Chapter 4
Periodical burst error correcting codes with weight distribution and error decoding probability

The contents of this chapter are based on the paper:

- Das, P. K. and Haokip, L. Periodical burst error correcting codes with decoding error probability. *Discrete Mathematics Letters*, 8:49-56, 2021.

Chapter 4

Periodical burst error correcting codes with weight distribution and error decoding probability

In this chapter, we study *periodical burst errors* that are found in some communication channels for known noise frequency (discussed in Section 1.2.4). By Definition 1.6, an s-periodical burst of length b is an error pattern where CT-burst of length b repeats in an n-tuple. The last burst may have less than b components. This chapter presents linear codes correcting periodical burst errors, (Hamming) weight distribution of the error pattern along with Plotkin's type of bound and error decoding probability of the codes. These works are organised as follows. Section 4.1 provides necessary and sufficient conditions for the existence of a linear code correcting periodical burst errors. Examples of such codes along with comparisons on the number of check digits of these codes with the $\boldsymbol{P_{(s,b),n,q}RC-codes}$ are also provided. In Section 4.2, we present the error weight distribution of periodical burst errors, followed by Plotkin's type of bound and the probability of decoding error of the code.

Like the previous chapters, we use the following notations for the specific studied error pattern and the corresponding error-correcting codes as follows:

$\boldsymbol{\psi_{(s,b),n,q}}$: set of all s-periodical burst errors of length b in an n-tuple over $GF(q)$.

$\boldsymbol{P_{(s,b),n,q}BC-code}$: length-n linear code correcting s-periodical burst errors of length b over $GF(q)$.

4.1 $P_{(s,b),n,q}BC - codes$ and comparison

In this section, we provide necessary and sufficient conditions for the existence of a $P_{(s,b),n,q}BC$-code along with comparisons on the number of check digits of these codes with the $\boldsymbol{P_{(s,b),n,q}RC - codes}$.

4.1.1 Conditions for existence of $P_{(s,b),n,q}BC - codes$

First, we give the necessary condition and then sufficient condition for the existence of a $P_{(s,b),n,q}BC$-code. Examples are included to justify the results. To prove our results, we first need the following lemma.

Lemma 4.1. *For given non-negative integers n, b and s $(n \geq b+s)$, the total number of vectors of $\psi_{(s,b),n,q}$ in a vector of length n is*

$$N_{s,b} = \sum_{i=1}^{n} (q-1)^{\lambda_i} q^{m_i - \lambda_i},$$

where $m_i = \left\lfloor \frac{n-i+1}{s+b} \right\rfloor b + \gamma\big((n-i+1) \mod (b+s)\big)$ *and* $\lambda_i = \left\lceil \frac{n-i+1}{s+b} \right\rceil$.

Proof. Error position in a vector of $\psi_{(s,b),n,q}$ can start from the i^{th} position $(i = 1, 2, \ldots, n)$. If m_i denotes the maximum number of nonzero positions in the error pattern and λ_i the number of sets in which the nonzero components are confined, then by Lemma 3.1

$$m_i = \left\lfloor \frac{n-i+1}{s+b} \right\rfloor b + \gamma\big((n-i+1) \mod (b+s)\big) \text{ and } \lambda_i = \left\lceil \frac{n-i+1}{s+b} \right\rceil.$$

Since there are λ_i sets in which the first component is always nonzero, there will be $m_i - \lambda_i$ positions where the components can be any of the q field elements. Therefore, the total number of vectors of $\psi_{(s,b),n,q}$ is

$$N_{s,b} = \sum_{i=1}^{n} (q-1)^{\lambda_i} q^{m_i - \lambda_i}.$$

□

Example 4.2. *Taking $n = 15, b = 2, s = 3$ and $q = 2$ in Lemma 4.1, we have $m_1 = m_2 = m_3 = m_4 = 6$, $m_5 = 5$, $m_6 = m_7 = m_8 = m_9 = 4$, $m_{10} = 3$,*

