Chapter 3
Correction of low-density periodic random errors with weight distribution and error decoding probability

The contents of this chapter are based on the paper:

- Haokip, L. and Das, P. K. Correction of low-density periodic random errors with weight distribution and error decoding probability. *Communicated.*

Chapter 3

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In this chapter, we extend the work done in the previous chapter and present a similar study for *low-density periodic random errors* defined in Definition 1.7. The (Hamming) weight distribution of low-density periodic random errors, their correction by a linear code, and probability of decoding error of such linear codes are studied here. We first, in Section 3.1, give the weight distribution of s-periodic random errors of length b with weight w ($\leq b$) in a q-ary n-tuple. Then we provide Plotkin's type of bound for the set of the error pattern. In Section 3.2, based on the weight distribution, necessary and sufficient conditions for a linear code correcting such errors are derived. Comparisons on the necessary and sufficient numbers of check digits of these codes with the $\boldsymbol{P_{(s,b),n,q}RC-codes}$ are provided. The section concludes with the probability of decoding error for such a code.

Throughout the thesis, we use the following notations for the specific error pattern and the corresponding error-correcting codes as follows:

$\boldsymbol{\xi_{(s,b|w),n,q}}$: set of all s-periodic random errors of length b with weight w in an n-tuple over $GF(q)$.

$\boldsymbol{LDP_{(s,b|w),n,q}RC-code}$: length-n linear code correcting s-periodic random errors of length b with weight w over $GF(q)$.

$\boldsymbol{\gamma}$: a function from $\{1, 2, \ldots, s+b-1\}$ to $\{1, 2, \ldots, b\}$ defined by

$$\gamma(r) = \begin{cases} r & if \quad 0 \leq r \leq b \\ b & if \quad b < r < b+s. \end{cases}$$

3.1 Hamming weight distribution of $\xi_{(s,b|w),n,q}$

In this section, we give the Hamming weight distribution of vectors of $\xi_{(s,b|w),n,q}$ and average Hamming weight of a vector from the set of vectors of $\xi_{(s,b|w),n,q}$. For this, we use the following two lemmas.

Lemma 3.1. *Consider any non-negative integers n, b and s ($n \geq b+s$) and a vector of $\xi_{(s,b|w),n,q}$ that starts from the i^{th} position ($i = 1, 2, \ldots, n$). Let λ_i denote the number of sets of nonzero positions and m_i the maximum number of nonzero positions in the error pattern. Then*

$$\lambda_i = \left\lceil \frac{n-i+1}{s+b} \right\rceil \text{ and } m_i = \left\lfloor \frac{n-i+1}{s+b} \right\rfloor b + \gamma\big((n-i+1) \mod (b+s)\big).$$

Proof. For any vector of $\xi_{(s,b|w),n,q}$ where error starts from the i^{th} position ($i = 1, 2, \ldots, s+b$), the nonzero components are confined to the last $n-i+1$ positions:

$$(\overbrace{00\ldots0}^{i-1}\overbrace{\underbrace{\bullet\bullet\bullet\cdots\bullet}_{b}\underbrace{00\ldots0}_{s}\underbrace{\bullet\bullet\bullet\cdots\bullet}_{b}\underbrace{00\ldots0}_{s}\underbrace{\bullet\bullet\bullet\cdots\bullet}_{b}}^{n-i+1}),$$

where $\bullet$ represents any field elements.

By Euclidean division algorithm, for integers $n-i+1$ and $s+b$, there exist integers λ'_i and r_i such that

$$n-i+1 = \lambda'_i(s+b) + r_i, \text{ where } 0 \leq r_i < s+b. \tag{3.1}$$

Then the number of sets, in which the nonzero components of the error pattern are confined to, is

$$\lambda_i = \lambda'_i + \left\lceil \frac{r_i}{s+b} \right\rceil = \left\lceil \frac{\lambda'_i(s+b)+r_i}{s+b} \right\rceil = \left\lceil \frac{n-i+1}{s+b} \right\rceil.$$

From Equation (3.1), we know that there are λ'_i sets of $s+b$ consecutive components and one set of consecutive r_i components in the vector of length $n-i+1$ in which the error components are confined. In each of the λ'_i sets, the first b positions can be filled by any nonzero component and in the last set of r_i consecutive components, there are $\gamma(r_i) = \gamma\left((n-i+1) \mod (b+s)\right)$ positions which can be filled by any nonzero component. Therefore, the number of nonzero positions in the periodic error pattern that starts from the i^{th} position is

$$\begin{aligned} m_i &= \lambda_i' b + \gamma\left((n-i+1) \mod (b+s)\right) \\ &= \left\lfloor \frac{n-i+1}{s+b} \right\rfloor b + \gamma\big((n-i+1) \mod (b+s)\big). \end{aligned}$$

□

Lemma 3.2. *Let p_i be the number of common nonzero positions of elements of $\xi_{(s,b|w),n,q}$ that start from the i^{th} $(i = s+2, \ldots, s+b)$ position with any error vector of $\xi_{(s,b|w),n,q}$ that starts from the 1^{st} position. Then p_i is given by*

(1) when $l = 0$ and $b-1 \leq l \leq s+b$: $\quad p_i = (i-s-1)\beta_{i-1}$ *and*

(2) when $1 \leq l < b-1$: $\quad p_i = \begin{cases} (i-s-1)\beta_{i-1} & \text{if } \; i = s+2 \\ (i-s-1)\beta_{i-1} + l & \text{if } \; i = s+3, \ldots, s+b, \end{cases}$

where $\beta_i = \left\lceil \frac{n-i-b+1}{s+b} \right\rceil$.

Proof. **Case 1:** When $l = 0$ and $b-1 \leq l \leq s+b$:

The common nonzero positions of the error pattern of $\xi_{(s,b|w),n,q}$ that starts from the $(s+2)^{th}$ position with the error pattern that starts from the first position are

$$s+b+1, 2(s+b)+1, \ldots, \beta_{s+1}(s+b)+1,$$

where $\beta_{s+1} = \left\lceil \frac{n-(s+1)-b+1}{s+b} \right\rceil$.

This number of common nonzero position is given by β_{s+1}.

The common nonzero positions of the error pattern of $\xi_{(s,b|w),n,q}$ that starts from the $(s+3)^{th}$ position with the error pattern that starts from the first position are

$$s+b+1, s+b+2, 2(s+b)+1, 2(s+b)+2, \ldots, \beta_{s+2}(s+b)+1, \beta_{s+2}(s+b)+2,$$

where $\beta_{s+2} = \left\lceil \frac{n-(s+2)-b+1}{s+b} \right\rceil$.

The number of these common nonzero positions is given by $2\beta_{s+2}$.

Continuing this, the common nonzero positions of the error pattern of $\xi_{(s,b|w),n,q}$

that starts from the $(s+b)^{th}$ position with the error pattern that starts from the first position are

$$s+b+1, s+b+2, \ldots, s+b+(b-1),\ 2(s+b)+1, 2(s+b)+2, \ldots, 2(s+b)+(b-1),$$
$$\ldots\ldots \beta_{s+b-1}(s+b)+1, \beta_{s+b-1}(s+b)+2, \ldots, \beta_{s+b-1}(s+b)+(b-1),$$

where $\beta_{s+b-1} = \left\lceil \dfrac{n-(s+b-1)-b+1}{s+b} \right\rceil$.

The number of these common nonzero positions is given by $(b-1)\beta_{s+b-1}$.

