Chapter 5

Various spectra and energies of commuting conjugacy class graph of groups

The commuting conjugacy class graph (or CCC-graph) of a group G, defined by $\mathcal{CCC}(G)$, is a simple undirected graph whose vertex set is the set of conjugacy classes of non-central elements of G and two distinct vertices x^G and y^G are adjacent if there exists some elements $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is abelian. In 2020, Salahshour and Ashrafi [86] have obtained the structure of $\mathcal{CCC}(G)$ considering G to be the groups $D_{2n} (n \geq 3)$, $Q_{4m} (m \geq 2)$, $U_{(n,m)} (m \geq 2$ and $n \geq 2)$, $V_{8n} (n \geq 2)$, $SD_{8n} (n \geq 2)$, $G(p, m, n)$, (where p is any prime, $m \geq 1$ and $n \geq 1$). In this chapter we compute various spectra and energies of commuting conjugacy class graph of these groups. Computation of various spectra is helpful to check whether $\mathcal{CCC}(G)$ is super integral. In Section 5.1, we shall compute various spectra and energies of CCC-graph of the above mentioned groups and observed that $\mathcal{CCC}(G)$ is super integral if $G = D_{2n}, Q_{4m}, U_{(n,m)}, V_{8n}, SD_{8n}$ and $G(p, m, n)$. In Section 5.2, we shall determine whether the inequalities in Conjecture 1.1.7 and Question 1.1.8 satisfy for $\mathcal{CCC}(G)$. In Section 5.3, we shall determine whether $\mathcal{CCC}(G)$ is hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic or Q-borderenergetic. This chapter is based on our papers [18] published in *Algebraic Structures and Their Applications* and [21] communicated for publication.

5.1 Various spectra and energies

In this section we compute various spectra and energies of commuting conjugacy class graphs of the groups mentioned in the introduction.

Theorem 5.1.1. *If $G = D_{2n}$ then*

(a) $$\operatorname{Spec}(\mathcal{CCC}(G)) = \begin{cases} \left\{(-1)^{\frac{n-3}{2}}, 0^1, \left(\frac{n-3}{2}\right)^1\right\}, & \text{if } n \text{ is odd} \\ \left\{(-1)^{\frac{n}{2}-2}, 0^2, \left(\frac{n}{2}-2\right)^1\right\}, & \text{if } n \text{ and } \frac{n}{2} \text{ are even} \\ \left\{(-1)^{\frac{n}{2}-1}, 1^1, \left(\frac{n}{2}-2\right)^1\right\}, & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd} \end{cases}$$

and $$E(\mathcal{CCC}(G)) = \begin{cases} n-3, & \text{if } n \text{ is odd} \\ n-4, & \text{if } n \text{ and } \frac{n}{2} \text{ are even} \\ n-2, & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd.} \end{cases}$$

(b) $$\operatorname{L-spec}(\mathcal{CCC}(G)) = \begin{cases} \left\{0^2, \left(\frac{n-1}{2}\right)^{\frac{n-3}{2}}\right\}, & \text{if } n \text{ is odd} \\ \left\{0^3, \left(\frac{n}{2}-1\right)^{\frac{n}{2}-2}\right\}, & \text{if } n \text{ and } \frac{n}{2} \text{ are even} \\ \left\{0^2, 2^1, \left(\frac{n}{2}-1\right)^{\frac{n}{2}-2}\right\}, & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd} \end{cases}$$

and $$LE(\mathcal{CCC}(G)) = \begin{cases} \frac{2(n-1)(n-3)}{n+1}, & \text{if } n \text{ is odd} \\ \frac{3(n-2)(n-4)}{n+2}, & \text{if } n \text{ and } \frac{n}{2} \text{ are even} \\ 4, & \text{if } n = 6 \\ \frac{(n-4)(3n-10)}{n+2}, & \text{if } n \text{ is even, } n \geq 10 \text{ and } \frac{n}{2} \text{ is odd.} \end{cases}$$

(c) $$\operatorname{Q-spec}(\mathcal{CCC}(G)) = \begin{cases} \left\{0^1, (n-3)^1, \left(\frac{n-5}{2}\right)^{\frac{n-3}{2}}\right\}, & \text{if } n \text{ is odd} \\ \left\{0^2, (n-4)^1, \left(\frac{n}{2}-3\right)^{\frac{n}{2}-2}\right\}, & \text{if } n \text{ and } \frac{n}{2} \text{ are even} \\ \left\{2^1, 0^1, (n-4)^1, \left(\frac{n}{2}-3\right)^{\frac{n}{2}-2}\right\}, & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd} \end{cases}$$

$$\text{and } LE^{+}(\mathcal{CCC}(G)) = \begin{cases} \frac{(n-3)(n+3)}{n+1}, & \text{if } n \text{ is odd} \\ \frac{(n-4)(n+6)}{n+2}, & \text{if } n = 4, 8 \\ \frac{2(n-2)(n-4)}{n+2}, & \text{if } n, \frac{n}{2} \text{ are even and } n \geq 12 \\ 4, & \text{if } n = 6 \\ \frac{22}{3}, & \text{if } n = 10 \\ \frac{2(n-2)(n-6)}{n+2}, & \text{if } n \text{ is even, } n \geq 14 \text{ and } \frac{n}{2} \text{ is odd.} \end{cases}$$

Proof. We shall prove the result by considering the following cases.

Case 1. n is odd.

By Result 1.3.18 we have $\mathcal{CCC}(G) = K_1 \sqcup K_{\frac{n-1}{2}}$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \left\{(-1)^{\frac{n-3}{2}}, 0^1, \left(\frac{n-3}{2}\right)^1\right\}, \quad \text{L-spec}(\mathcal{CCC}(G)) = \left\{0^2, \left(\frac{n-1}{2}\right)^{\frac{n-3}{2}}\right\}$$

$$\text{and Q-spec}(\mathcal{CCC}(G)) = \left\{0^1, (n-3)^1, \left(\frac{n-5}{2}\right)^{\frac{n-3}{2}}\right\}.$$

Hence, by (1.1.d), we get

$$E(\mathcal{CCC}(G)) = \frac{n-3}{2} + \frac{n-3}{2} = n-3.$$

We have $|V(\mathcal{CCC}(G))| = \frac{n+1}{2}$ and $|e(\mathcal{CCC}(G))| = \frac{(n-1)(n-3)}{8}$. Therefore, $\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{(n-1)(n-3)}{2(n+1)}$. Also,

$$\left|0 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0 - \frac{(n-1)(n-3)}{2(n+1)}\right| = \frac{(n-1)(n-3)}{2(n+1)} \quad \text{and}$$

$$\left|\frac{n-1}{2} - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|\frac{n-1}{2} - \frac{(n-1)(n-3)}{2(n+1)}\right| = \frac{2(n-1)}{n+1}.$$

Now, by (1.1.e), we have

$$LE(\mathcal{CCC}(G)) = 2 \times \frac{(n-1)(n-3)}{2(n+1)} + \frac{n-3}{2} \times \frac{2(n-1)}{n+1} = \frac{2(n-1)(n-3)}{n+1}.$$

Again,

$$\left|n-3 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|n-3 - \frac{(n-1)(n-3)}{2(n+1)}\right| = \frac{(n-3)(n+3)}{2(n+1)} \quad \text{and}$$

$$\left|\frac{n-5}{2} - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|\frac{n-5}{2} - \frac{(n-1)(n-3)}{2(n+1)}\right| = \left|\frac{-4}{n+1}\right| = \frac{4}{n+1}.$$

By (1.1.*f*), we have

$$LE^+(\mathcal{CCC}(G)) = \frac{(n-1)(n-3)}{2(n+1)} + \frac{(n-3)(n+3)}{2(n+1)} + \frac{n-3}{2} \times \frac{4}{n+1} = \frac{(n-3)(n+3)}{n+1}.$$

Case 2. n is even.

Consider the following subcases.

Subcase 2.1 $\frac{n}{2}$ is even.

By Result 1.3.18 we have $\mathcal{CCC}(G) = 2K_1 \sqcup K_{\frac{n}{2}-1}$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \left\{(-1)^{\frac{n}{2}-2}, 0^2, \left(\frac{n}{2}-2\right)^1\right\}, \quad \text{L-spec}(\mathcal{CCC}(G)) = \left\{0^3, \left(\frac{n}{2}-1\right)^{\frac{n}{2}-2}\right\}$$
and $\text{Q-spec}(\mathcal{CCC}(G)) = \left\{0^2, (n-4)^1, \left(\frac{n}{2}-3\right)^{\frac{n}{2}-2}\right\}$.

Hence, by (1.1.*d*), we get

$$E(\mathcal{CCC}(G)) = \frac{n}{2} - 2 + \frac{n}{2} - 2 = n - 4.$$

We have $|V(\mathcal{CCC}(G))| = \frac{n}{2} + 1$ and $|e(\mathcal{CCC}(G))| = \frac{(n-2)(n-4)}{8}$. So,

$$\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{(n-2)(n-4)}{2(n+2)}.$$

Also,

$$\left|0 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0 - \frac{(n-2)(n-4)}{2(n+2)}\right| = \frac{(n-2)(n-4)}{2(n+2)} \quad \text{and}$$
$$\left|\frac{n}{2} - 1 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|\frac{n}{2} - 1 - \frac{(n-2)(n-4)}{2(n+2)}\right| = \frac{3(n-2)}{n+2}.$$

Now, by (1.1.*e*), we have

$$LE(\mathcal{CCC}(G)) = 3 \times \frac{(n-2)(n-4)}{2(n+2)} + \left(\frac{n}{2} - 2\right) \times \frac{3(n-2)}{n+2} = \frac{3(n-2)(n-4)}{n+2}.$$

Again,

$$\left|n - 4 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|n - 4 - \frac{(n-2)(n-4)}{2(n+2)}\right| = \frac{(n-4)(n+6)}{2(n+2)} \quad \text{and}$$

$$\left|\frac{n}{2} - 3 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|\frac{n}{2} - 3 - \frac{(n-2)(n-4)}{2(n+2)}\right| = \left|\frac{n-10}{n+2}\right| = \begin{cases} \frac{-n+10}{n+2}, & \text{if } n = 4, 8 \\ \frac{n-10}{n+2}, & \text{if } n \geq 12. \end{cases}$$

By (1.1.f), we have

$$LE^+(\mathcal{CCC}(G)) = 2\times\frac{(n-2)(n-4)}{2(n+2)}+\frac{(n-4)(n+6)}{2(n+2)}+\left(\frac{n}{2}-2\right)\times\frac{-n+10}{n+2} = \frac{(n-4)(n+6)}{n+2},$$

if $n = 4, 8$. If $n \geq 12$ then

$$LE^+(\mathcal{CCC}(G)) = 2\times\frac{(n-2)(n-4)}{2(n+2)}+\frac{(n-4)(n+6)}{2(n+2)}+\left(\frac{n}{2}-2\right)\times\frac{n-10}{n+2} = \frac{2(n-2)(n-4)}{n+2}.$$

Subcase 2.2 $\frac{n}{2}$ is odd.

By Result 1.3.18 we have $\mathcal{CCC}(G) = K_2 \sqcup K_{\frac{n}{2}-1}$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \left\{(-1)^{\frac{n}{2}-1}, 1^1, \left(\frac{n}{2}-2\right)^1\right\}, \quad \text{L-spec}(\mathcal{CCC}(G)) = \left\{0^2, 2^1, \left(\frac{n}{2}-1\right)^{\frac{n}{2}-2}\right\}$$

$$\text{and Q-spec}(\mathcal{CCC}(G)) = \left\{2^1, 0^1, (n-4)^1, \left(\frac{n}{2}-3\right)^{\frac{n}{2}-2}\right\}.$$

Hence, by (1.1.d), we get

$$E(\mathcal{CCC}(G)) = \frac{n}{2} - 1 + 1 + \frac{n}{2} - 2 = n - 2.$$

We have $|V(\mathcal{CCC}(G))| = \frac{n}{2}+1$ and $|e(\mathcal{CCC}(G))| = \frac{(n-2)(n-4)+8}{8}$. Therefore, $\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{(n-2)(n-4)+8}{2(n+2)}$. Also,

$$\left|0 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0 - \frac{(n-2)(n-4)+8}{2(n+2)}\right| = \frac{(n-2)(n-4)+8}{2(n+2)},$$

$$\begin{aligned}\left|2 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|2 - \frac{(n-2)(n-4)+8}{2(n+2)}\right| \\ &= \left|\frac{-n^2+10n-8}{2(n+2)}\right| = \begin{cases} 1, & \text{if } n = 6 \\ \frac{n^2-10n+8}{2(n+2)}, & \text{if } n \geq 10 \end{cases}\end{aligned}$$

and

$$\left|\frac{n}{2} - 1 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|\frac{n}{2} - 1 - \frac{(n-2)(n-4)+8}{2(n+2)}\right| = \frac{3n-10}{n+2}.$$

Now, by (1.1.e), we have

$$LE(\mathcal{CCC}(G)) = 2 \times \frac{(n-2)(n-4)+8}{2(n+2)} + 1 + \left(\frac{n}{2}-2\right) \times \frac{3n-10}{n+2} = 4,$$

if $n = 6$. If $n \geq 10$ then

$$\begin{aligned}LE(\mathcal{CCC}(G)) &= 2 \times \frac{(n-2)(n-4)+8}{2(n+2)} + \frac{n^2-10n+8}{2(n+2)} + \left(\frac{n}{2}-2\right) \times \frac{3n-10}{n+2} \\ &= \frac{3n^2-22n-40}{n+2} = \frac{(n-4)(3n-10)}{n+2}.\end{aligned}$$

Again,

$$\left|n-4-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|n-4-\frac{(n-2)(n-4)+8}{2(n+2)}\right| = \frac{n^2+2n-32}{2(n+2)} \quad \text{and}$$

$$\begin{aligned}\left|\frac{n}{2}-3-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|\frac{n}{2}-3-\frac{(n-2)(n-4)+8}{2(n+2)}\right| \\ &= \left|\frac{n-14}{n+2}\right| = \begin{cases}\frac{-n+14}{n+2}, & \text{if } n = 6, 10 \\ \frac{n-14}{n+2}, & \text{if } n \geq 14.\end{cases}\end{aligned}$$

By (1.1.*f*), we have

$$LE^+(\mathcal{CCC}(G)) = 1 + \frac{(n-2)(n-4)+8}{2(n+2)} + \frac{n^2+2n-32}{2(n+2)} + \left(\frac{n}{2}-2\right) \times \frac{-n+14}{n+2} = 4,$$

if $n = 6$. If $n = 10$ then

$$\begin{aligned}LE^+(\mathcal{CCC}(G)) &= \frac{n^2-10n+8}{2(n+2)} + \frac{(n-2)(n-4)+8}{2(n+2)} + \frac{n^2+2n-32}{2(n+2)} + \left(\frac{n}{2}-2\right) \times \frac{-n+14}{n+2} \\ &= \frac{22}{3}.\end{aligned}$$

If $n \geq 14$ then

$$\begin{aligned}LE^+(\mathcal{CCC}(G)) &= \frac{n^2-10n+8}{2(n+2)} + \frac{(n-2)(n-4)+8}{2(n+2)} + \frac{n^2+2n-32}{2(n+2)} + \left(\frac{n}{2}-2\right) \times \frac{n-14}{n+2} \\ &= \frac{2(n-2)(n-6)}{n+2}.\end{aligned}$$

This completes the proof. □

Theorem 5.1.2. *If $G = Q_{4m}$ then*

(a) $\text{Spec}(\mathcal{CCC}(G)) = \begin{cases}\left\{(-1)^{m-1}, 1^1, (m-2)^1\right\}, & \textit{if } m \textit{ is odd} \\ \left\{(-1)^{m-2}, 0^2, (m-2)^1\right\}, & \textit{if } m \textit{ is even}\end{cases}$

and $E(\mathcal{CCC}(G)) = \begin{cases}2m-2, & \textit{if } m \textit{ is odd} \\ 2m-4, & \textit{if } m \textit{ is even.}\end{cases}$

(b) $\text{L-spec}(\mathcal{CCC}(G)) = \begin{cases}\left\{0^2, 2^1, (m-1)^{m-2}\right\}, & \textit{if } m \textit{ is odd} \\ \left\{0^3, (m-1)^{m-2}\right\}, & \textit{if } m \textit{ is even}\end{cases}$

$$\textit{and } LE(\mathcal{CCC}(G)) = \begin{cases} 4, & \textit{if } m=3 \\ \frac{2(m-2)(3m-5)}{m+1}, & \textit{if } m \textit{ is odd and } m \geq 5 \\ \frac{6(m-1)(m-2)}{m+1}, & \textit{if } m \textit{ is even.} \end{cases}$$

(c) $$\text{Q-spec}(\mathcal{CCC}(G)) = \begin{cases} \left\{2^1, 0^1, (2m-4)^1, (m-3)^{m-2}\right\}, & \textit{if } m \textit{ is odd} \\ \left\{0^2, (2m-4)^1, (m-3)^{m-2}\right\}, & \textit{if } m \textit{ is even} \end{cases}$$

$$\textit{and } LE^+(\mathcal{CCC}(G)) = \begin{cases} 4, & \textit{if } m=3 \\ \frac{22}{3}, & \textit{if } m=5 \\ \frac{4(m-1)(m-3)}{m+1}, & \textit{if } m \textit{ is odd and } m \geq 7 \\ \frac{2(m-2)(m+3)}{m+1}, & \textit{if } m=2,4 \\ \frac{4(m-1)(m-2)}{m+1}, & \textit{if } m \textit{ is even and } m \geq 6. \end{cases}$$

Proof. We shall prove the result by considering the following cases.

Case 1. m is odd.

By Result 1.3.19 we have $\mathcal{CCC}(G) = K_2 \sqcup K_{m-1}$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \left\{(-1)^{m-1}, 1^1, (m-2)^1\right\}, \quad \text{L-spec}(\mathcal{CCC}(G)) = \left\{0^2, 2^1, (m-1)^{m-2}\right\}$$

and $\text{Q-spec}(\mathcal{CCC}(G)) = \left\{2^1, 0^1, (2m-4)^1, (m-3)^{m-2}\right\}$.

Hence, by (1.1.d), we get

$$E(\mathcal{CCC}(G)) = m-1+1+m-2 = 2m-2.$$

We have $|V(\mathcal{CCC}(G))| = m+1$ and $|e(\mathcal{CCC}(G))| = \frac{(m-1)(m-2)+2}{2}$. Therefore,

$$\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{(m-1)(m-2)+2}{m+1}.$$

Also,

$$\left|0 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0 - \frac{(m-1)(m-2)+2}{m+1}\right| = \frac{(m-1)(m-2)+2}{m+1},$$

$$\left|2 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|2 - \frac{(m-1)(m-2)+2}{m+1}\right|$$

$$= \left|\frac{-m^2+5m-2}{m+1}\right| = \begin{cases} 1, & \text{if } m=3 \\ \frac{m^2-5m+2}{m+1}, & \text{if } m \geq 5 \end{cases}$$

and

$$\left|m-1-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|m-1-\frac{(m-1)(m-2)+2}{m+1}\right| = \frac{3m-5}{m+1}.$$

Now, by (1.1.e), we have

$$LE(\mathcal{CCC}(G)) = 2\times\frac{(m-1)(m-2)+2}{m+1}+1+(m-2)\times\frac{3m-5}{m+1} = 4,$$

if $m=3$. If $m\geq 5$ then

$$\begin{aligned}LE(\mathcal{CCC}(G)) &= 2\times\frac{(m-1)(m-2)+2}{m+1}+\frac{m^2-5m+2}{m+1}+(m-2)\times\frac{3m-5}{m+1}\\ &= \frac{2(m-2)(3m-5)}{m+1}.\end{aligned}$$

Again,

$$\left|2m-4-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|2m-4-\frac{(m-1)(m-2)+2}{m+1}\right| = \frac{m^2+m-8}{m+1}$$

and

$$\begin{aligned}\left|m-3-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|m-3-\frac{(m-1)(m-2)+2}{m+1}\right|\\ &= \left|\frac{m-7}{m+1}\right| = \begin{cases}\frac{-m+7}{m+1}, & \text{if } m=3,5\\ \frac{m-7}{m+1}, & \text{if } m\geq 7.\end{cases}\end{aligned}$$

By (1.1.f), we have

$$LE^+(\mathcal{CCC}(G)) = 1+\frac{(m-1)(m-2)+2}{m+1}+\frac{m^2+m-8}{m+1}+(m-2)\times\frac{-m+7}{m+1} = 4,$$

if $m=3$. If $m=5$ then

$$\begin{aligned}LE^+(\mathcal{CCC}(G)) &= \frac{m^2-5m+2}{m+1}+\frac{(m-1)(m-2)+2}{m+1}+\frac{m^2+m-8}{m+1}+(m-2)\times\frac{-m+7}{m+1}\\ &= \frac{22}{3}.\end{aligned}$$

If $m\geq 7$ then

$$\begin{aligned}LE^+(\mathcal{CCC}(G)) &= \frac{m^2-5m+2}{m+1}+\frac{(m-1)(m-2)+2}{m+1}+\frac{m^2+m-8}{m+1}+(m-2)\times\frac{m-7}{m+1}\\ &= \frac{4(m-1)(m-3)}{m+1}.\end{aligned}$$

Case 2. m is even.

