Chapter 5
Low-density periodical burst error correcting codes with decoding probability and weight distribution

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- Das, P. K. and Haokip, L. Low-density periodical burst correcting codes with decoding probability and detection capability. *Journal of Applied Mathematics and Computing*, 68:4537-4557, 2022.

Chapter 5

Low-density periodical burst error correcting codes with decoding probability and weight distribution

In this chapter, we again extend the work of Chapter 4. We study the periodical burst errors by putting weight constraint on them, i.e., we study in this chapter about the correction of *low-density periodical burst errors* defined in Definition 1.8. This chapter studies conditions for the existence of linear codes correcting low-density periodical burst errors, (Hamming) weight distribution of the error pattern, and error decoding probability of the codes. Further, we study weight distribution and Plotkin's type of bound for some periodical burst errors (other than correctable errors) which are detected by a low-density periodical burst correcting code. These works are arranged as follows. Section 5.1 presents necessary and sufficient conditions for the existence of a linear code correcting s-periodical burst errors of length b with weight w. This is followed by examples of such codes along with comparisons on the number of check digits of these codes with the $\boldsymbol{P_{(s,b),n,q}BC-codes}$. In Section 5.2, we present the error pattern's weight distribution and a bound on the largest attainable minimum weight by an element in the error set. This is followed by the probability of error decoding of the low-density periodical burst correcting codes in a binary symmetric channel. Section 5.3 gives some periodical burst errors (other than correctable errors) which will be detected by a low-density periodical burst correcting code, and then provides weight distribution and Plotkin's type of bound for these detected periodical burst errors.

Here, we use the following notations for the specific studied error pattern and the corresponding error-correcting codes:

$\boldsymbol{\psi_{(s,b|w),n,q}}$: set of all s-periodical burst errors of length b with weight w in an n-tuple over $GF(q)$.

$\boldsymbol{LDP_{(s,b|w),n,q}BC-code}$: length-n linear code correcting s-periodical burst errors of length b with weight w over $GF(q)$.

5.1 $LDP_{(s,b|w),n,q}BC$-codes and comparison

This section presents the conditions for the existence of a $LDP_{(s,b|w),n,q}BC$-code and then gives comparisons on the number of check digits of $P_{(s,b),n,q}BC$-codes with $LDP_{(s,b|w),n,q}BC$-codes.

5.1.1 Conditions for existence of $LDP_{(s,b|w),n,q}BC$-codes

We present here necessary and sufficient conditions for the existence of a $LDP_{(s,b|w),n,q}BC$-code. Examples are provided to support the results. We first prove with the following lemma, where γ is a function defined in Chapter 3.

Lemma 5.1. *For given non-negative integers n, b and s $(n \geq b+s)$, let $N_{(s,b|w),n,q} = |\psi_{(s,b|w),n,q}|$. Then*

$$N_{(s,b|w),n,q} = \sum_{i=1}^{n} \left\{ \left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j} \right]^{\left\lfloor \frac{n-i+1}{s+b} \right\rfloor} \times \sum_{j=0}^{\min\{w-1, g_i-1\}} \binom{g_i-1}{j} (q-1)^{1+j} \right\},$$

where $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$ and $\sum_{j=0}^{\min\{w-1,g_i-1\}} \binom{g_i-1}{j}(q-1)^{1+j} = 1$ if $g_i = 0$.

Proof. If periodical burst error starts from the i^{th} position $(1 \leq i \leq n)$ in a vector of length n, the number of sets (excluding the last set) in which nonzero components of vectors of $\psi_{(s,b|w),n,q}$ are confined, is (see Lemma 3.1)

$$\left\lfloor \frac{n-i+1}{s+b} \right\rfloor.$$

In each set, the first component is always nonzero and the remaining $b-1$ components can be chosen by $\sum_{j=1}^{w-1}\binom{b-1}{j}$ ways. As the number of complete sets of b consecutive components is $\left\lfloor\frac{n-i+1}{s+b}\right\rfloor$, the total number of vectors of $\psi_{(s,b|w),n,q}$ in these sets is

$$\left[\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^{1+j}\right]^{\left\lfloor\frac{n-i+1}{s+b}\right\rfloor}.$$

The last set contains $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$ components, out of which the first one is nonzero if $g_i > 0$. The number of ways the last set can be selected is

$$\begin{cases} 1 & \text{if} \quad g_i = 0 \\ \sum\limits_{j=0}^{\min\{w-1,g_i-1\}}\binom{g_i-1}{j}(q-1)^{1+j} & \text{otherwise.} \end{cases}$$

Therefore, the number of vectors of $\psi_{(s,b|w),n,q}$, if it starts from the i^{th} position, is

$$\left[\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^{1+j}\right]^{\left\lfloor\frac{n-i+1}{s+b}\right\rfloor} \times \sum_{j=0}^{\min\{w-1,g_i-1\}}\binom{g_i-1}{j}(q-1)^{1+j},$$

where $\sum\limits_{j=0}^{\min\{w-1,g_i-1\}}\binom{g_i-1}{j}(q-1)^{1+j} = 1$ if $g_i = 0$.

Summing over i from 1 to n, we get the total number of vectors in $\psi_{(s,b|w),n,q}$ as

$$N_{(s,b|w),n,q} = \sum_{i=1}^{n}\left\{\left[\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^{1+j}\right]^{\left\lfloor\frac{n-i+1}{s+b}\right\rfloor} \times \sum_{j=0}^{\min\{w-1,g_i-1\}}\binom{g_i-1}{j}(q-1)^{1+j}\right\},$$

where $\sum\limits_{j=0}^{\min\{w-1,g_i-1\}}\binom{g_i-1}{j}(q-1)^{1+j} = 1$ if $g_i = 0$. □

Example 5.2. *Taking $n = 15, b = 2, s = 3,$ $w = 1$ and $q = 2$ in Lemma 5.1, we have*

$$N_{(3,2|1),15,2} = \sum_{i=1}^{15}\left\{\left[\sum_{j=0}^{0}\binom{1}{j}1^{1+j}\right]^{\left\lfloor\frac{16-i}{5}\right\rfloor} \times \sum_{j=0}^{0}\binom{\gamma\big((16-i) \mod 5\big)-1}{j}1^{1+j}\right\} = 15.$$

Then, the total number of vectors of $\psi_{(3,2|1),15,2}$ in a vector of length 15 *are* 100001000010000, 010000100001000, 001000010000100, 000100001000010, 000010000100001, 000001000010000, 000000100001000, 000000010000100, 000000001000010, 000000000100001, 000000000010000, 000000000001000, 000000000000100, 000000000000010, 000000000000001.

Now, a necessary condition for the existence of a $LDP_{(s,b|w),n,q}BC$-code is given below.

Theorem 5.3. *For given non-negative integers n, b and s $(n \geq b+s)$, an (n,k) $LDP_{(s,b|w),n,q}BC$-code $(w \leq b)$ must satisfy*

$$q^{n-k} \geq 1 + N_{(s,b|w),n,q}, \tag{5.1}$$

where $N_{(s,b|w),n,q}$ is given by Lemma 5.1.

