Chapter 6
Codes with burst distance and periodical burst errors

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Chapter 6

Codes with burst distance and periodical burst errors

This chapter presents a study on the burst-b distance d_b of a linear code and provides periodical burst error detection and correction capability for a linear code equipped with minimum burst-b distance d_b. An (n, k) linear code with minimum burst-b distance d_b is written as (n, k, d_b) linear code. In [79], Villalba et al. extend the study on burst-b distance and derive an extended Reiger-Singleton bound for an (n, k, d_b) linear code as

$$b(d_b - 1) \leq n - k. \tag{6.1}$$

Codes with $b(d_b - 1) = n - k$ are called Maximum Distance Separable (MDS) codes. A class of Maximum Distance Separable (MDS) codes is presented in [79] as below.

Lemma 6.1. *[79] The code generated by the $b \times bd_b$ matrix G_b:*

$$G_b = \begin{array}{cc} & \begin{array}{cccc} \overbrace{\hphantom{1\ 0\ \dots\ 0}}^{b} & \overbrace{\hphantom{1\ 0\ \dots\ 0}}^{b} & \hphantom{\dots\dots} & \overbrace{\hphantom{1\ 0\ \dots\ 0}}^{b} \end{array} \\ b-1\left\{\vphantom{\begin{matrix}1\\0\\\vdots\\0\end{matrix}}\right. & \begin{bmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & \dots\dots & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 & \dots\dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots\ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1 & \dots\dots & 0 & 0 & \dots & 1 \end{bmatrix} \end{array} = \left[\underbrace{I_b\ I_b\ \dots\ I_b}_{d_b\ times}\right],$$

where 1's in each row are separated by exactly $b - 1$ zeros, is a (bd_b, b, d_b) MDS code.

Observe that the code given in Lemma 6.1 is a cyclic code and every codeword has same burst-b weight d_b. So, the code is a constant burst-b weight code. We denote the above (bd_b, b, d_b) code by $\boldsymbol{C_b}$ **code.**

This chapter investigates the periodical burst error detection and correction capability of the code C_b and its dual code along with a decoding procedure of the code C_b. The contents of this chapter are organised as follows. Section 6.1 gives bounds on d_b for a linear code and a constant burst-b weight linear code. Then, we give the connection between an (n, k, d_b) and $(n-b, k, d_b-1)$ codes along with the cardinality of (n, k, d_b) code. We also provide a connection between linearly independent columns of the parity check matrix of any MDS (n, k, d_b) codes. Section 6.2 presents periodical burst error detection and correction capability of a linear (n, k, d_b) code. The same is investigated for the MDS code C_b. Finally, Section 6.3 gives a decoding procedure for the code C_b in the case of periodical burst error.

6.1 Codes with burst-b distance

In this section, we give bounds on d_b for a linear code and a constant burst-b weight linear code. Then, we give the existence of a linear code with minimum distance $d_b - 1$ from a linear code with minimum distance d_b. Further, an upper bound on the cardinality of a linear code with distance d_b is derived. The connection between linearly independent columns of the parity check matrix of any MDS code with burst-b distance is derived.

Theorem 6.2. *Let C be an (n, k, d_b) linear code over $GF(q)$. Then*

$$d_b \leq \frac{nq^{k-1}(q-1)}{q^k - 1}.$$

Proof. From Result 1.12, the total number of nonzero components in all the codewords of C is

$$nq^{k-1}(q-1). \tag{6.2}$$

Again, the code has the minimum burst-b distance as d_b, so the number of nonzero components in each nonzero codeword is at least d_b. Therefore, the total number of nonzero components in all the codewords of C is at least

$$(q^k - 1)d_b.$$

Hence

$$(q^k - 1)d_b \leq nq^{k-1}(q-1).$$

This proves the theorem. $\square$

Remark 6.3. *This upper bound of d_b coincides with Result 1.12 when the distance is taken in Hamming sense for a q-ary (n, k, d) linear code.*

Theorem 6.4. *Let C be a constant burst-b weight (n, k, d_b) linear code over $GF(q)$. Then*

$$d_b \geq \frac{nq^{k-1}(q-1)}{(q^k-1)b}. \tag{6.3}$$

Proof. Since every codeword of the code C has burst-b distance d_b, the number of nonzero components in each nonzero codeword is at most bd_b. Therefore, the total number of nonzero components in all the codewords of C can be at most

$$(q^k - 1)bd_b.$$

But the total number of nonzero components in all the codewords of C is given by (6.2). Hence

$$(q^k - 1)bd_b \geq nq^{k-1}(q-1).$$

This proves the theorem. $\square$

Remark 6.5. *For constant burst-b weight (n, k, d_b) linear code over $GF(q)$, Theorem 6.2–6.4 give*

$$\frac{nq^{k-1}(q-1)}{(q^k-1)b} \leq d_b \leq \frac{nq^{k-1}(q-1)}{q^k-1}. \tag{6.4}$$