By Result 1.3.19 we have $\mathcal{CCC}(G) = 2K_1 \sqcup K_{m-1}$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \left\{(-1)^{m-2}, 0^2, (m-2)^1\right\}, \quad \text{L-spec}(\mathcal{CCC}(G)) = \left\{0^3, (m-1)^{m-2}\right\}$$
$$\text{and Q-spec}(\mathcal{CCC}(G)) = \left\{0^2, (2m-4)^1, (m-3)^{m-2}\right\}.$$

Hence, by (1.1.d), we get

$$E(\mathcal{CCC}(G)) = m - 2 + m - 2 = 2m - 4.$$

We have $|V(\mathcal{CCC}(G))| = m + 1$ and $|e(\mathcal{CCC}(G))| = \frac{(m-1)(m-2)}{2}$. Therefore, $\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{(m-1)(m-2)}{m+1}$. Also,

$$\left|0 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0 - \frac{(m-1)(m-2)}{m+1}\right| = \frac{(m-1)(m-2)}{m+1}$$

and

$$\left|m - 1 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|m - 1 - \frac{(m-1)(m-2)}{m+1}\right| = \frac{3(m-1)}{m+1}.$$

Now, by (1.1.e), we have

$$LE(\mathcal{CCC}(G)) = 3 \times \frac{(m-1)(m-2)}{m+1} + (m-2) \times \frac{3(m-1)}{m+1} = \frac{6(m-1)(m-2)}{m+1}.$$

Again,

$$\left|2m - 4 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|2m - 4 - \frac{(m-1)(m-2)}{m+1}\right| = \frac{(m-2)(m+3)}{m+1}$$

and

$$\left|m - 3 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|m - 3 - \frac{(m-1)(m-2)}{m+1}\right| = \left|\frac{m-5}{m+1}\right| = \begin{cases} \frac{-m+5}{m+1}, & \text{if } m = 2, 4 \\ \frac{m-5}{m+1}, & \text{if } m \geq 6. \end{cases}$$

By (1.1.f), we have

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) &= 2 \times \frac{(m-1)(m-2)}{m+1} + \frac{(m-2)(m+3)}{m+1} + (m-2) \times \frac{-m+5}{m+1} \\ &= \frac{2(m-2)(m+3)}{m+1}, \end{aligned}$$

if $m = 2, 4$. If $m \geq 6$ then

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) &= 2 \times \frac{(m-1)(m-2)}{m+1} + \frac{(m-2)(m+3)}{m+1} + (m-2) \times \frac{m-5}{m+1} \\ &= \frac{4(m-1)(m-2)}{m+1}. \end{aligned}$$

This completes the proof. □

Theorem 5.1.3. *If $G = U_{(n,m)}$ then*

(a) Spec$(\mathcal{CCC}(G)) =$

$$\begin{cases} \left\{(-1)^{\frac{n(m+1)-4}{2}}, \left(\frac{n(m-1)-2}{2}\right)^1, (n-1)^1\right\}, & \text{if } m \text{ is odd and } n \geq 2 \\ \left\{(-1)^{\frac{n(m+2)-6}{2}}, \left(\frac{n(m-2)-2}{2}\right)^1, (n-1)^2\right\}, & \text{if } m \text{ is even, } m \geq 4 \text{ and } n \geq 2 \end{cases}$$

and

$$E(\mathcal{CCC}(G)) = \begin{cases} n(m+1)-4, & \text{if } m \text{ is odd and } n \geq 2 \\ n(m+2)-6, & \text{if } m \text{ is even, } m \geq 4 \text{ and } n \geq 2. \end{cases}$$

(b)

$$\text{L-spec}(\mathcal{CCC}(G)) = \begin{cases} \left\{0^2, \left(\frac{n(m-1)}{2}\right)^{\frac{n(m-1)-2}{2}}, n^{n-1}\right\}, & \text{if } m \text{ is odd and } n \geq 2 \\ \left\{0^3, \left(\frac{n(m-2)}{2}\right)^{\frac{n(m-2)-2}{2}}, n^{2n-2}\right\}, & \text{if } m \text{ is even, } m \geq 4 \text{ and } n \geq 2 \end{cases}$$

and

$$LE(\mathcal{CCC}(G)) = \begin{cases} 4(n-1), & \text{if } m = 3 \text{ and } n \geq 2 \\ \frac{2(2n-1)(n+3)}{3}, & \text{if } m = 5 \text{ and } n \geq 2 \\ \frac{m^2n^2-4mn^2+m^2n+3n^2-2mn-2m+5n-2}{m+1}, & \text{if } m \text{ is odd, } m \geq 7 \\ & \text{and } n \geq 2 \\ 6(n-1), & \text{if } m = 4 \text{ and } n \geq 2 \\ \frac{2m^2n^2-12mn^2+m^2n+16n^2-4mn-2m+12n-4}{m+2}, & \text{if } m \text{ is even, } m \geq 6 \\ & \text{and } n \geq 2. \end{cases}$$

(c) Q-spec($\mathcal{CCC}(G)$)

$$= \begin{cases} \left\{(n(m-1)-2)^1, \left(\frac{n(m-1)-4}{2}\right)^{\frac{n(m-1)-2}{2}}, (2n-2)^1, (n-2)^{n-1}\right\}, \\ \qquad\qquad\qquad\qquad\qquad \textit{if } m \textit{ is odd and } n \geq 2 \\ \left\{(n(m-2)-2)^1, \left(\frac{n(m-2)-4}{2}\right)^{\frac{n(m-2)-2}{2}}, (2n-2)^2, (n-2)^{2n-2}\right\}, \\ \qquad\qquad\qquad\qquad\qquad \textit{if } m \textit{ is even, } m \geq 4 \textit{ and } n \geq 2 \end{cases}$$

$$\textit{and } LE^+(\mathcal{CCC}(G)) = \begin{cases} 4(n-1), & \textit{if } m = 3 \textit{ and } n \geq 2 \\ \frac{22}{3}, & \textit{if } m = 5 \textit{ and } n = 2 \\ \frac{2(2n+3)(n-1)}{3}, & \textit{if } m = 5 \textit{ and } n \geq 3 \\ \frac{n^2(m-1)(m-3)}{m+1}, & \textit{if } m \textit{ is odd, } m \geq 7 \textit{ and } n \geq 2 \\ 6(n-1), & \textit{if } m = 4 \textit{ and } n \geq 2 \\ 2(n+2)(n-1), & \textit{if } m = 6 \textit{ and } n \geq 2 \\ \frac{2n^2(m-2)(m-4)}{m+2}, & \textit{if } m \textit{ is even, } m \geq 8 \textit{ and } n \geq 2. \end{cases}$$

Proof. We shall prove the result by considering the following cases.

Case 1. m is odd.

By Result 1.3.20 we have $\mathcal{CCC}(G) = K_{\frac{n(m-1)}{2}} \sqcup K_n$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \left\{(-1)^{\frac{n(m+1)-4}{2}}, \left(\frac{n(m-1)-2}{2}\right)^1, (n-1)^1\right\},$$

$$\text{L-spec}(\mathcal{CCC}(G)) = \left\{0^2, \left(\frac{n(m-1)}{2}\right)^{\frac{n(m-1)-2}{2}}, n^{n-1}\right\}$$

and

$$\text{Q-spec}(\mathcal{CCC}(G)) = \left\{(n(m-1)-2)^1, \left(\frac{n(m-1)-4}{2}\right)^{\frac{n(m-1)-2}{2}}, (2n-2)^1, (n-2)^{n-1}\right\}.$$

Hence, by $(1.1.d)$, we get

$$E(\mathcal{CCC}(G)) = \frac{n(m+1)-4}{2} + \frac{n(m-1)-2}{2} + n - 1 = n(m+1) - 4.$$

We have $|V(\mathcal{CCC}(G))| = \frac{n(m+1)}{2}$ and $|e(\mathcal{CCC}(G))| = \frac{n^2(m-1)^2-2n(m-2n+1)}{8}$. Therefore, $\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{n(m-1)^2-2(m-2n+1)}{2(m+1)}$. Also,

$$\left|0 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0 - \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)}\right| = \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)},$$

since $n(m-1)^2 - 2(m-2n+1) = m^2n - 2m(n+1) + 5n - 2 > 0$;

$$\begin{aligned}\left|\frac{n(m-1)}{2} - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|\frac{n(m-1)}{2} - \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)}\right| \\ &= \frac{n(m-3)+m+1}{m+1}\end{aligned}$$

and

$$\left|n - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|n - \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)}\right| = \left|\frac{-f_1(m,n)}{2(m+1)}\right|,$$

where $f_1(m,n) = n(m^2+3) - (4mn+2m+2)$. For $m = 3$ and $n \geq 2$ we have $f_1(3,n) = -8$. For $m = 5$ and $n \geq 2$ we have $f_1(5,n) = 8n - 12 > 0$. For $m \geq 7$ and $n \geq 2$ we have $m^2 + 3 > m^2 > 4m + 2m + 2$. Therefore, $n(m^2+3) > 4mn + (2m+2)n > 4mn + 2m + 2$ and so $f_1(m,n) > 0$. Hence,

$$\left|\frac{-f_1(m,n)}{2(m+1)}\right| = \begin{cases} 1, & \text{if } m = 3 \text{ and } n \geq 2 \\ \frac{2n-3}{3}, & \text{if } m = 5 \text{ and } n \geq 2 \\ \frac{n(m^2+3)-(4mn+2m+2)}{2(m+1)}, & \text{if } m \geq 7 \text{ and } n \geq 2. \end{cases}$$

Now, by (1.1.e), we have

$$\begin{aligned}LE(\mathcal{CCC}(G)) = 2 \times \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)} &+ \frac{n(m-1)-2}{2} \times \frac{n(m-3)+m+1}{m+1} \\ &+ (n-1) \times 1 \\ = 4(n-1),\end{aligned}$$

if $m = 3$ and $n \geq 2$. If $m = 5$ and $n \geq 2$ then

$$\begin{aligned}LE(\mathcal{CCC}(G)) = 2 \times \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)} &+ \frac{n(m-1)-2}{2} \times \frac{n(m-3)+m+1}{m+1} \\ &+ (n-1) \times \frac{2n-3}{3} \\ = \frac{2(2n-1)(n+3)}{3}.\end{aligned}$$

If $m \geq 7$ and $n \geq 2$ then

$$\begin{aligned} LE(\mathcal{CCC}(G)) &= 2 \times \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)} + \frac{n(m-1)-2}{2} \times \frac{n(m-3)+m+1}{m+1} \\ &\quad + (n-1) \times \frac{n(m^2+3) - (4mn+2m+2)}{2(m+1)} \\ &= \frac{m^2n^2 - 4mn^2 + m^2n + 3n^2 - 2mn - 2m + 5n - 2}{m+1}. \end{aligned}$$

Again,

$$\begin{aligned} \left| n(m-1) - 2 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} \right| &= \left| n(m-1) - 2 - \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)} \right| \\ &= \left| \frac{n(m-1)(m+3) - 2(m+2n+1)}{2(m+1)} \right| \\ &= \frac{n(m-1)(m+3) - 2(m+2n+1)}{2(m+1)}, \end{aligned}$$

since $n(m-1)(m+3) - 2(m+2n+1) = n(m^2-4) - 2 + n(m-3) + m(n-2) > 0$;

$$\begin{aligned} \left| \frac{n(m-1)-4}{2} - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} \right| &= \left| \frac{n(m-1)-4}{2} - \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)} \right| \\ &= \left| \frac{f_2(m,n)}{2(m+1)} \right|, \end{aligned}$$

where $f_2(m,n) = n(m-6) - 2 + m(n-2)$. Clearly, for $m \geq 7$ and $n \geq 2$ we have $f_2(m,n) \geq 0$. For $m = 3$ and $n \geq 2$ we have $f_2(3,n) = -8$. Also for $m = 5$ and $n \geq 2$ we have $f_2(5,n) = 4n - 12$. Therefore, $f_2(5,2) = -4$ and $f_2(5,n) \geq 0$ for $n \geq 3$. Hence,

$$\left| \frac{f_2(m,n)}{2(m+1)} \right| = \begin{cases} 1, & \text{if } m = 3 \text{ and } n \geq 2 \\ \frac{1}{3}, & \text{if } m = 5 \text{ and } n = 2 \\ \frac{n-3}{3}, & \text{if } m = 5 \text{ and } n \geq 3 \\ \frac{n(m-3)-m-1}{m+1}, & \text{if } m \geq 7 \text{ and } n \geq 2. \end{cases}$$

$$\left| 2n - 2 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} \right| = \left| 2n - 2 - \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)} \right| = \left| -\frac{f_3(m,n)}{2(m+1)} \right|,$$

where $f_3(m,n) = mn(m-6) + 2m + n + 2$. Clearly, $f_3(m,n) > 0$ if $m \geq 7$ and $n \geq 2$. For $m = 3$ and $n \geq 2$ we have $f_3(3,n) = -8n + 8 < 0$. For $m = 5$ and $n \geq 2$ we have

$f_3(5, n) = -4n + 12$. Therefore, $f_3(5, 2) = 4$ and $f_3(5, n) \leq 0$ if $n \geq 3$. Hence,

$$\left|-\frac{f_3(m,n)}{2(m+1)}\right| = \begin{cases} n-1, & \text{if } m = 3 \text{ and } n \geq 2 \\ \frac{1}{3}, & \text{if } m = 5 \text{ and } n = 2 \\ \frac{n-3}{3}, & \text{if } m = 5 \text{ and } n \geq 3 \\ \frac{mn(m-6)+2m+n+2}{2(m+1)}, & \text{if } m \geq 7 \text{ and } n \geq 2. \end{cases}$$

$$\left|n - 2 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|n - 2 - \frac{n(m-1)^2 - 2(m-2n+1)}{2(m+1)}\right| = \left|-\frac{f_4(m,n)}{2(m+1)}\right|,$$

where $f_4(m, n) = mn(m-2) + 2 - (m(n-2) + n(m-3))$. For $m = 3$ and $n \geq 2$ we have $f_4(3, n) = 8$. Also, for $m \geq 5$ and $n \geq 2$ we have

$$mn(m-2) - 2mn + 2 = mn(m-4) + 2 > -2m - 3n.$$

Therefore,

$$mn(m-2) + 2 > 2mn - 2m - 3n = m(n-2) + n(m-3)$$

and so $f_4(m, n) > 0$ for $m \geq 5$ and $n \geq 2$. Hence,

$$\left|-\frac{f_4(m,n)}{2(m+1)}\right| = \frac{f_4(m,n)}{2(m+1)} = \frac{mn(m-2) + 2 - m(n-2) - n(m-3)}{2(m+1)}.$$

By (1.1.f), we have

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) = &\frac{n(m-1)(m+3) - 2(m+2n+1)}{2(m+1)} + \frac{n(m-1)-2}{2} \times 1 + (n-1) \\ &+ (n-1) \times \frac{mn(m-2) + 2 - m(n-2) - n(m-3)}{2(m+1)} \\ = &\, 4(n-1), \end{aligned}$$

if $m = 3$ and $n \geq 2$. If $m = 5$ and $n = 2$ then

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) = &\frac{n(m-1)(m+3) - 2(m+2n+1)}{2(m+1)} + \frac{n(m-1)-2}{2} \times \frac{1}{3} + \frac{1}{3} \\ &+ (n-1) \times \frac{mn(m-2) + 2 - m(n-2) - n(m-3)}{2(m+1)} \\ = &\, \frac{22}{3}. \end{aligned}$$

If $m = 5$ and $n \geq 3$ then

$$\begin{aligned}LE^+(\mathcal{CCC}(G)) &= \frac{n(m-1)(m+3)-2(m+2n+1)}{2(m+1)} + \frac{n(m-1)-2}{2} \times \frac{n-3}{3} + \frac{n-3}{3} \\ &\quad + (n-1) \times \frac{mn(m-2)+2-m(n-2)-n(m-3)}{2(m+1)} \\ &= \frac{2(2n^2+n-3)}{3} = \frac{2(2n+3)(n-1)}{3}.\end{aligned}$$

If $m \geq 7$ and $n \geq 2$ then

$$\begin{aligned}LE^+(\mathcal{CCC}(G)) &= \frac{n(m-1)(m+3)-2(m+2n+1)}{2(m+1)} + \frac{n(m-1)-2}{2} \times \frac{n(m-3)-m-1}{m+1} \\ &\quad + \frac{mn(m-6)+2m+n+2}{2(m+1)} \\ &\quad + (n-1) \times \frac{mn(m-2)+2-m(n-2)-n(m-3)}{2(m+1)} \\ &= \frac{n^2(m-1)(m-3)}{m+1}.\end{aligned}$$

Case 2. m is even.

By Result 1.3.20 we have $\mathcal{CCC}(G) = K_{\frac{n(m-2)}{2}} \sqcup 2K_n$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \left\{(-1)^{\frac{n(m+2)-6}{2}}, \left(\frac{n(m-2)-2}{2}\right)^1, (n-1)^2\right\},$$

$$\text{L-spec}(\mathcal{CCC}(G)) = \left\{0^3, \left(\frac{n(m-2)}{2}\right)^{\frac{n(m-2)-2}{2}}, n^{2n-2}\right\}$$

and

$$\text{Q-spec}(\mathcal{CCC}(G)) = \left\{(n(m-2)-2)^1, \left(\frac{n(m-2)-4}{2}\right)^{\frac{n(m-2)-2}{2}}, (2n-2)^2, (n-2)^{2n-2}\right\}.$$

We have $\left|\frac{n(m-2)-2}{2}\right| = \frac{n(m-2)-2}{2}$ if $m \geq 4$. Therefore, by $(1.1.d)$, we have

$$E(\mathcal{CCC}(G)) = \frac{n(m+2)-6}{2} + \frac{n(m-2)-2}{2} + 2(n-1) = n(m+2)-6.$$

if $m \geq 4$.