Thus

$$p_i = (i-s-1)\beta_{i-1} \quad for\ i = s+2, s+3, \ldots, s+b,$$

where $\beta_i = \left\lceil \dfrac{n-i-b+1}{s+b} \right\rceil$.

Case 2: When $1 \le l < b-1$:

The common nonzero positions of the error pattern of $\xi_{(s,b|w),n,q}$ that starts from the $(s+2)^{th}$ position with the error pattern that starts from the first position are

$$s+b+1, 2(s+b)+1, \ldots, \beta_{s+1}(s+b)+1,$$

where $\beta_{s+1} = \left\lceil \dfrac{n-(s+1)-b+1}{s+b} \right\rceil$.

The number of common nonzero positions is given by β_{s+1}.

If the error pattern starts from the $(s+3)^{th}$ position, the common nonzero positions with the error pattern that starts from the $(s+2)^{th}$ position with the error patterns that start from the first position, excluding the last set, are

$$s+b+1, s+b+2, 2(s+b)+1, 2(s+b)+2, \ldots, \beta_{s+2}(s+b)+1, \beta_{s+2}(s+b)+2,$$

where $\beta_{s+2} = \left\lceil \dfrac{n-(s+2)-b+1}{s+b} \right\rceil$.

The common positions with the last set are $\lambda_{s+b}(s+b)+1, \lambda_{s+b}(s+b)+2, \ldots, \lambda_{s+b}(s+b)+l$ whose number is l. Therefore, the total number of common nonzero positions is given by $2\beta_{s+2}+l$.

Continuing this, if the error pattern starts from the $(s+b)^{th}$ position, the common nonzero positions with the error patterns that start from the first position, excluding the last set, are

$s+b+1, s+b+2, \ldots, s+b+(b-1),\ 2(s+b)+1, 2(s+b)+2, \ldots, 2(s+b)+(b-1),$

$\ldots\ldots \beta_{s+b-1}(s+b)+1, \beta_{s+b-1}(s+b)+2, \ldots, \beta_{s+b-1}(s+b)+(b-1),$

where $\beta_{s+b-1} = \left\lceil \dfrac{n-(s+b-1)-b+1}{s+b} \right\rceil$.

The last set has common positions $\lambda_{s+b}(s+b)+1, \lambda_{s+b}(s+b)+2, \ldots, \lambda_{s+b}(s+b)+l$ whose number is l.

The number of these common nonzero positions is given by $(b-1)\beta_{s+b-1}+l$. Thus

$$p_i = \begin{cases} (i-s-1)\beta_{i-1} & \text{if} \quad i = s+2 \\ (i-s-1)\beta_{i-1}+l & \text{if} \quad i = s+3, \ldots, s+b. \end{cases}$$

□

Note that for $n = \lambda(b+s)+l$, where $0 \leq l < s+b$, the maximum weight w_{max} of the set $\xi_{(s,b|w),n,q}$ is given by

$$w_{max} = \begin{cases} w\lambda & \text{when} \quad l = 0 \\ w\lambda + \min\{l, w\} & \text{when} \quad 1 \leq l < b \\ w\,(\lambda+1) & \text{when} \quad b \leq l < s+b. \end{cases}$$

Theorem 3.3. *Let $R_{(s,b|w),n,q}(j)$ be the total number of vectors of $\xi_{(s,b|w),n,q}$ whose weight is j. Then*

For $j = 1$:

$$R_{(s,b|w),n,q}(1) = \sum_{i=1}^{s+1} \left[\binom{m_i}{1} - \binom{k_{i-1}}{1} \right] (q-1) + \sum_{i=s+2}^{s+b} \left[\binom{m_i}{1} - \binom{k_{i-1}}{1} - \binom{\beta_{i-1}}{1} \right] (q-1),$$

For $2 \leq j \leq w$:

$$R_{(s,b|w),n,q}(j) = \sum_{i=1}^{s+1} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} \right] (q-1)^j + \sum_{i=s+2}^{s+b} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{p_i}{j} + \binom{p_{i-1}}{j} \right] (q-1)^j,$$

For $w+1 \leq j \leq w_{max}-1$:

$$R_{(s,b|w),n,q}(j) = \sum_{i=1}^{s+1} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{(b-w)\beta_{i-1}}{1} \binom{m_i - b}{j-w-1} \right] (q-1)^j$$

$$+\sum_{i=s+2}^{s+b}\Bigg[\binom{m_i}{j}-\binom{k_{i-1}}{j}-\binom{(b-w)\beta_{i-1}}{1}\binom{m_i-b}{j-w-1}$$
$$-\binom{p_i}{j}+\binom{p_{i-1}}{j}\Bigg](q-1)^j,$$

For $j = w_{max}$:

$$R_{(s,b|w),n,q}(w_{max})=\begin{cases}\Big[(s+1)b^{\lambda}+b^{\lambda-1}-s-1\Big](q-1)^{w_{max}} & when \quad l=0\\ b^{\lambda}(q-1)^{w_{max}} & when \quad 1\le l<b\\ \Big[(l-b+1)b^{\lambda+1}+b^{\lambda}-l+b-1\Big](q-1)^{w_{max}} & when \quad b\le l<s+b,\end{cases}$$

where $p_{s+1} = 1$, $k_0 = 0$, $k_i = m_{i+1}-\beta_i$, $\beta_i = \left\lceil \frac{n-i-b+1}{s+b} \right\rceil$ *and* $m_i = b\left\lfloor \frac{n-i+1}{s+b} \right\rfloor +$ $\gamma\Big((n-i+1) \mod (s+b)\Big)$.

Proof. **Case I:** For $j = 1$.

The number of error patterns of weight 1 that start from the i^{th} position (where $i = 2, \ldots, s+1$) is given by $\binom{m_i}{1}(q-1)$. But in the calculation $\binom{m_{i+1}}{1}(q-1)$, the number of already counted nonzero components in $\binom{m_i}{1}(q-1)$ is $k_i = m_{i+1} - \beta_i$ for $i = 1, 2, 3, \ldots, s+1$, where $\beta_i = \left\lceil \frac{n-i-b+1}{s+b} \right\rceil$ represents the total number of complete sets of b consecutive positions that contain the nonzero positions in the error pattern. Therefore, the total number of the errors having weight 1 is the quantity $\sum_{i=1}^{s+1}\left[\binom{m_i}{1}-\binom{k_{i-1}}{1}\right](q-1)^1$ with $k_0 = 0$.

For error patterns whose starting position are $i = s+2, \ldots, s+b$, all the vectors of weight 1 are already counted in the error patterns that start from the first position. The number of these nonzero components is given by β_{i-1}. So, there are $\binom{\beta_{i-1}}{1}$ number of vectors of weight 1 which need to be substracted. Therefore, the number of vectors of weight 1 in these positions is given by the quantity $\binom{m_i}{1}-\binom{k_{i-1}}{1}-\binom{\beta_{i-1}}{1}$. Thus we have

$$R_{(s,b|w),n,q}(1)=\sum_{i=1}^{s+1}\left[\binom{m_i}{1}-\binom{k_{i-1}}{1}\right](q-1)$$
$$+\sum_{i=s+2}^{s+b}\left[\binom{m_i}{1}-\binom{k_{i-1}}{1}-\binom{\beta_{i-1}}{1}\right](q-1).$$

Case II: For $2 \le j \le w$.