Remark 6.6. *For constant burst-b weight code C_b, $k = b$ and $q = 2$. Then*

$$\frac{nq^{k-1}(q-1)}{(q^k-1)b} = \frac{bd_b 2^{b-1}}{(2^b-1)b} \leq d_b. \quad [\because \quad \frac{2^{b-1}}{(2^b-1)} \leq 1]$$

Further, for $b = 1$

$$\frac{nq^{k-1}(q-1)}{(q^k-1)} = \frac{d_1 2^{1-1}}{(2^1-1)} = d_1,$$

and for $b \geq 2$

$$\frac{nq^{k-1}(q-1)}{(q^k-1)} = \frac{bd_b 2^{b-1}(2-1)}{(2^b-1)} = \frac{bd_b 2^{b-1}}{(2^b-1)} \geq d_b. \quad [\because \quad \frac{b2^{b-1}}{(2^b-1)} \geq 1]$$

Thus

$$\frac{nq^{k-1}(q-1)}{(q^k-1)} \geq d_b.$$

Therefore, the bound 6.4 is satisfied by the code C_b.

Theorem 6.7. *If a q-ary (n, k, d_b) linear code C exists with $d_b \geq 2$, there also exists an $(n-b, k, d_b-1)$ linear code C'.*

Proof. There exist codewords x and y of C such that $d_b(x, y) = w_b(x-y) = d_b$. From the d_b sets in which nonzero components of $x-y$ are confined, we choose a set of b consecutive components. Delete the b consecutive components of the set in each codeword of C, the resultant vectors will have burst-b weight at least $d_b - 1(\geq 1)$ with one vector $x-y$ having exactly burst-b weight $d_b - 1$, and they will form a subspace C' of C with q^k elements. This code C' is the required $(n-b, k, d_b-1)$ linear code C'. □

Theorem 6.8. *The cardinality M of a linear code of length n with minimum burst-b distance d_b is bounded above by*

$$M \leq \frac{q^n}{\sum_{i=0}^{(d_b-1)/2} \binom{n-ib+i}{i} (q-1)^i q^{i(b-1)}}.$$

Proof. By [23], the number of $(d_b-1)/2$ bursts of length up to b is given by

$$\frac{q^n}{\sum_{i=0}^{(d_b-1)/2} \binom{n-ib+i}{i} (q-1)^i q^{i(b-1)}}$$

By Result 1.18, the code can correct all $(d_b-1)/2$ bursts of length up to b. As every codeword disturbed by such errors must produce distinct words, we have

$$q^n \geq M \sum_{i=0}^{(d_b-1)/2} \binom{n-ib+i}{i} (q-1)^i q^{i(b-1)}$$

i.e.

$$M \leq \frac{q^n}{\sum_{i=0}^{(d_b-1)/2} \binom{n-ib+i}{i} (q-1)^i q^{i(b-1)}}.$$

$\square$

Theorem 6.9. *Let C be an (n, k, d_b) code over $GF(q)$ and $n-k$ is a multiple of b. Let H be the parity check matrix of C. Then C is an MDS code if and only if every set of $n-k$ columns that are formed from $\frac{n-k}{b}$ sets of b consecutive columns of H are linearly independent.*

Proof. If C is an MDS code, then $b(d_b - 1) = n - k$, which implies $d_b - 1 = \frac{n-k}{b}$, an integer. As every vector of burst-b weight $d_b - 1$ cannot be a codeword, every linear combination of columns of H which consists of $d_b - 1$ sets of b consecutive columns is nonzero. Also, the total number of columns in $d_b - 1$ sets of b consecutive columns is $n-k$, so every set of $n-k$ columns that are formed from $\frac{n-k}{b}$ sets of b consecutive columns of H is linearly independent. Now, if every set of $n-k$ columns that are formed from $\frac{n-k}{b}$ sets of b consecutive columns are linearly independent, we cannot get a codeword whose nonzero components are confined to $\frac{n-k}{b}$ sets of b consecutive columns. The $\frac{n-k}{b}$ sets of b consecutive columns may be within the first $n-k$ columns of H or not. In either case, the burst-b distance of the code C is at least $\frac{n-k}{b} + 1$, i.e.

$$d_b \geq \frac{n-k}{b} + 1$$

$$\implies b(d_b - 1) \geq n - k.$$

Again from Equation (6.1), we have $b(d_b - 1) \leq n - k$. So, $b(d_b - 1) = n - k$ and hence C is an MDS code. $\square$

6.2 Burst-b distance and periodical burst errors

In this section, we present periodical burst detection and correction of an (n, k, d_b) linear code. Then we do the same investigation for code C_b and its dual code $C_b^{\perp}$.