We have $|V(\mathcal{CCC}(G))| = \frac{n(m+2)}{2}$ and $|e(\mathcal{CCC}(G))| = \frac{n^2(m-2)^2-2n(m-4n+2)}{8}$. Therefore, $\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{n(m-2)^2-2(m-4n+2)}{2(m+2)}$. Also,

$$\left|0 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0 - \frac{n(m-2)^2-2(m-4n+2)}{2(m+2)}\right| = \left|\frac{-f_5(m,n)}{2(m+2)}\right|,$$

where $f_5(m,n) = m(n(m-4)-2)+12n-4$. Note that for $m \geq 6$ we have $f_5(m,n) > 0$ since $n(m-4) > 2$ and $12n-4 > 0$. For $m = 4$ and $n \geq 2$ we have $f_5(4,n) = 12n-12 > 0$. Therefore, for all $m \geq 4$ and $n \geq 2$, we have

$$\left|\frac{-f_5(m,n)}{2(m+2)}\right| = \left|\frac{f_5(m,n)}{2(m+2)}\right| = \frac{m(n(m-4)-2)+12n-4}{2(m+2)}.$$

$$\left|\frac{n(m-2)}{2} - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|\frac{n(m-2)}{2} - \frac{n(m-2)^2-2(m-4n+2)}{2(m+2)}\right| = \left|\frac{f_6(m,n)}{m+2}\right|,$$

where $f_6(m,n) = 2n(m-4)+m+2$. Clearly, $f_6(m,n) > 0$ if $m \geq 4$ and $n \geq 2$. Therefore $\left|\frac{f_6(m,n)}{m+2}\right| = \frac{2n(m-4)+m+2}{m+2}$ if $m \geq 4$ and $n \geq 2$.

$$\left|n - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|n - \frac{n(m-2)^2-2(m-4n+2)}{2(m+2)}\right| = \left|\frac{-f_7(m,n)}{2(m+2)}\right|,$$

where $f_7(m,n) = mn(m-6)-2m+8n-4$. For $m = 4$ and $n \geq 2$ we have $f_7(4,n) = -12$. For $m = 6$ and $n \geq 2$ we have $f_7(6,n) = 8n-16 \geq 0$. Also, for $m \geq 8$ and $n \geq 2$ we have $m^2 \geq 8m$ which gives $m(m-6) \geq 2m$ and so $mn(m-6) \geq 2mn > 2m$. Therefore, $mn(m-6)-2m > 0$ and so $f_7(m,n) > 0$ since $8n-4 > 0$. Hence,

$$\left|\frac{-f_7(m,n)}{2(m+2)}\right| = \begin{cases} 1, & \text{if } m = 4 \text{ and } n \geq 2 \\ \frac{mn(m-6)-2m+8n-4}{2(m+2)}, & \text{if } m \geq 6 \text{ and } n \geq 2. \end{cases}$$

Now, by (1.1.e), we have

$$\begin{aligned} LE(\mathcal{CCC}(G)) &= 3 \times \frac{m(n(m-4)-2)+12n-4}{2(m+2)} + \frac{n(m-2)-2}{2} \times \frac{2n(m-4)+m+2}{m+2} \\ &\qquad + (2n-2) \times 1 \\ &= 6(n-1), \end{aligned}$$

if $m = 4$ and $n \geq 2$. If $m \geq 6$ and $n \geq 2$ then

$$\begin{aligned} LE(\mathcal{CCC}(G)) &= 3 \times \frac{m(n(m-4)-2)+12n-4}{2(m+2)} + \frac{n(m-2)-2}{2} \times \frac{2n(m-4)+m+2}{m+2} \\ &\qquad + (2n-2) \times \frac{mn(m-6)-2m+8n-4}{2(m+2)} \\ &= \frac{2m^2n^2-12mn^2+m^2n+16n^2-4mn-2m+12n-4}{m+2}. \end{aligned}$$

Again,

$$\left|n(m-2)-2-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|n(m-2)-2-\frac{n(m-2)^2-2(m-4n+2)}{2(m+2)}\right|$$

$$= \left|\frac{f_8(m,n)}{2(m+2)}\right|,$$

where $f_8(m,n) = n(m^2-20)+2m(n-1)+2mn-4$. For $m=4$ and $n \geq 2$ we have $f_8(4,n) = 12n-12 > 0$. For $m \geq 6$ and $n \geq 2$ we have $f_8(m,n) > 0$. Therefore,

$$\left|\frac{f_8(m,n)}{2(m+2)}\right| = \begin{cases} n-1, & \text{if } m=4 \text{ and } n \geq 2 \\ \frac{n(m^2-20)+2m(n-1)+2mn-4}{2(m+2)}, & \text{if } m \geq 6 \text{ and } n \geq 2. \end{cases}$$

$$\begin{aligned}\left|\frac{n(m-2)-4}{2} - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|\frac{n(m-2)-4}{2} - \frac{n(m-2)^2-2(m-4n+2)}{2(m+2)}\right| \\ &= \left|\frac{f_9(m,n)}{m+2}\right|,\end{aligned}$$

where $f_9(m,n) = n(m-8)+m(n-1)-2$. For $m=4$ and $n \geq 2$ we have $f_9(4,n) = -6$. For $m=6$ and $n \geq 2$ we have $f_9(6,n) = 4n-8 \geq 0$. Further, if For $m \geq 8$ and $n \geq 2$ then $f_9(m,n) > 0$ since $n(m-8) \geq 0$ and $m(n-1)-2 > 0$. Hence,

$$\left|\frac{f_9(m,n)}{m+2}\right| = \begin{cases} 1, & \text{if } m=4 \text{ and } n \geq 2 \\ \frac{n(m-8)+m(n-1)-2}{m+2}, & \text{if } m \geq 6 \text{ and } n \geq 2. \end{cases}$$

$$\left|2n-2-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|2n-2-\frac{n(m-2)^2-2(m-4n+2)}{2(m+2)}\right| = \left|\frac{-f_{10}(m,n)}{2(m+2)}\right|,$$

where $f_{10}(m,n) = n(m^2-8m+4)+2m+4$. Clearly, $f_{10}(m,n) > 0$ for $m \geq 8$ and $n \geq 2$. For $m=4$ and $n \geq 2$ we have $f_{10}(4,n) = -12n+12 < 0$. For $m=6$ and $n \geq 2$ we have $f_{10}(6,n) = -8n+16 \leq 0$. Hence,

$$\left|\frac{f_{10}(m,n)}{m+2}\right| = \begin{cases} n-1, & \text{if } m=4 \text{ and } n \geq 2 \\ \frac{n-2}{2}, & \text{if } m=6 \text{ and } n \geq 2 \\ \frac{n(m^2-8m+4)+2m+4}{2(m+2)}, & \text{if } m \geq 8 \text{ and } n \geq 2. \end{cases}$$

$$\left|n-2-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|n-2-\frac{n(m-2)^2-2(m-4n+2)}{2(m+2)}\right| = \left|\frac{-f_{11}(m,n)}{2(m+2)}\right|,$$

where $f_{11}(m,n) = n(m-2)(m-4)+2m+4$. Note that for $m \geq 4$ and $n \geq 2$ we have $f_{11}(m,n) > 0$. Therefore,

$$\left|\frac{-f_{11}(m,n)}{2(m+2)}\right| = \frac{f_{11}(m,n)}{2(m+2)} = \frac{n(m-2)(m-4)+2m+4}{2(m+2)}.$$

By (1.1.*f*), we have

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) &= n-1+\frac{n(m-2)-2}{2}\times 1+2\times(n-1) \\ &\quad +(2n-2)\times\frac{n(m-2)(m-4)+2m+4}{2(m+2)} \\ &= 6(n-1), \end{aligned}$$

if $m=4$ and $n\geq 2$. If $m=6$ and $n\geq 2$ then

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) &= \frac{n(m^2-20)+2m(n-1)+2mn-4}{2(m+2)} \\ &\quad +\frac{n(m-2)-2}{2}\times\frac{n(m-8)+m(n-1)-2}{m+2} \\ &\quad +2\times\frac{n-2}{2}+(2n-2)\times\frac{n(m-2)(m-4)+2m+4}{2(m+2)} \\ &= 2(n+2)(n-1). \end{aligned}$$

If $m\geq 8$ and $n\geq 2$ then

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) &= \frac{n(m^2-20)+2m(n-1)+2mn-4}{2(m+2)} \\ &\quad +\frac{n(m-2)-2}{2}\times\frac{n(m-8)+m(n-1)-2}{m+2} \\ &\quad +2\times\frac{n(m^2-8m+4)+2m+4}{2(m+2)} \\ &\quad +(2n-2)\times\frac{n(m-2)(m-4)+2m+4}{2(m+2)} \\ &= \frac{2n^2(m-2)(m-4)}{m+2}. \end{aligned}$$

This completes the proof. □

Theorem 5.1.4. *If $G=V_{8n}$ then*

(a) $\operatorname{Spec}(\mathcal{CCC}(G))=\begin{cases}\left\{(-1)^{2n-2},0^2,(2n-2)^1\right\}, & \text{if } n \text{ is odd}\\ \left\{(-1)^{2n-1},1^2,(2n-3)^1\right\}, & \text{if } n \text{ is even}\end{cases}$

and $E(\mathcal{CCC}(G))=\begin{cases}4n-4, & \text{if } n \text{ is odd}\\ 4n-2, & \text{if } n \text{ is even.}\end{cases}$

(b) $\text{L-spec}(\mathcal{CCC}(G))=\begin{cases}\left\{0^3,(2n-1)^{2n-2}\right\}, & \text{if } n \text{ is odd}\\ \left\{0^3,2^2,(2n-2)^{2n-3}\right\}, & \text{if } n \text{ is even}\end{cases}$

$$\textit{and } LE(\mathcal{CCC}(G)) = \begin{cases} \frac{6(2n-1)(2n-2)}{2n+1}, & \textit{if } n \textit{ is odd} \\ 6, & \textit{if } n = 2 \\ \frac{2(2n-3)(5n-7)}{n+1}, & \textit{if } n \textit{ is even and } n \geq 4. \end{cases}$$

$$\text{(c) Q-spec}(\mathcal{CCC}(G)) = \begin{cases} \left\{0^2, (4n-4)^1, (2n-3)^{2n-2}\right\}, & \textit{if } n \textit{ is odd} \\ \left\{2^2, 0^2, (4n-6)^1, (2n-4)^{2n-3}\right\}, & \textit{if } n \textit{ is even} \end{cases}$$

$$\textit{and } LE^+(\mathcal{CCC}(G)) = \begin{cases} \frac{4(2n-1)(2n-2)}{2n+1}, & \textit{if } n \textit{ is odd} \\ 6, & \textit{if } n = 2 \\ \frac{16(n-1)(n-2)}{n+1}, & \textit{if } n \textit{ is even and } n \geq 4. \end{cases}$$

Proof. We shall prove the result by considering the following cases.

Case 1. n is odd.

By Result 1.3.21 we have $\mathcal{CCC}(G) = 2K_1 \sqcup K_{2n-1}$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \left\{(-1)^{2n-2}, 0^2, (2n-2)^1\right\}, \quad \text{L-spec}(\mathcal{CCC}(G)) = \left\{0^3, (2n-1)^{2n-2}\right\}$$
$$\text{and Q-spec}(\mathcal{CCC}(G)) = \left\{0^2, (4n-4)^1, (2n-3)^{2n-2}\right\}.$$

Hence, by $(1.1.d)$, we get

$$E(\mathcal{CCC}(G)) = 2n-2+2n-2 = 4n-4.$$

We have $|V(\mathcal{CCC}(G))| = 2n+1$ and $|e(\mathcal{CCC}(G))| = \frac{(2n-1)(2n-2)}{2}$. Therefore,

$$\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{(2n-1)(2n-2)}{2n+1}.$$

Also,

$$\left|0 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0 - \frac{(2n-1)(2n-2)}{2n+1}\right| = \frac{(2n-1)(2n-2)}{2n+1}$$

and

$$\left|2n-1 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|2n-1-\frac{(2n-1)(2n-2)}{2n+1}\right| = \frac{3(2n-1)}{2n+1}.$$

Now, by $(1.1.e)$, we have

$$LE(\mathcal{CCC}(G)) = 3 \times \frac{(2n-1)(2n-2)}{2n+1} + (2n-2) \times \frac{3(2n-1)}{2n+1} = \frac{6(2n-1)(2n-2)}{2n+1}.$$

Again,

$$\left|4n-4-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|4n-4-\frac{(2n-1)(2n-2)}{2n+1}\right| = \frac{(2n-2)(2n+3)}{2n+1}$$

and

$$\left|2n-3-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|2n-3-\frac{(2n-1)(2n-2)}{2n+1}\right| = \frac{2n-5}{2n+1}.$$

By (1.1.*f*), we have

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) &= 2\times\frac{(2n-1)(2n-2)}{2n+1}+\frac{(2n-2)(2n+3)}{2n+1}+(2n-2)\times\frac{2n-5}{2n+1} \\ &= \frac{4(2n-1)(2n-2)}{2n+1}. \end{aligned}$$

Case 2. n is even.

By Result 1.3.21 we have $\mathcal{CCC}(G) = 2K_2 \sqcup K_{2n-2}$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \left\{(-1)^{2n-1}, 1^2, (2n-3)^1\right\}, \quad \text{L-spec}(\mathcal{CCC}(G)) = \left\{0^3, 2^2, (2n-2)^{2n-3}\right\}$$
$$\text{and Q-spec}(\mathcal{CCC}(G)) = \left\{2^2, 0^2, (4n-6)^1, (2n-4)^{2n-3}\right\}.$$

Hence, by (1.1.*d*), we get

$$E(\mathcal{CCC}(G)) = 2n-1+2+2n-3 = 4n-2.$$

We have $|V(\mathcal{CCC}(G))| = 2n+2$ and $|e(\mathcal{CCC}(G))| = \frac{(2n-2)(2n-3)+4}{2}$. Therefore,

$$\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{(n-1)(2n-3)+2}{n+1}.$$

Also,

$$\left|0-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0-\frac{(n-1)(2n-3)+2}{n+1}\right| = \frac{(n-1)(2n-3)+2}{n+1},$$

$$\begin{aligned} \left|2-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|2-\frac{(n-1)(2n-3)+2}{n+1}\right| \\ &= \left|\frac{-(2n-1)(n-3)}{n+1}\right| \\ &= \begin{cases} 1, & \text{if } n=2 \\ \frac{(2n-1)(n-3)}{n+1}, & \text{if } n\geq 4 \end{cases} \end{aligned}$$

and

$$\left|2n-2-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|2n-2-\frac{(n-1)(2n-3)+2}{n+1}\right| = \frac{5n-7}{n+1}.$$

Now, by (1.1.e), we have

$$LE(\mathcal{CCC}(G)) = 3\times\frac{(n-1)(2n-3)+2}{n+1} + 2\times 1 + (2n-3)\times\frac{5n-7}{n+1} = 6,$$

if $n=2$. If $n \geq 4$ then

$$\begin{aligned} LE(\mathcal{CCC}(G)) &= 3\times\frac{(n-1)(2n-3)+2}{n+1} + 2\times\frac{(2n-1)(n-3)}{n+1} + (2n-3)\times\frac{5n-7}{n+1} \\ &= \frac{2(10n^2-29n+21)}{n+1} = \frac{2(2n-3)(5n-7)}{n+1}. \end{aligned}$$

Again,

$$\left|4n-6-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|4n-6-\frac{(n-1)(2n-3)+2}{n+1}\right| = \frac{2n^2+3n-11}{n+1}$$

and

$$\begin{aligned} \left|2n-4-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|2n-4-\frac{(n-1)(2n-3)+2}{n+1}\right| \\ &= \left|\frac{3n-9}{n+1}\right| \\ &= \begin{cases} 1, & \text{if } n=2 \\ \frac{3n-9}{n+1}, & \text{if } n\geq 4. \end{cases} \end{aligned}$$

By (1.1.f), we have

$$LE^+(\mathcal{CCC}(G)) = 2\times 1 + 2\times\frac{(n-1)(2n-3)+2}{n+1} + \frac{2n^2+3n-11}{n+1} + (2n-3)\times 1 = 6,$$

if $n=2$. If $n\geq 4$ then

$$\begin{aligned} &LE^+(\mathcal{CCC}(G)) \\ &\quad = 2\times\frac{(2n-1)(n-3)}{n+1} + 2\times\frac{(n-1)(2n-3)+2}{n+1} + \frac{2n^2+3n-11}{n+1} \\ &\qquad + (2n-3)\times\frac{3n-9}{n+1} \\ &\quad = \frac{16(n-1)(n-2)}{n+1}. \end{aligned}$$

This completes the proof. □

Theorem 5.1.5. *If $G = SD_{8n}$ then*

(a) $\text{Spec}(\mathcal{CCC}(G)) = \begin{cases} \left\{(-1)^{2n}, 3^1, (2n-3)^1\right\}, & \textit{if } n \textit{ is odd} \\ \left\{(-1)^{2n-2}, 0^2, (2n-2)^1\right\}, & \textit{if } n \textit{ is even} \end{cases}$

and $E(\mathcal{CCC}(G)) = \begin{cases} 4n, & \textit{if } n \textit{ is odd} \\ 4n-4, & \textit{if } n \textit{ is even.} \end{cases}$

(b) $\text{L-spec}(\mathcal{CCC}(G)) = \begin{cases} \left\{0^2, 4^3, (2n-2)^{2n-3}\right\}, & \textit{if } n \textit{ is odd} \\ \left\{0^3, (2n-1)^{2n-2}\right\}, & \textit{if } n \textit{ is even} \end{cases}$

and $LE(\mathcal{CCC}(G)) = \begin{cases} 12, & \textit{if } n = 3 \\ \frac{2(2n-3)(5n-11)}{n+1}, & \textit{if } n \textit{ is odd and } n \geq 5 \\ \frac{6(2n-1)(2n-2)}{2n+1}, & \textit{if } n \textit{ is even.} \end{cases}$

(c) $\text{Q-spec}(\mathcal{CCC}(G)) = \begin{cases} \left\{6^1, 2^3, (4n-6)^1, (2n-4)^{2n-3}\right\}, & \textit{if } n \textit{ is odd} \\ \left\{0^2, (4n-4)^1, (2n-3)^{2n-2}\right\}, & \textit{if } n \textit{ is even} \end{cases}$

and $LE^+(\mathcal{CCC}(G)) = \begin{cases} 12, & \textit{if } n = 3 \\ 22, & \textit{if } n = 5 \\ \frac{16(n-1)(n-3)}{n+1}, & \textit{if } n \textit{ is odd and } n \geq 7 \\ \frac{28}{5}, & \textit{if } n = 2 \\ \frac{4(2n-1)(2n-2)}{2n+1}, & \textit{if } n \textit{ is even and } n \geq 4. \end{cases}$

Proof. We shall prove the result by considering the following cases.

Case 1. n is odd.

By Result 1.3.22 we have $\mathcal{CCC}(G) = K_4 \sqcup K_{2n-2}$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \left\{(-1)^{2n}, 3^1, (2n-3)^1\right\}, \quad \text{L-spec}(\mathcal{CCC}(G)) = \left\{0^2, 4^3, (2n-2)^{2n-3}\right\}$$
$$\text{and } \text{Q-spec}(\mathcal{CCC}(G)) = \left\{6^1, 2^3, (4n-6)^1, (2n-4)^{2n-3}\right\}.$$

Hence, by (1.1.d), we get

$$E(\mathcal{CCC}(G)) = 2n + 3 + 2n - 3 = 4n.$$

We have $|V(\mathcal{CCC}(G))| = 2n+2$ and $|e(\mathcal{CCC}(G))| = \frac{(2n-2)(2n-3)+12}{2}$. Therefore,

$$\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{(n-1)(2n-3)+6}{n+1}.$$

Also,

$$\left|0-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0-\frac{(n-1)(2n-3)+6}{n+1}\right| = \frac{(n-1)(2n-3)+6}{n+1},$$

$$\begin{aligned}\left|4-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|4-\frac{(n-1)(2n-3)+6}{n+1}\right|\\ &= \left|\frac{-2n^2+9n-5}{n+1}\right| = \begin{cases}1, & \text{if } n=3\\ \frac{2n^2-9n+5}{n+1}, & \text{if } n\geq 5\end{cases}\end{aligned}$$

and

$$\left|2n-2-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|2n-2-\frac{(n-1)(2n-3)+6}{n+1}\right| = \frac{5n-11}{n+1}.$$

Now, by (1.1.e), we have

$$LE(\mathcal{CCC}(G)) = 2\times\frac{(n-1)(2n-3)+6}{n+1} + 3\times 1 + (2n-3)\times\frac{5n-11}{n+1} = 12,$$

if $n=3$. If $n\geq 5$ then

$$\begin{aligned}LE(\mathcal{CCC}(G)) &= 2\times\frac{(n-1)(2n-3)+6}{n+1} + 3\times\frac{2n^2-9n+5}{n+1} + (2n-3)\times\frac{5n-11}{n+1}\\ &= \frac{2(10n^2-37n+33)}{n+1} = \frac{2(2n-3)(5n-11)}{n+1}.\end{aligned}$$

Again,

$$\begin{aligned}\left|6-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|6-\frac{(n-1)(2n-3)+6}{n+1}\right|\\ &= \left|\frac{-2n^2+11n-3}{n+1}\right|\\ &= \begin{cases}\frac{-2n^2+11n-3}{n+1}, & \text{if } n=3,5\\ \frac{2n^2-11n+3}{n+1}, & \text{if } n\geq 7,\end{cases}\end{aligned}$$

$$\left|2-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|2-\frac{(n-1)(2n-3)+6}{n+1}\right| = \frac{2n^2-7n+7}{n+1},$$

$$\left|4n-6-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|4n-6-\frac{(n-1)(2n-3)+6}{n+1}\right| = \frac{2n^2+3n-15}{n+1}$$

and

$$\begin{aligned}
\left|2n-4-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|2n-4-\frac{(n-1)(2n-3)+6}{n+1}\right| \\
&= \left|\frac{3n-13}{n+1}\right| \\
&= \begin{cases} 1, & \text{if } n=3 \\ \frac{3n-13}{n+1}, & \text{if } n\geq 5. \end{cases}
\end{aligned}$$

By (1.1.*f*), we have

$$\begin{aligned}
LE^+(\mathcal{CCC}(G)) &= \frac{-2n^2+11n-3}{n+1}+3\times\frac{2n^2-7n+7}{n+1}+\frac{2n^2+3n-15}{n+1}+(2n-3)\times 1 \\
&= 12,
\end{aligned}$$

if $n=3$. If $n=5$ then

$$\begin{aligned}
LE^+(\mathcal{CCC}(G)) &= \frac{-2n^2+11n-3}{n+1}+3\times\frac{2n^2-7n+7}{n+1}+\frac{2n^2+3n-15}{n+1} \\
&\quad+(2n-3)\times\frac{3n-13}{n+1} \\
&= 22.
\end{aligned}$$

If $n\geq 7$ then

$$\begin{aligned}
LE^+(\mathcal{CCC}(G)) &= \frac{2n^2-11n+3}{n+1}+3\times\frac{2n^2-7n+7}{n+1}+\frac{2n^2+3n-15}{n+1}+(2n-3)\times\frac{3n-13}{n+1} \\
&= \frac{16(n-1)(n-3)}{n+1}.
\end{aligned}$$

Case 2. n is even.

