let G be a group. Then for every maximal independent set S of $\mathcal{NS}(G)$ we have*

$$\bigcap_{x\in S} \mathrm{Sol}_G(x) = S \cup \mathrm{Sol}(G).$$

Proof. The result follows from the fact that S is maximal and $S \cup \mathrm{Sol}(G) \subset \mathrm{Sol}_G(x)$ for all $x \in S$. □

Remark 4.4.4. Let $R = \{(3,4,5),(1,4)(3,5),(2,5,3)\} \subset A_5$. Then R is an independent set of $\mathcal{NS}(A_5)$ and $\langle R\rangle \cong A_5$. This shows that a subgroup generated by an independent set may not be solvable. Also there exist a maximal independent set S, such that $R \subseteq S$. Since the edge set of $\mathcal{NS}(A_5)$ is non-empty, we have $S \neq A_5 \setminus \mathrm{Sol}(A_5)$, showing that $S \cup \mathrm{Sol}(A_5)$ is not a subgroup of A_5. Thus for a finite non-solvable group G and a maximal independent set S of $\mathcal{NS}(G)$, $S \cup \mathrm{Sol}(G)$ need not be a subgroup of G.

We conclude this section with the following result.

Theorem 4.4.5. *The order of a finite non-solvable group G is bounded by a function of the independence number of its non-solvable graph. Consequently, given a non-negative integer k, there are at the most finitely many finite non-solvable groups whose non-solvable graphs have independence number k.*

Proof. Let $x \in G$ such that $x, x^2 \notin \mathrm{Sol}(G)$. Then $x\,\mathrm{Sol}(G) \cup x^2\,\mathrm{Sol}(G)$ is an independent set of $\mathcal{NS}(G)$. Thus $|\mathrm{Sol}(G)| \leq \frac{k}{2}$. Let P be a Sylow subgroup of G then P is solvable. Thus it follows that $P \setminus \mathrm{Sol}(G)$ is an independent set of G. Hence $|P \setminus \mathrm{Sol}(G)| \leq k$, that is $|P| \leq \frac{3k}{2}$. Since, the number of primes less or equal to $\frac{3k}{2}$ is at most $\frac{3k}{4}$, we have $|G| \leq (\frac{3k}{2})^{\frac{3k}{4}}$. This completes the proof. □

4.5 Clique number of non-solvable graphs

In this section we prove the following results on clique number of $\mathcal{NS}(G)$.

Proposition 4.5.1. *Let G be a finite non-solvable group.*

(a) *If H is a non-solvable subgroup of G then $\omega(\mathcal{NS}(H)) \leq \omega(\mathcal{NS}(G))$.*

(b) *If $\frac{G}{N}$ is non-solvable then $\omega(\mathcal{NS}(\frac{G}{N})) \leq \omega(\mathcal{NS}(G))$. The equality holds when $N = \mathrm{Sol}(G)$.*

Proof. Part (a) follows from the fact that $\mathcal{NS}(H)$ is a subgraph of $\mathcal{NS}(G)$. For part (b), we shall show that $\mathcal{NS}(\frac{G}{N})$ is isomorphic to a subgraph of $\mathcal{NS}(G)$.