Theorem 6.10. *An (n, k, d_b) code can detect all vectors of $\psi_{(s,b),n,q}$ that start from the $\big(n + 1 - (d_b - 1)(s + b)\big)^{th}$ position.*

Proof. Consider a vector of the error set $\psi_{(s,b),n,q}$ defined in Chapter 4 whose nonzero components start from the j^{th} position ($j = 1, 2, \ldots, n$), so the errors are confined to the last $n - j + 1$ positions.

By Euclidean division algorithm, for integers $n-i+1$ and $s+b$, there exist integers λ_j and r_j such that

$$n - j + 1 = \lambda_j(s + b) + r_j, \text{ where } 0 \leq r_j < s + b. \tag{6.5}$$

So, every vector of $\psi_{(s,b),n,q}$, where the error starts from the j^{th} position, has burst-b weight

$$\begin{cases} \lfloor \frac{n-j+1}{s+b} \rfloor \text{ if } r_j + p \leq b & (p : \text{the last nonzero position of the first nonzero set}) \\ \lceil \frac{n-j+1}{s+b} \rceil & \text{otherwise.} \end{cases}$$

An (n, k, d_b) code can detect error vectors of $\psi_{(s,b),n,q}$ if its burst-b weight is less than or equal to $d_b - 1$. So, if

$$d_b - 1 \geq \lceil \frac{n-j+1}{s+b} \rceil$$

$$i.e., \quad (d_b - 1)(s + b) \geq n - j + 1$$

$$i.e., \quad j \geq n + 1 - (d_b - 1)(s + b),$$

the code detects any error vector of $\psi_{(s,b),n,q}$ that starts from the j^{th} position. Therefore, if the starting position of the error pattern of $\psi_{(s,b),n,q}$ is $n+1-(d_b-1)(s+b)$, the code detects such errors. $\square$

Taking $j = 1$, we have $1 = n+1-(d_b-1)(s+b)$, i.e., $d_b = \dfrac{n}{s+b} + 1$. Therefore, we have the following corollary.

Corollary 6.11. *An (n, k, d_b) code detects all error vectors of $\psi_{(s,b),n,q}$ provided $d_b \geq \lceil \frac{n}{s+b} \rceil + 1$.*

Theorem 6.12. *An (n, k, d_b) code corrects all error vectors of $\psi_{(s,b),n,q}$ that start from the $\left(\lceil n + 1 - \dfrac{(d_b-1)}{2}(s + b)\rceil\right)^{th}$ position.*

Proof. By Result 1.18, an (n, k, d_b) code can correct up to $(d_b - 1)/2$ bursts of length up to b each. Since every error vector of $\psi_{(s,b),n,q}$ that starts from j^{th} position ($j = 1, 2, \ldots, n$) has burst-b weight $\lceil \frac{n-j+1}{s+b} \rceil$ or less, an (n, k, d_b) code can correct an error vector of $\psi_{(s,b),n,q}$ if its burst-b weight is less than or equal to $(d_b - 1)/2$. Now

$$\frac{d_b - 1}{2} \geq \lceil \frac{n-j+1}{s+b} \rceil$$

$$\begin{aligned} & \textit{implies} \quad \frac{(d_b-1)}{2}(s+b) \geq n-j+1 \\ & \textit{implies} \quad j \geq n+1-\frac{(d_b-1)}{2}(s+b), \end{aligned}$$

Therefore, if the starting position of an error vector of $\psi_{(s,b),n,q}$ is $\lceil n+1-\frac{(d_b-1)}{2}(s+b)\rceil$, the code corrects such errors. $\square$

Again taking $j=1$, we have $1=\lceil n+1-\frac{(d_b-1)}{2}(s+b)\rceil$, i.e., $d_b \geq 2\lceil \frac{n}{s+b}\rceil+1$. This gives the following corollary:

Corollary 6.13. *An (n,k,d_b) code can correct vectors of $\psi_{(s,b),n,q}$ provided $d_b \geq 2\lceil \frac{n}{s+b}\rceil+1$.*

Theorem 6.14. *The code C_b $(d_b \geq 2)$ detects all*
(i) vectors of the error set $\psi_{(s,b),n,q}$ for any $s \geq b$, and
(ii) vectors of the error set $\psi_{(s,b),n,q}$ that start from $(b+1)^{th}$ position for any s.