By Result 1.3.22 we have $\mathcal{CCC}(G)=2K_1\sqcup K_{2n-1}$. Therefore, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G))=\left\{(-1)^{2n-2},0^2,(2n-2)^1\right\},\quad \text{L-spec}(\mathcal{CCC}(G))=\left\{0^3,(2n-1)^{2n-2}\right\}$$
$$\text{and Q-spec}(\mathcal{CCC}(G))=\left\{0^2,(4n-4)^1,(2n-3)^{2n-2}\right\}.$$

Hence, by (1.1.*d*), we get

$$E(\mathcal{CCC}(G))=2n-2+2n-2=4n-4.$$

We have $V(\mathcal{CCC}(G)) = 2n+1$ and $e(\mathcal{CCC}(G)) = \frac{(2n-1)(2n-2)}{2}$. So, $\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} = \frac{(2n-1)(2n-2)}{2n+1}$.

Also,
$$\left|0 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|0 - \frac{(2n-1)(2n-2)}{2n+1}\right| = \frac{(2n-1)(2n-2)}{2n+1}$$
and
$$\left|2n-1 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|2n-1 - \frac{(2n-1)(2n-2)}{2n+1}\right| = \frac{3(2n-1)}{2n+1}.$$
Now, by (1.1.e), we have
$$LE(\mathcal{CCC}(G)) = 3 \times \frac{(2n-1)(2n-2)}{2n+1} + (2n-2) \times \frac{3(2n-1)}{2n+1} = \frac{6(2n-1)(2n-2)}{2n+1}.$$

Again,
$$\left|4n-4 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| = \left|4n-4 - \frac{(2n-1)(2n-2)}{2n+1}\right| = \frac{(2n-2)(2n+3)}{2n+1} \quad \text{and}$$

$$\begin{aligned}\left|2n-3 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|2n-3 - \frac{(2n-1)(2n-2)}{2n+1}\right| \\ &= \left|\frac{2n-5}{2n+1}\right| = \begin{cases} \frac{1}{5}, & \text{if } n = 2 \\ \frac{2n-5}{2n+1}, & \text{if } n \geq 4. \end{cases}\end{aligned}$$

By (1.1.f), we have
$$LE^+(\mathcal{CCC}(G)) = 2 \times \frac{(2n-1)(2n-2)}{2n+1} + \frac{(2n-2)(2n+3)}{2n+1} + (2n-2) \times \frac{1}{5} = \frac{28}{5},$$
if $n = 2$. If $n \geq 4$ then
$$\begin{aligned}LE^+(\mathcal{CCC}(G)) &= 2 \times \frac{(2n-1)(2n-2)}{2n+1} + \frac{(2n-2)(2n+3)}{2n+1} + (2n-2) \times \frac{2n-5}{2n+1} \\ &= \frac{4(2n-1)(2n-2)}{2n+1}.\end{aligned}$$
This completes the proof. □

Theorem 5.1.6. *If $G = G(p, m, n)$ then*

(a) $\operatorname{Spec}(\mathcal{CCC}(G)) = \Big\{(-1)^{p^{m+n}-p^{m+n-2}-p^n+p^{n-1}-2}, (p^m - p^{m-1} - 1)^{p^{n-1}(p-1)},$
$(p^{m+n-1} - p^{m+n-2} - 1)^2\Big\}$ *and*
$$E(\mathcal{CCC}(G)) = 2(p^{m+n} - p^{m+n-2} - p^n + p^{n-1} - 2).$$

(b) $\text{L-spec}(\mathcal{CCC}(G)) = \Big\{0^{p^n-p^{n-1}+2}, (p^{m-1}(p-1))^{p^{n-2}(p-1)(p^{m+1}-p^m-p)}, (p^{m+n-2}(p-1))^{2((p-1)p^{m+n-2}-1)}\Big\}$ *and*

$$LE(\mathcal{CCC}(G)) = \begin{cases} \frac{2(p^{n+1}-p^n+2p)(2p^{m+n+1}-2p^{m+n}+p^{m+3}-2p^{m+2}+p^{m+1}-p^3-p^2)}{p^3(p+1)}, \\ \qquad \text{if } n=1, p\geq 2, m\geq 1; \text{ or } n=2, p=2, m=1 \\ \frac{4}{p^4(p+1)}\big(p^{2m+2n+3} - 3p^{2(m+n+1)} + 3p^{2m+2n+1} - p^{2(m+n)} \\ \quad -p^{2m+n+4} + 3p^{2m+n+3} - 3p^{2m+n+2} + p^{2m+n+1} \\ \quad +2p^{m+n+3} - 2p^{m+n+2} + p^{m+5} - 2p^{m+4} + p^{m+3} \\ \quad -p^5 - p^4\big), \qquad \text{otherwise.} \end{cases}$$

(c) $\text{Q-spec}(\mathcal{CCC}(G)) = \Big\{(2p^m-2p^{m-1}-2)^{p^{n-1}(p-1)}, (p^m-p^{m-1}-2)^{p^{n-2}(p-1)(p^{m+1}-p^m-p)}, (2p^{m+n-1}-2p^{m+n-2}-2)^2, (p^{m+n-1}-p^{m+n-2}-2)^{2(p^{m+n-1}-p^{m+n-2}-1)}\Big\}$

and

$$LE^+(\mathcal{CCC}(G)) = \begin{cases} 2(p^{m+1}-p^{m-1}-p-1), & \text{if } n=1, p\geq 2, m\geq 1 \\ \frac{2}{3}(7.2^m-6), & \text{if } n=2, p=2, m\leq 2 \\ \frac{2}{3}(4^m+2^m-6), & \text{if } n=2, p=2, m\geq 3 \\ \frac{4p^{2m+n-4}}{p+1}(p-1)^3(p^n-p), & \text{if } n=2, p\geq 3, m\geq 1; \\ & \text{or } n\geq 3, p\geq 2, m\geq 1. \end{cases}$$

Proof. By Result 1.3.23 we have

$$\mathcal{CCC}(G) = (p^n-p^{n-1})K_{p^{m-n}(p^n-p^{n-1})} \sqcup K_{p^{n-1}(p^m-p^{m-1})} \sqcup K_{p^{m-1}(p^n-p^{n-1})}.$$

Let $m_1 = p^n - p^{n-1}$, $m_2 = 1$, $m_3 = 1$, $n_1 = p^{m-n}(p^n-p^{n-1})$, $n_2 = p^{n-1}(p^m-p^{m-1})$ and $n_3 = p^{m-1}(p^n-p^{n-1})$. Then, by Result 1.1.6, it follows that

$$\text{Spec}(\mathcal{CCC}(G)) = \Big\{(-1)^{p^{m+n}-p^{m+n-2}-p^n+p^{n-1}-2}, (p^m-p^{m-1}-1)^{p^{n-1}(p-1)}, (p^{m+n-1}-p^{m+n-2}-1)^2\Big\},$$

$$\text{L-spec}(\mathcal{CCC}(G)) = \Big\{0^{p^n-p^{n-1}+2}, (p^{m-1}(p-1))^{p^{n-2}(p-1)(p^{m+1}-p^m-p)}, (p^{m+n-2}(p-1))^{2((-1+p)p^{m+n-2}-1)}\Big\}$$

and

$$\text{Q-spec}(\mathcal{CCC}(G)) = \Big\{(2p^m-2p^{m-1}-2)^{p^{n-1}(p-1)}, (p^m-p^{m-1}-2)^{p^{n-2}(p-1)(p^{m+1}-p^m-p)}, (2p^{m+n-1}-2p^{m+n-2}-2)^2, (p^{m+n-1}-p^{m+n-2}-2)^{2(p^{m+n-1}-p^{m+n-2}-1)}\Big\}.$$

Hence, by (1.1.d), we get

$$\begin{aligned}E(\mathcal{CCC}(G)) &= p^{m+n} - p^{m+n-2} - p^n + p^{n-1} - 2 + (p^{n-1}(p-1))(p^m - p^{m-1} - 1) \\ &\qquad + 2(p^{m+n-1} - p^{m+n-2} - 1) \\ &= 2(p^{m+n} - p^{m+n-2} - p^n + p^{n-1} - 2).\end{aligned}$$

We have $|V(\mathcal{CCC}(G))| = m_1n_1 + m_2n_2 + m_3n_3 = p^{m+n-2}(p^2-1)$ and

$$\begin{aligned}|e(\mathcal{CCC}(G))| &= \frac{m_1n_1(n_1-1)}{2} + \frac{m_2n_2(n_2-1)}{2} + \frac{m_3n_3(n_3-1)}{2} \\ &= \frac{p^{m+n-4}(p-1)}{2}(2p^{m+n+1} - 2p^{m+n} + p^{m+3} - 2p^{m+2} + p^{m+1} - p^3 - p^2).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} &= \frac{2p^{m+n+1} - 2p^{m+n} + p^{m+3} - 2p^{m+2} + p^{m+1} - p^3 - p^2}{p^2(p+1)} \\ &= \frac{p^{m+n}(p-2) + p^{m+2}(p-2) + (p^{m+n+1} - p^3) + (p^{m+1} - p^2)}{p^2(1+p)} \geq 0.\end{aligned}$$

Also,

$$\begin{aligned}\left|0 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|} \\ &= \frac{2p^{m+n+1} - 2p^{m+n} + p^{m+3} - 2p^{m+2} + p^{m+1} - p^3 - p^2}{p^2(p+1)},\end{aligned}$$

$$\begin{aligned}\left|p^{m-1}(p-1) - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|\frac{-2p^{m+n+1} + 2p^{m+n} + 2p^{m+2} - 2p^{m+1} + p^3 + p^2}{p^2 + p^3}\right| \\ &= \left|1 - \frac{2p^{m-1}(p^{n-1}-1)(p-1)}{p+1}\right| \\ &= \begin{cases} 1 - \frac{2p^{m-1}(p^{n-1}-1)(p-1)}{p+1}, & \text{if } n = 1, p \geq 2, m \geq 1; \\ & \text{or } n = 2, p = 2, m = 1 \\ \frac{2p^{m-1}(p^{n-1}-1)(p-1)}{p+1} - 1, & \text{otherwise} \end{cases}\end{aligned}$$

and

$$\begin{aligned}&\left|p^{m+n-2}(p-1) - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| \\ &\qquad = \left|\frac{p^{m+n+2} - 2p^{m+n+1} + p^{m+n} - p^{m+3} + 2p^{m+2} - p^{m+1} + p^3 + p^2}{p^2(p+1)}\right| \\ &\qquad = \left|\frac{p^{m+n}(p-1)^2 - p^{m+1}(p-1)^2 + p^3 + p^2}{p^2(p+1)}\right| \\ &\qquad = \frac{p^{m+n}(p-1)^2 - p^{m+1}(p-1)^2 + p^3 + p^2}{p^2(p+1)}.\end{aligned}$$

Now, by (1.1.e), we have

$$\begin{aligned}
LE(\mathcal{CCC}(G)) &= (p^n - p^{n-1} + 2) \times \frac{2p^{m+n+1} - 2p^{m+n} + p^{m+3} - 2p^{m+2} + p^{m+1} - p^3 - p^2}{p^2(p+1)} \\
&\quad + (p^{n-2}(p-1)(p^{m+1} - p^m - p)) \times \left(1 - \frac{2p^{m-1}(p^{n-1}-1)(p-1)}{p+1}\right) \\
&\quad + 2((p-1)p^{m+n-2} - 1) \times \frac{p^{m+n}(p-1)^2 - p^{m+1}(p-1)^2 + p^3 + p^2}{p^2(p+1)} \\
&= \frac{2(p^{n+1} - p^n + 2p)(2p^{m+n+1} - 2p^{m+n} + p^{m+3} - 2p^{m+2} + p^{m+1} - p^3 - p^2)}{p^3(p+1)},
\end{aligned}$$

if $n = 1, p \geq 2, m \geq 1$; or $n = 2, p = 2, m = 1$. Otherwise

$$\begin{aligned}
LE(\mathcal{CCC}(G)) &= (p^n - p^{n-1} + 2) \times \frac{2p^{m+n+1} - 2p^{m+n} + p^{m+3} - 2p^{m+2} + p^{m+1} - p^3 - p^2}{p^2(p+1)} \\
&\quad + (p^{n-2}(p-1)(p^{m+1} - p^m - p)) \times \left(\frac{2p^{m-1}(p^{n-1}-1)(p-1)}{p+1} - 1\right) \\
&\quad + 2((p-1)p^{m+n-2} - 1) \times \frac{p^{m+n}(p-1)^2 - p^{m+1}(p-1)^2 + p^3 + p^2}{p^2(p+1)} \\
&= \frac{4}{p^4(p+1)}\big(p^{2m+2n+3} - 3p^{2(m+n+1)} + 3p^{2m+2n+1} - p^{2(m+n)} - p^{2m+n+4} \\
&\qquad + 3p^{2m+n+3} - 3p^{2m+n+2} + p^{2m+n+1} + 2p^{m+n+3} - 2p^{m+n+2} \\
&\qquad + p^{m+5} - 2p^{m+4} + p^{m+3} - p^5 - p^4\big).
\end{aligned}$$

Again,

$$\begin{aligned}
&\left|2p^m - 2p^{m-1} - 2 - \frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| \\
&\qquad = \left|-\frac{2p^{m+n+1} - 2p^{m+n} - p^{m+3} - 2p^{m+2} + 3p^{m+1} + p^3 + p^2}{p^3 + p^2}\right| \\
&\qquad = \left|\frac{f_1(p,m,n)}{p^3 + p^2}\right|,
\end{aligned}$$

where $f_1(p,m,n) = -(2p^{m+n+1} - 2p^{m+n} - p^{m+3} - 2p^{m+2} + 3p^{m+1} + p^3 + p^2)$. For $n = 1, p \geq 2, m \geq 1$, we have $f_1(p,m,1) = p(p+1)(p^{m+1} - p^m - p) \geq 0$. For $n = 2, p \geq 2, m \geq 1$, we have

$$f_1(p,m,2) = -(p^{m+3} - 4p^{m+2} + 3p^{m+1} + p^3 + p^2) = -(p^{m+1}(p-1)(p-3) + p^3 + p^2).$$

So, $f_1(2,m,2) = 2(2^m - 6) > 0$ for $m \geq 3$ and $f_1(2,m,2) < 0$ for $m = 1, 2$. Also, $f_1(p,m,2) \leq 0$ for $p \geq 3$ and $m \geq 1$. For $n \geq 3, p \geq 2, m \geq 1$, we have

$$f_1(p,m,n) = -((p-1)(p^{m+2}(p^{n-2} - 1) + p^{m+1}(p^{n-1} - 3)) + p^3 + p^2) \leq 0.$$

Therefore

$$\begin{aligned}&|(2p^m- 2p^{m-1}-2)-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\Big|\\&=\begin{cases}-\frac{2p^{m+n+1}-2p^{m+n}-p^{m+3}-2p^{m+2}+3p^{m+1}+p^3+p^2}{p^3+p^2}, & \text{if } n=1, p\geq 2, m\geq 1;\\ & \text{or } n=2, p=2, m\geq 3\\ \frac{2p^{m+n+1}-2p^{m+n}-p^{m+3}-2p^{m+2}+3p^{m+1}+p^3+p^2}{p^3+p^2}, & \text{otherwise.}\end{cases}\end{aligned}$$

We have

$$\begin{aligned}\left|(p^m-p^{m-1}-2)-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\right| &= \left|-\frac{2p^{m+n+1}-2p^{m+n}-2p^{m+2}+2p^{m+1}+p^3+p^2}{p^3+p^2}\right|\\ &= \left|-\frac{2p^{m+n}(p-1)-2p^{m+1}(p-1)+p^3+p^2}{p^3+p^2}\right|\\ &= \frac{2p^{m+n}(p-1)-2p^{m+1}(p-1)+p^3+p^2}{p^3+p^2},\end{aligned}$$

$$\begin{aligned}&\Big|2p^{m+n-1}-2p^{m+n-2}-2-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\Big|\\ &= \left|-\frac{-2p^{m+n+1}+2p^{m+n}+p^{m+2}-2p^{m+1}+p^m+p^2+p}{p+p^2}\right|\\ &= \left|\frac{(p-1)(2p^{m+n-1}-p^{m+1}+p^m)}{p(p+1)}-1\right|\\ &= \left|(p-1)\frac{(p^{m+n-1}-p^m)+(p^{m+n-1}+p^{m-1})}{p+1}-1\right|\\ &= (p-1)\frac{(p^{m+n-1}-p^m)+(p^{m+n-1}+p^{m-1})}{p+1}-1\end{aligned}$$

and

$$\begin{aligned}&\Big|p^{m+n-1}-p^{m+n-2}-2-\frac{2|e(\mathcal{CCC}(G))|}{|V(\mathcal{CCC}(G))|}\Big|\\ &= \left|-\frac{-p^{m+n+2}+2p^{m+n+1}-p^{m+n}+p^{m+3}-2p^{m+2}+p^{m+1}+p^3+p^2}{p^2(p+1)}\right|\\ &= \left|-1+\frac{p^{m-1}(p^{n-1}-1)(p-1)^2}{p+1}\right|\\ &= \begin{cases}1-\frac{p^{m-1}(p^{n-1}-1)(p-1)^2}{p+1}, & \text{if } n=1, p\geq 2, m\geq 1;\\ & n=2, p=2, m\leq 2\\ -1+\frac{p^{m-1}(p^{n-1}-1)(p-1)^2}{p+1}, & \text{otherwise.}\end{cases}\end{aligned}$$

