Chapter 2
Correction and weight distribution of periodic random errors

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- Das, P. K. and Haokip, L. Correction and weight distribution of periodic random errors. *Science and Technology Asia*, 26(4):38-47, 2021.

Chapter 2

Correction and weight distribution of periodic random errors

In [19], necessary and sufficient conditions for existence of a linear code detecting periodic random errors are studied. In this chapter, we derive the conditions for a linear code correcting such periodic random errors. We also obtain the (Hamming) weight distribution of the error pattern and derive an upper bound on the total weight of all codewords in periodic random error-correcting codes. Section 2.1 provides necessary and sufficient conditions for the existence of periodic random error correcting linear code. Examples are followed. In Section 2.2, we first give the weight distribution of the error patterns followed by an upper bound on the weight of these error-correcting linear codes.

In what follows,

$\lfloor \boldsymbol{x} \rfloor$ means the floor function of x.

$\lceil \boldsymbol{x} \rceil$ means the ceiling function of x.

$\boldsymbol{\xi_{(s,b),n,q}}$ means the set of all s-periodic random errors of length b in an n-tuple over $GF(q)$.

$\boldsymbol{P_{(s,b),n,q}RC - code}$ means a length-n linear code correcting s-periodic random errors of length b over $GF(q)$.

2.1 Conditions for existence of $P_{(s,b),n,q}RC$-codes

In this section, we provide necessary and sufficient conditions for the existence of a $P_{(s,b),n,q}RC-code$, along with examples. Below result is a necessary condition, which is equivalent to Result 1.14. This is obtained by comparing the number of error patterns with the available number of cosets. The correctable errors must be in different cosets of a $P_{(s,b),n,q}RC-code$.

Theorem 2.1. *For given non-negative integers n, b and s $(n \geq b+s)$, let $n = \lambda(b+s)+l$ for some non-negative integers λ and l, where $0 \leq l < b+s$. Then a necessary condition for the existence of an (n,k) $P_{(s,b),n,q}RC-code$ is*

$$q^{n-k} \geq q^{\rho} + \sum_{i=1}^{s+b-1}(q^{\beta_i}-1)q^{(b-1)\beta_i+l_i} - \sum_{i'=1}^{b-1}(q^{\beta_{s+i'}}-1)q^{(i'-1)\beta_{s+i'}}, \qquad (2.1)$$

where $\beta_i = \lceil \frac{n-i-b+1}{s+b} \rceil$ and

(i) when $l=0$, $\rho = b\lambda$ and $l_i = \begin{cases} 0 & \text{for } 1 \leq i \leq s \\ s+b-i & \text{for } s+1 \leq i \leq s+b, \end{cases}$

(ii) when $1 \leq l \leq b$, $\rho = b\lambda + l$ and $l_i = \begin{cases} l-i & \text{for } 1 \leq i \leq l \\ 0 & \text{for } l+1 \leq i \leq s+b, \end{cases}$

(iii) when $b < l < s+b$, $\rho = b(\lambda+1)$ and $l_i = \begin{cases} 0 & \text{for } 1 \leq i \leq l-b,\ l < i \leq s+b \\ l-i & \text{for } l-b+1 \leq i \leq l. \end{cases}$

$$\left(Here \sum_{i'=1}^{b-1}(q^{\beta_{s+i'}}-1)q^{(i'-1)\beta_{s+i'}} = 0 \ for\ b=1 \right)$$

Proof. For the proof, we calculate the total number of vectors of $\xi_{(s,b),n,q}$ and compare it with the available number of cosets.

Let S_i $(1 \leq i \leq s+b)$ represent the collection of all vectors of $\xi_{(s,b),n,q}$ starting from the i^{th} position.

Then the total number of distinct vectors of $\xi_{(s,b),n,q}$ is

$$|S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_{s+b}|$$

$$=|S_1| + |S_2 \setminus S_1| + |S_3 \setminus (S_1 \cup S_2)| + \cdots + |S_{s+b} \setminus (S_1 \cup S_2 \cup \cdots \cup S_{s+b-1})|.$$

In calculating $|S_{i+1} \setminus (S_1 \cup S_2 \cup \cdots \cup S_i)|$, we note that the last position of each set of b consecutive positions of S_{i+1} $(i = 1, 2, \ldots, s)$ is not in S_i, but from S_{s+i} $(i = 2, 3, \ldots, b)$, the last position of each set is already considered in the set S_1.

Case-(i): When $l = 0$, we compute

$|S_1| = q^{b\lambda}$.

$$|S_2 \setminus S_1| = \sum_{r=1}^{\beta_1} \binom{\beta_1}{r} (q-1)^r q^{(b-1)\beta_1} = (q^{\beta_1} - 1) q^{(b-1)\beta_1},$$

where $\beta_1 = \lceil \frac{n-1-b+1}{s+b} \rceil$.

$$|S_3 \setminus (S_1 \cup S_2)| = (q^{\beta_2} - 1) q^{(b-1)\beta_2},$$

where $\beta_2 = \lceil \frac{n-2-b+1}{b+s} \rceil$.

$\vdots$

$\vdots$

$$|S_{s+1} \setminus (S_1 \cup S_2 \cup \cdots \cup S_s)| = (q^{\beta_s} - 1) q^{(b-1)\beta_s},$$

where $\beta_s = \lceil \frac{n-s-b+1}{s+b} \rceil$.

$$|S_{s+2} \setminus (S_1 \cup S_2 \cup \cdots \cup S_{s+1})| = (q^{\beta_{s+1}} - 1) q^{(b-1)\beta_{s+1}+(b-1)} - (q^{\beta_{s+1}} - 1),$$

where $\beta_{s+1} = \lceil \frac{n-(s+1)-b+1}{s+b} \rceil$.

$$|S_{s+3} \setminus (S_1 \cup S_2 \cup \cdots \cup S_{s+2})| = (q^{\beta_{s+2}} - 1) q^{(b-1)\beta_{s+2}+(b-2)} - (q^{\beta_{s+2}} - 1) q^{\beta_{s+2}},$$

where $\beta_{s+2} = \lceil \frac{n-(s+2)-b+1}{s+b} \rceil$.

Therefore

$$|S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_{s+b}| = q^{b\lambda} + \sum_{i=1}^{s+b-1} (q^{\beta_i} - 1) q^{(b-1)\beta_i + l_i} - \sum_{i'=1}^{b-1} (q^{\beta_{s+i'}} - 1) q^{(i'-1)\beta_{s+i'}},$$

where $\beta_i = \lceil \frac{n-i-b+1}{s+b} \rceil$ and $l_i = \begin{cases} 0 & \text{for} \quad 1 \le i \le s \\ s+b-i & \text{for} \quad s+1 \le i \le s+b. \end{cases}$

Case-(ii): When $1 \leq l \leq b$, we consider two subcases:

Subcase (i): If $s+2 < l$, then
$|S_1| = q^{b\lambda+l}$.

$|S_2 \setminus S_1| = (q^{\beta_1}-1)q^{(b-1)\beta_1+(l-1)}$,

where $\beta_1 = \lceil \frac{n-1-b+1}{s+b} \rceil$.