Let $V(\mathcal{NS}(\frac{G}{N})) = \{x_1N, x_2N, \dots, x_nN\}$ and $K = \{x_1, x_2, \dots, x_n\}$. Then, for $x_iN \in V(\mathcal{NS}(\frac{G}{N}))$, there exist $x_jN \in V(\mathcal{NS}(\frac{G}{N}))$ such that $\langle x_iN, x_jN\rangle$ is not solvable. Let $H = \langle x_i, x_j\rangle$. Then
$$\langle x_iN, x_jN\rangle = \frac{HN}{N}.$$
Suppose H is solvable. Then there exists a sub-normal series $\{1\} = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_n = H$, where H_i is normal in H_{i+1} and $\frac{H_{i+1}}{H_i}$ is abelian for all $i = 0, 1, \dots, n-1$. Consider the series $N = H_0N \leq H_1N \leq \cdots \leq H_nN = HN$. We have H_iN is normal in $H_{i+1}N$, for if $an \in H_iN$ and $bm \in H_{i+1}N$ then $bman(bm)^{-1} \in H_iN$. Also, $\frac{H_{i+1}N}{H_iN}$ is abelian, for if $a, b \in \frac{H_{i+1}N}{H_iN}$ then $a = kn_1(H_iN) = k(H_iN)$ and $b = ln_2(H_iN) = l(H_iN)$. Therefore, $ab = kl(H_iN)$. Since H_{i+1}/H_i is abelian, we have $kH_ilH_i = lH_ikH_i$, that is $klH_i = lkH_i$. Thus
$$ab = kl(H_iN) = (klH_i)N = (lkH_i)N = lk(H_iN) = ba.$$
Therefore, $\frac{H_{i+1}N}{H_iN}$ is abelian. Hence, HN is solvable and so $HN/N = \langle x_iN, x_jN\rangle$ is also solvable; which is a contradiction. Therefore, H is non-solvable. Let L be a graph such that $V(L) = K$ and two vertices x, y of L are adjacent if and only if xN and yN are adjacent in $\mathcal{NS}(\frac{G}{N})$. Then L is a subgraph of $\mathcal{NS}(G)[K]$ and hence a subgraph of $\mathcal{NS}(G)$. Define a map $\phi : V(\mathcal{NS}(\frac{G}{N})) \to V(L)$ by $\phi(x_iN) = x_i$. Then ϕ is one-one and onto. Also two vertices x_iN and x_jN are adjacent in $\mathcal{NS}(\frac{G}{N})$ if and only if x_i and x_j are adjacent in L. Thus $\mathcal{NS}(\frac{G}{N}) \cong L$.

If $N = \mathrm{Sol}(G)$ then, by Lemma 4.1.7, it follows that $\{x_1 \mathrm{Sol}(G), x_2 \mathrm{Sol}(G), \dots, x_t \mathrm{Sol}(G)\}$ is a clique of $\mathcal{NS}(\frac{G}{\mathrm{Sol}(G)})$ if and only if $\{x_1, x_2, \dots, x_t\}$ is a clique of $\mathcal{NS}(G)$. Hence, $\omega(\mathcal{NS}(\frac{G}{\mathrm{Sol}(G)})) = \omega(\mathcal{NS}(G))$. □

Theorem 4.5.2. *For any non-solvable group G and a solvable group S we have*

$$\omega(\mathcal{NS}(G)) = \omega(\mathcal{NS}(G \times S)).$$

Proof. Suppose C is a clique of $\mathcal{NS}(G)$. Let $a, b \in C$ then $\langle a, b\rangle$ is not solvable. Now, $\langle (a, e_s), (b, e_s)\rangle \cong \langle a, b\rangle$, where e_s is the identity element of S, and so $\langle (a, e_s), (b, e_s)\rangle$ is not solvable. Thus $C \times \{e_s\}$ is a clique of $\mathcal{NS}(G \times S)$. Now suppose D is a clique of $\mathcal{NS}(G \times S)$. Let $(x, s_1), (y, s_2) \in D$, where $x \neq y$. Then $\langle (x, s_1), (y, s_2)\rangle \subseteq \langle x, y\rangle \times \langle s_1, s_2\rangle$. Since S is solvable, we have $\langle s_1, s_2\rangle$ is solvable. Since $\langle (x, s_1), (y, s_2)\rangle$ is not solvable, we have $\langle x, y\rangle$ is not solvable. Thus $E = \{x : (x, s) \in D\}$ is a clique of $\mathcal{NS}(G)$. Hence, the result follows noting that $|D| = |E|$. □

The following lemma is useful in obtaining a lower bound for $\omega(\mathcal{NS}(G))$.