By (1.1.*f*), we have

$$
\begin{aligned}
&LE^+(\mathcal{CCC}(G))\\
&= (p^{n-1}(p-1)) \times \left(-\frac{2p^{m+n+1} - 2p^{m+n} - p^{m+3} - 2p^{m+2} + 3p^{m+1} + p^3 + p^2}{p^3 + p^2}\right)\\
&\quad + p^{n-2}(p-1)(p^{m+1} - p^m - p) \times \left(\frac{2p^{m+n}(p-1) - 2p^{m+1}(p-1) + p^3 + p^2}{p^3 + p^2}\right)\\
&\quad + 2 \times \left((p-1)\frac{(p^{m+n-1} - p^m) + (p^{m+n-1} + p^{m-1})}{p+1} - 1\right)\\
&\quad + 2(p^{m+n-1} - p^{m+n-2} - 1) \times \left(1 - \frac{p^{m-1}(p^{n-1} - 1)(p-1)^2}{p+1}\right)\\
&= 2(p^{m+1} - p^{m-1} - p - 1),
\end{aligned}
$$

if $n = 1, p \geq 2, m \geq 1$. If $n = 2, p = 2, m \leq 2$ then

$$
\begin{aligned}
&LE^+(\mathcal{CCC}(G))\\
&= (p^{n-1}(p-1)) \times \left(\frac{2p^{m+n+1} - 2p^{m+n} - p^{m+3} - 2p^{m+2} + 3p^{m+1} + p^3 + p^2}{p^3 + p^2}\right)\\
&\quad + p^{n-2}(p-1)(p^{m+1} - p^m - p) \times \left(\frac{2p^{m+n}(p-1) - 2p^{m+1}(p-1) + p^3 + p^2}{p^3 + p^2}\right)\\
&\quad + 2 \times \left((p-1)\frac{(p^{m+n-1} - p^m) + (p^{m+n-1} + p^{m-1})}{p+1} - 1\right)\\
&\quad + 2(p^{m+n-1} - p^{m+n-2} - 1) \times \left(1 - \frac{p^{m-1}(p^{n-1} - 1)(p-1)^2}{p+1}\right)\\
&= \frac{2}{3}(7.2^m - 6).
\end{aligned}
$$

If $n = 2, p = 2, m \geq 3$ then

$$
\begin{aligned}
&LE^+(\mathcal{CCC}(G))\\
&= (p^{n-1}(p-1)) \times \left(-\frac{2p^{m+n+1} - 2p^{m+n} - p^{m+3} - 2p^{m+2} + 3p^{m+1} + p^3 + p^2}{p^3 + p^2}\right)\\
&\quad + p^{n-2}(p-1)(p^{m+1} - p^m - p) \times \left(\frac{2p^{m+n}(p-1) - 2p^{m+1}(p-1) + p^3 + p^2}{p^3 + p^2}\right)\\
&\quad + 2 \times \left((p-1)\frac{(p^{m+n-1} - p^m) + (p^{m+n-1} + p^{m-1})}{p+1} - 1\right)\\
&\quad + 2(p^{m+n-1} - p^{m+n-2} - 1) \times \left(\frac{p^{m-1}(p^{n-1} - 1)(p-1)^2}{p+1} - 1\right)\\
&= \frac{2}{3}(4^m + 2^m - 6).
\end{aligned}
$$

If $n \geq 3, p \geq 2, m \geq 1$ then

$$\begin{aligned}
&LE^+(\mathcal{CCC}(G))\\
&\quad= (p^{n-1}(p-1)) \times \left(\frac{2p^{m+n+1} - 2p^{m+n} - p^{m+3} - 2p^{m+2} + 3p^{m+1} + p^3 + p^2}{p^3 + p^2}\right)\\
&\qquad + p^{n-2}(p-1)(p^{m+1} - p^m - p) \times \left(\frac{2p^{m+n}(p-1) - 2p^{m+1}(p-1) + p^3 + p^2}{p^3 + p^2}\right)\\
&\qquad + 2 \times \left((p-1)\frac{(p^{m+n-1} - p^m) + (p^{m+n-1} + p^{m-1})}{p+1} - 1\right)\\
&\qquad + 2(p^{m+n-1} - p^{m+n-2} - 1) \times \left(\frac{p^{m-1}(p^{n-1} - 1)(p-1)^2}{p+1} - 1\right)\\
&\quad= \frac{4p^{2m+n-4}}{p+1}(p-1)^3(p^n - p).
\end{aligned}$$

This completes the proof. □

We conclude this section with the following corollary.

Corollary 5.1.7. *If G is isomorphic to $D_{2n}, Q_{4m}, U_{(n,m)}, V_{8n}, SD_{8n}$ or $G(p, m, n)$ then $\mathcal{CCC}(G)$ is super integral.*

5.2 Comparing various energies

In this section we compare various energies of $\mathcal{CCC}(G)$ that are computed in Section 5.1 and derive the following relations.

Theorem 5.2.1. *Let $G = D_{2n}$.*

(a) *If $n = 3, 4, 6$ then $E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G))$.*
(b) *If $n = 5$ then $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G))$.*
(c) *If $n = 10$ then $LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.*
(d) *If $n \geq 7$ but $n \neq 10$ then $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.*

Proof. We shall prove the result by considering the following cases.
Case 1. n is odd.

If $n = 3$ then, by Theorem 5.1.1, we have

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 0.$$

If $n = 5$ then, by Theorem 5.1.1, we have

$$E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) = n - 3 - \frac{(n-3)(n+3)}{n+1} = -\frac{4}{5} < 0$$

and $LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = \frac{8}{3}$. Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G))$.

If $n \geq 7$ then, by Theorem 5.1.1, we have

$$E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) = n - 3 - \frac{(n-3)(n+3)}{n+1} = -\frac{2(n-3)}{n+1} < 0$$

and

$$LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) = \frac{(n-3)(n+3)}{n+1} - \frac{2(n-1)(n-3)}{n+1} = -\frac{(n-3)(n-5)}{n+1} < 0.$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

Case 2. n is even.

Consider the following subcases.

Subcase 2.1 $\frac{n}{2}$ is even.

If $n = 4$ then, by Theorem 5.1.1, we have

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 0.$$

If $n = 8$ then, by Theorem 5.1.1, we have

$$E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) = n - 4 - \frac{(n-4)(n+6)}{n+2} = -\frac{8}{5} < 0$$

and

$$LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) = \frac{(n-4)(n+6)}{n+2} - \frac{3(n-2)(n-4)}{n+2} = -\frac{8}{5} < 0.$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $n \geq 12$ then, by Theorem 5.1.1, we have

$$E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) = n - 4 - \frac{2(n-2)(n-4)}{n+2} = -\frac{(n-4)(n-6)}{n+2} < 0$$

and

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) &= \frac{2(n-2)(n-4)}{n+2} - \frac{3(n-2)(n-4)}{n+2} \\ &= -\frac{(n-2)(n-4)}{n+2} < 0. \end{aligned}$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

Subcase 2.2 $\frac{n}{2}$ is odd.

If $n = 6$ then, by Theorem 5.1.1, we have

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 4.$$

If $n = 10$ then, by Theorem 5.1.1, we have

$$LE^+(\mathcal{CCC}(G)) - E(\mathcal{CCC}(G)) = \frac{22}{3} - (n-2) = -\frac{2}{3} < 0$$

and

$$E(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) = n - 2 - \frac{(n-4)(3n-10)}{n+2} = -2 < 0.$$

Therefore, $LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $n \geq 14$ then, by Theorem 5.1.1, we have

$$\begin{aligned} E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) &= n - 2 - \frac{2(n-2)(n-6)}{n+2} \\ &= -\frac{(n-2)(n-10)}{n+2} < 0 \end{aligned}$$

and

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) &= \frac{2(n-2)(n-6)}{n+2} - \frac{(n-4)(3n-10)}{n+2} \\ &= -\frac{n^2 - 6n + 16}{n+2} \\ &= -\frac{n(n-14) + 8n + 10}{n+2} < 0. \end{aligned}$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$. Hence, the result follows. □

Theorem 5.2.2. *Let $G = Q_{4m}$.*

(a) *If $m = 2, 3$ then $E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G))$.*
(b) *If $m = 5$ then $LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.*
(c) *If $m = 7$ then $LE^+(\mathcal{CCC}(G)) = E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.*
(d) *If $m = 4, 6$ or $m \geq 8$ then $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.*

Proof. We shall prove the result by considering the following cases.

Case 1. m is odd.

If $m = 3$ then, by Theorem 5.1.2, we have

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 4.$$

If $m = 5$ then, by Theorem 5.1.2, we have

$$LE^+(\mathcal{CCC}(G)) - E(\mathcal{CCC}(G)) = \frac{22}{3} - (2m-2) = -\frac{2}{3} < 0$$

and

$$E(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) = 2m - 2 - \frac{2(m-2)(3m-5)}{m+1} = -2 < 0.$$

Therefore, $LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $m = 7$ then, by Theorem 5.1.2, we have $LE^+(\mathcal{CCC}(G)) = E(\mathcal{CCC}(G)) = 12$ and

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) &= \frac{4(m-1)(m-3)}{m+1} - \frac{2(m-2)(3m-5)}{m+1} \\ &= -\frac{2(m+4)(m-1)}{m+1} < 0. \end{aligned}$$

Therefore, $LE^+(\mathcal{CCC}(G)) = E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $m \geq 9$ then, by Theorem 5.1.2, we have

$$E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) = 2m - 2 - \frac{4(m-1)(m-3)}{m+1} = -\frac{2(m-1)(m-7)}{m+1} < 0$$

and

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) &= \frac{4(m-1)(m-3)}{m+1} - \frac{2(m-2)(3m-5)}{m+1} \\ &= -\frac{2(m+4)(m-1)}{m+1} < 0. \end{aligned}$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

Case 2. m is even.

If $m = 2$ then, by Theorem 5.1.2, we have

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 0.$$

If $m = 4$ then, by Theorem 5.1.2, we have

$$E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) = 2m - 4 - \frac{2(m-2)(m+3)}{m+1} = -\frac{8}{5} < 0$$

and

$$LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) = \frac{2(m-2)(m+3)}{m+1} - \frac{6(m-1)(m-2)}{m+1} = -\frac{8}{5} < 0.$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $m \geq 6$ then, by Theorem 5.1.2, we have

$$E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) = 2m - 4 - \frac{4(m-1)(m-2)}{m+1} = -\frac{2(m-2)(m-3)}{m+1} < 0$$

and

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) &= \frac{4(m-1)(m-2)}{m+1} - \frac{6(m-1)(m-2)}{m+1} \\ &= -\frac{2(m-1)(m-2)}{m+1}. \end{aligned}$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$. Hence, the result follows. □

Theorem 5.2.3. *Let $G = U_{(n,m)}$.*

(a) *If $m = 3, 4$ and $n \geq 2$ then $LE^+(\mathcal{CCC}(G)) = E(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G))$.*

(b) *If $m = 5$ and $n = 2, 3$; or $m = 6$ and $n = 2$ then*

$$LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

(c) *If $m = 5$ and $n \geq 4$; $m \geq 6$ and $n \geq 3$; or $m \geq 8$ and $n \geq 2$ then*

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

(d) *If $m = 7$ and $n = 2$ then $E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.*

Proof. We shall prove the result by considering the following cases.

Case 1. If m is odd and $n \geq 2$.

If $m = 3$ and $n \geq 2$ then, by Theorem 5.1.3, we have

$$LE^+(\mathcal{CCC}(G)) = E(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 4(n-1).$$

If $m = 5$ and $n = 2$ then, by Theorem 5.1.3, we have

$$LE^+(\mathcal{CCC}(G)) - E(\mathcal{CCC}(G)) = \frac{2n^2 + 10n - 6}{3} - (n(m+1) - 4) = -\frac{2}{3} < 0$$

and

$$E(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) = n(m+1) - 4 - \frac{2(2n-1)(n+3)}{3} = -2 < 0.$$

Therefore, $LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $m = 5$ and $n = 3$ then, by Theorem 5.1.3, we have

$$LE^+(\mathcal{CCC}(G)) - E(\mathcal{CCC}(G)) = \frac{2(2n+3)(n-1)}{3} - (n(m+1) - 4) = -2 < 0$$

and

$$E(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) = n(m+1) - 4 - \frac{2(2n-1)(n+3)}{3} = -4 < 0.$$

Therefore, $LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $m = 5$ and $n \geq 4$ then, by Theorem 5.1.3, we have

$$\begin{aligned} E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) &= n(m+1) - 4 - \frac{2(2n+3)(n-1)}{3} \\ &= \frac{-2(2n^2 - 8n + 3)}{3} \\ &= \frac{-2(2n(n-4) + 3)}{3} < 0 \end{aligned}$$

and

$$LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) = \frac{2(2n+3)(n-1)}{3} - \frac{2(2n-1)(n+3)}{3} = \frac{-8n}{3} < 0.$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $m \geq 7$ and $n \geq 2$ then, by Theorem 5.1.3, we have

$$E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) = n(m+1) - 4 - \frac{n^2(m-1)(m-3)}{m+1} = -\frac{f_1(m,n)}{m+1},$$

where $f_1(m,n) = mn(m-4)(n-3) + 2mn(m-7) + 3n(n-1) + 4(m+1)$. For $m \geq 7$ and $n = 2$ we have $f_1(m,n) = 2(m-1)(m-7) \geq 0$. Hence, $f_1(7,2) = 0$ and $f_1(m,2) > 0$ if $m \geq 9$. Thus, $E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G))$ and $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G))$ according as if $m = 7$, $n = 2$ and $m \geq 9$, $n = 2$. For $m \geq 7$ and $n \geq 3$ we have $f_1(m,n) > 0$ and so $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G))$.

If $m \geq 7$ and $n \geq 2$ then, by Theorem 5.1.3, we also have

$$\begin{aligned} &LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) \\ &\quad = \frac{n^2(m-1)(m-3)}{m+1} - \frac{m^2n^2 - 4mn^2 + m^2n + 3n^2 - 2mn - 2m + 5n - 2}{m+1} \\ &\quad = -\frac{m^2n - 2mn - 2m + 5n - 2}{m+1} \\ &\quad = -\frac{(mn-2)(m-2) + 5(n-2) + 4}{m+1} < 0. \end{aligned}$$

Therefore, $LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$. Thus, if $m = 7$ and $n = 2$ then

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$$

and if $m \geq 7$ and $n \geq 3$ or $m \geq 9$ and $n = 2$ then

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

Case 2. m is even and $n \geq 2$.

If $m = 4$ and $n \geq 2$ then, by Theorem 5.1.3, we have

$$LE^+(\mathcal{CCC}(G)) = E(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 6(n-1).$$

If $m = 6$ and $n = 2$ then, by Theorem 5.1.3, we have

$$LE^+(\mathcal{CCC}(G)) - E(\mathcal{CCC}(G)) = 2(n+2)(n-1) - (n(m+2) - 6) = -4 < 0$$

and

$$\begin{aligned} &E(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) \\ &= n(m+2) - 6 - \frac{2m^2n^2 - 12mn^2 + m^2n + 16n^2 - 4mn - 2m + 12n - 4}{m+2} \\ &= -2 < 0. \end{aligned}$$

Therefore, $LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $m = 6$ and $n \geq 3$ then, by Theorem 5.1.3, we have

$$E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) = n(m+2) - 6 - 2(n+2)(n-1) = 2n(3-n) - 2 < 0$$

and

$$\begin{aligned} &LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) \\ &= 2(n+2)(n-1) - \frac{2m^2n^2 - 12mn^2 + m^2n + 16n^2 - 4mn - 2m + 12n - 4}{m+2} \\ &= -(n+2) < 0. \end{aligned}$$

Therefore $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $m \geq 8$ and $n \geq 2$ then, by Theorem 5.1.3, we have

$$\begin{aligned} E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) &= n(m+2) - 6 - \frac{2n^2(m-2)(m-4)}{m+2} \\ &= -\frac{2m^2n^2 - 12mn^2 - m^2n + 16n^2 - 4mn + 6m - 4n + 12}{m+2} \\ &= -\frac{f_2(m,n)}{m+2}, \end{aligned}$$

where $f_2(m,n) = mn(2n-1)(m-8) + 2m(2n(n-3)+3) + 4n(4n-1) + 12$. For $n = 2$ and $m \geq 8$ we have $f_2(m,n) = (6m-2)(m-8) + 52 > 0$. For $n \geq 3$ and $m \geq 8$ we have $f_2(m,n) > 0$. Therefore, if $m \geq 8$ and $n \geq 2$ then $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G))$.

If $m \geq 8$ and $n \geq 2$ then, by Theorem 5.1.3, we also have

$$\begin{aligned}
LE^{+}(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) \\
&= \frac{2n^2(m-2)(m-4)}{m+2} \\
&\quad - \frac{2m^2n^2 - 12mn^2 + m^2n + 16n^2 - 4mn - 2m + 12n - 4}{m+2} \\
&= -\frac{m^2n - 4mn - 2m + 12n - 4}{m+2} \\
&= -\frac{mn(m-8) + 2m(2n-1) + 4(3n-1)}{m+2} < 0.
\end{aligned}$$

Therefore, $LE^{+}(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$. Thus, if $m \geq 8$ and $n \geq 2$ then

$$E(\mathcal{CCC}(G)) < LE^{+}(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

Hence, the result follows. □

Theorem 5.2.4. *If $G = V_{8n}$ then $E(\mathcal{CCC}(G)) \leq LE^{+}(\mathcal{CCC}(G)) \leq LE(\mathcal{CCC}(G))$. The equality holds if and only if $n = 2$.*

Proof. We shall prove the result by considering the following cases.

Case 1. n is odd.

By Theorem 5.1.4, we have

$$E(\mathcal{CCC}(G)) - LE^{+}(\mathcal{CCC}(G)) = 4n - 4 - \frac{4(2n-1)(2n-2)}{2n+1} = -\frac{4(n-1)(2n-3)}{2n+1}$$

and

$$\begin{aligned}
LE^{+}(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) &= \frac{4(2n-1)(2n-2)}{2n+1} - \frac{6(2n-1)(2n-2)}{2n+1} \\
&= -\frac{2(2n-1)(2n-2)}{2n+1} < 0.
\end{aligned}$$

Therefore, $E(\mathcal{CCC}(G)) < LE^{+}(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

Case 2. n is even.

If $n = 2$ then, by Theorem 5.1.4, we have

$$E(\mathcal{CCC}(G)) = LE^{+}(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 6.$$

If $n \geq 4$ then, by Theorem 5.1.4, we have

$$\begin{aligned}
E(\mathcal{CCC}(G)) - LE^{+}(\mathcal{CCC}(G)) &= 4n - 2 - \frac{16(n-1)(n-2)}{n+1} \\
&= -\frac{2(6n^2 + 25n - 17)}{n+1} = -\frac{2(6n(n-4) + 49n - 7)}{n+1} < 0
\end{aligned}$$

and

$$\begin{aligned} LE^{+}(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) &= \frac{16(n-1)(n-2)}{n+1} - \frac{2(2n-3)(5n-7)}{n+1} \\ &= -\frac{2(2n^2-5n+5)}{n+1} = -\frac{2(2n(n-4)+3n+5)}{n+1} < 0. \end{aligned}$$

Therefore, $E(\mathcal{CCC}(G)) < LE^{+}(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$. Hence, the result follows. □

Theorem 5.2.5. *If $G = SD_{8n}$ then $E(\mathcal{CCC}(G)) \leq LE^{+}(\mathcal{CCC}(G)) \leq LE(\mathcal{CCC}(G))$. The equality holds if and only if $n = 3$.*

Proof. We shall prove the result by considering the following cases.

Case 1. n is odd.

If $n = 3$ then, by Theorem 5.1.5, we have

$$E(\mathcal{CCC}(G)) = LE^{+}(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 12.$$

If $n = 5$ then, by Theorem 5.1.5, we have

$$E(\mathcal{CCC}(G)) - LE^{+}(\mathcal{CCC}(G)) = 4n - 22 = -2 < 0$$

and

$$LE^{+}(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) = 22 - \frac{2(2n-3)(5n-11)}{n+1} = -\frac{32}{3} < 0.$$

Therefore, $E(\mathcal{CCC}(G)) < LE^{+}(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $n \geq 7$ then, by Theorem 5.1.5, we have

$$\begin{aligned} E(\mathcal{CCC}(G)) - LE^{+}(\mathcal{CCC}(G)) &= 4n - \frac{16(n-1)(n-3)}{n+1} \\ &= -\frac{4(3n^2-17n+12)}{n+1} = -\frac{4(3n(n-7)+4n+12)}{n+1} < 0 \end{aligned}$$

and

$$\begin{aligned} LE^{+}(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) &= \frac{16(n-1)(n-3)}{n+1} - \frac{2(2n-3)(5n-11)}{n+1} \\ &= -\frac{2(2n^2-5n+9)}{n+1} = -\frac{2(2n(n-7)+9n+9)}{n+1} < 0. \end{aligned}$$

Therefore, $E(\mathcal{CCC}(G)) < LE^{+}(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

Case 2. n is even.