As above, we can find the total number of the errors having weight j that start from the i^{th} position, where $i = 1, 2, \ldots, s+1$, is the quantity $\sum_{i=1}^{s+1} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} \right] (q-1)^j$ with $k_0 = 0$. But, for error patterns that start from $i = s+2, \ldots, s+b$, there are some more common vectors with the already counted error vectors that start from the first position. By Lemma 3.2, p_i denotes the number of common nonzero components that start from the i^{th} ($i = s+2, \ldots, s+b$) position which are already present in the errors that start from the first position. So, $\binom{p_i}{j}(q-1)^j$ gives the number of error vectors of weight j that start from the i^{th} ($i = s+2, \ldots, s+b$) position, which are already present that start from the first position. This includes some vectors which are already deleted by using the term $\binom{k_{i-1}}{j}(q-1)^j$, thus the term $\binom{p_{i-1}}{j}(q-1)^j$ is added to include such already deleted error vectors. So, the exact number of common vectors that need to be excluded is $[\binom{p_i}{j} - \binom{p_{i-1}}{j}](q-1)^j$.

Therefore, we have

$$\begin{aligned} R_{(s,b|w),n,q}(j) = &\sum_{i=1}^{s+1} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} \right] (q-1)^j \\ &+ \sum_{i=s+2}^{s+b} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{p_i}{j} + \binom{p_{i-1}}{j} \right] (q-1)^j. \end{aligned}$$

Case III: For $w + 1 \leq j \leq w_{max} - 1$.

In this case also, we can similarly calculate the total number of all error vectors having weight j starting from the i^{th} position, where $i = 1, 2, \ldots, s+1$, after deleting the common vectors as the quantity

$$\sum_{i=1}^{s+1} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{(b-w)\beta_{i-1}}{1} \binom{m_i - b}{j-w-1} \right] (q-1)^j \quad \text{with } k_0 = 0.$$

Again, for error vectors having weight j starting from $(s+2)^{th}$ to $(s+b)^{th}$ positions, there are some more common vectors which we have calculated as the previous case to be $\binom{p_i}{j} - \binom{p_{i-1}}{j}$. Therefore

$$\begin{aligned} R_{(s,b|w),n,q}(j) = &\sum_{i=1}^{s+1} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{(b-w)\beta_{i-1}}{1} \binom{m_i - b}{j-w-1} \right] (q-1)^j \\ &+ \sum_{i=s+2}^{s+b} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{(b-w)\beta_{i-1}}{1} \binom{m_i - b}{j-w-1} \right. \end{aligned}$$

$$-\binom{p_i}{j}+\binom{p_{i-1}}{j}\Big](q-1)^j.$$

Case IV: For $j = w_{max}$.

In this case, the number of the error vectors having weight j is calculated and found to have different formulas for $l = 0$, $1 \leq l < b$ and $b \leq l < s + b$ as follows:

$$R_{(s,b|w),n,q}(w_{max}) = \begin{cases} \big[(s+1)b^{\lambda} + b^{\lambda-1} - s - 1\big](q-1)^{w_{max}} & \text{when} \quad l = 0 \\ b^{\lambda}(q-1)^{w_{max}} & \text{when} \quad 1 \leq l < b \\ \big[(l-b+1)b^{\lambda+1} + b^{\lambda} - l + b - 1\big](q-1)^{w_{max}} & \text{when} \quad b \leq l < s+b. \end{cases}$$

□

Remark 3.4. *The values of m_i and p_i in Lemma 2.7 are given by*

$$m_i = \begin{cases} b\lambda & \text{for } 1 \leq i \leq s+1 \\ b\lambda + s - i + 1 & \text{for } s+2 \leq i \leq s+b \end{cases} \quad \textbf{if} \quad l = 0,$$

$$m_i = \begin{cases} b\lambda + l - i + 1 & \text{for } 1 \leq i \leq l \\ b\lambda & \text{for } l+1 \leq i \leq s+l+1 \\ b\lambda + s + l - i + 1 & \text{for } s+l+2 \leq i \leq s+b \end{cases} \quad \textbf{if} \quad 1 \leq l < b,$$

$$m_i = \begin{cases} b(\lambda+1) & \text{for } 1 \leq i \leq l-b+1 \\ b(\lambda+1) + (l-b-i+1) & \text{for } l-b+1 < i \leq l \\ b\lambda & \text{for } 1+l \leq i \leq s+b \end{cases} \quad \textbf{if} \quad b \leq l < s+b,$$

$$p_i = i\beta_{i+s} \quad \textbf{if} \quad l = 0 \text{ or } b \leq l < s+b,$$

$$p_i = \begin{cases} i\beta_{i+s} & \text{for } 1 \leq i \leq l \\ i(p_1 - 1) + l & \text{for } l+1 \leq i \leq b-1 \end{cases} \quad \textbf{if} \quad 1 \leq l < b.$$

These values of m_i and p_i are simplified in Lemma 3.1 and Lemma 3.2.

Further for $b = w$, we have

$$w_{max} = \begin{cases} b\lambda & \text{when} \quad l = 0 \\ b\lambda + l & \text{when} \quad 1 \leq l < b \\ b\,(\lambda+1) & \text{when} \quad b \leq l < s+b \end{cases} \quad \text{and} \quad \binom{(b-w)\beta_{i-1}}{1}\binom{m_i - b}{j-w-1} = 0.$$

This gives us that Theorem 3.3 coincides with Lemma 2.7.

Example 3.5. *Consider the error set* $\xi_{(3,2|1),11,3}$ *in the ternary space of* 11*-tuples with weight* $w = 1$*, then* $\lambda = 2$*;* $l = 11 \mod 5 = 1$*;* $m_1 = 5, m_2 = \cdots = m_5 = 4$*;* $p_{s+1} = 1, p_{s+2} = 2$*;* $\beta_0 = \beta_1 = \cdots = \beta_4 = 2$*. Substituting these values in the above Theorem 3.3, we get*

$$\begin{aligned}
R_{(3,2|1),11,3}(1) &= \left[\binom{5}{1} - \binom{0}{1}\right](3-1)^1 + \left[\binom{4}{1} - \binom{2}{1}\right](3-1)^1 \\
&+ \left[\binom{4}{1} - \binom{2}{1}\right](3-1)^1 + \left[\binom{4}{1} - \binom{2}{1}\right](3-1)^1 \\
&+ \left[\binom{4}{1} - \binom{2}{1} - \binom{2}{1}\right](3-1)^1 \\
&= 5 \times 2 + 2 \times 3 \times 2 + 0 \\
&= 10 + 12 \\
&= 22.
\end{aligned}$$

$$\begin{aligned}
R_{(3,2|1),11,3}(2) &= \left[\binom{5}{2} - \binom{0}{2} - \binom{(3-2)\times 2}{1}\binom{3}{2-1-1}\right](3-1)^2 \\
&+ \left[\binom{4}{2} - \binom{2}{2} - \binom{(3-2)\times 2}{1}\binom{2}{2-1-1}\right](3-1)^2 \\
&+ \left[\binom{4}{2} - \binom{2}{2} - \binom{(3-2)\times 2}{1}\binom{2}{2-1-1}\right](3-1)^2 \\
&+ \left[\binom{4}{2} - \binom{2}{2} - \binom{(3-2)\times 2}{1}\binom{2}{2-1-1}\right](3-1)^2 \\
&+ \left[\binom{4}{2} - \binom{2}{2} - \binom{2}{2} + \binom{1}{2} - \binom{(3-2)\times 2}{1}\binom{2}{2-1-1}\right](3-1)^2 \\
&= 76.
\end{aligned}$$

$$R_{(3,2|1),11,3}(3) = 2^2(3-1)^3 = 32.$$

Here the maximum weight is $w_{max} = 3$*. This example can be verified by using Example 3.12 of the next section.*