Proof. For correction of errors by a linear code, all the errors should be in different cosets of the code. So, by Lemma 5.1, we have

$$q^{n-k} \geq 1 + N_{(s,b|w),n,q}.$$

□

Remark 5.4. *Inequality* (5.1) *gives*

$$q^k \leq \frac{q^n}{1 + N_{(s,b|w),n,q}}.$$

That is, the cardinality of an $(n,\, k)$ $LDP_{(s,b|w),n,q}BC$-code is at most $\dfrac{q^n}{1 + N_{(s,b|w),n,q}}$.

For a sufficient condition for the existence of a $LDP_{(s,b|w),n,q}BC$-code, we follow the same technique as followed in the previous chapters.

Theorem 5.5. *For given non-negative integers n, b and s $(n \geq b+s)$, we can always construct an $(n,\, k)$ $LDP_{(s,b|w),n,q}BC$-code $(w \leq b)$ provided*

$$q^{n-k} > \sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^j \times \left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j} \right]^{\left\lfloor \frac{n}{s+b} \right\rfloor - 1} \times \sum_{j=0}^{\min\{w-1,g-1\}} \binom{g-1}{j} (q-1)^{1+j} \times N_{(s,b|w),n-b,q},$$

where $g = \gamma\big(n \mod (s+b)\big)$, $\displaystyle\sum_{j=0}^{\min\{w-1,g-1\}} \binom{g-1}{j} (q-1)^{1+j} = 1$ if $g = 0$, and $N_{(s,b|w),n-b,q}$ is given by Lemma 5.1.

Proof. Take any nonzero $(n-k)$-tuple as the first column h_1 of the $(n-k) \times n$ parity-check matrix H of the code and suppose the columns $h_2, h_3, \ldots, h_{n-1}$ are added suitably to H. Then a nonzero column h_n can be added to H provided that it is not a linear combination of $w-1$ or less columns within the set of the immediately preceding $b-1$ columns, together with linear combinations of columns of previous sets of b consecutive columns with at most w columns from each set, along with a linear combination of w or less columns taken from the last set of b or less consecutive columns confined to the first $n-b$ columns with the condition that the sets are also at a gap of s columns and the first column in each set presents in the combination. This means

$$h_n \neq \left(\sum_{i=1}^{b-1} a_{i1}h_{n-i} + \sum_{i=0}^{b-1} b_{i1}h_{n-(s+b)-i} + \sum_{i=0}^{b-1} b_{i2}h_{n-2(s+b)-i} + \cdots + \sum_{i=0}^{g-1} b_{i\lambda}h_{n-\lambda(s+b)-i}\right)$$
$$+ \left(\sum_{i=0}^{b-1} \alpha_{i1}h_{j'-i} + \sum_{i=0}^{b-1} \beta_{i1}h_{j'-(s+b)-i} + \sum_{i=0}^{b-1} \beta_{i2}h_{j'-2(s+b)-i} + \cdots + \sum_{i=0}^{g'-1} \beta_{i\lambda'}h_{j'-\lambda'(s+b)-i}\right), \tag{5.2}$$

where $a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij} \in GF(q)$ such that with number of nonzero $a_{ij} \leq w-1$, and that of $b_{ij}, \alpha_{ij}, \beta_{ij} \leq w$ with $b_{0i}, \alpha_{0i}, \beta_{0i} \neq 0$; $j' \leq n-b$; $g = \gamma\big(n \mod (s+b)\big)$, $g' = \gamma\big((n-b-j'+1) \mod (s+b)\big)$, $\lambda = \left\lfloor \frac{n}{s+b} \right\rfloor$ and $\lambda' = \left\lceil \frac{n-b-j'+1}{s+b} \right\rceil$.

Note that in Expression (5.2), g and g' will be zero if n and $n-b-j'+1$ are multiples of $s+b$ and in that case we take $\sum_{i=0}^{g-1} b_{i\lambda}h_{n-\lambda(s+b)-i} = 0 = \sum_{i=0}^{g'-1} b_{i\lambda'}h_{j'-\lambda'(s+b)-i}$.

The condition (5.2) ensures that the syndromes of any two error patterns are distinct.

We now calculate the linear combinations on the right hand side (R.H.S.) of (5.2) as follows:

The number of ways a_{i1}'s in the first bracket on R.H.S. of (5.2) can be chosen is

$$\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^j.$$

The number of ways b_{ij}'s $(1 \leq j \leq \lambda - 1)$ can be chosen is

$$\left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j}\right]^{\lambda-1}.$$

The b_{ij}'s in the last summation of the first bracket can be chosen by

$$\begin{cases} 1 & \text{if} \quad g = 0 \\ \displaystyle\sum_{j=0}^{\min\{w-1, g-1\}} \binom{g-1}{j} (q-1)^{1+j} & \text{if} \quad g > 0. \end{cases}$$

Therefore, the total combination of the first bracket on R.H.S. of (5.2) is

$$\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^j \times \left[\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^{1+j}\right]^{\lambda-1} \times \sum_{j=0}^{\min\{w-1,g-1\}}\binom{g-1}{j}(q-1)^{1+j},$$

where $\displaystyle\sum_{j=0}^{\min\{w-1,g-1\}}\binom{g-1}{j}(q-1)^{1+j}=1$ if $g=0$.

The second bracket on R.H.S. of (5.2) gives the number of vectors of $\psi_{(s,b|w),n-b,q}$ in a vector of length $n-b$. This number, including the zero combination, is given by Lemma 5.1 as $N_{(s,b|w),n-b,q}$.

Thus, the total number of all possible linear combinations on R.H.S. of (5.2) is

$$\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^j \times \left[\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^{1+j}\right]^{\lambda-1}$$
$$\times \sum_{j=0}^{\min\{w-1,g-1\}}\binom{g-1}{j}(q-1)^{1+j} \times N_{(s,b|w),n-b,q},$$

where $\displaystyle\sum_{j=0}^{\min\{w-1,g-1\}}\binom{g-1}{j}(q-1)^{1+j}=1$ if $g=0$.

Since there are q^{n-k} available columns, we can add the n^{th} column provided

$$q^{n-k} > \sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^j \times \left[\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^{1+j}\right]^{\lambda-1}$$
$$\times \sum_{j=0}^{\min\{w-1,g-1\}}\binom{g-1}{j}(q-1)^{1+j} \times N_{(s,b|w),n-b,q},$$

where $\displaystyle\sum_{j=0}^{\min\{w-1,g-1\}}\binom{g-1}{j}(q-1)^{1+j}=1$ if $g=0$.

□

Now, we provide three examples of codes discussed in Theorem 5.5: two for binary case and one for ternary.