Proof. As $n=bd_b$ and $s \geq b$, then $\lceil \frac{n}{s+b}\rceil+1 \leq \lceil \frac{bd_b}{2b}\rceil+1=\lceil \frac{d_b}{2}\rceil+1 \leq d_b$ for $d_b \geq 2$. From Corollary 6.11, the code C_b detects all error vectors of $\psi_{(s,b),n,q}$ for any $s \geq b$. Further, if the error vector of $\psi_{(s,b),n,q}$ starts from the $(b+1)^{th}$ position for any s, then n becomes $(d_b-1)b$ and $\frac{b}{s+b} \leq 1$, so

$$\left\lceil \frac{n}{s+b}\right\rceil+1=\left\lceil \frac{b(d_b-1)}{s+b}\right\rceil+1 \leq d_b-1+1=d_b.$$

This proves the part (ii) by Corollary 6.11. $\square$

Theorem 6.15. *The number of vectors of the error set $\psi_{(s,b),n,q}$ that go undetected by the code C_b is 2^b.*

Proof. From Theorem 6.14, a vector of $\psi_{(s,b),n,q}$ may go undetected if it starts within the first b positions and $s<b$. Every codeword of C_b is a linear combination of rows of G_b, so every codeword can be written as

$$\begin{aligned} &(a_0+a_1X+a_2X^2+\cdots+a_jX^j)+X^b(a_0+a_1X+a_2X^2+\dots \\ &\quad +a_jX^j)+\cdots+X^{(d_b-1)b}(a_0+a_1X+a_2X^2+\cdots+a_jX^j), \end{aligned}$$

where $a_i \in GF(2)$ and $0 \leq j \leq b-1$.

Clearly, every nonzero codeword is a vector of $\psi_{(s,b),n,q}$ that starts within the first b

positions and $s < b$. Therefore, the number of vectors of $\psi_{(s,b),n,q}$ that go undetected by C_b is 2^b. $\square$

Theorem 6.16. *The code C_b corrects all*
(i) vectors of the error set $\psi_{(s,b),n,q}$ for any $s \geq 3b$, and
(ii) vectors of the set $\psi_{(s,b),n,q}$ that start from $\left(\lceil\frac{(d_b+1)b}{2}\rceil + 1\right)^{th}$ position for any s.

Proof. Since the difference between any two vectors of $\psi_{(s,b),n,q}$ for $s \geq 3b$ is a vector which contains at least b consecutive zeros, so the difference cannot be a codeword of C_b. Therefore, the code C_b corrects all error vectors of $\psi_{(s,b),n,q}$ for any $s \geq 3b$. This can be verified from Corollary 6.13 also:

As $n = bd_b$ and $s \geq 3b$, then $2\lceil\frac{n}{s+b}\rceil + 1 \leq 2\lceil\frac{bd_b}{4b}\rceil + 1 = 2\lceil\frac{d_b}{4}\rceil + 1 \leq d_b$ for $d_b \geq 3$.

Further, if an error vector of $\psi_{(s,b),n,q}$ starts from $\left(\lceil\frac{(d_b+1)b}{2}\rceil + 1\right)^{th}$ position, then n becomes $bd_b - \left\lceil\dfrac{(d_b+1)b}{2}\right\rceil \leq \dfrac{b(d_b-1)}{2}$. So

$$2\left\lceil\frac{n}{s+b}\right\rceil + 1 \leq 2\left\lceil\frac{b(d_b-1)}{2(s+b)}\right\rceil + 1 \leq d_b. \qquad (\because \frac{b}{s+b} \leq 1)$$

This proves the part (ii). $\square$

Theorem 6.17. *The dual code $C_b^{\perp}$ of C_b is also a $(bd_b, b(d_b-1), 2)$ MDS code.*

Proof. Since the code C_b is of order (bd_b, b), the order of its dual code $C_b^{\perp}$ is $(bd_b, b(d_b-1))$ and its generator matrix H_b is given by

$$H_b = {\scriptstyle d_b - 1}\left\{\begin{bmatrix} I_b & \\ I_b & \\ \vdots & I_{b(d_b-1)} \\ I_b & \\ I_b & \end{bmatrix}\right.,$$

where I_b represents the identity matrix of order b.