$|S_3 \setminus (S_1 \cup S_2)| = (q^{\beta_2}-1)q^{(b-1)\beta_2+(l-2)}$,

where $\beta_2 = \lceil \frac{n-2-b+1}{b+s} \rceil$.

$\vdots$

$\vdots$

$|S_{s+2} \setminus (S_1 \cup S_2 \cup \cdots \cup S_{s+1})| = (q^{\beta_{s+1}}-1)q^{(b-1)\beta_{s+1}+(l-s-1)} - (q^{\beta_{s+1}}-1)$,

where $\beta_{s+1} = \lceil \frac{n-(s+1)-b+1}{b+s} \rceil$.

$|S_{s+3} \setminus (S_1 \cup S_2 \cup \cdots \cup S_{s+2})| = (q^{\beta_{s+2}}-1)q^{(b-1)\beta_{s+2}+(l-s-2)} - (q^{\beta_{s+2}}-1)q^{\beta_{s+2}}$,

where $\beta_{s+2} = \lceil \frac{n-(s+2)-b+1}{b+s} \rceil$.

$\vdots$

$\vdots$

$|S_l \setminus (S_1 \cup S_2 \cup \cdots \cup S_{l-1})| = (q^{\beta_{l-1}}-1)q^{(b-1)\beta_{l-1}+1} - (q^{\beta_{l-1}}-1)q^{(l-s-2)\beta_{l-1}}$,

where $\beta_{l-1} = \left\lceil \frac{n-(l-1)-b+1}{b+s} \right\rceil$.

$|S_{l+1} \setminus (S_1 \cup S_2 \cup \cdots \cup S_l)| = (q^{\beta_l}-1)q^{(b-1)\beta_l} - (q^{\beta_l}-1)q^{(l-s-1)\beta_l}$,

where $\beta_l = \lceil \frac{n-l-b+1}{b+s} \rceil$.

$\vdots$

$\vdots$

$|S_{s+b} \setminus (S_1 \cup S_2 \cup \cdots \cup S_{s+b-1})| = (q^{\beta_{s+b-1}}-1)q^{(b-1)\beta_{s+b-1}} - (q^{\beta_{s+b-1}}-1)q^{(b-2)\beta_{s+b-1}}$,

where $\beta_{s+b-1} = \lceil \frac{n-(s+b-1)-b+1}{b+s} \rceil$.

Subcase (ii): If $s+2 \geq l$, we have

$|S_1| = q^{b\lambda+l}$.

$|S_2 \setminus S_1| = (q^{\beta_1} - 1)q^{(b-1)\beta_1+(l-1)}$,

where $\beta_1 = \lceil \frac{n-1-b+1}{s+b} \rceil$.

$|S_3 \setminus (S_1 \cup S_2)| = (q^{\beta_2} - 1)q^{(b-1)\beta_2+(l-2)}$,

where $\beta_2 = \lceil \frac{n-2-b+1}{b+s} \rceil$.

$\vdots$

$\vdots$

$|S_l \setminus (S_1 \cup S_2 \cup \cdots \cup S_{l-1})| = (q^{\beta_{l-1}} - 1)q^{(b-1)\beta_{l-1}+1}$,

where $\beta_{l-1} = \lceil \frac{n-(l-1)-b+1}{b+s} \rceil$.

$|S_{l+1} \setminus (S_1 \cup S_2 \cup \cdots \cup S_l)| = (q^{\beta_l} - 1)q^{(b-1)\beta_l}$,

where $\beta_l = \lceil \frac{n-l-b+1}{b+s} \rceil$.

$\vdots$

$\vdots$

$|S_{s+2} \setminus (S_1 \cup S_2 \cup \cdots \cup S_{s+1})| = (q^{\beta_{s+1}} - 1)q^{(b-1)\beta_{s+1}} - (q^{\beta_{s+1}} - 1)$,

where $\beta_{s+1} = \lceil \frac{n-(s+1)-b+1}{b+s} \rceil$.

$|S_{s+3} \setminus (S_1 \cup S_2 \cup \cdots \cup S_{s+2})| = (q^{\beta_{s+2}} - 1)q^{(b-1)\beta_{s+2}} - (q^{\beta_{s+2}} - 1)q^{\beta_{s+2}}$,

where $\beta_{s+2} = \lceil \frac{n-(s+2)-b+1}{b+s} \rceil$.

$\vdots$

$\vdots$

$|S_{s+b} \setminus (S_1 \cup S_2 \cup \cdots \cup S_{s+b-1})| = (q^{\beta_{s+b-1}} - 1)q^{(b-1)\beta_{s+b-1}} - (q^{\beta_{s+b-1}} - 1)q^{(b-2)\beta_{s+b-1}}$,

where $\beta_{s+b-1} = \lceil \frac{n-(s+b-1)-b+1}{b+s} \rceil$.

In either subcase, we have

$$|S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_{s+b}| = q^{b\lambda+l} + \sum_{i=1}^{s+b-1}(q^{\beta_i}-1)q^{(b-1)\beta_i+l_i} - \sum_{i'=1}^{b-1}(q^{\beta_{s+i'}}-1)q^{(i'-1)\beta_{s+i'}},$$

$$\text{where } \beta_i = \lceil \tfrac{n-i-b+1}{s+b} \rceil \text{ and } l_i = \begin{cases} l-i & \text{for } 1 \le i \le l \\ 0 & \text{for } l+1 \le i \le s+b. \end{cases}$$

Case-(iii): When $b < l \le s+b-1$. We can take $l = b+t$ and compute $|S_1| = q^{b(\lambda+1)}$.

$|S_{i+1} \setminus (S_1 \cup S_2 \cup \cdots \cup S_i)| = (q^{\beta_i}-1)q^{(b-1)\beta_i+l_i}$ for $1 \le i \le s$ and

$|S_{i+1} \setminus (S_1 \cup S_2 \cup \cdots \cup S_i)| = (q^{\beta_i}-1)q^{(b-1)\beta_i+l_i} - (q^{\beta_i}-1)q^{(i-s-1)\beta_i}$ for $s+1 \le i \le s+b-1$,

$$\text{where } \beta_i = \lceil \tfrac{n-i-b+1}{b+s} \rceil \text{ and } l_i = \begin{cases} 0 & \text{for } 1 \le i \le t,\ \ b+t < i < s+b \\ t+b-i & \text{for } t+1 \le i \le t+b. \end{cases}$$

Hence

$$|S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_{s+b}| = q^{b(\lambda+1)} + \sum_{i=1}^{s+b-1}(q^{\beta_i}-1)q^{(b-1)\beta_i+l_i} - \sum_{i'=1}^{b-1}(q^{\beta_{s+i'}}-1)q^{(i'-1)\beta_{s+i'}},$$

$$\text{where } \beta_i = \lceil \tfrac{n-i-b+1}{b+s} \rceil \text{ and } l_i = \begin{cases} 0 & \text{for } 1 \le i \le t,\ \ b+t < i < s+b \\ t+b-i & \text{for } t+1 \le i \le t+b. \end{cases}$$