Lemma 4.5.3. *Let G be a finite non-solvable group. Then there exists an element $x \in G \setminus \mathrm{Sol}(G)$ such that $o(x)$ is a prime greater or equal to 5.*

Proof. Suppose that $1 \neq o(x) = 2^\alpha 3^\beta$ for all $x \in G \setminus \mathrm{Sol}(G)$, where α and β are non-zero integers. Then $|G/\mathrm{Sol}(G)| = 2^m 3^n$ for some non-zero integers m, n. Therefore, $G/\mathrm{Sol}(G)$ is solvable and so, by Lemma 4.1.7, G is solvable; a contradiction. This proves the existence of an element $x \in G \setminus \mathrm{Sol}(G)$ such that $o(x)$ is a prime greater or equal to 5. □

Proposition 4.5.4. *Let G be a finite non-solvable group. Then $\omega(\mathcal{NS}(G)) \geq 6$.*

Proof. By Lemma 4.5.3, we have an element $x \in G \setminus \mathrm{Sol}(G)$ such that $o(x)$ is a prime greater or equal to 5. Let $y \in G \setminus \mathrm{Sol}(G)$ such that x is adjacent to y. Then $\{x, y, xy, x^2y, x^3y, x^4y\}$ is a clique of $\mathcal{NS}(G)$ and so $\omega(\mathcal{NS}(G)) \geq 6$. □

The following program in GAP [91] shows that

$$\omega(\mathcal{NS}(A_5)) = \omega(\mathcal{NS}(SL(2,5))) = \omega(\mathcal{NS}(\mathbb{Z}_2 \times A_5)) = 8 \text{ and } \omega(\mathcal{NS}(S_5)) = 16.$$

Note that $A_5 = \mathrm{SmallGroup}(60, 5)$, $SL(2, 5) = \mathrm{SmallGroup}(120, 5)$, $S_5 = \mathrm{SmallGroup}(120, 34)$ and $\mathbb{Z}_2 \times A_5 = \mathrm{SmallGroup}(120, 35)$. Also $G/\mathrm{Sol}(G) \cong A_5$ for $G = A_5, SL(2, 5)$ and $\mathbb{Z}_2 \times A_5$.

```
LoadPackage("GRAPE");
sol:=[60,120];
for n in sol do
 allg:=AllSmallGroups(n);
```

```
  for g in allg do
    if IsSolvable(g)=false then
    h:=Graph(g,Difference(g,RadicalGroup(g)), OnPoints,function(x,y) return
    IsSolvable(Subgroup(g,[x,y]))=false; end, true);
    k:=CompleteSubgraphs(h);
    cn:=[];
    for i in  k
      do
      AddSet(cn,Size(i));
      od;
      Print("\n",IdGroup(g),",  ",StructureDescription(g),", cliquenumber=",
      Maximum(cn),"\n");
     fi;
    od;
    n:=n+1;
od;
```

The following program in GAP [91] shows that the clique number of $\mathcal{NS}(G)$ for groups of order less or equal to 360 with $G/\operatorname{Sol}(G) \ncong A_5$ is greater or equal to 9.

```
n:=120;
while n<=504 do
allg:=AllSmallGroups(n);
for g in allg do
if IsSolvable(g)=false then
rad:=RadicalGroup(g);
m:=Size(rad);
l:=n/m;
if l>60 then
dif:=Difference(g,rad);
for x in dif do
clique:=[x];
p:=0;
for y in dif do
i:=0;
```

```
for z in clique do
if IsSolvable(Subgroup(g,[y,z]))=true then
i:=1;
break;
fi;
od;
if i=0 then
AddSet(clique,y);
fi;
if Size(clique)>9 then
p:=1;
break;
fi;
od;
if p=1 then
break;
fi;
od;
if p=1 then
Print(IdGroup(g), "Clique greater than 8", "\n","\n");
else
Print(IdGroup(g), "Not Clique greater than 8", "\n","\n");
fi;
fi;
fi;
od;
n:=n+1;
od;
```

We conclude this section with the following conjecture.