Example 5.6. *Consider* $n=18$*,* $s=5$*,* $b=4$*,* $w=2$ *and* $q=2$ *in Theorem 5.5, then* $\lambda=\lfloor\frac{18}{9}\rfloor=2, l=0$*. Then*

$$2^{n-k} > \sum_{j=0}^{1}\binom{3}{j} \times \left[\sum_{j=0}^{1}\binom{3}{j}\right]^{1} \times \sum_{j=0}^{\min\{1,g-1\}}\binom{g-1}{j} \times N_{(5,4|2),14,2}.$$

Now

$$N_{(5,4|2),14,2} = \sum_{i=1}^{14} \left\{ \left[\sum_{j=0}^{2-1} \binom{4-1}{j} (2-1)^{1+j} \right]^{\lfloor \frac{n-i+1}{s+b} \rfloor} \times \sum_{j=0}^{\min\{2-1, g_i-1\}} \binom{g_i-1}{j} (2-1)^{1+j} \right\}$$

$$= \sum_{i=1}^{14} \left\{ \left[\sum_{j=0}^{1} \binom{3}{j} (2-1)^{1+j} \right]^{\lfloor \frac{14-i+1}{9} \rfloor} \times \sum_{j=0}^{\min\{1, g_i-1\}} \binom{g_i-1}{j} (2-1)^{1+j} \right\}$$

$$= 4^{\lfloor \frac{14}{9} \rfloor} \times 4 + 4^{\lfloor \frac{13}{9} \rfloor} \times 4^1 + 4^{\lfloor \frac{12}{9} \rfloor} \times 3 + 4^{\lfloor \frac{11}{9} \rfloor} \times 2 + 4^{\lfloor \frac{10}{9} \rfloor} \times 1 + 4^{\lfloor \frac{9}{9} \rfloor} \times 1$$

$$+ 4^{\lfloor \frac{8}{9} \rfloor} \times 4 + 4^{\lfloor \frac{7}{9} \rfloor} \times 4 + 4^{\lfloor \frac{6}{9} \rfloor} \times 4 + 4^{\lfloor \frac{5}{9} \rfloor} \times 4 + 4^{\lfloor \frac{4}{9} \rfloor} \times 4 + 4^{\lfloor \frac{3}{9} \rfloor} \times 3$$

$$+ 4^{\lfloor \frac{2}{9} \rfloor} \times 2 + 4^{\lfloor \frac{1}{9} \rfloor} \times 1$$

$$= 4^2 + 4^2 + 4 \times 3 + 4 \times 2 + 4 \times 7 + 3 + 2 + 1$$

$$= 86,$$

where $g_i = \gamma\big((14-i+1) \pmod 9\big)$ *and* $\sum_{j=0}^{0} \binom{g_i-1}{j} (2-1)^{1+j} = 1$ *if* $g_i = 0$.

So

$$2^{n-k} > 4 \times 4 \times 86 = 1376$$

$$\Rightarrow n - k > 10.$$

We take $n-k = 11$ *and this gives rise to a binary* $(18,7)$ *linear code whose parity check matrix* H *of order* 11×18 *is given by*

$$H = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}_{11 \times 18}.$$

It can be verified from Table 5.1 that the syndromes of all vectors of $\psi_{(5,4|2),18,2}$ *are nonzero and distinct, showing that the code can correct all vectors of* $\psi_{(5,4|2),18,2}$. *So,* *the code is a* $(18, 7)$ *binary* $LDP_{(5,4|2),18,2}BC$*-code.*

Table 5.1: **Error Pattern-Syndrome**

Error Patterns	Syndromes	Error Patterns	Syndromes
1000 00000 1000 00000	10000000010	0 1010 00000 1010 0000	01110000101
1000 00000 1100 00000	10000000011	0 1010 00000 1001 0000	01000001000
1000 00000 1010 00000	11000111000	0 1001 00000 1000 0000	01001000001
1000 00000 1001 00000	10100000110	0 1001 00000 1100 0000	00001111011
1100 00000 1000 00000	11000000010	0 1001 00000 1010 0000	01101000101
1100 00000 1100 00000	11000000011	0 1001 00000 1001 0000	01011001000
1100 00000 1010 00000	10000111000	00 1000 00000 1000 000	01100111010
1100 00000 1001 00000	11100000110	00 1000 00000 1100 000	01000111110
1010 00000 1000 00000	10100000010	00 1000 00000 1010 000	01110110011
1010 00000 1100 00000	10100000011	00 1000 00000 1001 000	10011101010
1010 00000 1010 00000	11100111000	00 1100 00000 1000 000	01110111010
1010 00000 1001 00000	10000000110	00 1100 00000 1100 000	01010111110
1001 00000 1000 00000	10010000010	00 1100 00000 1010 000	01100110011
1001 00000 1100 00000	10010000011	00 1100 00000 1001 000	10001101010
1001 00000 1010 00000	11010111000	00 1010 00000 1000 000	01101111010
1001 00000 1001 00000	10110000110	00 1010 00000 1100 000	01001111110
0 1000 00000 1000 0000	01000000001	00 1010 00000 1010 000	01111110011
0 1000 00000 1100 0000	00000111011	00 1010 00000 1001 000	10010101010
0 1000 00000 1010 0000	01100000101	00 1001 00000 1000 000	01100011010
0 1000 00000 1001 0000	01010001000	00 1001 00000 1100 000	01000011110
0 1100 00000 1000 0000	01100000001	00 1001 00000 1010 000	01110010011
0 1100 00000 1100 0000	00100111011	00 1001 00000 1001 000	10011001010
0 1100 00000 1010 0000	01000000101	000 1000 00000 1000 00	00110000100
0 1100 00000 1001 0000	01110001000	000 1000 00000 1100 00	00100001101
0 1010 00000 1000 0000	01010000001	000 1000 00000 1010 00	11001010100
0 1010 00000 1100 0000	00010111011	000 1000 00000 1001 00	00001011110

Contd...

Table 5.1 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
000 1100 00000 1000 00	00111000100	00000 1000 00000 1000	11111110000
000 1100 00000 1100 00	00101001101	00000 1000 00000 1100	11000101010
000 1100 00000 1010 00	11000010100	00000 1000 00000 1010	10111111110
000 1100 00000 1001 00	00000011110	00000 1000 00000 1001	01111000101
000 1010 00000 1000 00	00110100100	00000 1100 00000 1000	11111100000
000 1010 00000 1100 00	00110101101	00000 1100 00000 1100	11000111010
000 1010 00000 1010 00	11001110100	00000 1100 00000 1010	10111101110
000 1010 00000 1001 00	00001111110	00000 1100 00000 1001	01111010101
000 1001 00000 1000 00	00110010100	00000 1010 00000 1000	11111111000
000 1001 00000 1100 00	00100011101	00000 1010 00000 1100	11000100010
000 1001 00000 1010 00	11001000100	00000 1010 00000 1010	10111110110
000 1001 00000 1001 00	00001001110	00000 1010 00000 1001	01111001101
0000 1000 00000 1000 0	00011001001	00000 1001 00000 1000	11111110100
0000 1000 00000 1100 0	11100011001	00000 1001 00000 1100	11000101110
0000 1000 00000 1010 0	00100010011	00000 1001 00000 1010	10111111010
0000 1000 00000 1001 0	01011000111	00000 1001 00000 1001	01111000001
0000 1100 00000 1000 0	00011101001	000000 1000 00000 100	00111001010
0000 1100 00000 1100 0	11100111001	000000 1000 00000 110	01111000100
0000 1100 00000 1010 0	00100110011	000000 1000 00000 101	10111111111
0000 1100 00000 1001 0	01011100111	000000 1100 00000 100	00111000010
0000 1010 00000 1000 0	00011011001	000000 1100 00000 110	01111001100
0000 1010 00000 1100 0	11100001001	000000 1100 00000 101	10111110111
0000 1010 00000 1010 0	00100000011	000000 1010 00000 100	00111001110
0000 1010 00000 1001 0	01011010111	000000 1010 00000 110	01111000000
0000 1001 00000 1000 0	00011000001	000000 1010 00000 101	10111111011
0000 1001 00000 1100 0	11100010001	000000 1001 00000 100	00111001000
0000 1001 00000 1010 0	00100011011	000000 1001 00000 110	01111000110
0000 1001 00000 1001 0	01011001111	000000 1001 00000 101	10111111101

Contd...