Example 3.6. *Consider the set* $\xi_{(3,2|1),12,2}$ *in the binary space of* 12*-tuples with weight* $w = 1$*, then* $\lambda = 2$*;* $l = 12 \mod 5 = 2$*;* $m_1 = 6, m_2 = 5, m_3 = \cdots = m_5 = 4$*;* $p_{s+1} = 1, p_{s+2} = 2$*;* $\beta_0 = 3$*,* $\beta_1 = \beta_2 = \cdots = \beta_4 = 2$*. Substituting the values in Theorem 3.3, we get*

$$\begin{aligned}
R_{(3,2|1),12,2}(1) &= \left[\binom{6}{1} - \binom{0}{1}\right] + \left[\binom{5}{1} - \binom{3}{1}\right] + \left[\binom{4}{1} - \binom{2}{1}\right] + \left[\binom{4}{1} - \binom{2}{1}\right] \\
&+ \left[\binom{4}{1} - \binom{2}{1} - \binom{2}{1}\right]
\end{aligned}$$

$$= 6 + 2 \times 3 + 0$$

$$= 12.$$

$$R_{(3,2|1),12,2}(2) = \left[\binom{6}{2} - \binom{0}{2} - \binom{(3-2)\times 3}{1}\binom{4}{2-1-1}\right]$$

$$+ \left[\binom{5}{2} - \binom{3}{2} - \binom{(3-2)\times 2}{1}\binom{3}{2-1-1}\right]$$

$$+ \left[\binom{4}{2} - \binom{2}{2} - \binom{(3-2)\times 2}{1}\binom{2}{2-1-1}\right]$$

$$+ \left[\binom{4}{2} - \binom{2}{2} - \binom{(3-2)\times 2}{1}\binom{2}{2-1-1}\right]$$

$$+ \left[\binom{4}{2} - \binom{2}{2} - \binom{2}{2} + \binom{1}{2} - \binom{(3-2)\times 2}{1}\binom{2}{2-1-1}\right]$$

$$= 12 + 5 + 3 \times 2 + 2$$

$$= 25.$$

$$R_{(3,2|1),12,2}(3) = (2-2+1) \times 2^3 + 2^2 - 2 + 2 - 1 = 11.$$

Note that the maximum weight is $w_{max} = 3$. *This example can be verified by Example 3.13.*

Example 3.7. *For the set* $\xi_{(4,3|2),14,2}$ *in the binary space of* 14*-tuples with weight* $w = 2$*, we have* $\lambda = 2$*;* $l = 14 \mod 7 = 0$*;* $m_1 = m_2 = \cdots = m_5 = 6, m_6 = 5, m_7 = 4$*;* $p_{s+1} = 1, p_{s+2} = 1, p_{s+3} = 2$*;* $\beta_0 = \beta_1 = \cdots = \beta_4 = 2$*,* $\beta_5 = \beta_6 = 1$*. From Theorem 3.3, we get*

$$R_{(4,3|2),14,2}(1) = \left[\binom{6}{1} - \binom{0}{1}\right] + \left[\binom{6}{1} - \binom{4}{1}\right]$$

$$+ \left[\binom{6}{1} - \binom{4}{1}\right] + \left[\binom{6}{1} - \binom{4}{1}\right]$$

$$+ \left[\binom{6}{1} - \binom{4}{1}\right] + \left[\binom{6}{1} - \binom{4}{1}\right]$$

$$+ \left[\binom{5}{1} - \binom{3}{1} - \binom{2}{1}\right] + \left[\binom{4}{1} - \binom{2}{1} - \binom{1}{1}\right]$$

$$= 6 + 4 \times 2 + 0$$

$$= 14.$$

$$R_{(4,3|2),14,2}(2) = \left[\binom{6}{2} - \binom{0}{2}\right] + \left[\binom{6}{2} - \binom{4}{2}\right]$$

$$+ \left[\binom{6}{2} - \binom{4}{2}\right] + \left[\binom{6}{2} - \binom{4}{2}\right] + \left[\binom{6}{2} - \binom{4}{2}\right]$$

$$
\begin{aligned}
&+\left[\binom{5}{2}-\binom{4}{2}-\binom{1}{2}+\binom{1}{2}\right]+\left[\binom{4}{2}-\binom{3}{2}-\binom{2}{2}+\binom{1}{2}\right]\\
&=15+9\times 4+4+2\\
&=57.
\end{aligned}
$$

$$
\begin{aligned}
R_{(4,3|2),14,2}(3) &= \left[\binom{6}{3}-\binom{0}{3}-\binom{(3-2)\times 2}{1}\binom{4}{3-2-1}\right]\\
&+\left[\binom{6}{3}-\binom{4}{3}-\binom{(3-2)\times 2}{1}\binom{4}{3-2-1}\right]\\
&+\left[\binom{6}{3}-\binom{4}{3}-\binom{(3-2)\times 2}{1}\binom{4}{3-2-1}\right]\\
&+\left[\binom{6}{3}-\binom{4}{3}-\binom{(3-2)\times 2}{1}\binom{4}{3-2-1}\right]\\
&+\left[\binom{5}{3}-\binom{4}{3}-\binom{1}{3}+\binom{1}{3}-\binom{(3-2)\times 1}{1}\binom{2}{3-2-1}\right]\\
&+\left[\binom{4}{3}-\binom{3}{3}-\binom{2}{3}+\binom{1}{3}-\binom{(3-2)\times 1}{1}\binom{1}{3-2-1}\right]\\
&=(20-2)+(20-4-2)\times 4+(10-4-1)+(4-1-1)\\
&=18+56+5+2\\
&=81.\\
R_{(4,3|2),14,2}(4) &= (4+1)\times 3^2+3^{2-1}-4-1=45-2=43.
\end{aligned}
$$

Here the maximum weight is $w_{max}=4$. This example can be verified by Example 3.14.

Now we give a bound on the minimum weight of a vector of the set $\xi_{(s,b|w),n,q}$, an equivalent result to Plotkin's bound (Result 1.12).

Theorem 3.8. *The minimum weight of a vector of the set $\xi_{(s,b|w),n,q}$ is at most*

$$
\frac{\sum_{j=1}^{w_{max}} jR_{(s,b|w),n,q}(j)}{\sum_{j=1}^{w_{max}} R_{(s,b|w),n,q}(j)},
$$

where $R_{(s,b|w),n,q}(j)$ is given by Theorem 3.3.

Proof. By Theorem 3.3, the number of vectors of $\xi_{(s,b|w),n,q}$ having Hamming weight j is $R_{(s,b|w),n,q}(j)$ and the total weight of vectors of $\xi_{(s,b|w),n,q}$ is given by $\sum_{j=1}^{w_{max}} jR_{(s,b|w),n,q}(j)$.

As the minimum weight of a vector can be at most the average weight, the minimum weight of a vector of $\xi_{(s,b|w),n,q}$ is at most

$$\frac{\sum_{j=1}^{w_{max}} jR_{(s,b|w),n,q}(j)}{\sum_{j=1}^{w_{max}} R_{(s,b|w),n,q}(j)}.$$

This proves the result. □

3.2 $LDP_{(s,b|w),n,q}RC$-codes, comparison and error decoding probability

In this section, we derive necessary and sufficient conditions for the existence of $LDP_{(s,b|w),n,q}RC$-codes just like in the previous chapter, followed by comparisons on the necessary and sufficient numbers of check digits of $LDP_{(s,b|w),n,q}RC$-codes with $P_{(s,b),n,q}RC$-codes along with the probability of decoding error of $LDP_{(s,b|w),n,q}RC$-codes.