Therefore, combining all the three cases and taking the number of correctable errors less than or equal to the available number of cosets (q^{n-k}), we must get the following for an $(n = \lambda(b+s)+l, k)$ $P_{(s,b),n,q}RC-code$:

$$q^{n-k} \ge q^{\rho} + \sum_{i=1}^{s+b-1}(q^{\beta_i}-1)q^{(b-1)\beta_i+l_i} - \sum_{i'=1}^{b-1}(q^{\beta_{s+i'}}-1)q^{(i'-1)\beta_{s+i'}}$$

$$\Rightarrow n-k \ge \log_q\left[q^{\rho} + \sum_{i=1}^{s+b-1}(q^{\beta_i}-1)q^{(b-1)\beta_i+l_i} - \sum_{i'=1}^{b-1}(q^{\beta_{s+i'}}-1)q^{(i'-1)\beta_{s+i'}}\right].$$

□

Remark 2.2. *The right hand side (R.H.S.) of* (2.1) *gives the total number of elements (including zero vector) in the set* $\xi_{(s,b),n,q}$.

Example 2.3. *Taking* $n = 12, b = 2, s = 3$ *and* $q = 2$ *in Theorem 2.1, we have* $l = 2 = b$ *and* $\lambda = 2$. *Then* $\rho = b\lambda + l = 6$, $\beta_1 = \left\lceil \frac{12-1-2+1}{5} \right\rceil = 2$, $\beta_2 = \left\lceil \frac{12-2-2+1}{5} \right\rceil = 2$, $\beta_3 = \left\lceil \frac{12-3-2+1}{5} \right\rceil = 2$, $\beta_4 = \left\lceil \frac{12-4-2+1}{5} \right\rceil = 2$, $l_1 = 1$, $l_2 = 0$, $l_3 = 0$, $l_4 = 0$.

The total number of 3*-periodic random errors of length 2, i.e., the cardinality of the set* $\xi_{(3,2),12,2}$ *(including zero vector) is*

$$
\begin{aligned}
& q^\rho + \sum_{i=1}^{s+b-1} (q^{\beta_i} - 1) q^{(b-1)\beta_i + l_i} - \sum_{i'=1}^{b-1} (q^{\beta_{s+i'}} - 1) q^{(i'-1)\beta_{s+i'}} \\
&= 2^6 + \sum_{i=1}^{4} (2^{\beta_i} - 1) 2^{\beta_i + l_i} - \sum_{i'=1}^{1} (2^{\beta_{3+i'}} - 1) 2^{(i'-1)\beta_{3+i'}} \\
&= 2^6 + (2^2-1)2^3 + (2^2-1)2^2 + (2^2-1)2^2 + (2^2-1)2^2 - (2^2-1) \\
&= 121.
\end{aligned}
$$

These 120 error patterns (excluding zero vector) are given below:

000000000001, 000000000010, 000000000011, 000000100000, 000001000000, 000001100000, 000000100001, 000000100010, 000000100011, 000001000001, 000001000010, 000001000011, 000001100001, 000001100010, 000001100011, 010000000000, 010000000001, 010000000010, 010000000011, 010000100000, 010001000000, 010001100000, 010000100001, 010000100010, 010000100011, 010001000001, 010001000010, 010001000011, 010001100001, 010001100010, 010001100011, 100000000000, 100000000001, 100000000010, 100000000011, 100000100000, 100001000000, 100001100000, 100000100001, 100000100010, 100000100011, 100001000001, 100001000010, 100001000011, 100001100001, 100001100010, 100001100011, 110000000000, 110000000001, 110000000010, 110000000011, 110000100000, 110001000000, 110001100000, 110000100001, 110000100010, 110000100011, 110001000001, 110001000010, 110001000011, 110001100001, 110001100010, 110001100011, 000000010000, 000000110000, 000000010001, 000000110001, 001000000000, 001000000001, 001000010000, 001000100000, 001000110000, 001000010001, 001000100001, 001000110001, 010000010000, 010000110000, 010000010001, 010000110001, 011000000000, 011000000001, 011000010000, 011000100000, 011000110000, 011000010001, 011000100001, 011000110001, 000000001000, 000000011000, 000100000000, 000100001000, 000100010000, 000100011000, 001000001000, 001000011000, 001100000000, 001100001000, 001100010000, 001100011000, 000000000100, 000000001100, 000010000000, 000010000100, 000010001000, 000010001100, 000100000100, 000100001100, 000110000000, 000110000100, 000110001000, 000110001100, 000000000110, 000001000100, 000001000110, 000010000010, 000010000110, 000011000000, 000011000010, 000011000100, 000011000110.

The following theorem gives a sufficient condition for the existence of a $P_{(s,b),n,q}RC-$ *code*. The proof follows the technique used in proving Result 1.13–1.15.

Theorem 2.4. *For given non-negative integers n, b and s $(n \geq b+s)$, let $n = \lambda(b+s)+l$ and $n-b = \lambda'(b+s)+L$ for some non-negative integers λ, λ', l and L, where $0 \leq l, L < b+s$. Then a sufficient condition for the existence of an (n,k) $P_{(s,b),n,q}RC - code$ is as follows:*

$$q^{n-k} > q^{\delta}\left[q^{\eta} + \sum_{i=1}^{s+b-1}(q^{\beta'_i}-1)q^{(b-1)\beta'_i+l'_i} - \sum_{i'=1}^{b-1}(q^{\beta'_{s+i'}}-1)q^{(i'-1)\beta'_{s+i'}}\right], \tag{2.2}$$

where

$$\beta'_i = \left\lceil \frac{n-i-2b+1}{s+b} \right\rceil \quad (1 \leq i \leq s+b-1),$$

$$\delta = \begin{cases} \lambda b - 1 & \text{for} \quad l = 0 \\ \lambda b - 1 + l & \text{for} \quad 1 \leq l \leq b \\ \lambda b - 1 + b & \text{for} \quad b < l < s+b, \end{cases}$$

$$\eta = \begin{cases} b\lambda' & \text{for} \quad L = 0 \\ b\lambda' + L & \text{for} \quad 1 \leq L \leq b \\ b(\lambda'+1) & \text{for} \quad b+1 \leq L < s+b, \end{cases}$$

and

$$l'_i = \begin{cases} 0 & \text{for} \quad 1 \leq i \leq s \\ s+b-i & \text{for} \quad s+1 \leq i \leq s+b \end{cases} \quad \text{if} \quad L = 0,$$

$$l'_i = \begin{cases} L-i & \text{for} \quad 1 \leq i \leq L \\ 0 & \text{for} \quad L+1 \leq i \leq s+b \end{cases} \quad \text{if} \quad 1 \leq L \leq b,$$

$$l'_i = \begin{cases} 0 & for \;\; 1 \le i \le L-b, \;\; L+1 \le i \le s+b \\ L-i & for \;\; L-b+1 \le i \le L \end{cases} \quad if \quad b+1 \le L < s+b.$$

$$\left(\; Here \sum_{i'=1}^{b-1} (q^{\beta'_{s+i'}} - 1) q^{(i'-1)\beta'_{s+i'}} = 0 \; for \; b = 1 \; \right)$$