Table 5.1 – **Error Pattern-Syndrome**

Error Patterns	Syndroms	Error Patterns	Syndromes
0000000 1000 00000 10	01000000110	00000000000 1100 000	01100111110
0000000 1000 00000 11	11000110011	00000000000 1010 000	01010110011
0000000 1100 00000 10	01000000010	00000000000 1001 000	10111101010
0000000 1100 00000 11	11000110111	000000000000 1000 00	00100000100
0000000 1010 00000 10	01000000100	000000000000 1100 00	00110001101
0000000 1010 00000 11	11000110001	000000000000 1010 00	11011010100
0000000 1001 00000 10	01000000111	000000000000 1001 00	00011011110
0000000 1001 00000 11	11000110010	0000000000000 1000 0	00010001001
00000000 1000 00000 1	10000110001	0000000000000 1100 0	11101011001
00000000 1100 00000 1	10000110011	0000000000000 1010 0	00101010011
00000000 1010 00000 1	10000110000	0000000000000 1001 0	01010000111
00000000 1001 00000 1	11000001011	00000000000000 1000	11111010000
000000000 1000 00000	00000000010	00000000000000 1100	11000001010
000000000 1100 00000	00000000011	00000000000000 1010	10111011110
000000000 1010 00000	01000111000	00000000000000 1001	01111100101
000000000 1001 00000	00100000110	000000000000000 100	00111011010
0000000000 1000 0000	00000000001	000000000000000 110	01111010100
0000000000 1100 0000	01000111011	000000000000000 101	10111101111
0000000000 1010 0000	00100000101	0000000000000000 10	01000001110
0000000000 1001 0000	00010001000	0000000000000000 11	11000111011
00000000000 1000 000	01000111010	00000000000000000 1	10000110101

Example 5.7. *Consider* $n = 20$, $s = 5$, $b = 4$, $w = 3$ *and* $q = 2$ *in Theorem 5.5, then* $\lambda = \lfloor \frac{20}{9} \rfloor = 2, l = 2$. *Then Theorem 5.5 gives*

$$2^{n-k} > \sum_{j=0}^{2} \binom{4-1}{j} (2-1)^j \times \left[\sum_{j=0}^{2} \binom{4-1}{j} (2-1)^{1+j} \right]^{\lambda-1}$$
$$\times \sum_{j=0}^{\min\{3-1,2-1\}} \binom{2-1}{j} (2-1)^{1+j} \times N_{(5,4|3),16,2}$$

$$\Rightarrow 2^{n-k} > 7 \times 7 \times 2 \times 294 = 28812$$

$$\Rightarrow n-k \geq 15.$$

Taking $n-k=18$*, we get a binary* $(20,2)$ *linear code whose parity check matrix* H *of order* 18×20 *is given by*

$$H = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0
\end{bmatrix}_{18\times 20}.$$

Here also, the syndromes of all vectors of $\psi_{(5,4|3),20,2}$ *are found to be nonzero and distinct, showing that the code is a binary* $(20,2)$ $LDP_{(5,4|3),20,2}BC$*-code.*

Example 5.8. *Consider* $n=23$*,* $s=5$*,* $b=4$*,* $w=1$ *and* $q=3$ *in Theorem 5.5, then* $\lambda = \lfloor \frac{23}{9} \rfloor = 2, l=5$*. Then Theorem 5.5 gives*

$$3^{n-k} > 1 \times 2 \times 2 \times 62$$

$$\Rightarrow n-k > 5.$$

Considering $n-k=6$*, we get a ternary* $(23,17)$ *linear code whose parity check* *matrix* H *of order* 6×23 *is given by*

$$H = \begin{bmatrix} 1&0&1&0&0&0&0&0&2&0&0&0&0&0&0&1&1&0&0&1&0&0&1 \\ 0&1&0&0&0&0&1&0&0&1&0&0&0&0&1&0&1&1&0&0&0&0&0 \\ 0&0&2&0&0&0&2&0&0&0&1&0&0&2&0&0&1&0&1&0&0&0&0 \\ 0&0&0&1&1&0&1&2&0&0&0&2&0&0&0&0&0&0&2&0&0&0&0 \\ 1&2&0&0&1&0&1&0&0&0&0&0&2&0&1&0&0&0&0&1&1&0&0 \\ 0&0&0&0&1&2&0&1&0&1&0&0&0&0&0&1&0&0&0&1&0&1&0 \end{bmatrix}_{6\times 23}.$$

Here also, the syndromes of all vectors of $\psi_{(5,4|1),23,3}$ *are nonzero and distinct, showing that the code is a* $(23, 17)$ *ternary* $LDP_{(5,4|1),23,3}BC$*-code.*

5.1.2 Comparison

In this subsection also, we provide comparisons on the necessary and sufficient numbers of check digits for $LDP_{(s,b|w),n,q}BC$-codes with $P_{(s,b),n,q}BC$-codes. The $LDP_{(s,b|w),n,q}BC$-codes take less number of check digits than $P_{(s,b),n,q}BC$-codes. So, $LDP_{(s,b|w),n,q}BC$-codes are more efficient than $P_{(s,b),n,q}BC$-codes in terms of code rate.

Table 5.2: **Necessary number of check digits**

n	s	b	w	q	$n-k$ ($P_{(s,b),n,q}BC$-codes)	$n-k$ ($LDP_{(s,b\|w),n,q}BC$-codes)
10	3	2	1	3	4	3
10	3	2	1	4	4	2
11	3	2	1	3	5	3
12	4	2	1	2	5	3
13	4	2	1	2	5	3
14	4	3	2	2	6	6
18	5	4	2	2	8	7
20	5	4	3	2	9	9
23	5	4	1	3	11	4

Table 5.3: **Sufficient number of check digits**

n	s	b	w	q	$n-k$ ($P_{(s,b),n,q}BC$-codes)	$n-k$ ($LDP_{(s,b\|w),n,q}BC$-codes)
10	3	2	1	3	7	3
10	3	2	1	4	8	3
11	3	2	1	3	8	4
12	4	2	1	2	7	4
13	4	2	1	2	7	4
14	4	3	2	2	11	9
18	5	4	2	2	14	11
20	5	4	3	2	16	15
23	5	4	1	3	19	6

5.2 Weight distribution and error decoding probability

This section presents the weight distribution of the vectors of the error set $\psi_{(s,b|w),n,q}$. One may refer to [21, 24, 66] and their references for weight distribution of other types of error pattern. We provide an upper bound on the minimum weight of the error set $\psi_{(s,b|w),n,q}$. We then derive the total probability of the error pattern and decoding error probability of a $LDP_{(s,b|w),n,q}BC$-code in a binary symmetric channel.