$m_{11} = m_{12} = m_{13} = m_{14} = 2$, $m_{15} = 1$ *and* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 3$, $\lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = 2$, $\lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_{15} = 1$. *Then, the total number of vectors of* $\psi_{(3,2),15,2}$ *in a vector of length* 15 *is*

$$N_{3,2} = \sum_{i=1}^{15} 2^{m_i - \lambda_i} = 63.$$

These 63 *error patterns are:*

100001000010000, 100001000011000, 100001100010000, 100001100011000, 110001000010000, 110001000011000, 110001100010000, 110001100011000, 010000100001000, 010000100001100, 010000110001000, 010000110001100, 011000100001000, 011000100001100, 011000110001000, 011000110001100, 001000010000100, 001000010000110, 001000011000100, 001000011000110, 001100010000100, 001100010000110, 001100011000100, 001100011000110, 000100001000010, 000100001000011, 000100001100010, 000100001100011, 000110001000010, 000110001000011, 000110001100010, 000110001100011, 000010000100001, 000010000110001, 000011000100001, 000011000110001, 000001000010000, 000001000011000, 000001100010000, 000001100011000, 000000100001000, 000000110001000, 000000100001100, 000000110001100, 000000010000100, 000000010000110, 000000011000100, 000000011000110, 000000001000010, 000000001100010, 000000001000011, 000000001100011, 000000000100001, 000000000110001, 000000000010000, 000000000011000, 000000000001000, 000000000001100, 000000000000100, 000000000000110, 000000000000010, 000000000000011, 000000000000001.

Now, we give a necessary condition for the existence of a $P_{(s,b),n,q}BC$-code (equivalent to Result 1.14).

Theorem 4.3. *For given non-negative integers* n, b *and* s $(n \geq b + s)$, *a necessary condition for an* (n, k) $P_{(s,b),n,q}BC$*-code is*

$$q^{n-k} \geq 1 + N_{s,b}, \tag{4.1}$$

where $N_{s,b}$ *is given by Lemma 4.1.*

Proof. As the code corrects all vectors of $\psi_{(s,b),n,q}$, all the errors must be in different cosets of the code. Thus, by Lemma 4.1, we get

$$q^{n-k} \geq 1 + N_{s,b}.$$

□

Remark 4.4. *From Inequality* (4.1), *we get*

$$q^k \leq \frac{q^n}{1+N_{s,b}}.$$

This implies that the number of codewords of an $(n,\, k)$ $P_{(s,b),n,q}BC$*-code is bounded above by* $\dfrac{q^n}{1+N_{s,b}}$.

For a sufficient condition of a $P_{(s,b),n,q}BC$-code, we use the same technique used in the previous chapters by adding the columns in the parity check matrix one after another, keeping in mind that the syndromes of the errors should be all nonzero and distinct.

Theorem 4.5. *For given non-negative integers* n, b *and* s $(n \geq b+s)$, *let* $n = \lambda(b+s)+l$ *for some non-negative integers* λ *and* l *(where* $0 \leq l < b+s$*), a sufficient condition for an* $(n,\, k)$ $P_{(s,b),n,q}BC$*-code is*

$$q^{n-k} > \begin{cases} \left[(q-1)^{\lambda-1}q^{\lambda(b-1)}\right]\left[1+\displaystyle\sum_{i=1}^{n-b}(q-1)^{\lambda_i'}q^{m_i'-\lambda_i'}\right] & \text{if} \quad l=0 \\ \left[(q-1)^{\lambda}q^{\lambda(b-1)+l-1}\right]\left[1+\displaystyle\sum_{i=1}^{n-b}(q-1)^{\lambda_i'}q^{m_i'-\lambda_i'}\right] & \text{if} \quad 1 \leq l \leq b \\ \left[(q-1)^{\lambda}q^{(\lambda+1)(b-1)}\right]\left[1+\displaystyle\sum_{i=1}^{n-b}(q-1)^{\lambda_i'}q^{m_i'-\lambda_i'}\right] & \text{if} \quad b < l < s+b, \end{cases}$$

where $m_i' = \left\lfloor \dfrac{n-b-i+1}{s+b} \right\rfloor b + \gamma\big((n-b-i+1) \mod (b+s)\big)$ *and* $\lambda_i' = \left\lceil \dfrac{n-b-i+1}{s+b} \right\rceil$.