If $n = 2$ then, by Theorem 5.1.5, we have

$$E(\mathcal{CCC}(G)) - LE^{+}(\mathcal{CCC}(G)) = 4n - 4 - \frac{28}{5} = -\frac{8}{5} < 0$$

and

$$LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) = \frac{28}{5} - \frac{6(2n-1)(2n-2)}{2n+1} = -\frac{8}{5} < 0.$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $n \geq 4$ then, by Theorem 5.1.5, we have

$$E(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) = 4n - 4 - \frac{4(2n-1)(2n-2)}{2n+1} = -\frac{4(n-1)(2n-3)}{2n+1} < 0$$

and

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) - LE(\mathcal{CCC}(G)) &= \frac{4(2n-1)(2n-2)}{2n+1} - \frac{6(2n-1)(2n-2)}{2n+1} \\ &= -\frac{2(2n-1)(2n-2)}{2n+1} < 0. \end{aligned}$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$. Hence, the result follows. □

Theorem 5.2.6. *Let $G = G(p, m, n)$.*

(a) *If $n = 1, p \geq 2$ and $m \geq 1$ then $E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G))$.*

(b) *If $n = 2, p = 2$ and $m = 1$ then $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G))$.*

(c) *If $n = 2$, $p = 2$ and $m = 2$ then $LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.*

(d) *If $n = 2$, $p = 2$, $m \geq 3$; $n = 2, p \geq 3$, $m \geq 1$; or $n \geq 3, p \geq 2$, $m \geq 1$ then*

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

Proof. We shall proof the result by considering the following cases.

Case 1. $n = 1$, $p \geq 2$ and $m \geq 1$.

By Theorem 5.1.6, we have

$$E(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = 2(p^{m+1} - p^{m-1} - p - 1).$$

Case 2. $n = 2, p = 2$ and $m \geq 1$.

If $n = 2, p = 2$ and $m = 1$ then, by Theorem 5.1.6, we have

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) - E(\mathcal{CCC}(G)) &= \frac{2}{3}(7.2^m - 6) - 2(p^{m+n} - p^{m+n-2} - p^n + p^{n-1} - 2) \\ &= \frac{16}{3} - 4 = \frac{4}{3} > 0 \end{aligned}$$

and

$$LE(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = \frac{16}{3}.$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G))$.

If $n = 2, p = 2$ and $m = 2$ then, by Theorem 5.1.6, we have

$$\begin{aligned}
&LE(\mathcal{CCC}(G)) - E(\mathcal{CCC}(G)) \\
&\qquad = \frac{4}{p^4(p+1)}\big(p^{2m+2n+3} - 3p^{2(m+n+1)} + 3p^{2m+2n+1} - p^{2(m+n)} - p^{2m+n+4} + 3p^{2m+n+3} \\
&\qquad\quad - 3p^{2m+n+2} + p^{2m+n+1} + 2p^{m+n+3} - 2p^{m+n+2} + p^{m+5} - 2p^{m+4} + p^{m+3} - p^5 - p^4\big) \\
&\qquad\quad - 2(p^{m+n} - p^{m+n-2} - p^n + p^{n-1} - 2) \\
&\qquad = 20 - 16 = 4 > 0
\end{aligned}$$

and

$$\begin{aligned}
LE^+(\mathcal{CCC}(G)) - E(\mathcal{CCC}(G)) &= \frac{2}{3}(7.2^m - 6) - 2(p^{m+n} - p^{m+n-2} - p^n + p^{n-1} - 2) \\
&= \frac{44}{3} - 16 = -\frac{4}{3} < 0.
\end{aligned}$$

Therefore, $LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

If $n = 2, p = 2$ and $m \geq 3$ then, by Theorem 5.1.6, we have

$$\begin{aligned}
&LE^+(\mathcal{CCC}(G)) - E(\mathcal{CCC}(G)) \\
&\qquad = \frac{2}{3}(4^m + 2^m - 6) - 2(p^{m+n} - p^{m+n-2} - p^n + p^{n-1} - 2) \\
&\qquad = \frac{2}{3}(4^m + 2^m - 6) - 2(3.2^m - 4) \\
&\qquad = -\frac{2}{3}(2^{2m} - 2^{m+3} + 6) \\
&\qquad = -\frac{2}{3}(2^m(2^m - 8) + 6) < 0
\end{aligned}$$

and

$$\begin{aligned}
&LE(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G)) \\
&\quad = \frac{4}{p^4(p+1)}\big(p^{2m+2n+3} - 3p^{2(m+n+1)} + 3p^{2m+2n+1} - p^{2(m+n)} - p^{2m+n+4} + 3p^{2m+n+3} \\
&\quad\quad - 3p^{2m+n+2} + p^{2m+n+1} + 2p^{m+n+3} - 2p^{m+n+2} + p^{m+5} - 2p^{m+4} + p^{m+3} - p^5 - p^4\big) \\
&\quad\quad - \frac{2}{3}(4^m + 2^m - 6) \\
&\quad = \frac{2}{3}(4^m + 5.2^m - 6) - \frac{2}{3}(4^m + 2^m - 6) \\
&\quad = \frac{2^{m+3}}{3} > 0.
\end{aligned}$$

Therefore, $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.

Case 3. $n = 2, p \geq 3$, $m \geq 1$; or $n \geq 3, p \geq 2$, $m \geq 1$.

By Theorem 5.1.6, we have

$$\begin{aligned} LE^+(\mathcal{CCC}(G)) - E(\mathcal{CCC}(G)) \\ &= \frac{4p^{2m+n-4}}{p+1}(p-1)^3(p^n - p) - 2(p^{m+n} - p^{m+n-2} - p^n + p^{n-1} - 2) \\ &= \frac{2}{p^4(p+1)}(2p^{2m+2n+3} - 6p^{2(m+n+1)} + 6p^{2m+2n+1} - 2p^{2(m+n)} - 2p^{2m+n+4} \\ &\quad + 6p^{2m+n+3} - 6p^{2m+n+2} + 2p^{2m+n+1} - p^{m+n+5} - p^{m+n+4} + p^{m+n+3} \\ &\quad + p^{m+n+2} + p^{n+5} - p^{n+3} + 2p^5 + 2p^4) \\ &:= f(p, m, n). \end{aligned}$$

Therefore, for $n = 2$, we have

$$\begin{aligned} f(p, m, 2) = &\frac{2}{p(p+1)}(2p^{2m+4} - 8p^{2m+3} + 12p^{2m+2} - 8p^{2m+1} + 2p^{2m} - p^{m+4} \\ &- p^{m+3} + p^{m+2} + p^{m+1} + p^4 + p^2 + 2p) \end{aligned}$$

and so $f(3, m, 2) = \frac{16}{3}(9^m - 3^{m+1} + 3) > 0$ for $m \geq 1$. Also,

$$\begin{aligned} f(p, m, 2) = &\frac{2}{p(p+1)}(p^{2m+3}(2p-9) + (p^{2m+3} - p^{m+4}) + (p^{2m+2} - p^{m+3}) + p^{2m+1}(11p - 8) \\ &+ 2p^{2m} + p^{m+2} + p^{m+1} + p^4 + p^2 + 2p) > 0, \end{aligned}$$

if $m \geq 1$ and $p \geq 5$. For $p = 2$, we have

$$f(2, m, n) = \frac{1}{12}(2^{m+n}\left((2^{n-1} - 1)2^{m+1} - 18\right) + 3.2^{n+2} + 48\big) > 0,$$

if $n \geq 4, p = 2, m \geq 1$. For $n = 3, p = 2, m \geq 1$ we have $f(2, m, 3) = 48(4^m - 3.2^m + 3) = 48((2^m - 2)^2 + 2^m - 1) > 0$.

For $p \geq 3, n \geq 3, m \geq 1$, we have

$$\begin{aligned} f(p, m, n) = &\frac{2}{p^4(p+1)}(2p^{2m+2n+2}(p-3) + (p^{2m+2n+1} - p^{m+n+5}) + 2p^{2m+n+1} + 4p^{2m+n+3} \\ &+ (p^{2m+2n+1} - 2p^{2m+2n}) + (p^{2m+2n+1} - 2p^{2m+n+4}) + 2p^{2m+2n+1} \\ &+ 2p^{2m+n+2}(p-3) + (p^{2m+2n+1} - p^{m+n+4}) + p^{m+n+3} \\ &+ p^{m+n+2} + p^{n+3}(p^2 - 1) + 2p^5 + 2p^4\big) > 0. \end{aligned}$$

Therefore, $LE^+(\mathcal{CCC}(G)) > E(\mathcal{CCC}(G))$ if $n = 2, p \geq 3,\ m \geq 1$; or $n \geq 3, p \geq 2,\ m \geq 1$.

Again,

$$\begin{aligned}
&LE(\mathcal{CCC}(G)) - LE^+(\mathcal{CCC}(G))\\
&\quad= \frac{4}{p^4(p+1)}\big(p^{2m+2n+3} - 3p^{2(m+n+1)} + 3p^{2m+2n+1} - p^{2(m+n)} - p^{2m+n+4} + 3p^{2m+n+3}\\
&\quad - 3p^{2m+n+2} + p^{2m+n+1} + 2p^{m+n+3} - 2p^{m+n+2} + p^{m+5} - 2p^{m+4} + p^{m+3}\\
&\quad - p^5 - p^4\big) - \frac{4p^{2m+n-4}}{p+1}(p-1)^3(p^n - p)\\
&\quad= \frac{4(2p^{m+n+1} - 2p^{m+n} + p^{m+3} - 2p^{m+2} + p^{m+1} - p^3 - p^2)}{p^2(p+1)}\\
&\quad= \frac{4\big((p^{m+n}(p-2) + (p^{m+n+1} - p^3) + p^{m+2}(p-2) + p^{m+1} - p^2)\big)}{p^2(1+p)} > 0,
\end{aligned}$$

if $p \geq 2, n \geq 2, m \geq 1$. Therefore, $LE(\mathcal{CCC}(G)) > LE^+(\mathcal{CCC}(G))$, if $n = 2, p \geq 3,\ m \geq 1$; or $n \geq 3, p \geq 2,\ m \geq 1$. Hence,

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)),$$

if $n = 2, p \geq 3,\ m \geq 1$; or $n \geq 3, p \geq 2,\ m \geq 1$. This completes the proof. □

Note that Theorems 5.2.1–5.2.6 can be summarized in the following way.

Theorem 5.2.7. *Let G be a finite non-abelian group. Then we have the following.*

(a) *If G is isomorphic to $D_6, D_8, D_{12}, Q_8, Q_{12}$, $U_{(n,2)}, U_{(n,3)}, U_{(n,4)} (n \geq 2)$, V_{16}, SD_{24} or $G(p, m, 1)(p \geq 2, m \geq 1)$ then*

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)).$$

(b) *If G is isomorphic to D_{20}, Q_{20}, $U_{(2,5)}, U_{(3,5)}$, $U_{(2,6)}$ or $G(2, 2, 2)$ then*

$$LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

(c) *If G is isomorphic to $D_{14}, D_{16}, D_{18}, D_{2n}(n \geq 11)$, $Q_{16}, Q_{24}, Q_{4m}(m \geq 8)$, $U_{(n,5)}, (n \geq 4)$, $U_{(n,m)}(m \geq 6$ and $n \geq 3)$, $U_{(n,m)}(m \geq 8$ and $n \geq 2)$, $V_{8n}(n \geq 3)$, SD_{16}, $SD_{8n}(n \geq 4)$, $G(2, m, 2)(m \geq 3)$, $G(p, m, 2)(p \geq 3, m \geq 1)$ or $G(p, m, n)(n \geq 3, p \geq 2, m \geq 1)$ then*

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

(d) *If G is isomorphic to Q_{28} or $U_{(2,7)}$ then $E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$.*

(e) *If G is isomorphic to D_{10} and $G(2,1,2)$ then* $E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G))$.

We conclude this section with the following remark regarding Conjecture 1.1.7 and Question 1.1.8.

Remark 5.2.8. By Theorem 5.2.7, it follows that

$$E(\mathcal{CCC}(G)) \leq LE(\mathcal{CCC}(G)) \text{ and } LE^+(\mathcal{CCC}(G)) \leq LE(\mathcal{CCC}(G))$$

for commuting conjugacy class graph of the groups D_{2n}, Q_{4m}, $U_{(n,m)}$, V_{8n}, SD_{8n} and $G(p,m,n)$. Therefore, Conjecture 1.1.7 holds for commuting conjugacy class graph of these groups whereas the inequality in Question 1.1.8 does not. However, $LE(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G))$ if $G = D_6, D_8, D_{10}, D_{12}$, Q_8, Q_{12}, V_{16}, SD_{24}, $U_{(n,m)}$ where $m = 3, 4; n \geq 2$ and $G(p,m,n)$ where $n = 1, 2; p \geq 2; m \geq 1$.

5.3 Hyperenergetic and borderenergetic graph

In this section we consider CCC-graph for the groups considered in Section 5.1 and determine whether they are hyperenergetic, L-hyperenergetic or Q-hyperenergetic. We shall also determine whether they are borderenergetic, L-borderenergetic or Q-borderenergetic.

Theorem 5.3.1. *Let $G = D_{2n}$.*

(a) *If n is odd or $n = 4, 6$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(b) *If $n = 8, 10, 12, 14$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(c) *If n is even and $n \geq 16$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.*

Proof. We shall prove the result by considering the following cases.

Case 1. n is odd.

We have $|V(\mathcal{CCC}(G))| = \frac{n+1}{2}$. Using (1.1.$g$), we get

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^+(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = n - 1 \tag{5.3.a}$$

If $n = 3$ then, by Theorem 5.2.1 and Theorem 5.1.1, we get

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 0 < 2 = E(K_{|V(\mathcal{CCC}(G))|}). \tag{5.3.b}$$

If $n = 5$ then, by Theorem 5.2.1 and Theorem 5.1.1, we get

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = \frac{8}{3} < 4 = E(K_{|V(\mathcal{CCC}(G))|}). \tag{5.3.c}$$

Therefore $\mathcal{CCC}(G)$ is neither hyperenergetic nor L-hyperenergetic nor Q-hyperenergetic for $n = 5$.

If $n \geq 7$ then, by Theorem 5.2.1 and Theorem 5.1.1, we get

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)) = \frac{(n-3)(n+3)}{n+1}.$$

Again,

$$\frac{(n-3)(n+3)}{n+1} - (n-1) = -\frac{8}{n+1} < 0$$

Therefore,

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)) < n - 1 = E(K_{|V(\mathcal{CCC}(G))|}). \tag{5.3.d}$$

Hence, in view of (5.3.a)–(5.3.d), it follows that $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

Case 2. n is even.

We have $|V(\mathcal{CCC}(G))| = \frac{n}{2} + 1$. Using (1.1.$g$), we get

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^+(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = n. \tag{5.3.e}$$

Subcase 2.1. $\frac{n}{2}$ is even.

If $n = 4$ then, by Theorem 5.2.1 and Theorem 5.1.1, we get

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 0 < 4 = E(K_{|V(\mathcal{CCC}(G))|}). \tag{5.3.f}$$

Therefore, by (5.3.e) and by (5.3.f), it follows that $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $n = 8$ then, by Theorem 5.2.1 and Theorem 5.1.1, we get

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = 6 < 8 = E(K_{|V(\mathcal{CCC}(G))|}).$$

Also,

$$LE(\mathcal{CCC}(G)) = 9 > 8 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

So, $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $n \geq 12$ then, by Theorem 5.1.1, we get

$$LE(\mathcal{CCC}(G)) = \frac{3(n-2)(n-4)}{n+2}.$$

We have

$$n - \frac{3(n-2)(n-4)}{n+2} = -\frac{2(n(n-12)+2n+12)}{n+2} < 0.$$

Therefore, $LE(K_{|V(\mathcal{CCC}(G))|}) < LE(\mathcal{CCC}(G))$ and so $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic.

By Theorem 5.2.1 and Theorem 5.1.1, we also get

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = \frac{2(n-2)(n-4)}{n+2}.$$

We have

$$\frac{2(n-2)(n-4)}{n+2} - n = \frac{n^2-14n+16}{n+2} = \frac{n(n-16)+2n+16}{n+2} := f_1(n) \qquad (5.3.g)$$

Therefore, for $n = 12$, we have $f_1(n) < 0$ and so

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = \frac{2(n-2)(n-4)}{n+2} < n = LE^+(K_{|V(\mathcal{CCC}(G))|}).$$

Thus, if $n = 12$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $n \geq 16$ then, by (5.3.g), we have $f_1(n) > 0$ and so $LE^+(\mathcal{CCC}(G)) > n = LE^+(K_{|V(\mathcal{CCC}(G))|})$. Therefore, $\mathcal{CCC}(G)$ is Q-hyperenergetic but not Q-borderenergetic. Also,

$$E(\mathcal{CCC}(G)) = n - 4 < n = E(K_{|V(\mathcal{CCC}(G))|})$$

and so $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic. Thus, if $n \geq 16$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.

Subcase 2.2. $\frac{n}{2}$ is odd.

If $n = 6$ then, by Theorem 5.2.1 and Theorem 5.1.1, we get

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 4 < 6 = E(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $n = 10$ then, by Theorem 5.2.1 and Theorem 5.1.1, we get

$$LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)) = 10 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

So, $\mathcal{CCC}(G)$ is L-bordererenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $n \geq 14$ then, by Theorem 5.1.1, we get

$$LE(\mathcal{CCC}(G)) = \frac{(n-4)(3n-10)}{n+2}.$$

We have

$$n - \frac{(n-4)(3n-10)}{n+2} = -\frac{2n(n-14)+4n+40}{n+2} < 0.$$

So, $LE(K_{|V(\mathcal{CCC}(G))|}) < LE(\mathcal{CCC}(G))$ and so $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic.

By Theorem 5.2.1 and Theorem 5.1.1, we also get

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = \frac{2(n-2)(n-6)}{n+2}.$$

We have

$$\frac{2(n-2)(n-6)}{n+2} - n = \frac{n^2-18n+24}{n+2} = \frac{n(n-18)+24}{n+2} := f_2(n). \tag{5.3.h}$$

Therefore, for $n = 14$, we have $f_2(n) < 0$ and so

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = \frac{2(n-2)(n-4)}{n+2} < n = LE^+(K_{|V(\mathcal{CCC}(G))|}).$$

Thus, if $n = 14$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic. If $n \geq 18$ then, by (5.3.h), we have $f_2(n) > 0$ and so $LE^+(\mathcal{CCC}(G)) > n = LE^+(K_{|V(\mathcal{CCC}(G))|})$. Therefore, $\mathcal{CCC}(G)$ is Q-hyperenergetic but not Q-borderenergetic. Also, $E(\mathcal{CCC}(G)) = n-2 < n = E(K_{|V(\mathcal{CCC}(G))|})$ and so $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic. Thus, if $n \geq 18$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic. □

Theorem 5.3.2. *Let $G = Q_{4m}$.*

(a) *If $m = 2, 3, 4$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(b) *If $m = 5$ then $\mathcal{CCC}(G)$ is L-borderenergetic but neither hyperenergetic, borderenergetic, L-hyperenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(c) *If $m = 6, 7$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(d) *If $m \geq 8$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.*

Proof. We shall prove the result by considering the following cases.

Case 1. m is odd.