Lemma 5.9. *For $0 \le j \le n$, let $N_{(s,b|w),n,q}(j) = |\psi_{(s,b|w),n,q}(j)|$. Then*

$$N_{(s,b|w),n,q}(j) = \sum_{i=1}^{n} \sum_{j_1,j_2,\dots,j_{l_i},j_{l'}} \binom{b-1}{j_1}\binom{b-1}{j_2}\cdots\binom{b-1}{j_{l_i}}\binom{g_i-1}{j_{l'}}(q-1)^{\lambda_i+j_1+j_2+\cdots+j_{l_i}+j_{l'}},$$

where $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$, $\lambda_i = \left\lceil \frac{n-i+1}{s+b} \right\rceil$, $l_i = \left\lfloor \frac{n-i+1}{s+b} \right\rfloor$ and $j_1, j_2, \dots, j_{l_i}, j_{l'}$ are nonnegative integers such that $\lambda_i + j_1 + j_2 + \cdots + j_{l_i} + j_{l'} = j$, $0 \le j_1, j_2, \dots, j_{l_i} \le w-1$ and $0 \le j_{l'} \le \min\{g_i - 1, w-1\}$.

Proof. The nonzero components of the error pattern that starts from the i^{th} position $(1 \leq i \leq n)$ are confined to $l_i = \left\lfloor \frac{n-i+1}{s+b} \right\rfloor$ sets of b consecutive components followed by the last set consisting of $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$ consecutive components, first position of each set is nonzero. Then we can select any j_i positions $(i = 1, 2, \ldots, l_i)$ from $b-1$ positions for nonzero components by $\binom{b-1}{j_i}$ ways and $j_{l'}$ positions from the last set by $\binom{g_i-1}{j_{l'}}$ ways, where $j_i \leq w-1$ and $j_{l'} \leq \min\{g_i - 1, w-1\}$. Therefore, the total number of vectors of $\psi_{(s,b|w),n,q}$ that have weight j is

$$\binom{b-1}{j_1}(q-1)^{1+j_1}\binom{b-1}{j_2}(q-1)^{1+j_2}\cdots\binom{b-1}{j_{l_i}}(q-1)^{1+j_{l_i}}\binom{g_i-1}{j_{l'}}(q-1)^{\delta+j_{l'}}$$

$$= \binom{b-1}{j_1}\binom{b-1}{j_2}\cdots\binom{b-1}{j_{l_i}}\binom{g_i-1}{j_{l'}}(q-1)^{l_i+\delta+j_1+j_2+\cdots+j_{l_i}+j_{l'}},$$

where $l_i+\delta+j_1+j_2+\cdots+j_{l_i}+j_{l'} = j$, $\delta = \begin{cases} 0 \text{ if } g_i = 0 \\ 1 \text{ otherwise} \end{cases}$, $0 \leq j_1, j_2, \cdots, j_{l_i} \leq w-1$ and $0 \leq j_{l'} \leq \min\{g_i - 1, w-1\}$.

So, the total number of vectors in $\psi_{(s,b|w),n,q}(j)$ is

$$N_{(s,b|w),n,q}(j) = \sum_{i=1}^{n} \sum_{j_1,j_2,\ldots,j_{l_i},j_{l'}} \binom{b-1}{j_1}\binom{b-1}{j_2}\cdots\binom{b-1}{j_{l_i}}\binom{g_i-1}{j_{l'}} \times (q-1)^{\lambda_i+j_1+j_2+\cdots+j_{l_i}+j_{l'}},$$

where $\lambda_i = l_i + \delta = \left\lceil \frac{n-i+1}{s+b} \right\rceil$, and $j_1, j_2, \ldots, j_{l_i}, j_{l'}$ are nonnegative integers such that $\lambda_i + j_1 + j_2 + \cdots + j_{l_i} + j_{l'} = j$, $0 \leq j_1, j_2, \ldots, j_{l_i} \leq w-1$ and $0 \leq j_{l'} \leq \min\{g_i - 1, w-1\}$.

□

Remark 5.10. *Observe that for given non-negative integers n, b and s $(n \geq s+b)$, the maximum number of nonzero components in a vector of $\psi_{(s,b|w),n,q}$ can be found when the periodical burst starts from the first position. This number, W_{max}, is given by*

$$W_{max} = \left\lfloor \frac{n}{s+b} \right\rfloor w + \gamma'(g), \text{ where } \gamma'(g) = \begin{cases} g & \text{if } g \leq w \\ w & \text{otherwise.} \end{cases}$$

So, $N_{(s,b|w),n,q}(j) = 0$ for $W_{max} < j \leq n$.

Theorem 5.11. *The minimum weight of a vector in the set* $\psi_{(s,b|w),n,q}$ *is at most*

$$\frac{\sum_{j=1}^{w_{max}} jN_{(s,b|w),n,q}(j)}{N_{(s,b|w),n,q}},$$

where $N_{(s,b|w),n,q}$ *is given by Lemma 5.1 and* $N_{(s,b|w),n,q}(j)$ *by Lemma 5.9.*

Proof. By Lemma 5.1 and Lemma 5.9, the average weight of a vector in $\psi_{(s,b|w),n,q}$ is

$$\frac{\sum_{j=1}^{w_{max}} jN_{(s,b|w),n,q}(j)}{N_{(s,b|w),n,q}}.$$

As the minimum weight of a vector in a set can be at most the average weight, this follows the theorem. □

Remark 5.12. *If w is odd, we consider the two vectors of* $\psi_{(s,b|w),n,q}$*:*

$$\big(\underbrace{x_1'0x_3'0x_5'0\ldots0x_w'00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ \underbrace{x_1''0x_3''0x_5''0\ldots0x_w''00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ldots\ldots\big) \text{ and}$$

$$\big(\underbrace{0x_2'0x_4'0x_6'0\ldots x_{w-1}'00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ \underbrace{0x_2''0x_4''0x_6''0\ldots x_{w-1}''00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ldots\ldots\big),$$

where $x_i', x_i'' \in GF(q)\setminus\{0\}$.

If w is even, then consider the two vectors of $\psi_{(s,b|w),n,q}$*:*

$$\big(\underbrace{x_1'0x_3'0x_5'0\ldots x_{w-1}'00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ \underbrace{x_1''0x_3''0x_5''0\ldots x_{w-1}''00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ldots\ldots\big) \text{ and}$$

$$\big(\underbrace{0x_2'0x_4'0x_6'0\ldots0x_w'00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ \underbrace{0x_2''0x_4''0x_6''0\ldots0x_w''00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ldots\ldots\big),$$

where $x_i', x_i'' \in GF(q)\setminus\{0\}$.

In both cases, the difference between the two vectors is a vector of $\psi_{(s,b|W_{max}),n,q}$. *So, the minimum distance of the set* $\psi_{(s,b|w),n,q} \le W_{max}$ *and the maximum distance of the set* $\psi_{(s,b|w),n,q} \ge W_{max}$.

Now, total probability of occurring of vectors of $\psi_{(s,b|w),n,2}$ in a binary symmetric channel is given in the following theorem.