Proof. The proof is shown by constructing an appropriate $(n-k)\times n$ parity-check matrix H of the code. Take the first column h_1 as any nonzero $(n-k)$-tuple and suppose the columns $h_2, h_3, \ldots, h_{n-1}$ are added suitably to H. Then any (nonzero) column h_n is added to H provided that it is not a linear combination of immediately preceding $b-1$ columns together with previous sets of b consecutive columns which are at a gap of s columns (the last set may contain less than b columns), along with a linear combination of sets of b consecutive columns which are at a gap of s columns confined to the first $n-b$ columns (the last set may contain less than b columns). This can be written as

$$h_n \neq \left(\sum_{i=1}^{b-1} a_{i1}h_{n-i} + \sum_{i=0}^{b-1} b_{i1}h_{n-(s+b)-i} + \sum_{i=0}^{b-1} b_{i2}h_{n-2(s+b)-i} + \cdots + \sum_{i=0}^{g_1-1} b_{i\lambda}h_{n-\lambda(s+b)-i}\right)$$

$$+\left(\sum_{i=0}^{b-1}\alpha_{i1}h_{j'-i}+\sum_{i=0}^{b-1}\beta_{i1}h_{j'-(s+b)-i}+\sum_{i=0}^{b-1}\beta_{i2}h_{j'-2(s+b)-i}+\cdots+\sum_{i=0}^{g_2-1}\beta_{i\lambda'}h_{j'-\lambda'(s+b)-i}\right), \tag{4.2}$$

where $a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij} \in GF(q)$; $b_{0i}, \alpha_{0i}, \beta_{0i} \neq 0$; $j' \leq n-b$; $g_1 = \gamma\big(n \mod (s+b)\big)$, $g_2 = \gamma\big((n-b-j'+1) \mod (s+b)\big)$, $\lambda = \left\lfloor \frac{n}{s+b} \right\rfloor$ and $\lambda' = \left\lceil \frac{n-b-j'+1}{s+b} \right\rceil$.

In Expression (4.2), it is convenient to assume that $\sum_{i=0}^{g_1-1} b_{i\lambda}h_{n-\lambda(s+b)-i} = 0$ and $\sum_{i=0}^{g_2-1} b_{i\lambda'}h_{j'-\lambda'(s+b)-i} = 0$ when n and $n-b-j'+1$ are multiples of $s+b$.

The condition (4.2) ensures that syndromes of any two vectors of $\psi_{(s,b),n,q}$ are distinct. The number of linear combinations in the first bracket on the right hand side (R.H.S.) of (4.2) is calculated as follows:

The number of ways a_{i1}'s can be chosen is q^{b-1}. The number of ways b_{ij}'s ($j = 1, 2, \ldots, \lambda-1$) can be chosen is $(q-1)q^{b-1}$. For the last summation of the first bracket, we get different combinations depending on l. For $l = 0$, there is no term in the last summation of the first bracket. For $1 \leq l \leq b$, g_1 will be l and $b_{i\lambda}$'s can be chosen by $(q-1)q^{l-1}$ ways. For $b < l < s+b$, g_1 will be b and $b_{i\lambda}$'s can be chosen by $(q-1)q^{b-1}$ ways. Therefore, the total number of combinations of the first bracket on R.H.S. of (4.2) is

$$\begin{cases} q^{b-1}[(q-1)q^{b-1}]^{\lambda-1} = (q-1)^{\lambda-1}q^{\lambda(b-1)} & \text{if} \quad l = 0 \\ q^{b-1}[(q-1)q^{b-1}]^{\lambda-1}(q-1)q^{l-1} = (q-1)^{\lambda}q^{\lambda(b-1)+l-1} & \text{if} \quad 1 \leq l \leq b \\ q^{b-1}[(q-1)q^{b-1}]^{\lambda-1}(q-1)q^{b-1} = (q-1)^{\lambda}q^{(\lambda+1)(b-1)} & \text{if} \quad b < l < s+b. \end{cases}$$