Conjecture 4.5.5. Let G be a non-solvable group such that $\omega(\mathcal{NS}(G)) = 8$. Then $G/\operatorname{Sol}(G) \cong A_5$.

4.6 Groups with the same non-solvable graphs

In [72], Moghaddamfar et al. conjectured that if G and H are two non-abelian finite groups such that their non-commuting graphs are isomorphic then $|G| = |H|$. This conjecture was verified for several classes of finite groups in [1, 72]. However, the conjecture was refuted by Moghaddamfar [71] in the year 2006. Recently, Nongsiang and Saikia [80] posed similar conjecture for non-nilpotent graphs of finite groups. In this section, we consider the following problem.

Problem 4.6.1. Let G and H be two non-solvable groups such that $\mathcal{NS}(G) \cong \mathcal{NS}(H)$. Determine whether $|G| = |H|$.

We begin the section with the following theorem.

Theorem 4.6.2. *Let G and H be two non-solvable groups such that $\mathcal{NS}(G) \cong \mathcal{NS}(H)$. If G is finite then H is also finite. Moreover, $|\operatorname{Sol}(H)|$ divides*

$$\gcd(|G| - |\operatorname{Sol}(G)|, |G| - |\operatorname{Sol}_G(x)|, |\operatorname{Sol}_G(g)| - |\operatorname{Sol}(G)|),$$

where $g \in G \setminus \operatorname{Sol}(G)$, and hence $|H|$ is bounded by a function of G.

Proof. Since $\mathcal{NS}(G) \cong \mathcal{NS}(H)$, we have $|H \setminus \operatorname{Sol}(H)| = |G \setminus \operatorname{Sol}(G)|$ and so $|H \setminus \operatorname{Sol}(H)|$ is finite. If $h \in H \setminus \operatorname{Sol}(H)$ then $\{aha^{-1} : a \in H\} \subset H \setminus \operatorname{Sol}(H)$, since $\operatorname{Sol}(H)$ is closed under conjugation. Thus every element in $H \setminus \operatorname{Sol}(H)$ has finitely many conjugates in H. It follows that $K = C_H(H \setminus \operatorname{Sol}(H))$ has finite index in H. Since $\mathcal{NS}(H)$ has no isolated vertex, there exist two adjacent vertices u and v in $\mathcal{NS}(H)$. Now, if $s \in K$ then $s \in C_H(\{u, v\})$ and so $\langle su, v\rangle$ is not solvable since $\langle su, v\rangle \cong \langle u, v\rangle \times \langle s\rangle$. Therefore $Ku \subset H \setminus \operatorname{Sol}(H)$ and so K is finite. Hence, H is finite.