3.2.1 Conditions for existence of $LDP_{(s,b|w),n,q}RC$-codes

Out of the necessary and sufficient conditions for the existence of a $LDP_{(s,b|w),n,q}RC$-code, we first prove the necessary condition by considering the weight distribution of the error pattern derived in Section 3.1.

Theorem 3.9. *Every (n,k) $LDP_{(s,b|w),n,q}RC$-code satisfies*

$$n-k \geq \log_q \Big[1 + \sum_{j=1}^{w_{max}} R_{(s,b|w),n,q}(j)\Big],$$

where $R_{(s,b|w),n,q}(j)$ is given by Theorem 3.3.

Proof. By Theorem 3.3, the number of error vectors in the set $\xi_{(s,b|w),n,q}$, including the zero vector, is $1+|\xi_{(s,b|w),n,q}| = 1+\sum_{j=1}^{w_{max}} R_{(s,b|w),n,q}(j)$. As the maximum available coset is q^{n-k} and $LDP_{(s,b|w),n,q}RC$-code corrects all such errors, we have

$$q^{n-k} \geq 1 + \sum_{j=1}^{w_{max}} R_{(s,b|w),n,q}(j)$$
$$\Longrightarrow n-k \geq \log_q \Big[1 + \sum_{j=1}^{w_{max}} R_{(s,b|w),n,q}(j)\Big].$$

□

Remark 3.10. *The maximum number of codewords of an* (n,k) $LDP_{(s,b|w),n,q}RC$-*code is*

$$M \leq \frac{q^n}{1 + \sum_{j=1}^{w_{max}} R_{(s,b|w),n,q}(j)}.$$

The following is the sufficient condition for the existence of a $LDP_{(s,b|w),n,q}RC$-code.

Theorem 3.11. *For the existence of an* (n,k) $LDP_{(s,b|w),n,q}RC$*-code, the following condition is sufficient:*

$$q^{n-k} > \sum_{j=0}^{w-1} \binom{b-1}{j}(q-1)^j \left(\sum_{j=0}^{w} \binom{b-1}{j}(q-1)^j\right)^{\lambda-1} \sum_{j=0}^{\min\{w,g\}} \binom{g}{j}(q-1)^j \left(1 + \sum_{j=1}^{w_{max}} R_{(s,b|w),n-b,q}(j)\right), \tag{3.2}$$

where $g = \gamma(l)$ *and* $R_{(s,b|w),n-b,q}(j)$ *is given by Theorem 3.3.*

$$\left(Here \sum_{j=0}^{\min\{w,g\}} \binom{g}{j}(q-1)^j = 1 \quad for \quad g = 0\right)$$

Proof. The proof is shown by constructing an appropriate $(n-k) \times n$ parity-check matrix H of the code. Suppose that $h_1, h_2, h_3, \ldots, h_{n-1}$ columns are added suitably to H. Any (nonzero) column h_n is added to H as n^{th} column provided that it is not a linear combination of $w-1$ or less columns within the set of the immediately preceding $b-1$ columns, together with linear combinations of columns of previous sets of b consecutive columns with at most w columns from each set, along with a linear combination of w or less columns taken from the last set of b or less consecutive columns confined to the first $n-b$ columns with the condition that the sets are also at a gap of s columns. This can be written as

$$h_n \neq \left(\sum_{i=1}^{b-1} a_{i1} h_{n-i} + \sum_{i=0}^{b-1} b_{i1} h_{n-(s+b)-i} + \sum_{i=0}^{b-1} b_{i2} h_{n-2(s+b)-i} + \cdots + \sum_{i=0}^{g-1} b_{i\lambda} h_{n-\lambda(s+b)-i} \right)$$
$$+ \left(\sum_{i=0}^{b-1} \alpha_{i1} h_{j'-i} + \sum_{i=0}^{b-1} \beta_{i1} h_{j'-(s+b)-i} + \sum_{i=0}^{b-1} \beta_{i2} h_{j'-2(s+b)-i} + \cdots + \sum_{i=0}^{g'-1} \beta_{i\lambda'} h_{j'-\lambda'(s+b)-i} \right), \tag{3.3}$$

where $a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij} \in GF(q)$ such that the number of nonzero a_{ij}'s is at most $w-1$, and that of b_{ij}'s, α_{ij}'s, β_{ij}'s are at most w; $j' \leq n-b$; $g = \gamma\big(n \mod (s+b)\big) = \gamma(l)$, $g' = \gamma\big((n-b-j'+1) \mod (s+b)\big)$ and $\lambda' = \lfloor \frac{n-b}{s+b} \rfloor$.

The number of coefficients a_{i1}'s is $\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^j$.

The number of coefficients b_{ij}'s is $\left(\sum_{j=0}^{w} \binom{b-1}{j} (q-1)^j \right)^{\lambda-1} \times \sum_{j=0}^{\min\{w,g\}} \binom{g}{j} (q-1)^j$.

So, the number of all possible linear combinations in the first bracket on R.H.S. of (3.3) is

$$\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^j \times \left(\sum_{j=0}^{w} \binom{b-1}{j} (q-1)^j \right)^{\lambda-1} \times \sum_{j=0}^{\min\{w,g\}} \binom{g}{j} (q-1)^j.$$

The second bracket of (3.3) gives the number of low-density periodic random errors in a vector of length $n-b$. This is given by Theorem 3.9 as

$$1 + \sum_{j=1}^{w_{max}} R_{(s,b|w),n-b,q}(j).$$

Therefore, the total number of all the possible linear combinations of the R.H.S of (3.3) is

$$\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^j \left(\sum_{j=0}^{w} \binom{b-1}{j} (q-1)^j \right)^{\lambda-1} \sum_{j=0}^{\min\{w,g\}} \binom{g}{j} (q-1)^j$$
$$\times \left(1 + \sum_{j=1}^{w_{max}} R_{(s,b|w),n-b,q}(j) \right). \tag{3.4}$$

Since we can have at most q^{n-k} columns, taking q^{n-k} greater than or equal to the term computed in (3.4) gives the sufficient condition for the existence of the required code. That is

$$q^{n-k} > \sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^j \left(\sum_{j=0}^{w} \binom{b-1}{j} (q-1)^j \right)^{\lambda-1}$$

$$\times \sum_{j=0}^{\min\{w,g\}} \binom{g}{j}(q-1)^j \left(1+\sum_{j=1}^{w_{max}} R_{(s,b|w),n-b,q}(j)\right). \tag{3.5}$$

This proves the theorem.