The second bracket on R.H.S. of (4.2) gives the number of vectors of $\psi_{(s,b),n-b,q}$ in a vector of length $n-b$. This number, including the zero combination, is given by Lemma 4.1 as

$$1+\sum_{i=1}^{n-b}(q-1)^{\lambda'_i}q^{m'_i-\lambda'_i},$$

where $m'_i = \left\lfloor \frac{n-b-i+1}{s+b} \right\rfloor b + \gamma\big((n-b-i+1) \mod (b+s)\big)$ and $\lambda'_i = \left\lceil \frac{n-b-i+1}{s+b} \right\rceil$.

Therefore, the total number of all possible linear combinations on R.H.S. of (4.2) is

$$\begin{cases} \left[(q-1)^{\lambda-1}q^{\lambda(b-1)}\right]\left[1+\sum_{i=1}^{n-b}(q-1)^{\lambda'_i}q^{m'_i-\lambda'_i}\right] & \text{if} \quad l=0 \\ \left[(q-1)^{\lambda}q^{\lambda(b-1)+l-1}\right]\left[1+\sum_{i=1}^{n-b}(q-1)^{\lambda'_i}q^{m'_i-\lambda'_i}\right] & \text{if} \quad 1\le l\le b \\ \left[(q-1)^{\lambda}q^{(\lambda+1)(b-1)}\right]\left[1+\sum_{i=1}^{n-b}(q-1)^{\lambda'_i}q^{m'_i-\lambda'_i}\right] & \text{if} \quad b<l<s+b. \end{cases}$$

Since we can have at most $q^{n-k}-1$ nonzero columns, the sufficient condition for the existence of the required code is given by

$$q^{n-k} > \begin{cases} \left[(q-1)^{\lambda-1}q^{\lambda(b-1)}\right]\left[1+\sum_{i=1}^{n-b}(q-1)^{\lambda'_i}q^{m'_i-\lambda'_i}\right] & \text{if} \quad l=0 \\ \left[(q-1)^{\lambda}q^{\lambda(b-1)+l-1}\right]\left[1+\sum_{i=1}^{n-b}(q-1)^{\lambda'_i}q^{m'_i-\lambda'_i}\right] & \text{if} \quad 1\le l\le b \\ \left[(q-1)^{\lambda}q^{(\lambda+1)(b-1)}\right]\left[1+\sum_{i=1}^{n-b}(q-1)^{\lambda'_i}q^{m'_i-\lambda'_i}\right] & \text{if} \quad b<l<s+b. \end{cases}$$

□

Remark 4.6. *For $q=2$, the sufficient condition of Theorem 4.5 becomes*

$$2^{n-k} > \begin{cases} 2^{\lambda(b-1)}\left[1+\sum_{i=1}^{n-b}2^{m'_i-\lambda'_i}\right] & \textit{if} \quad l=0 \\ 2^{\lambda(b-1)+l-1}\left[1+\sum_{i=1}^{n-b}2^{m'_i-\lambda'_i}\right] & \textit{if} \quad 1\le l\le b \\ 2^{(\lambda+1)(b-1)}\left[1+\sum_{i=1}^{n-b}2^{m'_i-\lambda'_i}\right] & \textit{if} \quad b<l<s+b. \end{cases}$$

Now, we give three examples to justify Theorem 4.5 corresponding to $l=0, 1\le l\le b$ and $b<l<s+b$.