It follows that $\operatorname{Sol}(H)$ is a subgroup of H and so $|\operatorname{Sol}(H)|$ divides $|H| - |\operatorname{Sol}(H)|$. Since $|H| - |\operatorname{Sol}(H)| = |G| - |\operatorname{Sol}(G)|$, we have $|\operatorname{Sol}(H)|$ divides $|G| - |\operatorname{Sol}(G)|$. Let $x' \in H \setminus \operatorname{Sol}(H)$ and $y \in \operatorname{Sol}_H(x')$. Then, by Lemma 4.1.6(a), $\langle x', yz\rangle$ is solvable for all $z \in \operatorname{Sol}(H)$. Thus $\operatorname{Sol}_H(x') = \operatorname{Sol}(H) \cup y_1 \operatorname{Sol}(H) \cup \cdots \cup y_n \operatorname{Sol}(H)$, for some $y_i \in H$. Therefore $|\operatorname{Sol}(H)|$ divides $|\operatorname{Sol}_H(x')|$ and so $|\operatorname{Sol}(H)|$ divides $|H| - |\operatorname{Sol}_H(x')|$. We have $\deg(\mathcal{NS}(G)) = \deg(\mathcal{NS}(H))$ since $\mathcal{NS}(G) \cong \mathcal{NS}(H)$. Also $\deg_{\mathcal{NS}(G)}(g) = |G| - |\operatorname{Sol}_G(g)|$ for any $g \in V(\mathcal{NS}(G))$ and $\deg_{\mathcal{NS}(H)}(h) = |H| - |\operatorname{Sol}_H(h)|$ for any $h \in V(\mathcal{NS}(H))$. Therefore $|\operatorname{Sol}(H)|$ divides $|H| - |\operatorname{Sol}_H(h)|$ and hence $|G| - |\operatorname{Sol}_G(g)|$ for any $g \in G \setminus \operatorname{Sol}(G)$. Since $|\operatorname{Sol}(H)|$ divides $|G| - |\operatorname{Sol}(G)|$ and $|G| - |\operatorname{Sol}_G(g)|$, it divides $|G| - |\operatorname{Sol}(G)| - (|G| - |\operatorname{Sol}_G(g)|) = |\operatorname{Sol}_G(g)| - |\operatorname{Sol}(G)|$. This completes the proof. □

Proposition 4.6.3. *Let G be a non-solvable group such that $\mathcal{NS}(G)$ is finite. Then G is a finite group.*

Proof. It follows directly from the first paragraph of the proof of Theorem 4.6.2. $\square$

Proposition 4.6.4. *Let G be a group such that $\mathcal{NS}(G) \cong \mathcal{NS}(A_5)$ then $G \cong A_5$.*

Proof. Since $\mathcal{NS}(G) \cong \mathcal{NS}(A_5)$, we have G is a finite non-solvable group and

$$|G \setminus \mathrm{Sol}(G)| = |A_5 \setminus \mathrm{Sol}(A_5)| = 59.$$

Therefore, $|G| = |\mathrm{Sol}(G)| + 59$. Since $\mathrm{Sol}(G)$ is a subgroup of G, we have $|\mathrm{Sol}(G)| \leq \frac{|G|}{2}$ and so $|G| \leq 118$. Hence, the result follows. $\square$

Remark 4.6.5. Using the following program in GAP [91], one can see that the non-solvable graphs of $SL(2,5)$ and $\mathbb{Z}_2 \times A_5$ are isomorphic. It follows that non-solvable graphs of two groups are isomorphic need not implies that their corresponding groups are isomorphic.

```
LoadPackage("GRAPE");
g:=SmallGroup(120,5);
solg:=RadicalGroup(g);
gmc:= Difference(g,solg);
m:=Size(gmc);
h:=SmallGroup(120,35);
hmc:=Difference(h,RadicalGroup(h));
if m=Size(hmc) then
   gg:=Graph(g,gmc,OnPoints,function(x,y) return
   IsSolvable(Subgroup(g,[x,y]))=false; end, true);
   gh:=Graph(h,hmc,OnPoints,function(x,y) return
   IsSolvable(Subgroup(h,[x,y]))=false; end, true);
   if IsIsomorphicGraph(gg,gh)=true then
     Print("\n","\n","an example of G and H isomorphic
     but not of same order.doc","G= ",
     StructureDescription(g), ", ", " Id=", IdGroup(g)," H = ",
     StructureDescription(h)," Id=", IdGroup(h),"\n","\n");
   fi;
fi;
```

Proposition 4.6.6. *Let G and H be two finite non-solvable groups. If $\mathcal{NS}(G) \cong \mathcal{NS}(H)$ then $\mathcal{NS}(G \times A) \cong \mathcal{NS}(H \times B)$, where A and B are two solvable groups having equal order.*