In the following examples, λ', p_i' and β_i' represent the values of λ, p_i and β_i respectively when n is replaced by $n-b$. □

Example 3.12. *Consider $n = 11$, $s = 3$, $b = 2$, $w = 1$ and $q = 3$ in Theorem 3.11, then $\lambda = 2$; $l = 11 \mod 5 = 1$; and $\lambda' = 1$; $p'_{s+1} = 1, p'_{s+2} = 1$; $\beta'_0 = \beta'_1 = \beta'_2 = 2, \beta'_3 = \beta'_4 = 1$. Putting these values in the above inequality (3.2), we get*

$$\begin{aligned}
3^{n-k} &> \sum_{j=0}^{0}\binom{1}{j}(3-1)^j\left(\sum_{j=0}^{1}\binom{2-1}{j}(3-1)^j\right)^1 \\
&\times \sum_{j=0}^{\min\{w=1,g=1\}}\binom{1}{j}(3-1)^j\left(1+\sum_{j=1}^{3}R_{(3,2|1),9,3}(j)\right) \\
&= 1\times 3\times 3\times\left[1+66\right] \\
&= 9\times 67 \\
&= 603.
\end{aligned}$$

This implies $n-k \geq 6$. Thus, we can construct a parity check matrix H of order 6×11, which gives rise to a $(11,5)$ ternary $LDP_{(3,2|1),11,3}RC$-code:

$$H = \begin{bmatrix}
1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2
\end{bmatrix}_{6\times 11}.$$

It can be verified from the Error Pattern-Syndrome Table 3.1 that the syndromes of all vectors of $\xi_{(3,2|1),11,3}$ are nonzero and distinct and the ternary $(11,5)$ code (the null space of H) is a $LDP_{(3,2|1),11,3}RC$-code.

Table 3.1: **Error Pattern-Syndrome**

Error Patterns	Syndromes	Error Patterns	Syndromes
00 000 00 000 1	001002	00 000 01 000 2	001001
00 000 00 000 2	002001	00 000 02 000 1	002002
00 000 01 000 0	002000	00 000 02 000 2	000001
00 000 02 000 0	001000	00 000 10 000 1	101002
00 000 10 000 0	100000	00 000 10 000 2	102001
00 000 20 000 0	200000	00 000 20 000 1	201002
00 000 01 000 1	000002	00 000 20 000 2	202001
10 000 00 000 0	120001	01 000 02 000 1	012102
10 000 00 000 1	121000	01 000 02 000 2	010101
10 000 00 000 2	122002	20 000 00 000 0	210002
10 000 10 000 0	220001	20 000 00 000 1	211001
10 000 20 000 0	020001	20 000 00 000 2	212000
10 000 01 000 0	122001	20 000 10 000 0	010002
10 000 02 000 0	121001	20 000 20 000 0	110002
10 000 10 000 1	221000	20 000 01 000 0	212002
10 000 10 000 2	222002	20 000 02 000 0	211002
10 000 20 000 1	021000	20 000 10 000 1	011001
10 000 20 000 2	022002	20 000 10 000 2	012000
10 000 01 000 1	120000	20 000 01 000 1	210001
10 000 01 000 2	121002	20 000 01 000 2	211000
10 000 02 000 1	122000	20 000 20 000 1	111001
10 000 02 000 2	120002	20 000 20 000 2	112000
01 000 00 000 0	010100	20 000 02 000 1	212001
01 000 00 000 1	011102	20 000 02 000 2	210000
01 000 00 000 2	012101	02 000 00 000 0	020200
01 000 10 000 0	110100	02 000 00 000 1	021202
01 000 20 000 0	210100	02 000 00 000 2	022201

Contd...

Table 3.1 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
01 000 01 000 0	012100	02 000 10 000 0	120200
01 000 02 000 0	011100	02 000 20 000 0	220200
01 000 10 000 1	111102	02 000 01 000 0	022200
01 000 10 000 2	112101	02 000 02 000 0	021200
01 000 01 000 1	010102	02 000 10 000 1	121202
01 000 01 000 2	011101	02 000 10 000 2	122201
01 000 20 000 1	211102	02 000 01 000 1	020202
01 000 20 000 2	212101	02 000 01 000 2	021201
02 000 20 000 1	221202	00 01 000 02 00	022000
02 000 20 000 2	222201	00 20 000 01 00	211101
02 000 02 000 1	022202	00 20 000 02 00	022212
02 000 02 000 2	020201	00 02 000 00 00	200222
0 00 000 01 000	000100	00 02 000 01 00	011000
0 00 000 02 000	000200	00 02 000 02 00	122111
0 10 000 01 000	010200	00 02 000 10 00	200022
0 10 000 02 000	010000	00 02 000 20 00	200122
0 01 000 00 000	200010	000 00 000 01 0	000010
0 01 000 10 000	202010	000 00 000 02 0	000020
0 01 000 20 000	201010	000 10 000 01 0	100121
0 01 000 01 000	200110	000 10 000 02 0	100101
0 01 000 02 000	200210	000 01 000 00 0	000200
0 20 000 01 000	020000	000 01 000 10 0	111011
0 20 000 02 000	020100	000 01 000 20 0	222122
0 02 000 00 000	100020	000 01 000 01 0	000210
0 02 000 01 000	100120	000 01 000 02 0	000220
0 02 000 02 000	100220	000 20 000 01 0	200202
0 02 000 10 000	102020	000 20 000 02 0	200212
0 02 000 20 000	101020	000 02 000 00 0	000100

Contd...

Table 3.1 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
00 00 000 01 00	111111	000 02 000 10 0	111211
00 00 000 02 00	222222	000 02 000 20 0	222022
00 10 000 01 00	011121	000 02 000 01 0	000110
00 10 000 02 00	122202	000 02 000 02 0	000120
00 01 000 00 00	100111	0000 10 000 01	001202
00 01 000 10 00	100211	0000 10 000 02	002201
00 01 000 20 00	100011	0000 01 000 10	100010
00 01 000 01 00	211222	0000 01 000 20	100020
0000 20 000 01	001102	0000 02 000 10	200010
0000 20 000 02	002101	0000 02 000 20	200020

Example 3.13. *Consider* $n = 12$, $s = 3$, $b = 2$, $w = 1$ *and* $q = 2$ *in Theorem 3.11, then* $\lambda = 2$*;* $l = 12 \mod 5 = 2$*; and* $\lambda' = 2$*;* $p'_{s+1} = 1, p'_{s+2} = 1$*;* $\beta'_0 = \beta'_1 = \cdots = \beta'_3 = 2, \beta'_4 = 1$*. Putting these values in inequality (3.2), we get*

$$\begin{aligned}
2^{n-k} > & \sum_{j=0}^{0} \binom{2-1}{j} (2-1)^j \left(\sum_{j=0}^{1} \binom{2-1}{j} (2-1)^j \right)^1 \\
& \times \sum_{j=0}^{\min\{w=1,g=2\}} \binom{2}{j} (2-1)^j \left(1 + \sum_{j=1}^{3} R_{(3,2|1),10,2}(j) \right) \\
& = 1 \times 2^1 \times 3 \times \left[1 + 24\right] \\
& = 6 \times 25 = 150.
\end{aligned}$$

This implies $n - k \geq 8$*. Thus, we can construct a parity check matrix* H *of order*

8×12, *which gives rise to a* $(12,4)$ *binary* $LDP_{(3,2|1),12,2}RC$*-code:*

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}_{8\times 12}.$$

It can be verified from the Error Pattern-Syndrome Table 3.2 that the syndromes of all vectors of $\xi_{(3,2|1),12,2}$ *are nonzero and distinct, showing that the* $(12,4)$ *code is a binary* $LDP_{(3,2|1),12,2}RC$*-code.*

Table 3.2: **Error Pattern-Syndrome**

Error Patterns	Syndromes	Error Patterns	Syndromes
00 000 00 000 01	11010011	10 000 10 000 01	01010111
00 000 00 000 10	10101010	10 000 10 000 10	00101110
00 000 01 000 00	00000010	0 00 000 01 000 0	00000001
00 000 10 000 00	00000100	0 00 000 01 000 1	11010010
00 000 01 000 01	11010001	0 01 000 00 000 0	00100000
00 000 01 000 10	10101000	0 01 000 00 000 1	11110011
00 000 10 000 01	11010111	0 01 000 01 000 0	00100001
00 000 10 000 10	10101110	0 01 000 10 000 0	00100010
01 000 00 000 00	01000000	0 01 000 01 000 1	11110010
01 000 00 000 01	10010011	0 01 000 10 000 1	11110001
01 000 00 000 10	11101010	0 10 000 01 000 0	01000001
01 000 01 000 00	01000010	0 10 000 01 000 1	10010010
01 000 10 000 00	01000100	00 00 000 01 000	00110100
01 000 01 000 01	10010001	00 01 000 00 000	00010000
01 000 01 000 10	11101000	00 01 000 01 000	00100100

Contd...