Clearly, the minimum burst-b distance of $C_b^{\perp}$ is $d'_b = 2$. So, for the dual code $C_b^{\perp}$, $n = bd_b$, $k = b(d_b-1)$ and $d'_b = 2$. Now $b(d'_b-1) = b = n-k$. Therefore, the dual code $C_b^{\perp}$ is also an MDS code. $\square$

Theorem 6.18. *If $d_b = 2$, the dual code $C_b^{\perp}$ of C_b*

(i) detects all vectors of the error set $\psi_{(b,b),n,q}$ and

(ii) corrects all vectors of the set $\psi_{(b,b),n,q}$ that start from the $(b+1)^{th}$ position.

Proof. From the previous theorem, we have that the length and the minimum burst-b distance of $C_b^{\perp}$ are bd_b and $d'_b = 2$ respectively. Now taking $s = b$, we have

$$\lceil \frac{n}{s+b} \rceil + 1 = \lceil \frac{bd_b}{2b} \rceil + 1 = \lceil \frac{d_b}{2} \rceil + 1 \leq d'_b.$$

So, by Corollary 6.11, the code $C_b^{\perp}$ detects all vectors of $\psi_{(b,b),n,q}$ provided $d_b = 2$. Again

$$\lceil n + 1 - \frac{(d'_b - 1)}{2}(s+b) \rceil = \lceil 2b + 1 - \frac{(2-1)}{2}(b+b) \rceil = b + 1.$$

So, by Theorem 6.12, the dual code $C_b^{\perp}$ corrects all vectors of $\psi_{(b,b),n,q}$ that start from the $(b+1)^{th}$ position. □

6.3 Decoding of the MDS code C_b

In this section, we give a decoding method for periodical burst error by the code C_b. Suppose that v is a sent codeword of C_b and $w = (w_1, w_2, \ldots, w_{bd_b})$ is received after any vector of $\psi_{(s,b),n,q}$ (where $s \geq 3b$) which is $e = (e_1, e_2, \ldots, e_{bd_b})$. Then the sent codeword is $v = w - e$. In the following, we describe how to get the error from the received vector w. Now the syndrome of w is

$$\begin{aligned} S =& wH_b^{\perp} \\ =& eH_b^{\perp} \\ =& \Big(e_1 + e_{b+1}, e_2 + e_{b+2}, \ldots, e_b + e_{2b}, e_1 + e_{2b+1}, e_2 + e_{2b+2}, \ldots, e_b + e_{3b}, \\ & \ldots, \ldots, e_1 + e_{(d_b-1)b+1}, e_2 + e_{(d_b-1)b+2}, \ldots, e_b + e_{d_b b} \Big) \\ =& \Big(s_1, s_2, \ldots, s_b, s_{b+1}, s_{b+2}, \ldots, s_{2b}, \ldots, \ldots, s_{(d_b-2)b+1}, s_{(d_b-2)b+2}, \ldots, s_{(d_b-1)b} \Big). \end{aligned}$$

If the periodical burst error starts from j^{th} position ($1 \leq j \leq b$), all the three b-tuples $(s_1, s_2, \ldots, s_b), (s_{b+1}, s_{b+2}, \ldots, s_{2b}), (s_{2b+1}, s_{2b+2}, \ldots, s_{3b})$ of b consecutive components of S will be nonzero tuples and also $e_{j+b} = e_{j+b+1} = \cdots = e_{j+4b-1} = 0$. Taking the majority of the three sets of b consecutive components gives us $(e_1, e_2, \ldots, e_b)$. Then subtracting $(e_1, e_2, \ldots, e_b)$ from each set of b consecutive components of S gives the remaining $(d_b - 1)b$ components of e, i.e.