Proof. The existence of this type of linear code is shown by constructing an appropriate $(n-k) \times n$ parity-check matrix H. Take the first column h_1 as any nonzero $(n-k)$-tuple and the remaining columns $h_2, h_3, \ldots, h_n$ are added to H one after another such that h_n is not a linear combination of immediately preceding $b-1$ columns together with previous sets of b consecutive columns which are at a gap of s columns (the last set may contain less than b columns), along with a linear combination of sets of b consecutive columns which are at a gap of s columns confined to the first $n-b$ columns (the last set may contain less than b columns). That is

$$\begin{aligned} h_n \neq & \left(\sum_{i=1}^{b-1} a_{i1} h_{n-i} + \sum_{i=0}^{b-1} b_{i1} h_{n-(s+b)-i} + \sum_{i=0}^{b-1} b_{i2} h_{n-2(s+b)-i} + \cdots + \sum_{i=0}^{g_1-1} b_{i\lambda} h_{n-\lambda(s+b)-i} \right) \\ & + \left(\sum_{i=0}^{b-1} a_{i1} h_{j'-i} + \sum_{i=0}^{b-1} b_{i1} h_{j'-(s+b)-i} + \sum_{i=0}^{b-1} b_{i2} h_{j'-2(s+b)-i} + \cdots + \sum_{i=0}^{g_2-1} b_{i\lambda'} h_{j'-\lambda'(s+b)-i} \right), \end{aligned} \tag{2.3}$$

where $a_i, b_i \in GF(q)$, $j' \le n-b$, $g_1 = \begin{cases} 0 & if \;\; l = 0 \\ l & if \;\; 1 \le l \le b \\ b & if \;\; b < l < s+b \end{cases}$ and

$$g_2 = \begin{cases} 0 & if \;\; L = 0 \\ L & if \;\; 1 \le L \le b \\ b & if \;\; b < L < s+b. \end{cases}$$

$$\left(\text{We take} \sum_{i=0}^{g_i-1} b_{i\lambda} h_{n-\lambda(s+b)-i} = 0 \text{ for } g_i = 0 \right)$$

The number of linear combinations in first bracket on R.H.S. of (2.3) is

$$\begin{cases} q^{\lambda b-1} & \textit{if} \quad l=0 \\ q^{\lambda b-1+l} & \textit{if} \quad 1 \leq l \leq b \\ q^{\lambda b-1+b} & \textit{if} \quad b < l < s+b. \end{cases}$$

The second bracket of (2.3) gives the number of vectors of $\xi_{(s,b),n-b,q}$ in a vector of length $n-b$. This is given by Theorem 2.1 as

$$q^{\eta}+\sum_{i=1}^{s+b-1}(q^{\beta'_i}-1)q^{(b-1)\beta'_i+l'_i}-\sum_{i'=1}^{b-1}(q^{\beta'_{s+i'}}-1)q^{(i'-1)\beta'_{s+i'}}.$$

Therefore, the total number of all possible linear combinations on R.H.S. of (2.3) is

$$q^{\delta}\left[q^{\eta}+\sum_{i=1}^{s+b-1}(q^{\beta'_i}-1)q^{(b-1)\beta'_i+l'_i}-\sum_{i'=1}^{b-1}(q^{\beta'_{s+i'}}-1)q^{(i'-1)\beta'_{s+i'}}\right], \tag{2.4}$$

where $\delta = \begin{cases} \lambda b-1 & \text{for} \quad l=0 \\ \lambda b-1+l & \text{for} \quad 1 \leq l \leq b \\ \lambda b-1+b & \text{for} \quad b < l < s+b, \end{cases}$

$$\eta = \begin{cases} b\lambda' & \text{for} \quad L=0 \\ b\lambda'+L & \text{for} \quad 1 \leq L \leq b \\ b\,(\lambda'+1) & \text{for} \quad b+1 \leq L < s+b, \end{cases}$$

and

$$l'_i = \begin{cases} 0 & \text{for} \quad 1 \leq i \leq s \\ s+b-i & \text{for} \quad s+1 \leq i \leq s+b \end{cases} \qquad \textit{if} \quad L=0,$$

$$l'_i = \begin{cases} L-i & \text{for} \quad 1 \leq i \leq L \\ 0 & \text{for} \quad L+1 \leq i \leq s+b \end{cases} \qquad \textit{if} \quad 1 \leq L \leq b,$$

$$l'_i = \begin{cases} 0 & \text{for} \quad 1 \leq i \leq L-b, \;\; L+1 \leq i \leq s+b \\ L-i & \text{for} \quad L-b+1 \leq i \leq L \end{cases} \qquad \textit{if} \quad b+1 \leq L < s+b.$$

Since we can have at most q^{n-k} columns, so taking q^{n-k} greater than the term computed in (2.4) with different values of δ, η and l'_i gives the sufficient condition for the existence of the required code. This proves the theorem. □

Example 2.5. *Consider* $n = 12$*,* $s = 4$*,* $b = 2$ *and* $q = 2$ *in Theorem 2.4. Here* $\lambda = 2$*,* $l = 0$*,* $\lambda' = 1$*,* $L = 4$*. Then* $\delta = 3$*,* $\eta = 4$*,* $\beta'_1 = \left\lceil \frac{12-1-4+1}{6} \right\rceil = \left\lceil \frac{8}{6} \right\rceil = 2$*,* $\beta'_2 = \left\lceil \frac{7}{6} \right\rceil = 2$*,* $\beta'_3 = \left\lceil \frac{6}{6} \right\rceil = 1$*,* $\beta'_4 = \left\lceil \frac{5}{6} \right\rceil = 1$*,* $\beta'_5 = \left\lceil \frac{4}{6} \right\rceil = 1$*,* $l'_1 = 0$*,* $l'_2 = 0$*,* $l'_3 = 1$*,* $l'_4 = 0$*,* $l'_5 = 0$*.*

By the inequality (2.2) *of Theorem 2.4, we have*

$$2^{n-k} > 2^{\delta}\left[q^{\eta} + \sum_{i=1}^{s+b-1}(q^{\beta'_i}-1)q^{(b-1)\beta'_i+l'_i} - \sum_{i'=1}^{b-1}(q^{\beta'_{s+i'}}-1)q^{(i'-1)\beta'_{s+i'}}\right]$$

$$\implies 2^{n-k} > 2^{3}\Big[2^4 + (2^2-1)2^{2+0} + (2^2-1)2^{2+0} + (2^1-1)2^{1+1} + (2^1-1)2^{1+0} + (2^1-1)2^{1+0} - (2^1-1)2^{0}\Big]$$

$$\implies 2^{n-k} > 376.$$

This implies $n-k \geq 9$*. So,* $n-k = 9$ *gives rise to a binary* $(12,3)$ *linear code whose parity check matrix* H*, constructed by the procedure mentioned in the proof of Theorem 2.4, is given below:*

$$H = \begin{bmatrix} 1&1&0&1&0&0&0&0&0&1&1&0 \\ 0&1&1&0&0&0&0&0&0&0&0&1 \\ 0&1&1&0&0&0&0&0&0&1&0&0 \\ 0&1&0&0&1&0&0&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&1&0&0&1&0&0 \\ 1&1&0&0&0&0&0&1&0&0&0&0 \\ 1&1&0&1&0&0&0&0&1&1&0&0 \end{bmatrix}_{9\times 12}.$$