Proof. Let $\varphi : \mathcal{NS}(G) \to \mathcal{NS}(H)$ be a graph isomorphism and $\psi : A \to B$ be a bijective map. Then $(g, a) \mapsto (\varphi(g), \psi(a))$ defines a graph isomorphism between $\mathcal{NS}(G \times A)$ and $\mathcal{NS}(H \times B)$. □

A non-solvable group G is called an *Fs-group* if for every two elements $x, y \in G \setminus \mathrm{Sol}(G)$ such that $\mathrm{Sol}_G(x) \neq \mathrm{Sol}_G(y)$ implies $\mathrm{Sol}_G(x) \not\subset \mathrm{Sol}_G(y)$ and $\mathrm{Sol}_G(y) \not\subset \mathrm{Sol}_G(x)$.

Proposition 4.6.7. *Let G be an Fs-group. If H is a non-solvable group such that $\mathcal{NS}(G) \cong \mathcal{NS}(H)$ then H is also an Fs-group.*

Proof. Let $\psi : \mathcal{NS}(H) \to \mathcal{NS}(G)$ be a graph isomorphism. Let $x, y \in H \setminus \mathrm{Sol}(H)$ such that $\mathrm{Sol}_H(x) \subseteq \mathrm{Sol}_H(y)$. Then $\psi(\mathrm{Sol}_H(x) \setminus \mathrm{Sol}(H)) \subseteq \psi(\mathrm{Sol}_H(y) \setminus \mathrm{Sol}(H))$. We have

$$\psi(\mathrm{Sol}_H(z) \setminus \mathrm{Sol}(H)) = \mathrm{Sol}_G(\psi(z)) \setminus \mathrm{Sol}(G) \text{ for all } z \in H \setminus \mathrm{Sol}(H).$$

Therefore, $\mathrm{Sol}_G(\psi(x)) \setminus \mathrm{Sol}(G) \subseteq \mathrm{Sol}_G(\psi(y)) \setminus \mathrm{Sol}(G)$. Since G is an Fs-group, we have

$$\mathrm{Sol}_G(\psi(x)) \setminus \mathrm{Sol}(G) = \mathrm{Sol}_G(\psi(y)) \setminus \mathrm{Sol}(G).$$

It follows that $\mathrm{Sol}_H(x) \setminus \mathrm{Sol}(H) = \mathrm{Sol}_H(y) \setminus \mathrm{Sol}(H)$ and so $\mathrm{Sol}_H(x) = \mathrm{Sol}_H(y)$. Hence, H is an Fs-group. □

4.7 Genus of non-solvable graph

In Result 1.3.16, it was shown that $\mathcal{NS}(G)$ is not planar for finite non-solvable group G. In this section, we extent Result 1.3.16 and show that $\mathcal{NS}(G)$ is neither planar, toroidal, double-toroidal nor triple-toroidal. We also obtain the following bound for $|\mathrm{Sol}(G)|$ in terms of genus of $\mathcal{NS}(G)$.

Proposition 4.7.1. *Let G be a finite non-solvable group. Then*

$$|\mathrm{Sol}(G)| \leq \sqrt{2\gamma(\mathcal{NS}(G))} + 2.$$

Proof. Assume that $Z = \mathrm{Sol}(G)$. By Proposition 4.5.4, we have $\omega(\mathcal{NS}(G)) \geq 3$. So, there exist $u, v, w \in G \setminus Z$ such that they are adjacent to each other. Then, by Lemma 4.1.6(b), $\mathcal{NS}(G)[uZ \cup vZ \cup wZ]$ is isomorphic to $K_{|Z|,|Z|,|Z|}$. We have

$$\gamma(\mathcal{NS}(G)) \geq \gamma(K_{|Z|,|Z|,|Z|}) = \frac{(|Z|-2)(|Z|-1)}{2} \geq \frac{(|Z|-2)(|Z|-2)}{2}$$

and hence the result follows. □

Theorem 4.7.2. *Let G be a finite non-solvable graph. Then $\gamma(\mathcal{NS}(G)) \geq 4$. In particular, $\mathcal{NS}(G)$ is neither planar, toroidal, double-toroidal nor triple-toroidal.*