Example 4.7. *Consider $n=13$, $s=3$, $b=2$ and $q=2$ in Theorem 4.5, then $\lambda=2$; $l=3$; and $\lambda'_1=3, \lambda'_2=\cdots=\lambda'_6=2$, $\lambda'_7=\cdots=\lambda'_{11}=1$; $m'_1=5, m'_2=\cdots=m'_5=4, m'_6=3, m'_7=\cdots=m'_{10}=2, m'_{11}=1$. From Theorem 4.5, we have*

$$\begin{aligned} 2^{n-k} &> \left[(q-1)^{\lambda}q^{(\lambda+1)(b-1)}\right]\left(1+\sum_{i=1}^{n-b}(q-1)^{\lambda'_i}q^{m'_i-\lambda'_i}\right) \\ &= 2^{(2+1)(2-1)}\left(1+\sum_{i=1}^{11}2^{m'_i-\lambda'_i}\right) \end{aligned}$$

$$= 8\left(1 + 2^{5-3} + 2^{4-2} \times 4 + 2^{3-2} + 2^{2-1} \times 4 + 2^{0}\right) = 8 \times 32 = 256.$$

This implies that we can construct a parity check matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}_{9\times 13}$$

of order 9×13 *which gives rise to a* $(13, 4)$ *binary linear code.*

It can be verified from Error Pattern-Syndrome Table 4.1 that the syndromes of all vectors of $\psi_{(3,2),13,2}$ *are nonzero and distinct, showing that the code can correct all vectors of* $\psi_{(3,2),13,2}$*. So, the code is a* $(13, 4)$ $P_{(3,2),13,2}BC$*-code.*

Table 4.1: **Error Pattern-Syndrome**

Error Patterns	Syndromes	Error Patterns	Syndromes
10 000 10 000 10 0	101011011	0 11 000 10 000 11	111101101
10 000 10 000 11 0	101111011	0 11 000 11 000 10	011100110
10 000 11 000 10 0	101011111	0 11 000 11 000 11	111101111
10 000 11 000 11 0	101111111	00 10 000 10 000 1	101001011
11 000 10 000 10 0	111011011	00 10 000 11 000 1	000011110
11 000 10 000 11 0	111111011	00 11 000 10 000 1	101101011
11 000 11 000 10 0	111011111	00 11 000 11 000 1	000111110
11 000 11 000 11 0	111111111	000 10 000 10 000	101110101
0 10 000 10 000 10	010100100	000 10 000 11 000	111110101
0 10 000 10 000 11	110101101	000 11 000 10 000	101100101
0 10 000 11 000 10	010100110	000 11 000 11 000	111100101

Contd...

Table 4.1 - **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
0 10 000 11 000 11	110101111	0000 10 000 10 00	010010000
0 11 000 10 000 10	011100100	0000 10 000 11 00	011000011
0000 11 000 10 00	010011000	0000000 11 000 1	001011110
0000 11 000 11 00	011001011	00000000 10 000	101010101
00000 10 000 10 0	001011011	00000000 11 000	111010101
00000 10 000 11 0	001111011	000000000 10 00	010000000
00000 11 000 10 0	001011111	000000000 11 00	011010011
00000 11 000 11 0	001111111	0000000000 10 0	001010011
000000 10 000 10	000100100	0000000000 11 0	001110011
000000 10 000 11	100101101	00000000000 10	000100000
000000 11 000 10	000100110	00000000000 11	100101001
000000 11 000 11	100101111	000000000000 1	100001001
0000000 10 000 1	100001011		

Example 4.8. *Consider $n = 11$, $s = 3$, $b = 2$ and $q = 3$ in Theorem 4.5, then $\lambda = 2$; $l = 1$; and $\lambda'_1 = \cdots = \lambda'_4 = 2$, $\lambda'_5 = \cdots = \lambda'_9 = 1$; $m'_1 = m'_2 = m'_3 = 4, m'_4 = 3, m'_5 = \cdots = m'_8 = 2, m'_9 = 1$. Theorem 4.5 gives*

$$\begin{aligned}
3^{n-k} &> \left[(q-1)^{\lambda} q^{\lambda(b-1)+l-1}\right]\left(1+\sum_{i=1}^{n-b}(q-1)^{\lambda'_i} q^{m'_i-\lambda'_i}\right) \\
&= (3-1)^2 3^{2(2-1)}\left(1+\sum_{i=1}^{9}(3-1)^{\lambda'_i} 3^{m'_i-\lambda'_i}\right) \\
&= 4\times 9\left(1+4\times 3^{4-2}\times 3+4\times 3^{3-2}+2\times 3^{2-1}\times 4+2\times 3^{0}\right) \\
&= 5292,
\end{aligned}$$

which implies $n-k \geq 8$. This gives rise to a ternary $(11, 3)$ linear code whose parity check matrix H of order 8×11 is given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{8\times 11}.$$

Here also, the syndromes of all vectors of $\psi_{(3,2),11,3}$ are found to be nonzero and distinct, showing that the code is a $(11,3)$ $P_{(3,2),11,3}BC$-code.