From the following Error Pattern-Syndrome Table 2.1, we can verify that the syndromes of all 66 *periodic random errors are nonzero and distinct. Therefore, the null space of this matrix is a* $(12,3)$ $P_{(4,2),12,2}RC-$ *code.*

Table 2.1: **Error Pattern-Syndrome**

Error Patterns	Syndromes	Error Patterns	Syndromes
00 0000 01 0000	000000010	11 0000 01 0000	011100110
00 0000 10 0000	000000100	11 0000 10 0000	011100000
00 0000 11 0000	000000110	11 0000 11 0000	011100010
01 0000 00 0000	111111111	0 00 0000 01 000	000000001
01 0000 01 0000	111111101	0 00 0000 11 000	000000011
01 0000 10 0000	111111011	0 01 0000 00 000	011000000
01 0000 11 0000	111111001	0 01 0000 01 000	011000001
10 0000 00 0000	100011011	0 01 0000 10 000	011000010
10 0000 01 0000	100011001	0 01 0000 11 000	011000011
10 0000 10 0000	100011111	0 10 0000 01 000	111111110
10 0000 11 0000	100011101	0 10 0000 11 000	111111100
11 0000 00 0000	011100100	0 11 0000 00 000	100111111
0 11 0000 01 000	100111110	000 10 0000 01 0	000010001
0 11 0000 10 000	100111101	000 10 0000 11 0	101010100
0 11 0000 11 000	100111100	000 11 0000 00 0	100110001
00 00 0000 01 00	101000101	000 11 0000 01 0	000110001
00 00 0000 11 00	101000100	000 11 0000 10 0	001110100
00 01 0000 00 00	100010001	000 11 0000 11 0	101110100
00 01 0000 01 00	001010100	0000 00 0000 01	010000000
00 01 0000 10 00	100010000	0000 00 0000 11	110000000
00 01 0000 11 00	001010101	0000 01 0000 00	000001000
00 10 0000 01 00	110000101	0000 01 0000 01	010001000
00 10 0000 11 00	110000100	0000 01 0000 10	100001000
00 11 0000 00 00	111010001	0000 01 0000 11	110001000
00 11 0000 01 00	010010100	0000 10 0000 01	010100000
00 11 0000 10 00	111010000	0000 10 0000 11	110100000
00 11 0000 11 00	010010101	0000 11 0000 00	000101000

Contd...

Table 2.1 : **Error Pattern-Syndrome**

Error Patterns	Syndromes	Error Patterns	Syndromes
000 00 0000 01 0	100000000	0000 11 0000 01	010101000
000 00 0000 11 0	001000101	0000 11 0000 10	100101000
000 01 0000 00 0	000100000	0000 11 0000 11	110101000
000 01 0000 01 0	100100000	00000 01 0000 1	010000100
000 01 0000 10 0	101100101	00000 11 0000 0	000001100
000 01 0000 11 0	001100101	00000 11 0000 1	010001100

Example 2.6. *Consider* $n = 10$, $s = 3$, $b = 2$ *and* $q = 3$ *in Theorem 2.4. So* $\lambda = 2$, $l = 0$, $\lambda' = 1$, $L = 3$. *Then* $\delta = 3$, $\eta = 4$, $\beta'_1 = \left\lceil \frac{10-1-4+1}{5} \right\rceil = \left\lceil \frac{6}{5} \right\rceil = 2$, $\beta'_2 = \left\lceil \frac{5}{5} \right\rceil = 1$, $\beta'_3 = \left\lceil \frac{4}{5} \right\rceil = 1$, $\beta'_4 = \left\lceil \frac{3}{5} \right\rceil = 1$, $l'_1 = 0$, $l'_2 = 1$, $l'_3 = 0$, $l'_4 = 0$.
From the inequality (2.2), *we have*

$$3^{n-k} > 3^{\delta}\left[q^{\eta} + \sum_{i=1}^{s+b-1}(q^{\beta'_i} - 1)q^{(b-1)\beta'_i + l'_i} - \sum_{i'=1}^{b-1}(q^{\beta'_{s+i'}} - 1)q^{(i'-1)\beta'_{s+i'}}\right]$$

$$\implies 3^{n-k} > 3^3\Big[3^4 + (3^2-1)3^{2+0} + (3^1-1)3^{1+1} + (3^1-1)3^{1+0}$$
$$+ (3^1-1)3^{1+0} - (3^1-1)3^0\Big]$$

$$\implies 3^{n-k} > 4887.$$

This implies $n - k \geq 8$. *So,* $n - k = 8$ *gives rise to a* $(10, 2)$ *ternary linear code whose parity check matrix* H, *constructed by the procedure mentioned in the proof* *of Theorem 2.4, is given below:*

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}_{8\times 10}.$$

Here also, the syndromes of all vectors of $\xi_{(3,2),10,3}$ are nonzero and distinct (see Table 2.2 below), showing that the code is a $(10, 2)$ $P_{(3,2),10,3}RC$-code.

Table 2.2: **Error Pattern-Syndrome**

Error Patterns	Syndromes	Error Patterns	Syndromes
00 000 10 000	00000100	10 000 10 000	10000100
00 000 01 000	00000020	10 000 01 000	10000020
00 000 11 000	00000120	10 000 11 000	10000120
00 000 20 000	00000200	10 000 20 000	10000200
00 000 02 000	00000010	10 000 02 000	10000010
00 000 12 000	00000110	10 000 12 000	10000110
00 000 21 000	00000220	10 000 21 000	10000220
00 000 22 000	00000210	10 000 22 000	10000210
10 000 00 000	10000000	01 000 00 000	02000000
01 000 10 000	02000100	02 000 01 000	01000020
01 000 01 000	02000020	02 000 11 000	01000120
01 000 11 000	02000120	02 000 20 000	01000200
01 000 20 000	02000200	02 000 02 000	01000010
01 000 02 000	02000010	02 000 12 000	01000110
01 000 12 000	02000110	02 000 21 000	01000220
01 000 21 000	02000220	02 000 22 000	01000210

Contd...