Proof. By Lemma 4.5.3, we have an element $x \in G \setminus \mathrm{Sol}(G)$ such that $o(x)$ is a prime greater or equal to 5. Clearly, $\mathrm{Nbd}_{\mathcal{NS}(G)}(x) \neq \emptyset$. Assume that $o(y) = 2$ for all $y \in \mathrm{Nbd}_{\mathcal{NS}(G)}(x)$. Then $xy \in \mathrm{Nbd}_{\mathcal{NS}(G)}(x)$ and so $o(xy) = 2$. Thus $\langle x, y \rangle = \langle y, xy \rangle$ is isomorphic to a dihedral group, which is a contradiction. Therefore, there exist $y \in \mathrm{Nbd}_{\mathcal{NS}(G)}(x)$ such that $o(y) \geq 3$. Let $1 \neq j \in \mathbb{N}$ and $\gcd(j, o(x)) = 1$. Consider the subsets $H = \{x, x^2, x^3, x^4\}$, $K = \{y^i x^j : i = 1, 2, j = 0, 1, 2, 3, 4\}$ of $G \setminus \mathrm{Sol}(G)$ and the induced graph $\mathcal{NS}(G)[H \cup K]$. Notice that $\mathcal{NS}(G)[H \cup K]$ has a subgraph isomorphic to $K_{4,10}$ and hence

$$\gamma(\mathcal{NS}(G)) \geq \gamma(\mathcal{NS}(G)[H \cup K]) \geq \gamma(K_{4,10}) = 4.$$

This completes the proof. □

Remark 4.7.3. By GAP [91], using the following program, we see that $\mathcal{NS}(A_5)$ has 1140 edges and 59 vertices. Thus by Result 1.1.2, we have $\gamma(\mathcal{NS}(A_5)) \geq \frac{1140}{6} - \frac{59}{2} + 1 = 161.5$ and so $\gamma(\mathcal{NS}(A_5)) \geq 162$.

```
LoadPackage("GRAPE");
g:=AlternatingGroup(5);
solg:=RadicalGroup(g);
h:=Graph(g,Difference(g,solg),OnPoints,function(x,y) return
IsSolvable(Subgroup(g,[x,y]))= false; end, true);
k:=Vertices(h);
i:=0;
for x in k do
    i:=i+VertexDegree(h,x);
od;
Print("Number of Edges=",i/2);
```

Similarly $\mathcal{NS}(S_5), \mathcal{NS}(SL(2,5))$ and $\mathbb{Z}_2 \times A_5$ has 4560 edges and 119 vertices. So their genera are at least 732.

It is shown in [49] that $2K_5$ is not projective. Hence, any graph containing a subgraph isomorphic to $2K_5$ is not projective. We conclude this chapter with the following result.

Theorem 4.7.4. *Let G be a finite non-solvable group. Then $\mathcal{NS}(G)$ is not projective.*

Proof. As shown in the proof of Theorem 4.7.2, there exist $x, y \in G\setminus\mathrm{Sol}(G)$ such that $o(x)$ is a prime greater or equal to 5, $o(y) \geq 3$ and they are adjacent. Let $1 \neq j \in \mathbb{N}, \gcd(j, o(y)) = 1$. Consider the subsets $H = \{y, xy, x^2y, x^3y, x^4y\}$ and $K = \{y^j, xy^j, x^2y^j, x^3y^j, x^4y^j\}$ of $G \setminus \mathrm{Sol}(G)$. Then $H \cap K = \emptyset$ and $\mathcal{NS}(G)[H] \cong \mathcal{NS}(G)[K] \cong K_5$. It follows that $\mathcal{NS}(G)$ has a subgraph isomorphic to $2K_5$. Hence, $\mathcal{NS}(G)$ is not projective. □