We have $|V(\mathcal{CCC}(G))| = m + 1$. Using (1.1.$g$), we get

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^+(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = 2m. \qquad (5.3.i)$$

If $m = 3$ then, by Theorem 5.2.2 and Theorem 5.1.2, we get

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 4 < 6 = E(K_{|V(\mathcal{CCC}(G))|}). \qquad (5.3.j)$$

So, by (5.3.i) and (5.3.j), $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m = 5$ then, by Theorem 5.2.2 and Theorem 5.1.2, we get

$$LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)) = 10 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

So, $\mathcal{CCC}(G)$ is L-borderenergetic but neither hyperenergetic, borderenergetic, L-hyperenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m = 7$ then, by Theorem 5.2.2 and Theorem 5.1.2, we get

$$LE^+(\mathcal{CCC}(G)) = E(\mathcal{CCC}(G)) = 12 < 14 = E(K_{|V(\mathcal{CCC}(G))|}).$$

Also,

$$LE(\mathcal{CCC}(G)) = 20 > 14 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

So, $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m \geq 9$ then, by Theorem 5.2.2 and Theorem 5.1.2, we get

$$\frac{4(m-1)(m-3)}{m+1} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

We have

$$2m - \frac{4(m-1)(m-3)}{m+1} = -\frac{2(m(m-9)+6)}{m+1} < 0$$

and so $LE^+(K_{|V(\mathcal{CCC}(G))|}) = 2m < \frac{4(m-1)(m-3)}{m+1} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$. Hence, $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither L-borderenergetic nor Q-borderenergetic. Also,

$$E(\mathcal{CCC}(G)) = 2m - 2 < 2m = E(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic. Thus, if $m \geq 9$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.

Case 2. m is even.

We have $|V(\mathcal{CCC}(G))| = m+1$. Using (1.1.$g$), we get

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^{+}(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = 2m. \tag{5.3.k}$$

If $m = 2$ then, by Theorem 5.2.2 and Theorem 5.1.2, we get

$$E(\mathcal{CCC}(G)) = LE^{+}(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 0 < 4 = E(K_{|V(\mathcal{CCC}(G))|}). \tag{5.3.l}$$

So, by (5.3.k) and (5.3.l), $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m = 4$ then, by Theorem 5.2.2 and Theorem 5.1.2, we get

$$E(\mathcal{CCC}(G)) < LE^{+}(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)) = \frac{36}{5} < 8 = E(K_{|V(\mathcal{CCC}(G))|}). \tag{5.3.m}$$

So, by (5.3.k) and (5.3.m), $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m \geq 6$ then, by Theorem 5.1.2, we get

$$LE(\mathcal{CCC}(G)) = \frac{6(m-1)(m-2)}{m+1}.$$

We have

$$2m - \frac{6(m-1)(m-2)}{m+1} = -\frac{4(m^2-5m+3)}{m+1} = -\frac{4(m(m-6)+m+3)}{m+1} < 0$$

and so

$$LE(K_{|V(\mathcal{CCC}(G))|}) = 2m < \frac{6(m-1)(m-2)}{m+1} = LE(\mathcal{CCC}(G)).$$

Hence, $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic.

By Theorem 5.2.2 and Theorem 5.1.2, we also get

$$E(\mathcal{CCC}(G)) < LE^{+}(\mathcal{CCC}(G)) = \frac{4(m-1)(m-2)}{m+1}.$$

We have

$$\frac{4(m-1)(m-2)}{m+1} - 2m = \frac{2(m^2-7m+4)}{m+1} = \frac{2(m(m-8)+m+4)}{m+1} = f(m). \tag{5.3.n}$$

Therefore, for $m = 6$, we have $f(m) < 0$ and so

$$E(\mathcal{CCC}(G)) < LE^{+}(\mathcal{CCC}(G)) = \frac{4(m-1)(m-2)}{m+1} < 2m = LE^{+}(K_{|V(\mathcal{CCC}(G))|}).$$

Thus, if $m = 6$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $n \geq 8$ then, by (5.3.n), we have $f(m) > 0$ and so

$$LE^+(\mathcal{CCC}(G)) > 2m = LE^+(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is Q-hyperenergetic but not Q-borderenergetic. Also, $E(\mathcal{CCC}(G)) = 2m - 4 < 2m = E(K_{|V(\mathcal{CCC}(G))|})$ and so $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic. Thus, if $n \geq 8$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic. □

Theorem 5.3.3. *Let* $G = U_{(n,m)}$.

(a) *If* $m = 3, 4$ *and* $n \geq 2$ *or* $m = 6$ *and* $n = 2$ *then* $\mathcal{CCC}(G)$ *is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(b) *If* $m = 5$ *and* $n = 2$ *then* $\mathcal{CCC}(G)$ *is L-borderenergetic but neither hyperenergetic, borderenergetic, L-hyperenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(c) *If* $m = 5$ *and* $n = 3$, $m = 6$ *and* $n = 3$ *or* $m = 7$ *and* $n = 2$ *then* $\mathcal{CCC}(G)$ *is L-hyper-energetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(d) *If* $m = 5, 6$ *and* $n \geq 4$; $m = 7$ *and* $n \geq 3$ *or* $m \geq 8$ *and* $n \geq 2$ *then* $\mathcal{CCC}(G)$ *is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-border-energetic nor Q-borderenergetic.*

Proof. We shall prove the result by considering the following cases.

Case 1. m is odd and $n \geq 2$.

We have $|V(\mathcal{CCC}(G))| = \frac{n(m+1)}{2}$. Using (1.1.$g$), we get

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^+(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = mn + n - 2. \tag{5.3.o}$$

By Theorem 5.1.3 we get

$$E(\mathcal{CCC}(G)) = mn + n - 4 < mn + n - 2.$$

Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic.

If $m = 3$ and $n \geq 2$ then, by Theorem 5.1.3, we get

$$LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 4n - 4 < 4n - 2 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic. Thus, if $m = 3$ and $n \geq 2$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m = 5$ and $n = 2$ then, by Theorem 5.2.3 and Theorem 5.1.3, we get

$$LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)) = 10 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is L-borderenergetic but neither L-hyperenergetic, Q-hyperenergetic nor Q-borderenergetic. Thus, if $m = 5$ and $n = 2$ then $\mathcal{CCC}(G)$ is L-borderenergetic but neither hyperenergetic, borderenergetic, L-hyperenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m = 5$ and $n = 3$ then, by Theorem 5.1.3, we get

$$LE(\mathcal{CCC}(G)) = 20 > 16 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic. Also,

$$LE^+(\mathcal{CCC}(G)) = 12 < 16 = LE^+(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither Q-hyperenergetic nor Q-borderenergetic. Thus, if $m = 5$ and $n = 3$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m = 5$ and $n \geq 4$ then, by Theorem 5.2.3 and Theorem 5.1.3, we get

$$\frac{2(2n+3)(n-1)}{3} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

We have

$$6n - 2 - \frac{2(2n+3)(n-1)}{3} = -\frac{4(n^2-n-1)}{3} = -\frac{4(n(n-4)+3n-1)}{3} < 0$$

and so $LE^+(K_{|V(\mathcal{CCC}(G))|}) = 6n - 2 < \frac{2(2n+3)(n-1)}{3} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$. Therefore, $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither L-borderenergetic nor Q-border-energetic. Thus, if $m = 5$ and $n \geq 4$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.

If $m \geq 7$ and $n \geq 2$ then, by Theorem 5.1.3, we get

$$LE(\mathcal{CCC}(G)) = \frac{m^2n^2 - 4mn^2 + m^2n + 3n^2 - 2mn - 2m + 5n - 2}{m+1}.$$

We have

$$\begin{aligned} mn + n - 2 - LE(\mathcal{CCC}(G)) &= -\frac{m^2n^2 - 4mn^2 - 4mn + 3n^2 + 4n}{m+1} \\ &= -\frac{mn^2(m-7) + 2mn(n-2) + mn^2 + 3n^2 + 4n}{m+1} < 0 \end{aligned}$$

and so $LE(K_{|V(\mathcal{CCC}(G))|}) = mn+n-2 < LE(\mathcal{CCC}(G))$. Therefore, $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic. By Theorem 5.2.3 and Theorem 5.1.3, we also get

$$\frac{n^2(m-1)(m-3)}{m+1} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

Let $f_1(m,n) = \frac{n^2(m-1)(m-3)}{m+1} - (mn+n-2)$. Then

$$\begin{aligned} f_1(m,n) &= \frac{2+2m-2mn-m^2n-n+3n^2-4mn^2+m^2n^2}{m+1} \\ &= \frac{mn^2(m-11)+mn^2+m^2n(n-2)+2mn(n-2)+2n(3n-1)+2(m+1)}{2(m+1)}. \end{aligned}$$

For $m = 7$ and $n = 2$ we have $f_1(m,n) = -2 < 0$ and so

$$LE^+(\mathcal{CCC}(G)) = \frac{n^2(m-1)(m-3)}{m+1} < mn+n-2 = LE^+(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither Q-hyperenergetic nor Q-borderenergetic. Thus, if $m = 7$ and $n = 2$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m = 7$ and $n \geq 3$ then $f_1(m,n) = \frac{n(3n-8)+16}{8} > 0$. Therefore,

$$LE^+(K_{|V(\mathcal{CCC}(G))|}) = mn+n-2 < \frac{n^2(m-1)(m-3)}{m+1} = LE^+(\mathcal{CCC}(G))$$

and so $\mathcal{CCC}(G)$ is Q-hyperenergetic but not Q-borderenergetic. Thus, if $m = 7$ and $n \geq 3$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.

Now, for $m = 9$ and $n = 2$ we have $f_1(m,n) = \frac{6}{5} > 0$. For $m = 9$ and $n \geq 3$ we have $f_1(m,n) = \frac{2n(12n-25)+10}{5} > 0$. For $m \geq 11$ and $n \geq 2$ we have $f_1(m,n) > 0$. Therefore, for $m \geq 9$ and $n \geq 2$ we have

$$LE^+(K_{|V(\mathcal{CCC}(G))|}) = mn+n-2 < \frac{n^2(m-1)(m-3)}{m+1} = LE^+(\mathcal{CCC}(G))$$

and so $\mathcal{CCC}(G)$ is Q-hyperenergetic but not Q-borderenergetic. Thus, if $m \geq 9$ and $n \geq 2$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.

Case 2. m is even and $n \geq 2$.

We have $|V(\mathcal{CCC}(G))| = \frac{n(m+2)}{2}$. Using (1.1.$g$), we get

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^+(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = mn+2n-2. \qquad (5.3.p)$$

By Theorem 5.1.3 we get

$$E(\mathcal{CCC}(G)) = mn + 2n - 6 < mn + 2n - 2 = E(K_{|V(\mathcal{CCC}(G))|}),$$

if $m \geq 4$. Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic.

If $m = 4$ and $n \geq 2$ then, by Theorem 5.1.3, we get

$$LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 6n - 6 < 6n - 2.$$

Therefore, $\mathcal{CCC}(G)$ is neither L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic. Thus, if $m = 4$ and $n \geq 2$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m = 6$ and $n = 2$ then, Theorem 5.2.3 and Theorem 5.1.3, we get

$$LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)) = 12 < 14 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic. Thus, if $m = 6$ and $n = 2$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $m = 6$ and $n \geq 3$ then, by Theorem 5.1.3, we get

$$LE(\mathcal{CCC}(G)) = 2n^2 + 3n - 2.$$

We have

$$8n - 2 - (2n^2 + 3n - 2) = -n(2n - 5) < 0.$$

Therefore,

$$LE(K_{|V(\mathcal{CCC}(G))|}) = 8n - 2 < 2n^2 + 3n - 2 < LE(\mathcal{CCC}(G))$$

and so $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic. By Theorem 5.1.3, we also get

$$LE^+(\mathcal{CCC}(G)) = 2(n + 2)(n - 1).$$

Let $g(n) = 2(n+2)(n-1) - (8n-2)$. Then $g(n) = 2(n(n-4) + n - 1)$. Therefore, if $n = 3$ then $g(n) = -2 < 0$ and so

$$LE^+(\mathcal{CCC}(G)) = 2(n + 2)(n - 1) < 8n - 2 = LE^+(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither Q-hyperenergetic nor Q-borderenergetic. Thus, if $m = 6$ and $n = 3$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $n \geq 4$ then $g(n) > 0$ and so

$$LE^{+}(K_{|V(\mathcal{CCC}(G))|}) = 8n - 2 < 2(n+2)(n-1) = LE^{+}(\mathcal{CCC}(G)).$$

Therefore, $\mathcal{CCC}(G)$ is Q-hyperenergetic but not Q-borderenergetic. Thus, if $m = 6$ and $n \geq 4$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.

If $m \geq 8$ and $n \geq 2$ then, by Theorem 5.2.3 and Theorem 5.1.3, we get

$$\frac{2n^2(m-2)(m-4)}{m+2} = LE^{+}(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

We have

$$\begin{aligned} mn + 2n - 2 - \frac{2n^2(m-2)(m-4)}{m+2} &= -\frac{4 + 2m - m^2n - 4n - 4mn + 16n^2 - 12mn^2 + 2m^2n^2}{m+2} \\ &= -f_2(m,n), \end{aligned}$$

where $f_2(m,n) = \frac{mn^2(m-12) + m^2n(n-2) + mn(m-6) + 2n(m-2) + 16n^2 + 2m + 4}{m+2}$.

For $m = 8$ and $n = 2$ we have $f_2(m,n) = \frac{6}{5} > 0$. For $m = 8$ and $n \geq 3$ we have $f_2(m,n) = \frac{2}{5}(12n^2 - 25n + 5) = \frac{2}{5}(12n(n-3) + 11n + 5) > 0$. For $m = 10$ and $n \geq 2$ we have $f_2(m,n) = 2(4n^2 - 6n + 1) = 2(4n(n-2) + 2n + 1) > 0$. For $m \geq 12$ and $n \geq 2$ we have $f_2(m,n) > 0$. Therefore,

$$LE^{+}(K_{|V(\mathcal{CCC}(G))|}) = mn + 2n - 2 < \frac{2n^2(m-2)(m-4)}{m+2} = LE^{+}(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$$

and so $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither L-borderenergetic nor Q-borderenergetic. Thus, if $m \geq 8$ and $n \geq 2$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic. □

Theorem 5.3.4. *Let $G = V_{8n}$.*

(a) *If $n = 2$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-border-energetic, Q-hyperenergetic nor Q-borderenergetic.*

(b) *If $n = 3, 4$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(c) *If $n \geq 5$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.*

Proof. We shall prove the result by considering the following cases.

Case 1. n is odd.

We have $|V(\mathcal{CCC}(G))| = 2n+1$. Using (1.1.$g$), we get

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^+(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = 4n. \tag{5.3.q}$$

By Theorem 5.1.4 we get

$$LE(\mathcal{CCC}(G)) = \frac{6(2n-1)(2n-2)}{2n+1}.$$

We have

$$4n - \frac{6(2n-1)(2n-2)}{2n+1} = -\frac{4(4n^2-10n+3)}{2n+1} = -\frac{4(4n(n-3)+2n+3)}{2n+1} < 0$$

and so $LE(K_{|V(\mathcal{CCC}(G))|}) = 4n < \frac{6(2n-1)(2n-2)}{2n+1} = LE(\mathcal{CCC}(G))$. Hence, $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic.

By Theorem 5.2.4 and Theorem 5.1.4, we also get

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = \frac{4(2n-1)(2n-2)}{2n+1}.$$

We have

$$\frac{4(2n-1)(2n-2)}{2n+1} - 4n = \frac{4(2n^2-7n+2)}{2n+1} = \frac{4(2n(n-5)+3n+2)}{2n+1} := g_1(n). \tag{5.3.r}$$

Therefore, for $n = 3$, we have $g_1(n) < 0$ and so

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = \frac{4(2n-1)(2n-2)}{2n+1} < 4n = LE^+(K_{|V(\mathcal{CCC}(G))|}).$$

Thus, if $n = 3$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic. If $n \geq 5$ then, by (5.3.r), we have $g_1(n) > 0$ and so $LE^+(\mathcal{CCC}(G)) > 4n = LE^+(K_{|V(\mathcal{CCC}(G))|})$. Therefore, $\mathcal{CCC}(G)$ is Q-hyperenergetic but not Q-borderenergetic. Also, $E(\mathcal{CCC}(G)) = 4n-4 < 4n = E(K_{|V(\mathcal{CCC}(G))|})$ and so $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic. Thus, if $n \geq 5$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.

Case 2. n is even.

We have $|V(\mathcal{CCC}(G))| = 2n+2$. Using (1.1.$g$), we get

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^+(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = 4n+2. \tag{5.3.s}$$

If $n = 2$ then, by Theorem 5.2.4 and Theorem 5.1.4, we get

$$E(\mathcal{CCC}(G)) = LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 6 < 10 = E(K_{|V(\mathcal{CCC}(G))|}). \tag{5.3.t}$$

Therefore, by (5.3.s) and (5.3.t), we have $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $n \geq 4$ then, Theorem 5.1.4, we get

$$E(\mathcal{CCC}(G)) = 4n - 2 < 4n + 2 = E(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic.

By Theorem 5.2.4 and Theorem 5.1.4, we also get

$$\frac{16(n-1)(n-2)}{n+1} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

We have

$$\frac{16(n-1)(n-2)}{n+1} - (4n+2) = \frac{6(2n^2 - 9n + 5)}{n+1} = \frac{6(2n(n-6) + 3n + 5)}{n+1} := g_2(n). \tag{5.3.u}$$

Therefore, for $n = 4$ we have $g_2(n) < 0$ and so

$$LE^+(\mathcal{CCC}(G)) = \frac{16(n-1)(n-2)}{n+1} < 4n + 2 = LE^+(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither Q-hyperenergetic nor Q-borderenergetic. Also,

$$LE(\mathcal{CCC}(G)) = \frac{130}{5} = 26 > 18 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic. Thus, if $n = 4$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.

If $n \geq 6$ then, by (5.3.u), we have $g_2(n) > 0$ and so

$$LE^+(K_{|V(\mathcal{CCC}(G))|}) = 4n + 2 < \frac{16(n-1)(n-2)}{n+1} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

Therefore, $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither L-borderenergetic nor Q-borderenergetic. Thus, if $n \geq 6$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic. □

Theorem 5.3.5. *Let $G = SD_{8n}$.*

(a) *If $n = 2, 3$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(b) *If $n = 5$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-borderenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-hyperenergetic.*

(c) *If $n = 4$ or $n \geq 6$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.*

Proof. We shall prove the result by considering the following cases.

Case 1. n is odd.

We have $|V(\mathcal{CCC}(G))| = 2n + 2$. Using (1.1.$g$), we get

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^+(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = 4n + 2. \tag{5.3.v}$$

By Theorem 5.1.5 we get

$$E(\mathcal{CCC}(G)) = 4n < 4n + 2.$$

Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic.

If $n = 3$ then, by Theorem 5.1.5, we get

$$LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = 12 < 14 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic. Thus, if $n = 3$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-border-energetic, Q-hyperenergetic nor Q-borderenergetic.

If $n = 5$ then, by Theorem 5.2.5 and Theorem 5.1.5, we get

$$LE^+(K_{|V(\mathcal{CCC}(G))|}) = 22 = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

Therefore, $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-borderenergetic but neither L-borderenergetic nor Q-hyperenergetic. Thus, if $n = 5$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-borderenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-hyperenergetic.

If $n \geq 7$ then, by Theorem 5.2.5 and Theorem 5.1.5, we get

$$\frac{16(n-1)(n-3)}{n+1} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

We have

$$4n + 2 - \frac{16(n-1)(n-3)}{n+1} = -\frac{2(6n^2 - 35n + 23)}{n+1} = -\frac{2(6n(n-7) + 7n + 23)}{n+1} < 0.$$

So, $LE^+(K_{|V(\mathcal{CCC}(G))|}) = 4n+2 < \frac{16(n-1)(n-3)}{n+1} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$ and so $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither L-borderenergetic nor Q-borderenergetic. Thus, if $n \geq 7$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.

Case 2. n is even.

We have $|V(\mathcal{CCC}(G))| = 2n+1$. Using (1.1.$g$), we get

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^+(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = 4n. \tag{5.3.w}$$

By Theorem 5.1.5 we get

$$E(\mathcal{CCC}(G)) = 4n - 4 < 4n.$$

Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic.

If $n = 2$ then, by Theorem 5.2.5 and Theorem 5.1.5, we get

$$LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)) = \frac{36}{5} < 8 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic. Thus, if $n = 2$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-border-energetic, Q-hyperenergetic nor Q-bordere-nergetic.