Table 2.2 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
01 000 22 000	02000210	12 000 00 000	11000000
11 000 00 000	12000000	12 000 10 000	11000100
11 000 10 000	12000100	12 000 01 000	11000020
11 000 01 000	12000020	12 000 11 000	11000120
11 000 11 000	12000120	12 000 20 000	11000200
11 000 20 000	12000200	12 000 02 000	11000010
11 000 02 000	12000010	12 000 12 000	11000110
11 000 12 000	12000110	12 000 21 000	11000220
11 000 21 000	12000220	12 000 22 000	11000210
11 000 22 000	12000210	21 000 00 000	22000000
20 000 00 000	20000000	21 000 10 000	22000100
20 000 10 000	20000100	21 000 01 000	22000020
20 000 01 000	20000020	21 000 11 000	22000120
20 000 11 000	20000120	21 000 20 000	22000200
20 000 20 000	20000200	21 000 02 000	22000010
20 000 02 000	20000010	21 000 12 000	22000110
20 000 12 000	20000110	21 000 21 000	22000220
20 000 21 000	20000220	21 000 22 000	22000210
20 000 22 000	20000210	22 000 00 000	21000000
02 000 00 000	01000000	22 000 10 000	21000100
02 000 10 000	01000100	22 000 01 000	21000020
22 000 11 000	21000120	0 11 000 10 00	02100020
22 000 20 000	21000200	0 11 000 01 00	02100001
22 000 02 000	21000010	0 11 000 11 00	02100021
22 000 12 000	21000110	0 11 000 20 00	02100010
22 000 21 000	21000220	0 11 000 02 00	02100002
22 000 22 000	21000210	0 11 000 12 00	02100022
0 00 000 01 00	00000001	0 11 000 21 00	02100011

Contd...

Table 2.2 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
0 00 000 11 00	00000021	0 11 000 22 00	02100012
0 00 000 02 00	00000002	0 20 000 01 00	01000001
0 00 000 12 00	00000022	0 20 000 11 00	01000021
0 00 000 21 00	00000011	0 20 000 02 00	01000002
0 00 000 22 00	00000012	0 20 000 12 00	01000022
0 10 000 01 00	02000001	0 20 000 21 00	01000011
0 10 000 11 00	02000021	0 20 000 22 00	01000012
0 10 000 02 00	02000002	0 02 000 00 00	00200000
0 10 000 12 00	02000022	0 02 000 10 00	00200020
0 10 000 21 00	02000011	0 02 000 01 00	00200001
0 10 000 22 00	02000012	0 02 000 11 00	00200021
0 01 000 00 00	00100000	0 02 000 20 00	00200010
0 01 000 10 00	00100020	0 02 000 02 00	00200002
0 01 000 01 00	00100001	0 02 000 12 00	00200022
0 01 000 11 00	00100021	0 02 000 21 00	00200012
0 01 000 20 00	00100010	0 02 000 22 00	00200012
0 01 000 02 00	00100002	0 12 000 00 00	02200000
0 01 000 12 00	00100022	0 12 000 10 00	02200020
0 01 000 21 00	00100011	0 12 000 01 00	02200001
0 01 000 22 00	00100012	0 12 000 11 00	02200021
0 11 000 00 00	02100000	0 12 000 20 00	02200010
0 12 000 02 00	02200002	00 10 000 01 0	01001210
0 12 000 12 00	02200022	00 10 000 11 0	01001211
0 12 000 21 00	02200011	00 10 000 02 0	02202120
0 12 000 22 00	02200012	00 10 000 12 0	02202121
0 21 000 00 00	01100000	00 10 000 21 0	01001212
0 21 000 10 00	01100020	00 10 000 22 0	02102122
0 21 000 01 00	01100001	00 01 000 00 0	00020000

Contd...

Table 2.2 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
0 21 000 11 00	01100021	00 01 000 10 0	00020001
0 21 000 20 00	01100010	00 01 000 01 0	01221210
0 21 000 02 00	01100002	00 01 000 11 0	01221211
0 21 000 12 00	01100022	00 01 000 20 0	00020002
0 21 000 21 00	01100011	00 01 000 02 0	02122120
0 21 000 22 00	01100012	00 01 000 12 0	02122121
0 22 000 00 00	01200000	00 01 000 21 0	01221212
0 22 000 10 00	01200020	00 01 000 22 0	02122122
0 22 000 01 00	01200001	00 11 000 00 0	00120000
0 22 000 11 00	01200021	00 11 000 10 0	00120001
0 22 000 20 00	01200010	00 11 000 01 0	01021210
0 22 000 02 00	01200002	00 11 000 11 0	01021211
0 22 000 12 00	01200022	00 11 000 20 0	00120002
0 22 000 21 00	01200011	00 11 000 02 0	02222120
0 22 000 22 00	01200012	00 11 000 12 0	02222121
00 00 000 01 0	01201210	00 11 000 21 0	01021212
00 00 000 11 0	01201211	00 11 000 22 0	02122122
00 00 000 02 0	02102120	00 20 000 01 0	01101210
00 00 000 12 0	02102121	00 20 000 11 0	01101211
00 00 000 21 0	01201212	00 20 000 02 0	02002120
00 00 000 22 0	02102122	00 20 000 12 0	02002121
00 20 000 21 0	01101212	00 21 000 22 0	02022122
00 20 000 22 0	02002122	00 22 000 00 0	00210000
00 02 000 00 0	00010000	00 22 000 10 0	00210001
00 02 000 10 0	00010001	00 22 000 01 0	01111210
00 02 000 01 0	01211210	00 22 000 11 0	01111211
00 02 000 11 0	01211211	00 22 000 20 0	00210002
00 02 000 20 0	00010002	00 22 000 02 0	02012120

Contd...

Table 2.2 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
00 02 000 02 0	02112120	00 22 000 12 0	02012121
00 02 000 12 0	01211212	00 22 000 21 0	01111212
00 02 000 21 0	01211212	00 22 000 22 0	02012122
00 02 000 22 0	02112122	000 00 000 01	20000002
00 12 000 00 0	00110000	000 00 000 11	21201212
00 12 000 10 0	00110001	000 00 000 02	10000001
00 12 000 01 0	01211210	000 00 000 12	11201211
00 12 000 11 0	01011211	000 00 000 21	21102122
00 12 000 20 0	00110002	000 00 000 22	12102121
00 12 000 02 0	02212120	000 10 000 01	20020002
00 12 000 12 0	02212121	000 10 000 11	21221212
00 12 000 21 0	01011212	000 10 000 02	10020001
00 12 000 22 0	02112122	000 10 000 12	11221211
00 21 000 00 0	00220000	000 10 000 21	22122122
00 21 000 10 0	00220001	000 10 000 22	12122121
00 21 000 01 0	01121210	000 01 000 00	00002000
00 21 000 11 0	01121211	000 01 000 10	01200210
00 21 000 20 0	00220002	000 01 000 01	20002002
00 21 000 02 0	02022120	000 01 000 11	21202212
00 21 000 12 0	02022121	000 01 000 20	02101120
00 21 000 21 0	01121212	000 01 000 02	10002001
000 01 000 12	11200211	000 12 000 10	01222210
000 01 000 21	22101112	000 12 000 01	20021002
000 01 000 22	12101121	000 12 000 11	21222212
000 11 000 00	00022000	000 12 000 20	02120120
000 11 000 10	01220210	000 12 000 02	10021001
000 11 000 01	20022002	000 12 000 12	11222211
000 11 000 11	21220212	000 12 000 21	22120122

Contd...