Example 4.9. *Consider $n = 10$, $s = 3$, $b = 2$ and $q = 4$ in Theorem 4.5, then $\lambda = 2$; $l = 0$; and $\lambda'_1 = \cdots = \lambda'_3 = 2, \lambda'_4 = \cdots = \lambda'_8 = 1$; $m'_1 = m'_2 = 4, m'_3 = 3, m'_4 = \cdots = m'_7 = 2, m'_8 = 1$. From Theorem 4.5, we have*

$$\begin{aligned} 4^{n-k} &> \left[(q-1)^{\lambda-1}q^{\lambda(b-1)}\right]\left(1+\sum_{i=1}^{n-b}(q-1)^{\lambda'_i}q^{m'_i-\lambda'_i}\right) \\ &= (4-1)\times 4^{2(2-1)}\left[1+\sum_{i=1}^{8}(4-1)^{\lambda'_i}4^{m'_i-\lambda'_i}\right] \\ &= 18048. \end{aligned}$$

This implies $n-k \geq 8$. Thus, we can construct a parity check matrix H of order 8×10, which gives rise to a $(10,2)$ $P_{(3,2),10,4}BC$-code.

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}_{8\times 10}.$$

4.1.2 Comparison

This subsection gives comparisons on the necessary and sufficient numbers of check digits for $P_{(s,b),n,q}BC$-codes with $P_{(s,b),n,q}RC$-codes. We find that the $P_{(s,b),n,q}BC$-codes take less number of check digits than $P_{(s,b),n,q}RC$-codes. So, $P_{(s,b),n,q}BC$-codes are more efficient than $P_{(s,b),n,q}RC$-codes in terms of code rate.

Table 4.2: **Necessary number of check digits**

n	s	b	q	$n-k$ ($P_{(s,b),n,q}RC$-codes)	$n-k$ ($P_{(s,b),n,q}BC$-codes)
10	3	2	3	5	4
10	3	2	4	4	4
11	3	2	3	5	5
12	4	2	2	6	5
13	4	2	2	6	5
14	4	3	2	8	6
18	5	4	2	10	8
20	5	4	2	11	9
23	5	4	3	12	11

Table 4.3: **Sufficient number of check digits**

n	s	b	q	$n-k$ ($P_{(s,b),n,q}RC$-codes)	$n-k$ ($P_{(s,b),n,q}BC$-codes)
10	3	2	3	8	7
10	3	2	4	8	8
11	3	2	3	10	8
12	4	2	2	9	7
13	4	2	2	10	7

Contd...

Table 4.3 – **Sufficient number of check digits**

n	s	b	q	$n-k$ ($P_{(s,b),n,q}RC$-codes)	$n-k$ ($P_{(s,b),n,q}BC$-codes)
14	4	3	2	13	11
18	5	4	2	17	14
20	5	4	2	19	16
23	5	4	3	23	19

4.2 Weight distribution and error decoding probability

In this section, we obtain the weight distribution of vectors of $\psi_{(s,b),n,q}$. Then, we give a Plotkin's type of bound for the error set $\psi_{(s,b),n,q}$. We also give the total probability of the set $\psi_{(s,b),n,2}$ followed by the probability of decoding error of $P_{(s,b),n,2}BC$-code over a binary symmetric channel. The following lemma gives the weight distribution of the error vectors.