Theorem 5.13. *The total probability* $P_w(E)$ *of the error set* $\psi_{(s,b|w),n,2}$ *over a mem-*

oryless binary symmetric channel with transition probability ϵ is given by

$$P_w(E) = \sum_{i=1}^{n} \Bigg[\sum_{j_1,j_2,\dots,j_{l_i},j_{l'}} \binom{b-1}{j_1}\binom{b-1}{j_2}\dots\binom{b-1}{j_{l_i}}\binom{g_i-1}{j_{l'}} \times$$
$$\epsilon^{\lambda_i+(j_1+j_2+\dots+j_{l_i}+j_{l'})}(1-\epsilon)^{n-\lambda_i-j_1-j_2-\dots-j_{l_i}-j_{l'}} \Bigg],$$

where $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$, $\lambda_i = \left\lceil \frac{n-i+1}{s+b} \right\rceil$, $l_i = \left\lfloor \frac{n-i+1}{s+b} \right\rfloor$, $0 \le j_1, j_2, \dots, j_{l_i} \le w-1$ *and* $0 \le j_{l'} \le \min\{g_i-1, w-1\}$.

Proof. As the first position of each set of the error pattern has a nonzero component, the number of always nonzero components in a periodical burst that starts from the i^{th} position $(1 \le i \le n)$ is $\lambda_i = \left\lceil \frac{n-i+1}{s+b} \right\rceil$. The other nonzero components come from the remaining positions such that each set contains no more than w nonzero components. As the nonzero components are confined to $l_i = \left\lfloor \frac{n-i+1}{s+b} \right\rfloor$ sets of b consecutive components followed by a set of $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$ consecutive components, the total probability of occurring of the vectors of $\psi_{(s,b|w),n,2}$ that start from i^{th} position is given by

$$\sum_{j_1,j_2,\dots,j_{l_i},j_{l'}} \binom{b-1}{j_1}\binom{b-1}{j_2}\dots\binom{b-1}{j_{l_i}}\binom{g_i-1}{j_{l'}} \times$$
$$\epsilon^{\lambda_i+(j_1+j_2+\dots+j_{l_i}+j_{l'})}(1-\epsilon)^{n-\lambda_i-j_1-j_2-\dots-j_{l_i}-j_{l'}},$$

where $0 \le j_1, j_2, \dots, j_{l_i} \le w-1$ and $0 \le j_{l'} \le \min\{g_i-1, w-1\}$.

Taking $i = 1, 2, \dots, n$ gives the result. □

Next, we give the probability of decoding error for a $LDP_{(s,b|w),n,2}BC$-code as follows.

Theorem 5.14. *Let $PD_w(E)$ be the probability of decoding error of an (n,k) binary $LDP_{(s,b|w),n,2}BC$-code on a memoryless binary symmetric channel with transition probability ϵ, then*

$$PD_w(E) = 1 - \sum_{j=1}^{W_{max}} N_{(s,b|w),n,2}(j).\epsilon^j(1-\epsilon)^{n-j},$$

where $N_{(s,b|w),n,2}(j)$ is given by Lemma 5.9.

Proof. We know that the probability of correcting an error is the probability that the error is a coset leader in the standard array for the code and the probability of a vector of $\psi_{(s,b|w),n,2}(j)$ being one of the coset leaders is

$$N_{(s,b|w),n,2}(j).\epsilon^j(1-\epsilon)^{n-j}.$$

Therefore, the probability $PD_w(E)$ of decoding error of the code is given by

$$PD_w(E) = 1 - \sum_{j=1}^{W_{max}} N_{(s,b|w),n,2}(j).\epsilon^j(1-\epsilon)^{n-j}.$$

□

5.3 Detection and weight distribution of some periodical bursts

In this section, we give detection of some periodical burst errors (other than correctable errors) by a $LDP_{(s,b|w),n,q}BC$-code. The weight distribution of those errors and upper bound on the minimum weight of the set of such errors are given. For this, we first define two sets.

For $s > b$, let $\mathbb{A}$ be the collection of all vectors of $\psi_{(s-b,2b),n,q}$ of the form:

$$(\underbrace{x_1 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_2 \bullet\bullet\cdots\bullet}_{b}\underbrace{00\ldots0}_{s-b}\underbrace{x_3 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_4 \bullet\bullet\cdots\bullet}_{b}\underbrace{00\ldots0}_{s-b}\underbrace{x_5 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_6 \bullet\bullet\cdots\bullet}_{b}\ldots\ldots),$$

and for $s \le b$, $\mathbb{A}'$ be the collection of all vectors of $\psi_{(1,b+s-1),n,q}$ of the form:

$$(\underbrace{x_1 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_2 \bullet\bullet\cdots\bullet}_{s-1}0\underbrace{x_3 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_4 \bullet\bullet\cdots\bullet}_{s-1}0\underbrace{x_5 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_6 \bullet\bullet\cdots\bullet}_{s-1}0\ldots\ldots),$$

where $x_i \in GF(q)\setminus\{0\}$ and $\bullet \in GF(q)$ such that consecutive $b-1$ bullets have at most $w-1$ nonzero components.

Theorem 5.15. *A $LDP_{(s,b|w),n,q}BC$-code detects all periodical burst errors from the sets $\mathbb{A}$ and $\mathbb{A}'$.*

Proof. Since every member of $\mathbb{A}$ or $\mathbb{A}'$ can be expressed as the sum (difference) of two vectors of $\psi_{(s,b|w),n,q}$, no element of $\mathbb{A}$ or $\mathbb{A}'$ can be a codeword of $LDP_{(s,b|w),n,q}BC$-code. This proves the theorem. □

Now we give the weight distribution of the vectors of the sets $\mathbb{A}$ and $\mathbb{A}'$.

Lemma 5.16. *If A_j be the collections of the vectors of the set $\mathbb{A}$ having weight j and $g^{(1)} = \gamma_1\big(n \mod (b+s)\big)$, where* $\gamma_1(r) = \begin{cases} r & \text{if } \ n \pmod{(b+s)} \leq 2b \\ 2b & \text{otherwise.} \end{cases}$

Then

1. *if $g^{(1)} = \gamma_1\big(n \mod (b+s)\big) = 0$,*
$$|A_j| = \sum_{j_1, j_2, \ldots, j_{2l}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] (q-1)^{2l+j_1+j_2+\cdots+j_{2l}},$$
where $2l + j_1 + j_2 + \cdots + j_{2l} = j$ such that $0 \leq j_1, j_2, \ldots, j_{2l} \leq w-1$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

2. *if $1 \leq g^{(1)} = \gamma_1\big(n \mod (b+s)\big) \leq b$,*
$$|A_j| = \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] \binom{g^{(1)}-1}{j_{l'}} (q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+1},$$
where $2l + j_1 + j_2 + \cdots + j_{2l} + j_{l'} + 1 = j$ such that $0 \leq j_1, j_2, \ldots, j_{2l} \leq w-1$, $0 \leq j_{l'} \leq \min\{g^{(1)}-1, w-1\}$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

3. *if $b+1 \leq g^{(1)} = \gamma_1\big(n \mod (b+s)\big) \leq 2b$,*
$$|A_j| = \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}, j_{l''}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] \times \binom{b-1}{j_{l'}} \binom{g^{(1)}-b-1}{j_{l''}} (q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2},$$
where $2l + j_1 + j_2 + \cdots + j_{2l} + j_{l'} + j_{l''} + 2 = j$ such that $0 \leq j_1, j_2, \ldots, j_{2l}, j_{l'} \leq w-1$, $0 \leq j_{l''} \leq \min\{g^{(1)}-b-1, w-1\}$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Further, maximum weight of the set $\mathbb{A}$ is
$$W^1_{max} = 2w \left\lfloor \frac{n}{s+b} \right\rfloor + \gamma''_1(g^{(1)}),$$
where $\gamma''_1(g^{(1)}) = \begin{cases} g^{(1)} & \text{if } \ 0 \leq g^{(1)} \leq w \\ w & \text{if } \ w+1 \leq g^{(1)} \leq b \\ w + g^{(1)} - b & \text{if } \ b+1 \leq g^{(1)} \leq b+w \\ 2w & \text{if } \ b+w+1 \leq g^{(1)} \leq 2b. \end{cases}$