$$\begin{aligned}
&\Big(e_{b+1}, e_{b+2}, \ldots, e_{2b}, e_{2b+1}, e_{2b+2}, \ldots, e_{3b}, \ldots\ldots, e_{(d_b-1)b+1}, e_{(d_b-1)b+2}, \ldots, e_{d_b b}\Big)\\
=&\Big(\underbrace{s_1, s_2, \ldots, s_b}_{b}, \underbrace{s_{b+1}, s_{b+2}, \ldots, s_{2b}}_{b}, \ldots\ldots, \underbrace{s_{(d_b-2)b+1}, s_{(d_b-2)b+2}, \ldots, s_{(d_b-1)b}}_{b}\Big)\\
&-\Big(\underbrace{e_1, e_2, \ldots, e_b}_{b}, \underbrace{e_1, e_2, \ldots, e_b}_{b}, \ldots\ldots, \underbrace{e_1, e_2, \ldots, e_b}_{b}\Big)\\
=&\Big(\underbrace{s_1 - e_1, s_2 - e_2, \ldots, s_b - e_b}_{b}, \underbrace{s_{b+1} - e_1, s_{b+2} - e_2, \ldots, s_{2b} - e_b}_{b}, \ldots\ldots\ldots,\\
&\underbrace{s_{(d_b-2)b+1} - e_1, s_{(d_b-2)b+2} - e_2, \ldots, s_{(d_b-1)b} - e_b}_{b}\Big).
\end{aligned}$$

If the periodical burst error starts from j^{th} position $(j \geq b+1)$, then

$$\begin{aligned}
S &= wH_b^{\perp}\\
&= eH_b^{\perp}\\
&= \Big(e_{b+1}, e_{b+2}, \ldots, e_{2b}, e_{2b+1}, e_{2b+2}, \ldots, e_{3b}, \ldots, \ldots, e_{(d_b-1)b+1}, e_{(d_b-1)b+2}, \ldots, e_{d_b b}\Big)\\
&= \Big(s_1, s_2, \ldots, s_b, s_{b+1}, s_{b+2}, \ldots, s_{2b}, \ldots, \ldots, s_{(d_b-2)b+1}, s_{(d_b-2)b+2}, \ldots, s_{(d_b-1)b}\Big).
\end{aligned}$$

Thus, if the periodical burst starts from j^{th} position $(j > b)$, at least one b-tuple of $\{(s_1, s_2, \ldots, s_b), (s_{b+1}, s_{b+2}, \ldots, s_{2b}), (s_{2b+1}, s_{2b+2}, \ldots, s_{3b})\}$ of b consecutive components of S will be a zero tuple. Then, the error vector e will be $e = \big(\underbrace{00\ldots0}_{b} s_1 s_2 \ldots s_{b(d_b-1)}\big)$.

Example 6.19. *For $b = 4$ and $d_4 = 6$, the parity check matrix of C_4 is given by*

$$H_4 = \begin{bmatrix} 100010000000000000000000 \\ 010001000000000000000000 \\ 001000100000000000000000 \\ 000100010000000000000000 \\ 100000001000000000000000 \\ 010000000100000000000000 \\ 001000000010000000000000 \\ 000100000001000000000000 \\ 100000000000100000000000 \\ 010000000000010000000000 \\ 001000000000001000000000 \\ 000100000000000100000000 \\ 100000000000000010000000 \\ 010000000000000001000000 \\ 001000000000000000100000 \\ 000100000000000000010000 \\ 100000000000000000001000 \\ 010000000000000000000100 \\ 001000000000000000000010 \\ 000100000000000000000001 \end{bmatrix}_{20\times 24}.$$

(i) Let the received vector be $w = 101101011101110110001101$ after an error vector of $\psi_{(3b=12,3),24,2}$. Now the syndrome of w is $wH_4^T = 11100110011000110110$. As each of the first 3 tuples of $b = 4$ consecutive components is not all zero, the error starts within the first $b = 4$ positions. From the majority of first three sets of four consecutive components, we can have $(e_1, e_2, e_3, e_4) = (0110)$. So

$$\begin{aligned}(e_5, e_6, \ldots, e_{24}) &= 11100110011000110110 - 01100110011001100110 \\ &= (10000000000001010000).\end{aligned}$$

Therefore, the error vector is $e = (011010000000000001010000)$ and the sent *codeword is*

$$v = w - e = (101101011101110110001101) - (011010000000000001010000)$$
$$= (110111011101110111011101).$$

(ii) Let the received vector be $w = (110101001101110111010001)$ *after an error vector of* $\psi_{(3b=12,3),24,2}$. *Now the syndrome of* w *is* $wH_4^T = (10010000000000001100)$. *As the second and third tuples of* b *components are all zero, the error starts after* $b = 4$ *positions.*

Therefore the error vector is $e = (000010010000000000001100)$ *and the sent codeword is*

$$v = w - e = (110101001101110111010001) - (000010010000000000001100)$$
$$= 110111011101110111011101.$$