Lemma 4.10. *For $0 \leq j \leq n$, let $N_{s,b}(j)$ denote the number of vectors of $\psi_{(s,b),n,q}$ with weight j in a vector of length n over $GF(q)$. Then*

$$N_{s,b}(j) = \sum_{i=1}^{n} \binom{m_i - \lambda_i}{j - \lambda_i} (q-1)^j,$$

where m_i and λ_i are given by Lemma 3.1.

Proof. By Lemma 3.1, λ_i $(i = 1, 2, \ldots, n)$ represents number of sets where the nonzero components are confined, and m_i represents the number of the nonzero components if vectors of $\psi_{(s,b),n,q}$ that start from the i^{th} position. As first component of each set is nonzero, there will be at least λ_i nonzero positions, that is, $m_i \geq \lambda_i$ and the weight of the error pattern is at least λ_i. Then there will be $m_i - \lambda_i$ positions where any field element can be chosen.

Therefore, to obtain vectors of $\psi_{(s,b),n,q}$ with weight j, we need to choose any $j - \lambda_i$ positions from $m_i - \lambda_i$ positions. Note that the weight j of the error pattern that starts from the i^{th} position is at least λ_i, so

$$m_i - \lambda_i \geq j - \lambda_i \geq 0.$$

We can choose $j - \lambda_i$ positions from $m_i - \lambda_i$ positions by $\binom{m_i - \lambda_i}{j - \lambda_i}$ ways. Therefore, the total number of vectors of $\psi_{(s,b),n,q}$ with weight j is

$$N_{s,b}(j) = \sum_{i=1}^{n} \binom{m_i - \lambda_i}{j - \lambda_i} (q-1)^j.$$

□

Observe that for given non-negative integers n, b and s $(n \geq s + b)$, the maximum number of nonzero components in a vector of $\psi_{(s,b),n,q}$ can be found when the error pattern starts from the first position. By Lemma 3.1, this number is

$$m_1 = \left\lfloor \frac{n}{s+b} \right\rfloor b + \gamma\big(n \mod (b+s)\big).$$

So, the maximum weight of vectors of $\psi_{(s,b),n,q}$ can be at most m_1. We denote it by w_{max}. Therefore, $N_{s,b}(j) = 0$ for $w_{max} < j \leq n$.

Now, we give the Plotkin's type of bound (equivalent to Result 1.12) for the vectors of the set $\psi_{(s,b),n,q}$.

Theorem 4.11. *The minimum weight of a vector having vectors of $\psi_{(s,b),n,q}$ in the space of n-tuples over $GF(q)$ is at most*

$$\frac{\sum_{j=1}^{w_{max}} j N_{s,b}(j)}{N_{s,b}},$$

where $N_{s,b}$ is given by Lemma 4.1 and $N_{s,b}(j)$ by Lemma 4.10.

Proof. By Lemma 4.1, the number of all vectors of $\psi_{(s,b),n,q}$ in the space of n-tuples over $GF(q)$ is $N_{s,b}$.

From Lemma 4.10, the total weight of all vectors of $\psi_{(s,b),n,q}$ in the space of n-tuples over $GF(q)$ is given by $\sum_{j=1}^{w_{max}} j N_{s,b}(j)$.

As the minimum weight of a vector can be at most the average weight, the minimum weight of a vector having vectors of $\psi_{(s,b),n,q}$ in the space of n-tuples over $GF(q)$ is at most

$$\frac{\sum_{j=1}^{w_{max}} jN_{s,b}(j)}{N_{s,b}}.$$

$\square$

Remark 4.12. *The difference of two vectors:*

$$(\underbrace{x0x0x0\ldots\ldots}_{b}\underbrace{00\ldots0}_{s}\underbrace{x0x0x0\ldots\ldots}_{b}\underbrace{00\ldots0}_{s}\ldots\ldots) \text{ and}$$

$$(\underbrace{0x0x0x\ldots\ldots}_{b}\underbrace{00\ldots0}_{s}\underbrace{0x0x0x\ldots\ldots}_{b}\underbrace{00\ldots0}_{s}\ldots\ldots),$$

where $x \in GF(q) \setminus \{0\}$*,*

in $\psi_{(s,b),n,q}$ *gives vectors of* $\psi_{(s,b),n,q}$ *with maximum weight* w_{max}*. So, the minimum distance of the set with all vectors of* $\psi_{(s,b),n,q}$ *is less than or equal to* w_{max}*, and the maximum distance of the set is more than or equal to* w_{max}*.*

The following result gives the total probability of vectors of $\psi_{(s,b),n,2}$ over a binary symmetric channel.