Proof. Observe that if $n \pmod{(b+s)} = 0$, all nonzero components in any vector of $\mathbb{A}$ are confined to $\left\lfloor \frac{n}{s+b} \right\rfloor$ sets that are separated by $s-b$ consecutive zeros. If $n \pmod{(b+s)} \neq 0$, nonzero components in a vector of $\mathbb{A}$ are confined to $\left\lfloor \frac{n}{s+b} \right\rfloor + 1$ sets that are separated by $s-b$ consecutive zeros, where each of $\left\lfloor \frac{n}{s+b} \right\rfloor$ sets has $2b$ consecutive components and the last set has $g^{(1)} = \gamma_1\big(n \mod (b+s)\big)$ components, where $\gamma_1(r) = \begin{cases} r & \text{if} \quad n \pmod{(b+s)} \leq 2b \\ 2b & \text{otherwise.} \end{cases}$

Then the cardinality of the set A_j of the vectors of $\mathbb{A}$ having weight j is calculated as follows.

Sub-case (i). If $g^{(1)} = 0$, the number $|A_j|$ of the vectors of $\mathbb{A}$ having weight j is given by

$$\sum_{j_1, j_2, \ldots, j_{2l}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{b-1}{j_2}(q-1)^{1+j_2} \times \cdots \times \binom{b-1}{j_{2l}}(q-1)^{1+j_{2l}}$$
$$= \sum_{j_1, j_2, \ldots, j_{2l}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right](q-1)^{2l+j_1+j_2+\cdots+j_{2l}},$$

where $2l + j_1 + j_2 + \cdots + j_{2l} = j$ such that $0 \leq j_1, j_2, \ldots, j_{2l} \leq w-1$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Sub-case (ii). If $1 \leq g^{(1)} \leq b$, the number $|A_j|$ of the vectors of $\mathbb{A}$ having weight j is given by

$$\sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{b-1}{j_2}(q-1)^{1+j_2} \times \cdots \times \binom{b-1}{j_{2l}}(q-1)^{1+j_{2l}}$$
$$\times \binom{g^{(1)}-1}{j_{l'}}(q-1)^{1+j_{l'}}$$
$$= \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] \binom{g^{(1)}-1}{j_{l'}}(q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+1},$$

where $2l + j_1 + j_2 + \cdots + j_{2l} + j_{l'} + 1 = j$ such that $0 \leq j_1, j_2, \ldots, j_{2l} \leq w-1$, $0 \leq j_{l'} \leq \min\{g^{(1)}-1, w-1\}$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Sub-case (iii). If $b+1 \leq g^{(1)} \leq 2b$, the number $|A_j|$ of the vectors of $\mathbb{A}$ having weight j is given by

$$\sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}, j_{l''}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{b-1}{j_2}(q-1)^{1+j_2} \times \cdots \times \binom{b-1}{j_{2l}}(q-1)^{1+j_{2l}}$$
$$\times \binom{b-1}{j_{l'}}(q-1)^{1+j_{l'}} \times \binom{g^{(1)}-b-1}{j_{l''}}(q-1)^{1+j_{l''}}$$

$$= \sum_{j_1,j_2,\ldots,j_{2l},j_{l'},j_{l''}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] \binom{b-1}{j_{l'}} \binom{g^{(1)}-b-1}{j_{l''}} (q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2},$$

where $2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2=j$ such that $0 \le j_1, j_2, \ldots, j_{2l}, j_{l'} \le w-1$, $0 \le j_{l''} \le \min\{g^{(1)}-b-1, w-1\}$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Maximum weight of the set $\mathbb{A}$ is the weight of the vector which can be calculated by taking $2w$ weight in each $\left\lfloor \frac{n}{s+b} \right\rfloor$ set of complete $2b$ components and the last set having the maximum weight

$$\gamma_1''(g^{(1)}) = \begin{cases} g^{(1)} & \text{if } \ 0 \le g^{(1)} \le w \\ w & \text{if } \ w+1 \le g^{(1)} \le b \\ w+g^{(1)}-b & \text{if } \ b+1 \le g^{(1)} \le b+w \\ 2w & \text{if } \ b+w+1 \le g^{(1)} \le 2b. \end{cases}$$

This shows that

$$W^1_{max} = 2w\left\lfloor \frac{n}{s+b} \right\rfloor + \gamma_1''(g^{(1)}).$$

□

Lemma 5.17. *Let A'_j be the set of all vectors of the set $\mathbb{A}'$ whose weight is j and* $g^{(2)} = \gamma_2\big(n \mod (b+s)\big)$ *where* $\gamma_2(r) = \begin{cases} r & \text{if } \ n \pmod{(b+s)} \le b+s-1 \\ b+s-1 & \text{otherwise.} \end{cases}$

Then

1. *if* $g^{(2)} = \gamma_2\big(n \mod (b+s)\big) = 0$,

$$|A'_j| = \sum_{j_1,j_2,\ldots,j_{2l}} \left[\prod_{\rho=1}^{l} \binom{b-1}{j_{2\rho-1}} \prod_{\rho=1}^{l} \binom{s-2}{j_{2\rho}}\right] (q-1)^{2l+j_1+j_2+\cdots+j_{2l}},$$

where $2l+j_1+j_2+\cdots+j_{2l}=j$ *such that* $0 \le j_1, j_3, \ldots, j_{2l-1} \le w-1$, $0 \le j_2, j_4, \ldots, j_{2l} \le \min\{w-1, s-2\}$ *and* $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

2. *if* $1 \le g^{(2)} = \gamma_2\big(n \mod (b+s)\big) \le b$,

$$|A'_j| = \sum_{j_1,j_2,\ldots,j_{2l},j_{l'}} \left[\prod_{\rho=1}^{l} \binom{b-1}{j_{2\rho-1}} \prod_{\rho=1}^{l} \binom{s-2}{j_{2\rho}}\right] \binom{g^{(2)}-1}{j_{l'}} (q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+1},$$

where $2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+1=j$ *such that* $0 \le j_1, j_3, \ldots, j_{2l-1} \le w-1$, $0 \le j_2, j_4, \ldots, j_{2l} \le \min\{w-1, s-2\}$, $0 \le j_{l'} \le \min\{w-1, g^{(2)}-1\}$ *and*

$$l = \left\lfloor \frac{n}{s+b} \right\rfloor.$$

3. if $b+1 \leq g^{(2)} = \gamma_2\big(n \mod (b+s)\big) \leq b+s-1$,

$$|A'_j| = \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}, j_{l''}} \left[\prod_{\rho=1}^{l} \binom{b-1}{j_{2\rho-1}} \prod_{\rho=1}^{l} \binom{s-2}{j_{2\rho}} \right] \times \binom{b-1}{j_{l'}} \binom{g^{(2)}-b-1}{j_{l''}} (q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2},$$

where $2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2 = j$ *such that* $0 \leq j_1, j_3, \ldots, j_{2l-1}, j_{l'} \leq w-1$, $0 \leq j_2, j_4, \ldots, j_{2l} \leq \min\{w-1, s-2\}$, $0 \leq j_{l''} \leq \min\{w-1, g^{(2)}-b-1\}$ *and* $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Further, maximum weight of the set $\mathbb{A}'$ *is*

$$W^2_{max} = 2w\left\lfloor \frac{n}{s+b} \right\rfloor + \gamma''_2(g^{(2)}),$$

$$\textit{where } \gamma''_2(g^{(2)}) = \begin{cases} g^{(2)} & \textit{if} \quad 0 \leq g^{(2)} \leq w \\ w & \textit{if} \quad w+1 \leq g^{(2)} \leq b \\ w+g^{(2)}-b & \textit{if} \quad b+1 \leq g^{(2)} \leq b+w \\ 2w & \textit{if} \quad b+w+1 \leq g^{(2)} \leq b+s-1. \end{cases}$$