Table 3.2 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
01 000 10 000 01	10010111	00 01 000 10 000	00010001
01 000 10 000 10	11101110	00 10 000 01 000	00010100
10 000 00 000 00	10000000	000 00 000 01 00	11111111
10 000 00 000 01	01010011	000 01 000 00 00	00001000
10 000 00 000 10	00101010	000 01 000 01 00	11110111
10 000 01 000 00	10000010	000 01 000 10 00	00111100
10 000 10 000 00	10000100	000 10 000 01 00	11101111
10 000 01 000 01	01010001	0000 01 000 10 0	11111011
10 000 01 000 10	00101000	0000 10 000 01 0	10100010

Example 3.14. *Consider $n = 14$, $s = 4$, $b = 3$, $w = 2$ and $q = 2$ in Theorem 3.11, then $\lambda = 2$; $l = 14 \mod 7 = 0$; and $\lambda' = 1$; $p'_{s+1} = 1, p'_{s+2} = 1, p'_{s+3} = 2$; $\beta'_0 = \beta'_1 = 2, \beta'_2 = \beta'_3 = \beta'_4 = \beta'_5 = \beta'_6 = 1$. Putting these values in the above inequality (3.2), we get*

$$
\begin{aligned}
2^{n-k} &> \sum_{j=0}^{2-1} \binom{3-1}{j} (2-1)^j \left(\sum_{j=0}^{2} \binom{3-1}{j} (2-1)^j \right)^1 \\
&\times \sum_{j=0}^{\min\{w=2,g=0\}} \binom{g_1}{j} (2-1)^j \left(1 + \sum_{j=1}^{4} R_{(4,3|2),11,2}(j) \right) \\
&= 3 \times 4^1 \times 1 \times \left[1 + 135 \right] \\
&= 12 \times 136 \\
&= 1632.
\end{aligned}
$$

This implies $n - k \geq 11$. Thus, we can construct a parity check matrix H of *order* 11×14, *which gives rise to a* $(14, 3)$ *binary* $LDP_{(4,3|2),14,2}RC$*-code:*

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}_{11 \times 14}.$$

Error Pattern-Syndrome Table 3.3 shows that the syndromes of all vectors of $\xi_{(4,3|2),14,2}$ are nonzero and distinct, showing that the $(14, 3)$ binary code is a $LDP_{(4,3|2),14,2}RC$-code.

Table 3.3: **Error Pattern-Syndrome**

Error Patterns	Syndromes	Error Patterns	Syndromes
000 0000 001 0000	00000000010	011 0000 000 0000	01100000000
000 0000 010 0000	00000000100	011 0000 001 0000	01100000010
000 0000 100 0000	00000001000	011 0000 010 0000	01100000100
000 0000 011 0000	00000000110	011 0000 100 0000	01100001000
000 0000 110 0000	00000001100	011 0000 011 0000	01100000110
000 0000 101 0000	00000001010	011 0000 110 0000	01100001100
100 0000 000 0000	10000000000	011 0000 101 0000	01100001010
100 0000 001 0000	10000000010	110 0000 000 0000	11000000000
100 0000 010 0000	10000000100	110 0000 001 0000	11000000010
100 0000 100 0000	10000001000	110 0000 010 0000	11000000100
100 0000 011 0000	10000000110	110 0000 100 0000	11000001000
100 0000 110 0000	10000001100	110 0000 011 0000	11000000110

Contd...

Table 3.3 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
100 0000 101 0000	10000001010	110 0000 110 0000	11000001100
010 0000 000 0000	01000000000	110 0000 101 0000	11000001010
010 0000 001 0000	01000000010	101 0000 000 0000	10100000000
010 0000 010 0000	01000000100	101 0000 001 0000	10100000010
010 0000 100 0000	01000001000	101 0000 010 0000	10100000100
010 0000 011 0000	01000000110	101 0000 100 0000	10100001000
010 0000 110 0000	01000001100	101 0000 011 0000	10100000110
010 0000 101 0000	01000001010	101 0000 110 0000	10100001100
001 0000 000 0000	00100000000	101 0000 101 0000	10100001010
001 0000 001 0000	00100000010	0 000 0000 001 000	00000000001
001 0000 010 0000	00100000100	0 000 0000 011 000	00000000011
001 0000 100 0000	00100001000	0 000 0000 101 000	00000000101
001 0000 011 0000	00100000110	0 100 0000 001 000	01000000001
001 0000 110 0000	00100001100	0 100 0000 011 000	01000000011
001 0000 101 0000	00100001010	0 100 0000 101 000	01000000101
0 010 0000 001 000	00100000001	00 000 0000 011 00	10000110101
0 010 0000 011 000	00100000011	00 000 0000 101 00	10000110110
0 010 0000 101 000	00100000101	00 100 0000 001 00	10100110100
0 001 0000 000 000	00010000000	00 100 0000 011 00	10100110101
0 001 0000 001 000	00010000001	00 100 0000 101 00	10100110110
0 001 0000 010 000	00010000010	00 010 0000 001 00	10010110100
0 001 0000 100 000	00010000100	00 010 0000 011 00	10010110101
0 001 0000 011 000	00010000011	00 010 0000 101 00	10010110110
0 001 0000 110 000	00010000110	00 001 0000 000 00	00001000000
0 001 0000 101 000	00010000101	00 001 0000 001 00	10001110100
0 110 0000 001 000	01100000001	00 001 0000 010 00	00001000001
0 110 0000 011 000	01100000011	00 001 0000 100 00	00001000010
0 110 0000 101 000	01100000101	00 001 0000 011 00	10001110101

Contd...