Theorem 4.13. *Let* ϵ *be the transition probability of a memoryless binary symmetric channel. Then the total probability* $P(E)$ *of vectors of* $\psi_{(s,b),n,2}$ *in a vector of length* n *over* $GF(2)$ *is given by*

$$\sum_{i=0}^{n} \epsilon^{\lambda_i}(1-\epsilon)^{n-m_i},$$

where m_i *and* λ_i *are given by Lemma* 3.1.

Proof. With the usual meaning of m_i and λ_i $(i = 1, 2, \ldots, n)$ by Lemma 3.1, the number of nonzero positions in a vector of $\psi_{(s,b),n,2}$ that starts from i^{th} position is m_i, and other nonzero positions can be from the remaining $m_i - \lambda_i$ positions. Therefore, the total probability of vectors of $\psi_{(s,b),n,2}$ that start from the i^{th} position is given by

$$\sum_{j=0}^{m_i-\lambda_i} \binom{m_i-\lambda_i}{j} \epsilon^{\lambda_i+j}(1-\epsilon)^{n-\lambda_i-j}$$

$$= \epsilon^{\lambda_i}(1-\epsilon)^{-\lambda_i}(1-\epsilon)^{n-(m_i-\lambda_i)} \sum_{j=0}^{m_i-\lambda_i} \binom{m_i-\lambda_i}{j} \epsilon^j (1-\epsilon)^{m_i-\lambda_i-j}$$

$$= \epsilon^{\lambda_i}(1-\epsilon)^{n-m_i}.$$

Varying i from 1 to n gives the result. □

Finally, we give the probability of decoding error for a $P_{(s,b),n,2}BC$-code, which is equivalent to Result 1.17.

Theorem 4.14. *Let C be an (n,k) binary $P_{(s,b),n,2}BC$-code. If $P_D(E)$ is the probability of decoding error of the code C on a memoryless binary symmetric channel with transition probability ϵ, then*

$$P_D(E) = 1 - \sum_{j=0}^{w_{max}} N_{s,b}(j).\epsilon^j(1-\epsilon)^{n-j},$$

where $N_{s,b}(j)$ is given by Lemma 4.10.

Proof. For the proof, we follow the same technique as done in Theorem 3.15. In this case, the probability of vectors of $\psi_{(s,b),n,2}$ with weight j forming one of the coset leaders is

$$N_{s,b}(j).\epsilon^j(1-\epsilon)^{n-j}.$$

where $N_{s,b}(j)$ is given by Lemma 4.10.

Therefore, the probability $P_D(E)$ of decoding error of the code C is the probability that the error is not one of the coset leaders. So

$$P_D(E) = 1 - \sum_{j=0}^{w_{max}} N_{s,b}(j).\epsilon^j(1-\epsilon)^{n-j}.$$

□

Remark 4.15. *For $s = 3, b = 2$ and $\epsilon = 0.01$, we determine the probability of decoding error $P_D(E)$ of $P_{(s,b),n,2}BC$-codes of different lengths as follows.*

Table 4.4: **Table of $P_D(E)$**

n	λ	l	$P_D(E)$
11	2	1	0.060
12	2	2	0.064
13	2	3	0.074
14	2	4	0.085
15	3	0	0.095
16	3	1	0.106
17	3	2	0.116
18	3	3	0.127
19	3	4	0.130
20	4	0	0.148
21	4	1	0.158
22	4	2	0.169
23	4	3	0.179

We find that the probability of decoding error of a $P_{(s,b),n,2}BC$-code increases as the length of the code increases. So, a shorter length of code is more efficient.