Proof. In this case also, if $n \pmod{(b+s)} = 0$, all nonzero components in any vector of $\mathbb{A}'$ are confined to $\left\lfloor \frac{n}{s+b} \right\rfloor$ sets that are separated by one zero. If $n \pmod{(b+s)} \neq 0$, nonzero components in a vector of $\mathbb{A}'$ are confined to $\left\lfloor \frac{n}{s+b} \right\rfloor + 1$ sets that are separated by one zero, where each of $\left\lfloor \frac{n}{s+b} \right\rfloor$ sets has $b+s-1$ consecutive components and the last set has $g^{(2)} = \gamma_2\big(n \mod (b+s)\big)$ components, where
$$\gamma_2(r) = \begin{cases} r & \text{if} \quad n \pmod{(b+s)} \leq b+s-1 \\ b+s-1 & \text{otherwise.} \end{cases}$$
Then the cardinality of the set A'_j of the vectors of $\mathbb{A}'$ having weight j is calculated as follows.

Sub-case (i). If $g^{(2)} = 0$, the number $|A'_j|$ of the vectors of $\mathbb{A}'$ having weight j is given by

$$\sum_{j_1, j_2, \ldots, j_{2l}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{s-2}{j_2}(q-1)^{1+j_2} \times \binom{b-1}{j_3}(q-1)^{1+j_3}$$

$$\times \binom{s-2}{j_4}(q-1)^{1+j_4} \times \cdots \times \binom{b-1}{j_{2l-1}}(q-1)^{1+j_{2l-1}} \times \binom{s-2}{j_{2l}}(q-1)^{1+j_{2l}}$$
$$= \sum_{j_1,j_2,\ldots,j_{2l}} \left[\prod_{\rho=1}^{l}\binom{b-1}{j_{2\rho-1}}\prod_{\rho=1}^{l}\binom{s-2}{j_{2\rho}}\right](q-1)^{2l+j_1+j_2+\cdots+j_{2l}},$$

where $2l + j_1 + j_2 + \cdots + j_{2l} = j$ such that $0 \leq j_1, j_3, \ldots, j_{2l-1} \leq w-1$, $0 \leq j_2, j_4, \ldots, j_{2l} \leq \min\{w-1, s-2\}$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Sub-case (ii). If $1 \leq g^{(2)} \leq b$, the number $|A'_j|$ of the vectors of $\mathbb{A}'$ having weight j is given by

$$\sum_{j_1,j_2,\ldots,j_{2l},j_{l'}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{s-2}{j_2}(q-1)^{1+j_2} \times \binom{b-1}{j_3}(q-1)^{1+j_3}$$
$$\times \binom{s-2}{j_4}(q-1)^{1+j_4} \times \cdots \times \binom{b-1}{j_{2l-1}}(q-1)^{1+j_{2l-1}} \times \binom{s-2}{j_{2l}}(q-1)^{1+j_{2l}}$$
$$\times \binom{g^{(2)}-1}{j_{l'}}(q-1)^{1+j_{l'}}$$
$$= \sum_{j_1,j_2,\ldots,j_{2l},j_{l'}} \left[\prod_{\rho=1}^{l}\binom{b-1}{j_{2\rho-1}}\prod_{\rho=1}^{l}\binom{s-2}{j_{2\rho}}\right]\binom{g^{(2)}-1}{j_{l'}}(q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+1},$$

where $2l + j_1 + j_2 + \cdots + j_{2l} + j_{l'} + 1 = j$ such that $0 \leq j_1, j_3, \ldots, j_{2l-1} \leq w-1$, $0 \leq j_2, j_4, \ldots, j_{2l} \leq \min\{w-1, s-2\}$, $0 \leq j_{l'} \leq \min\{w-1, g^{(2)}-1\}$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Sub-case (iii). If $b+1 \leq g^{(2)} \leq b+s-1$, the number $|A'_j|$ of the vectors of $\mathbb{A}'$ having weight j is given by

$$\sum_{j_1,j_2,\ldots,j_{2l},j_{l'},j_{l''}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{s-2}{j_2}(q-1)^{1+j_2} \times \binom{b-1}{j_3}(q-1)^{1+j_3}$$
$$\times \binom{s-2}{j_4}(q-1)^{1+j_4} \times \cdots \times \binom{b-1}{j_{2l-1}}(q-1)^{1+j_{2l-1}} \times \binom{s-2}{j_{2l}}(q-1)^{1+j_{2l}}$$
$$\times \binom{b-1}{j_{l'}}(q-1)^{1+j_{l'}} \times \binom{g^{(2)}-b-1}{j_{l''}}(q-1)^{1+j_{l''}}$$
$$= \sum_{j_1,j_2,\ldots,j_{2l},j_{l'},j_{l''}} \left[\prod_{\rho=1}^{l}\binom{b-1}{j_{2\rho-1}}\prod_{\rho=1}^{l}\binom{s-2}{j_{2\rho}}\right]$$
$$\times \binom{b-1}{j_{l'}}\binom{g^{(2)}-b-1}{j_{l''}}(q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2},$$

where $2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2 = j$ such that $0 \leq j_1, j_3, \ldots, j_{2l-1}, j_{l'} \leq w-1$, $0 \leq j_2, j_4, \ldots, j_{2l} \leq \min\{w-1, s-2\}$, $0 \leq j_{l''} \leq \min\{w-1, g^{(2)}-b-1\}$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Maximum weight of the set $\mathbb{A}'$ can be calculated in a similar way as Lemma

5.16. □

Finally, we give Plotkin's type of bound for the sets $\mathbb{A}$ and $\mathbb{A}'$ whose proof is similar to Theorem 5.11.

Theorem 5.18. *The minimum weight of a vector in the sets $\mathbb{A}$ and $\mathbb{A}'$ is bounded above by* $\dfrac{\sum_{j=1}^{W^1_{max}} jA(j)}{\sum_{j=1}^{W^1_{max}} A(j)}$ *and* $\dfrac{\sum_{j=1}^{W^2_{max}} jA'(j)}{\sum_{j=1}^{W^2_{max}} A'(j)}$ *respectively, where $A(j)$, $A'(j)$, W^1_{max} and W^2_{max} are given by Lemma 5.16–5.17.*