Table 3.3 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
0 011 0000 000 000	00110000000	00 001 0000 110 00	00001000011
0 011 0000 001 000	00110000001	00 001 0000 101 00	10001110110
0 011 0000 010 000	00110000010	00 110 0000 001 00	10110110100
0 011 0000 100 000	00110000100	00 110 0000 011 00	10110110101
0 011 0000 011 000	00110000011	00 110 0000 101 00	10110110110
0 011 0000 110 000	00110000110	00 011 0000 000 00	00011000000
0 011 0000 101 000	00110000101	00 011 0000 001 00	10011110100
0 101 0000 000 000	01010000000	00 011 0000 010 00	00011000001
0 101 0000 001 000	01010000001	00 011 0000 100 00	00011000010
0 101 0000 100 000	01010000100	00 011 0000 011 00	10011110101
0 101 0000 011 000	01010000011	00 011 0000 110 00	00011000011
0 101 0000 010 000	01010000010	00 011 0000 101 00	10011110110
0 101 0000 110 000	01010000110	00 101 0000 000 00	00101000000
0 101 0000 101 000	01010000101	00 101 0000 001 00	10101110100
00 000 0000 001 00	10000110100	00 101 0000 010 00	00101000001
00 101 0000 100 00	00101000010	000 011 0000 110 0	10001010101
00 101 0000 011 00	10101110101	000 011 0000 101 0	11110011110
00 101 0000 110 00	00101000011	000 101 0000 000 0	00010100000
00 101 0000 101 00	10101110110	000 101 0000 001 0	11101011111
000 000 0000 001 0	11111111111	000 101 0000 010 0	10010010100
000 000 0000 011 0	01111001011	000 101 0000 100 0	00010100001
000 000 0000 101 0	11111111110	000 101 0000 011 0	01101101011
000 100 0000 001 0	11101111111	000 101 0000 110 0	10010010101
000 100 0000 011 0	01101001011	000 101 0000 101 0	11101011110
000 100 0000 101 0	11101111110	0000 000 0000 001	11001001011
000 010 0000 001 0	11110111111	0000 000 0000 011	00110110100
000 010 0000 011 0	01110001011	0000 000 0000 101	01001111111
000 010 0000 101 0	11110111110	0000 100 0000 001	11000001011

Contd...

Table 3.3 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
000 001 0000 000 0	00000100000	0000 100 0000 011	00111110100
000 001 0000 001 0	11111011111	0000 100 0000 101	01000111111
000 001 0000 010 0	10000010100	0000 010 0000 001	11001101011
000 001 0000 100 0	00000100001	0000 010 0000 011	00110010100
000 001 0000 011 0	01111101011	0000 010 0000 101	01001011111
000 001 0000 110 0	10000010101	0000 001 0000 000	00000010000
000 001 0000 101 0	11111011110	0000 001 0000 001	11001011011
000 110 0000 001 0	11100111111	0000 001 0000 010	11111101111
000 110 0000 011 0	01100001011	0000 001 0000 100	10000100100
000 110 0000 101 0	11100111110	0000 001 0000 011	00110100100
000 011 0000 000 0	00001100000	0000 001 0000 110	01111011011
000 011 0000 001 0	11110011111	0000 001 0000 101	01001101111
000 011 0000 010 0	10001010100	0000 110 0000 001	11000101011
000 011 0000 100 0	00001100001	0000 110 0000 011	00111010100
000 011 0000 011 0	01110101011	0000 110 0000 101	01000011111
0000 011 0000 000	00000110000	00000 001 0000 10	11111110111
0000 011 0000 001	11001111011	00000 001 0000 11	00110111100
0000 011 0000 010	11111001111	00000 011 0000 00	00000011000
0000 011 0000 100	10000000100	00000 011 0000 01	11001010011
0000 011 0000 011	00110000100	00000 011 0000 10	11111100111
0000 011 0000 110	01111111011	00000 011 0000 11	00110101100
0000 011 0000 101	01001001111	00000 101 0000 00	00000101000
0000 101 0000 000	00001010000	00000 101 0000 01	11001100011
0000 101 0000 001	11000011011	00000 101 0000 10	11111010111
0000 101 0000 010	11110101111	00000 101 0000 11	00110011100
0000 101 0000 100	10001100100	000000 001 0000 1	11001001111
0000 101 0000 011	00111100100	000000 011 0000 1	11001000111
0000 101 0000 110	01110011011	000000 101 0000 0	00000010100

Contd...

Table 3.3 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
0000 101 0000 101	01000101111	000000 101 0000 1	11001011111
00000 001 0000 01	11001000011		

3.2.2 Comparison

In this subsection, we provide comparisons on the necessary and sufficient numbers of check digits for $LDP_{(s,b|w),n,q}RC$-codes with $P_{(s,b),n,q}RC$-codes. In Table 3.4-3.5, some of these check digit numbers are listed and we see that the $LDP_{(s,b|w),n,q}RC$-codes take fewer number of check digits than $P_{(s,b),n,q}RC$-codes. This is true for other cases also. Therefore, $LDP_{(s,b|w),n,q}RC$-codes are more efficient than $P_{(s,b),n,q}RC$-codes in terms of code rate.

Table 3.4: **Necessary number of check digits**

n	s	b	q	w	$n-k$ ($P_{(s,b),n,q}RC$-codes)	$n-k$ ($LDP_{(s,b\|w),n,q}RC$-codes)
10	3	2	3	1	5	3
10	3	2	4	1	4	3
11	3	2	3	1	5	4
12	4	2	2	1	6	4
13	4	2	2	1	6	5
14	4	3	2	2	8	7
18	5	4	2	2	10	9
20	5	4	2	3	11	11
23	5	4	3	1	12	6

Table 3.5: **Sufficient number of check digits**

n	s	b	q	w	$n-k$ ($P_{(s,b),n,q}RC$-codes)	$n-k$ ($LDP_{(s,b\|w),n,q}RC$-codes)
10	3	2	3	1	8	5
10	3	2	4	1	8	5
11	3	2	3	1	10	6
12	4	2	2	1	9	8
13	4	2	2	1	10	8
14	4	3	2	2	13	11
18	5	4	2	2	17	13
20	5	4	2	3	19	18
23	5	4	3	1	23	8

3.2.3 Error decoding probability

The chapter concludes with the probability of decoding error for a $LDP_{(s,b|w),n,q}RC$-code (equivalent to Result 1.17) over a symmetric channel along with a remark.

Theorem 3.15. *Suppose $PD_R(E)$ is the probability of decoding error of an (n,k) $LDP_{(s,b|w),n,2}RC$-code on a binary symmetric channel with transition probability ϵ, then*

$$PD_R(E) = 1 - \sum_{j=1}^{w_{max}} R_{(s,b|w),n,2}(j).\epsilon^j(1-\epsilon)^{n-j},$$

where $R_{(s,b|w),n,2}(j)$ is given by Theorem 3.3.

Proof. An error pattern can be corrected if and only if it is a coset leader in the standard array for the code [72]. So, the probability of correcting an error is the probability that the error is a coset leader. Since the binary symmetric channel has the transition probability ϵ, the probability of occurring of any one of the error vectors of weight j being coset leader is $\epsilon^j(1-\epsilon)^{n-j}$. So, the probability of occurring of any error vector from the set $\xi_{(s,b|w),n,2}$ is

$$\sum_{j=1}^{w_{max}} R_{(s,b|w),n,2}(j).\epsilon^j(1-\epsilon)^{n-j},$$

where $R_{(s,b|w),n,2}(j)$ is given by Theorem 3.3.

As the code corrects all such error patterns, the probability $PD_R(E)$ of decoding error of the code is

$$PD_R(E) = 1 - \sum_{j=1}^{w_{max}} R_{(s,b|w),n,2}(j).\epsilon^j(1-\epsilon)^{n-j}.$$

□

Remark 3.16. *For $s = 3, b = 2$, and $\epsilon = 0.1$, we determine the probability of decoding error $PD_R(E)$ of binary $LDP_{(s,b|w),n,2}RC$-codes of different lengths as follows.*

Table 3.6: Values of $PD_R(E)$

n	λ	l	$PD_R(E)$
10	2	0	0.19
11	2	1	0.21
12	2	2	0.23
13	2	3	0.29
14	2	4	0.31
15	3	0	0.33

We find that the probability of decoding error of $LDP_{(s,b|w),n,2}RC$-codes increases as the length of the code increases. So, a shorter length code is more efficient than a longer one.