Table 2.2 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
000 11 000 20	02121120	000 12 000 22	12120121
000 11 000 02	10022001	000 21 000 00	00012000
000 11 000 12	11220211	000 21 000 10	01210210
000 11 000 21	22121122	000 21 000 01	20012002
000 11 000 22	12121121	000 21 000 11	21210212
000 20 000 01	20010002	000 21 000 20	02111120
000 20 000 11	21211212	000 21 000 02	10012001
000 20 000 02	10010001	000 21 000 12	11210211
000 20 000 12	11211211	000 21 000 21	22111122
000 20 000 21	22112122	000 21 000 22	12111121
000 20 000 22	12112121	000 22 000 00	00011000
000 02 000 00	00001000	000 22 000 10	01212210
000 02 000 10	01202210	000 22 000 01	20011002
000 02 000 01	20001002	000 22 000 11	21212212
000 02 000 11	20002212	000 22 000 20	02110120
000 02 000 20	02100120	000 22 000 02	10011001
000 02 000 02	10001001	000 22 000 12	11212211
000 02 000 12	11202211	000 22 000 21	22110122
000 02 000 21	22100122	000 22 000 22	12110121
000 02 000 22	12100121	0000 01 000 1	20000102
000 12 000 00	00021000	0000 01 000 2	10000101
0000 11 000 0	00002100	0000 12 000 2	10002201
0000 11 000 1	20002102	0000 21 000 0	00001100
0000 11 000 2	10002101	0000 21 000 1	20001102
0000 02 000 1	20000202	0000 21 000 2	10001101
0000 02 000 2	10000201	0000 22 000 0	00001200
0000 12 000 0	00002200	0000 22 000 1	20001202
0000 12 000 1	20002202	0000 22 000 2	10001201

2.2 Weight distribution of vectors of $\xi_{(s,b),n,q}$

In this section, we first give the weight distribution of vectors of $\xi_{(s,b),n,q}$, and then we provide the upper bounds on the total weight of the code words of an (n,k) $P_{(s,b),n,q}RC-code$.

For $n = \lambda(b+s)+l$ $(where\ 0 \leq l < s+b)$, the three cases arise: $l = 0$, $1 \leq l < b$ and $b \leq l < b+s$. The maximum weight of a vector of $\xi_{(s,b),n,q}$ is at most

$$\begin{cases} b\lambda & \text{when} \quad l = 0 \\ b\lambda + l & \text{when} \quad 1 \leq l < b \\ b\,(\lambda+1) & \text{when} \quad b \leq l < s+b. \end{cases}$$

In the calculation of the number of vectors of $\xi_{(s,b),n,q}$ in an n-tuple having weight t, we take note of the situation that vectors of $\xi_{(s,b),n,q}$ that start after $(s+1)^{th}$ position, there are some vectors that are already counted in the number of errors that start from the first position. So, we need to exclude them. The number of vectors of $\xi_{(s,b),n,q}$ having weight t is given in the following lemma.

Lemma 2.7. *Let $n = \lambda(b+s)+l$ $(where\ 0 \leq l < s+b)$ and $W_{s,b}(t)$ be the total number of vectors of $\xi_{(s,b),n,q}$ having weight t. Then*

$$W_{s,b}(1) = \sum_{i=1}^{s+1}\left[\binom{m_i}{1} - \binom{k_{i-1}}{1}\right](q-1) + \sum_{i=s+2}^{s+b}\left[\binom{m_i}{1} - \binom{k_{i-1}}{1} - \binom{\beta_{i-1}}{1}\right](q-1)$$

and

$$W_{s,b}(t) = \sum_{i=1}^{s+1}\left[\binom{m_i}{t} - \binom{k_{i-1}}{t}\right](q-1)^t$$
$$+ \sum_{i=s+2}^{s+b}\left[\binom{m_i}{t} - \binom{k_{i-1}}{t} - \binom{p_{i-s-1}}{t} + \binom{p_{i-s-2}}{t}\right](q-1)^t \ for\ t \neq 1,$$

where $k_0 = 0$, $p_0 = 1$, $k_i = m_{i+1} - \beta_i$, $\beta_i = \left\lceil \dfrac{n-i-b+1}{s+b} \right\rceil$,

$$m_i = \begin{cases} b\lambda & for\ 1 \leq i \leq s+1 \\ b\lambda + s - i + 1 & for\ s+2 \leq i \leq s+b \end{cases} \quad if \quad l = 0,$$

$$m_i = \begin{cases} b\lambda + l - i + 1 & \textit{for } 1 \le i \le l \\ b\lambda & \textit{for } l+1 \le i \le s+l+1 \quad \textit{if} \quad 1 \le l < b, \\ b\lambda + s + l - i + 1 & \textit{for } s+l+2 \le i \le s+b \end{cases}$$

$$m_i = \begin{cases} b(\lambda+1) & \textit{for } 1 \le i \le l-b+1 \\ b(\lambda+1) + (l-b-i+1) & \textit{for } l-b+1 < i \le l \quad \textit{if} \quad b \le l < s+b, \\ b\lambda & \textit{for } 1+l \le i \le s+b \end{cases}$$

$$p_i = i\beta_{i+s} \quad \textit{if} \quad l = 0 \textit{ or } b \le l < s+b,$$

$$p_i = \begin{cases} i\beta_{i+s} & \textit{for } 1 \le i \le l \\ i(p_1 - 1) + l & \textit{for } l+1 \le i \le b-1 \end{cases} \quad \textit{if} \quad 1 \le l < b.$$

Next, we give two examples on weight distribution of periodic random errors.

Example 2.8. *Taking* $n = 12$, $s = 3$, $b = 2$ *in Lemma 2.7, we have* $\lambda = 2$ *and* $l = 2$. *Then* $m_1 = 2(2+1) = 6$, $m_2 = 2(2+1) + 0 - 2 + 1 = 5$, $m_3 = m_4 = m_5 = 2 \times 2 = 4$, $\beta_1 = \left\lceil \frac{12-1-2+1}{5} \right\rceil = \left\lceil \frac{10}{5} \right\rceil = 2$, *similarly,* $\beta_2 = \beta_3 = \beta_4 = 2$.
Also, $p_1 = \beta_4 = 2$, $k_1 = m_2 - \beta_1 = 5 - 2 = 3$ *and similarly* $k_2 = k_3 = k_4 = 2$. *Then*

$$\begin{aligned} W_{3,2}(1) &= \sum_{i=1}^{4} \left[\binom{m_i}{1} - \binom{k_{i-1}}{1} \right] (2-1) + \sum_{i=5}^{5} \left[\binom{m_i}{1} - \binom{k_{i-1}}{1} - \binom{\beta_{i-1}}{1} \right] (2-1) \\ &= \left[\binom{6}{1} - \binom{0}{1} \right] + \left[\binom{5}{1} - \binom{3}{1} \right] + \left[\binom{4}{1} - \binom{2}{1} \right] + \left[\binom{4}{1} - \binom{2}{1} \right] \\ &\quad + \left[\binom{4}{1} - \binom{2}{1} - \binom{2}{1} \right] \\ &= 12. \end{aligned}$$

$$\begin{aligned} W_{3,2}(2) &= \sum_{i=1}^{4} \left[\binom{m_i}{2} - \binom{k_{i-1}}{2} \right] (2-1)^2 \\ &\quad + \sum_{i=5}^{5} \left[\binom{m_i}{2} - \binom{k_{i-1}}{2} - \binom{p_{i-s-1}}{2} + \binom{p_{i-s-2}}{2} \right] (2-1)^2 \\ &= \left[\binom{6}{2} - \binom{0}{2} \right] + \left[\binom{5}{2} - \binom{3}{2} \right] + \left[\binom{4}{2} - \binom{2}{2} \right] + \left[\binom{4}{2} - \binom{2}{2} \right] \\ &\quad + \left[\binom{4}{2} - \binom{2}{2} - \binom{2}{2} + \binom{1}{2} \right] \\ &= 36. \end{aligned}$$