If $n \geq 4$ then, by Theorem 5.2.5 and Theorem 5.1.5, we get

$$\frac{4(2n-1)(2n-2)}{2n+1} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)).$$

We have

$$4n - \frac{4(2n-1)(2n-2)}{2n+1} = -\frac{4(2n^2-7n+2)}{2n+1} = -\frac{4(2n(n-4)+n+2)}{2n+1} < 0.$$

Therefore, $LE^+(K_{|V(\mathcal{CCC}(G))|}) = 4n < \frac{4(2n-1)(2n-2)}{2n+1} = LE^+(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G))$ and so $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither L-borderenergetic nor Q-borderenergetic. Thus, if $n \geq 4$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic. □

Theorem 5.3.6. *Let $G = G(p, m, n)$.*

(a) *If $n = 1$, $p \geq 2$, $m \geq 1$; $n = 2$, $p = 2$, $m = 1, 2$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic, nor Q-borderenergetic.*

(b) *If $n = 2$, $p = 2$, $m = 3$; or $n = 3$, $p = 2$, $m = 1$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic.*

(c) *If $n = 2$, $p = 2$, $m \geq 4$; $n = 2, p \geq 3, m \geq 1$; $n = 3, p = 2, m \geq 2$; or $n \geq 4, p \geq 2, m \geq 1$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.*

Proof. We have $|V(\mathcal{CCC}(G))| = p^{m+n-2}(p^2 - 1)$ and so

$$E(K_{|V(\mathcal{CCC}(G))|}) = LE^+(K_{|V(\mathcal{CCC}(G))|}) = LE(K_{|V(\mathcal{CCC}(G))|}) = 2(p^{m+n} - p^{m+n-2} - 1),$$

noting that $E(K_n) = LE(K_n) = LE^+(K_n) = 2(n-1)$.

We shall prove the result by considering the following cases.

Case 1. $n = 1$, $p \geq 2$ and $m \geq 1$.

If $n = 1, p \geq 2$ and $m \geq 1$ then, by Theorem 5.2.6, we get

$$LE^+(\mathcal{CCC}(G)) = E(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)).$$

By Theorem 5.2.6, we also have

$$\begin{aligned} LE(\mathcal{CCC}(G)) - 2(p^{m+n} - p^{m+n-2} - 1) \\ &= \frac{2(p^{n+1} - p^n + 2p)(2p^{m+n+1} - 2p^{m+n} + p^{m+3} - 2p^{m+2} + p^{m+1} - p^3 - p^2)}{p^3(1+p)} \\ &\quad - 2(p^{m+n} - p^{m+n-2} - 1) \\ &= -2p < 0. \end{aligned}$$

Therefore, $LE(\mathcal{CCC}(G)) < LE(K_{|V(\mathcal{CCC}(G))|})$ and so $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic, nor Q-borderenergetic. Thus, if $n = 1$, $p \geq 2$ and $m \geq 1$ then $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic, nor Q-borderenergetic.

Case 2. $n = 2, p = 2$ and $m \geq 1$.

If $n = 2$, $p = 2$ and $m = 1$ then, by Theorem 5.2.6 and Theorem 5.1.6, we get

$$E(\mathcal{CCC}(G)) < LE^+(\mathcal{CCC}(G)) = LE(\mathcal{CCC}(G)) = \frac{16}{3} < 10 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic, nor Q-borderenergetic.

If $n = 2$, $p = 2$ and $m = 2$ then, by Theorem 5.2.6 and Theorem 5.1.6, we get

$$LE^+(\mathcal{CCC}(G)) < E(\mathcal{CCC}(G)) < LE(\mathcal{CCC}(G)) = 20 < 22 = LE(K_{|V(\mathcal{CCC}(G))|}).$$

Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic, nor Q-borderenergetic.

If $n = 2$, $p = 2$ and $m \geq 3$ then, by Theorem 5.1.6, we get $E(\mathcal{CCC}(G)) = 2(3.2^m - 4)$ and $E(K_{|V(\mathcal{CCC}(G))|}) = 2(3.2^m - 1)$ and so $E(\mathcal{CCC}(G)) - E(K_{|V(\mathcal{CCC}(G))|}) = 2(3.2^m - 4) - 6.2^m + 2 = -6 < 0$. Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic.

By Theorem 5.1.6, we also get $LE(\mathcal{CCC}(G)) = \frac{2}{3}(4^m + 5.2^m - 6)$ for $n = 2$, $p = 2$ and $m \geq 3$. Therefore, $LE(\mathcal{CCC}(G)) - (3.2^m - 1) = \frac{2}{3}(4^m - 2^{m+2} - 3) > 0$ and so $LE(\mathcal{CCC}(G)) > LE(K_{|V(\mathcal{CCC}(G))|})$. Hence, $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic for $n = 2$, $p = 2$ and $m \geq 3$.

If $n = 2$, $p = 2$ and $m \geq 3$ then, by Theorem 5.1.6, we get $LE^+(\mathcal{CCC}(G)) = \frac{2}{3}(2^m - 2)(2^m + 3)$. Therefore, $LE^+(\mathcal{CCC}(G)) - (3.2^m - 1) = \frac{2}{3}(2^m(2^m - 8) - 3)$ and so $LE^+(\mathcal{CCC}(G)) < LE^+(K_{|V(\mathcal{CCC}(G))|})$ or $LE^+(\mathcal{CCC}(G)) > LE^+(K_{|V(\mathcal{CCC}(G))|})$ according as $m = 3$ or $m \geq 4$. Hence, $\mathcal{CCC}(G)$ is neither Q-hyperenergetic nor Q-borderenergetic if $m = 3$; and if $m \geq 4$ then $\mathcal{CCC}(G)$ is Q-hyperenergetic but not Q-borderenergetic. Thus, if $n = 2$, $p = 2$ and $m = 3$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic; and if $n = 2$, $p = 2$ and $m \geq 4$ then $\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic.

Case 3. $n = 2$, $p \geq 3$, $m \geq 1$; or $n \geq 3$, $p \geq 2$, $m \geq 1$.

By Theorem 5.1.6, we get

$$E(\mathcal{CCC}(G)) = 2(p^{m+n} - p^{m+n-2} - p^n + p^{n-1} - 2),$$

$$LE^+(\mathcal{CCC}(G)) = \frac{4p^{2m+n-4}}{p+1}(p-1)^3(p^n - p)$$

and

$$\begin{aligned} LE(\mathcal{CCC}(G)) = \frac{4}{p^4(p+1)}\big(&p^{2m+2n+3} - 3p^{2(m+n+1)} + 3p^{2m+2n+1} - p^{2(m+n)} - p^{2m+n+4} \\ &+ 3p^{2m+n+3} - 3p^{2m+n+2} + p^{2m+n+1} + 2p^{m+n+3} - 2p^{m+n+2} \\ &+ p^{m+5} - 2p^{m+4} + p^{m+3} - p^5 - p^4\big). \end{aligned}$$

We have

$$\begin{aligned} E(\mathcal{CCC}(G)) - 2(p^{m+n} - p^{m+n-2} - 1) &= 2(p^{m+n} - p^{m+n-2} - p^n + p^{n-1} - 2) \\ &\quad - 2(p^{m+n} - p^{m+n-2} - 1) \\ &= -2p^n + 2p^{n-1} - 2 < 0. \end{aligned}$$

Therefore, $\mathcal{CCC}(G)$ is neither hyperenergetic nor borderenergetic. Also,

$$\begin{aligned} &LE^+(\mathcal{CCC}(G)) - 2(p^{m+n} - p^{m+n-2} - 1) \\ &\quad = \frac{4p^{2m+n-4}}{p+1}(p-1)^3(p^n - p) - 2(p^{m+n} - p^{m+n-2} - 1) \\ &\quad = \frac{2}{p^4(p+1)}(2p^{2m+2n+3} - 6p^{2m+2n+2} + 6p^{2m+2n+1} - 2p^{2m+2n} - 2p^{2m+n+4} + 6p^{2m+n+3} \\ &\qquad - 6p^{2m+n+2} + 2p^{2m+n+1} - p^{m+n+5} - p^{m+n+4} + p^{m+n+3} + p^{m+n+2} + p^5 + p^4) \\ &\quad := f_1(p, m, n). \end{aligned}$$

Now, for $p = 2$, $n \geq 3$ and $m \geq 2$, we have

$$f_1(2, m, n) = \frac{1}{12}\left(2^{m+n}(2^m(2^n - 2) - 18) + 24\right) > 0.$$

For $p = 2, n = 3, m = 1$ we have $f_{23}(2, 1, 3) = -6 < 0$. For $n = 2, p = 3$ we have

$$f_1(3, m, 2) = 4^{m+1} - 3.2^{m+2} + 2 > 0,$$

if $m \geq 2$. For $n = 2, p = 3$ and $m = 1$ we have $f_1(3, 1, 2) = 2 > 0$. For $n = 2$ we have

$$\begin{aligned} f_1(p, m, 2) &= \frac{2}{p(p+1)}(2p^{2m+4} - 8p^{2m+3} + 12p^{2m+2} - 8p^{2m+1} + 2p^{2m} - p^{m+4} - p^{m+3} \\ &\quad + p^{m+2} + p^{m+1} + p^2 + p) \\ &= \frac{2}{p(p+1)}(2p^{2m+3}(p-5) + (p^{2m+3} - p^{m+4}) + (p^{2m+3} - p^{m+3}) \\ &\quad + (12p^{2m+2} - 8p^{2m+1}) + 2p^{2m} + p^{m+2} + p^{m+1} + p^2 + p\big) > 0, \end{aligned}$$

if $p \geq 5, m \geq 1$. For $n \geq 3, p \geq 3, m \geq 1$ we have

$$\begin{aligned} f_1(p,m,n) &= \frac{2}{p^4(p+1)}(2p^{2m+2n+3} - 6p^{2m+2n+2} + 6p^{2m+2n+1} - 2p^{2m+2n} - 2p^{2m+n+4} \\ &\quad + 6p^{2m+n+3} - 6p^{2m+n+2} + 2p^{2m+n+1} - p^{m+n+5} - p^{m+n+4} + p^{m+n+3} \\ &\quad + p^{m+n+2} + p^5 + p^4) \\ &= \frac{2}{p^4(p+1)}((2p^{2m+2n+3} - 6p^{2(m+n+1)}) + (2p^{2m+2n+1} - 2p^{2m+n+4}) + (6p^{2m+n+3} \\ &\quad - 6p^{2m+n+2}) + 2p^{2m+n+1} + (p^{2m+2n+1} - p^{m+n+5}) + (2p^{2m+2n+1} - p^{m+n+4}) \\ &\quad + (p^{2m+2n+1} - 2p^{2(m+n)}) + p^{m+n+3} + p^{m+n+2} + p^5 + p^4\big) > 0. \end{aligned}$$

Therefore, $\mathcal{CCC}(G)$ is neither Q-hyperenergetic nor Q-borderenergetic for $n = 3, p = 2, m = 1$. For $n = 2, p \geq 3, m \geq 1$; $n = 3, p = 2, m \geq 2$; $n \geq 4, p \geq 2, m \geq 1$; $\mathcal{CCC}(G)$ is Q-hyperenergetic but not Q-borderenergetic.

We have

$$\begin{aligned} &LE(\mathcal{CCC}(G)) - 2(p^{m+n} - p^{m+n-2} - 1) \\ &\quad = \frac{4}{p^4(p+1)}\big(p^{2m+2n+3} - 3p^{2(m+n+1)} + 3p^{2m+2n+1} - p^{2(m+n)} - p^{2m+n+4} + 3p^{2m+n+3} \\ &\quad - 3p^{2m+n+2} + p^{2m+n+1} + 2p^{m+n+3} - 2p^{m+n+2} + p^{m+5} - 2p^{m+4} \\ &\quad + p^{m+3} - p^5 - p^4\big) - 2(p^{m+n} - p^{m+n-2} - 1) \\ &\quad = \frac{2}{p^4(p+1)}(2p^{2m+2n+3} - 6p^{2(m+n+1)} + 6p^{2m+2n+1} - 2p^{2(m+n)} - 2p^{2m+n+4} + 6p^{2m+n+3} \\ &\quad - 6p^{2m+n+2} + 2p^{2m+n+1} - p^{m+n+5} - p^{m+n+4} + 5p^{m+n+3} - 3p^{m+n+2} + 2p^{m+5} \\ &\quad - 4p^{m+4} + 2p^{m+3} - p^5 - p^4) := f_2(p,m,n). \end{aligned}$$

For $n = 2$, $p \geq 3$ and $m \geq 1$, we have

$$\begin{aligned} f_2(p,m,n) &= \frac{2}{p(p+1)}(2p^{2m+4} - 8p^{2m+3} + 12p^{2m+2} - 8p^{2m+1} + 2p^{2m} - p^{m+4} - p^{m+3} \\ &\quad + 7p^{m+2} - 7p^{m+1} + 2p^m - p^2 - p). \end{aligned}$$

Therefore, for $n = 2$, $p \geq 3$ and $m = 1$ we have $f_2(p,m,n) = \frac{2}{p+1}(2p^5 - 9p^4 + 11p^3 - p^2 - 6p + 1) = \frac{2}{p+1}(p^3(p-3)^2 + p^4(p-3) + 3p^2(p-1) + 2p(p-3) + 1) > 0$. For $n = 2$, $p \geq 3$ and $m \geq 2$ we have

$$\begin{aligned} f_2(p,m,n) &= \frac{2}{p(p+1)}\big((2p^{2m+2}(p-2)^2 - p^{m+3}) + p^{2m+1}(3p-8) + p^{m+2}(p^m - p^2) \\ &\quad + (2p^{2m} - p^2) + 7p^{m+1}(p-1) + (2p^m - p)\big) > 0. \end{aligned}$$

Therefore, if $n = 2$, $p \geq 3$ and $m \geq 1$ then $LE(\mathcal{CCC}(G)) > LE(K_{|V(\mathcal{CCC}(G))|})$ and so $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic.

For $n \geq 3$, $p = 2$ and $m \geq 1$ we have

$$\begin{aligned} f_2(p,m,n) &= \frac{1}{12}(2^{2(m+n)} - 2^{2m+n+1} - 5.2^{m+n+1} + 2^{m+3} - 24) \\ &\geq \frac{1}{12}(4.2^{2m+n+1} - 2^{2m+n+1} - 5.2^{m+n+1} + 2^{m+3} - 24) \\ &= \frac{1}{12}\big((2^{m+n+1}(3.2^m - 5) - 24) + 2^{m+3}\big) > 0. \end{aligned}$$

Also, for $n \geq 3$, $p \geq 3$ and $m \geq 1$ we have

$$\begin{aligned} f_2(p,m,n) = \frac{4}{p^4(p+1)}&(2p^{2m+2n+2}(p-3) + p^{m+n+1}(p^{m+n} - p^4) + p^{m+n+1}(p^{m+n} - p^3) \\ &+ 2p^{2m+2n}(p-1) + 6p^{2m+n+2}(p-1) + 2p^{2m+n+1}(p^n - p^3) + p^{m+n+1}(2p^m \\ &- 3p) + 2p^{m+4}(p-2) + p^4(2p^{m-1} - 1) + p^3(5p^{m+n} - p^2)\big) > 0. \end{aligned}$$

Therefore, if $n \geq 3$, $p \geq 2$ and $m \geq 1$ then $LE(\mathcal{CCC}(G)) > LE(K_{|V(\mathcal{CCC}(G))|})$ and so $\mathcal{CCC}(G)$ is L-hyperenergetic but not L-borderenergetic. Thus, if $n = 2$, $p \geq 3$ and $m \geq 1$ or $n \geq 3$, $p \geq 2$ and $m \geq 1$ then $\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic. □

We conclude this chapter with the following characterization of commuting conjugacy class graph.

Theorem 5.3.7. *Let G be a finite non-abelian group. Then*

(a) *$\mathcal{CCC}(G)$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic if G is isomorphic to D_8, D_{12}, D_{2n} (n is odd), Q_8, Q_{12}, Q_{16}, $U_{(2,6)}$, $U_{(n,3)}$, $U_{(n,4)}$ ($n \geq 2$), V_{16}, SD_{16}, SD_{24}, $G(p,m,1)$ ($p \geq 2$ and $m \geq 1$), $G(2,1,2)$ or $G(2,2,2)$.*

(b) *$\mathcal{CCC}(G)$ is L-borderenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic if G is isomorphic to Q_{20} or $U_{(2,5)}$.*

(c) *$\mathcal{CCC}(G)$ is L-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic if G is isomorphic to D_{16}, D_{20}, D_{24}, D_{28}, Q_{24}, Q_{28}, $U_{(3,5)}$, $U_{(3,6)}$, $U_{(2,7)}$, V_{24}, V_{32}, $G(2,3,2)$ or $G(2,1,3)$.*

(d) *$\mathcal{CCC}(G)$ is L-hyperenergetic and Q-borderenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-hyperenergetic if G is isomorphic to SD_{40}.*

(e) *$\mathcal{CCC}(G)$ is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic if G is isomorphic to* D_{2n} *(n is even,* $n \geq 16$*),* Q_{4m} $(m \geq 8)$, $U_{(n,5)}$ $(n \geq 4)$, $U_{(n,6)}$ $(n \geq 4)$, $U_{(n,7)}$ $(n \geq 3)$, $U_{(n,m)}$ $(n \geq 2$ *and* $m \geq 8)$, V_{8n} $(n \geq 5)$, SD_{32}, SD_{8n} $(n \geq 6)$, $G(2,m,2)$ $(m \geq 4)$, $G(p,m,2)$ $(p \geq 3$ *and* $m \geq 1)$, $G(2,m,3)$ $(m \geq 2)$ *or* $G(p,m,n)$ $(n \geq 4, p \geq 2$ *and* $m \geq 1)$.

Theorem 5.3.8. *Let G be a finite non-abelian group. Then*

(a) *If G is isomorphic to* D_{2n}, Q_{4m}, $U_{(n,m)}$, V_{8n}, SD_{8n} *or* $G(p,m,n)$ *then* $\mathcal{CCC}(G)$ *is neither hyperenergetic nor borderenergetic.*

(b) *If G is isomorphic to* D_{2n} *(n is even,* $n \geq 8$*),* Q_{4m} $(m \geq 6)$, $U_{(n,5)}$ $(n \geq 3)$, $U_{(n,6)}$ $(n \geq 3)$, $U_{(n,m)}$ $(n \geq 2$ *and* $m \geq 7)$, V_{8n} $(n \geq 3)$, SD_{8n} $(n \geq 4)$, $G(2,m,2)$ $(m \geq 3)$, $G(p,m,2)$ $(p \geq 3$ *and* $m \geq 1)$, $G(2,m,3)$ $(m \geq 1)$ *or* $G(p,m,n)$ $(n \geq 4, p \geq 2$ *and* $m \geq 1)$ *then* $\mathcal{CCC}(G)$ *is L-hyperenergetic*

(c) *If G is isomorphic to* Q_{20} *or* $U_{(2,5)}$ *then* $\mathcal{CCC}(G)$ *is L-borderenergetic.*

(d) *If G is isomorphic to* D_{2n} *(n is even,* $n \geq 16$*),* Q_{4m} $(m \geq 8)$, $U_{(n,5)}$ $(n \geq 4)$, $U_{(n,6)}$ $(n \geq 4)$, $U_{(n,7)}$ $(n \geq 3)$, $U_{(n,m)}$ $(n \geq 2$ *and* $m \geq 8)$, V_{8n} $(n \geq 5)$, SD_{32}, SD_{8n} $(n \geq 6)$, $G(2,m,2)$ $(m \geq 4)$, $G(p,m,2)$ $(p \geq 3$ *and* $m \geq 1)$, $G(2,m,3)$ $(m \geq 2)$ *or* $G(p,m,n)$ $(n \geq 4,$ $p \geq 2$ *and* $m \geq 1)$ *then* $\mathcal{CCC}(G)$ *is Q-hyperenergetic.*

(e) *If G is isomorphic to* SD_{40} *then* $\mathcal{CCC}(G)$ *is Q-borderenergetic.*