Similarly, we can find $W_{3,2}(3) = 41$, $W_{3,2}(4) = 23$, $W_{3,2}(5) = 7$ *and* $W_{3,2}(6) = 1$.

Note that the maximum weight of a 3-periodic random error of length 2 is $b(\lambda+1)=6$ *and these weight distributions can be verified from Example 2.3.*

Example 2.9. *Put* $n=12$, $s=4$, $b=2$ *(so* $l=0$*) in Lemma 2.7. Then* $m_1=m_2=m_3=m_4=m_5=2\times 2=4$, $m_6=(2\times 2)+4-6+1=3$, $\beta_1=\left\lceil\frac{10}{6}\right\rceil=2$, *similarly,* $\beta_2=\beta_3=\beta_4=2$ *and* $\beta_5=1$.
Also, $p_1=\beta_5=1$, $k_1=m_2-\beta_1=4-2=2$ *and similarly* $k_2=k_3=k_4=k_5=2$. *So*

$$\begin{aligned}W_{4,2}(1)=&\sum_{i=1}^{5}\left[\binom{m_i}{1}-\binom{k_{i-1}}{1}\right](2-1)+\sum_{i=6}^{6}\left[\binom{m_i}{1}-\binom{k_{i-1}}{1}-\binom{\beta_{i-1}}{1}\right](2-1)\\ =&\left[\binom{4}{1}-\binom{0}{1}\right]+\left[\binom{4}{1}-\binom{2}{1}\right]+\left[\binom{4}{1}-\binom{2}{1}\right]+\left[\binom{4}{1}-\binom{2}{1}\right]\\ &+\left[\binom{4}{1}-\binom{2}{1}\right]+\left[\binom{3}{1}-\binom{2}{1}-\binom{1}{1}\right]\\ =&\ 12.\end{aligned}$$

$$\begin{aligned}W_{4,2}(2)=&\sum_{i=1}^{5}\left[\binom{m_i}{2}-\binom{k_{i-1}}{2}\right](2-1)^2\\ &+\sum_{i=6}^{6}\left[\binom{m_i}{2}-\binom{k_{i-1}}{2}-\binom{p_{i-s-1}}{2}+\binom{p_{i-s-2}}{2}\right](2-1)^2,\\ =&\left[\binom{4}{2}-\binom{0}{2}\right]+\left[\binom{4}{2}-\binom{2}{2}\right]+\left[\binom{4}{2}-\binom{2}{2}\right]+\left[\binom{4}{2}-\binom{2}{2}\right]\\ &+\left[\binom{4}{2}-\binom{2}{2}\right]+\left[\binom{3}{2}-\binom{2}{2}-\binom{1}{2}+\binom{1}{2}\right]\\ =&\ 28.\end{aligned}$$

Similarly, $W_{4,2}(3)=21$ *and* $W_{4,2}(4)=5$.

Theorem 2.10. *The total weight of all codewords of an* $(n=\lambda(s+b)+l,k)$ $P_{(s,b),n,q}RC-$ *code is at most*

$$\begin{cases}\displaystyle\sum_{i=1}^{n}\binom{n}{i}i(q-1)^i-\sum_{t_1}t_1W_{s-b,2b}(t_1) & \text{if}\quad s>b\\ \displaystyle\sum_{i=1}^{n}\binom{n}{i}i(q-1)^i-\sum_{t_2}t_2W_{1,b+s-1}(t_2) & \text{if}\quad s\le b,\end{cases}$$

where $W_{s-b,2b}(t_1)$ *and* $W_{1,b+s-1}(t_2)$ *are given by Lemma 2.7,*

t_1 *varies from 1 to* $\begin{cases}2b\lambda+l & \text{if}\quad 0\le l\le 2b-1\\ 2b(\lambda+1) & \text{if}\quad 2b\le l<s+b\end{cases}$ *and*

$$t_2 \textit{ varies from 1 to } \begin{cases} (b+s-1)\lambda + l & \textit{if} \quad 0 \le l \le b+s-2 \\ (b+s-1)(\lambda+1) & \textit{if} \quad b+s-1 \le l < s+b. \end{cases}$$

Proof. *Case (i):* If $s > b$, then any vector of $\xi_{(s-b,2b),n,q}$ can be expressed as the sum (difference) of two vectors of $\xi_{(s,b),n,q}$. So, the code can detect any vector of $\xi_{(s-b,b),n,q}$. A codevector can not be any element of $\xi_{(s-b,2b),n,q}$. Therefore, the total weight of all codevectors of a $P_{(s,b),n,q}RC-code$ is at most

$$[\ \textit{Total weight on all n-tuples} - \textit{total weight of all vectors of } \xi_{(s-b,2b),n,q}]$$
$$= \sum_{i=1}^{n} \binom{n}{i} i(q-1)^i - \sum_{t_1} t_1 W_{s-b,2b}(t_1),$$

where t_1 varies from 1 to $\begin{cases} 2b\lambda + l & \textit{if} \quad 0 \le l \le 2b-1 \\ 2b(\lambda+1) & \textit{if} \quad 2b \le l < s+b. \end{cases}$

Case (ii): If $s \le b$, then any vector of $\xi_{(1,b+s-1),n,q}$ can be expressed as the sum (difference) of two vectors of $\xi_{(s,b),n,q}$. So, any vector of $\xi_{(1,b+s-1),n,q}$ can not be a codevector. Therefore, the total weight of all codevectors of the $P_{(s,b),n,q}RC-code$ is at most

$$[\textit{Total weight on all n-tuples} - \textit{total weight of all vectors of } \xi_{(1,b+s-1),n,q}]$$
$$= \sum_{i=1}^{n} \binom{n}{i} i(q-1)^i - \sum_{t_2} t_2 W_{1,b+s-1}(t_2),$$

where t_2 varies from 1 to $\begin{cases} (b+s-1)\lambda + l & \textit{if} \quad 0 \le l \le b+s-2 \\ (b+s-1)(\lambda+1) & \textit{if} \quad b+s-1 \le l < s+b. \end{cases}$ □