Chapter 6

On r-noncommuting graphs of finite rings

In this chapter, we introduce and study r-noncommuting graph of a finite ring R for any given element $r \in R$ analogous to g-noncommuting graph of a finite group. The *r-noncommuting graph* of R, denoted by Γ_R^r , is a simple undirected graph whose vertex set is R and two vertices x and y are adjacent if $[x, y] \neq r$ and $[x, y] \neq -r$. Clearly, $\Gamma_R^r = \Gamma_R^{-r}$. If $r = 0$ then the induced subgraph of Γ_R^r with vertex set $R \setminus Z(R)$, denoted by Δ_R^r, is nothing but the non-commuting graph of R. Note that Γ_R^r is 0-regular graph if $r = 0$ and R is commutative. Also, Γ_R^r is complete if $r \notin K(R)$. Thus for $r \notin K(R)$, Γ_R^r is n-regular if and only if R is of order $n + 1$. Therefore throughout the chapter we consider $r \in K(R)$.

In Section 6.1, we first compute degree of any vertex of Γ_R^r in terms of its centralizers. Then we characterize R if Γ_R^r is a tree, in particular a star graph. We further show that Γ_R^r is not a regular graph (if $r \in K(R)$) or a lollipop graph for any non-commutative ring R. We conclude this section by showing that $\Gamma_{R_1}^r$ is isomorphic to $\Gamma_{R_2}^{\psi(r)}$ if (ϕ, ψ) is an isoclinism between two finite rings R_1 and R_2 such that $|Z(R_1)| = |Z(R_2)|$. In Section 6.2, we consider the induced subgraph Δ_R^r of Γ_R^r, induced by $R \setminus Z(R)$, and obtain results on clique number and diameter of Δ_R^r along with certain characterizations of finite non-commutative rings such that Δ_R^r is n-regular for some positive integer n. More precisely, we characterize certain finite non-commutative rings such that their non-commuting graphs are n-regular for $n \leq 6$. This chapter is based on our paper [74] published in *Axioms*.

It has been shown in [42] that there are only two non-commutative rings (up to isomorphism) having order p^2, where p is a prime, and the rings are given by

$$E(p^2) = \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle$$

$$\text{and } F(p^2) = \langle x, y : px = py = 0, x^2 = x, y^2 = y, xy = y, yx = x \rangle.$$

Following figures show the graphs $\Gamma^r_{E(p^2)}$ for $p = 2, 3$.

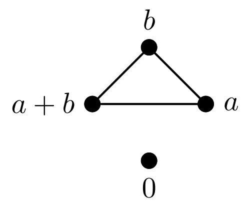

Figure 6.1: $\Gamma^0_{E(4)}$

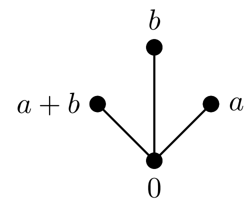

Figure 6.2: $\Gamma^{a+b}_{E(4)}$

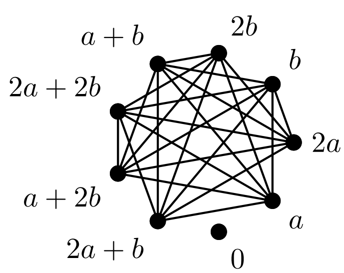

Figure 6.3: $\Gamma^0_{E(9)}$

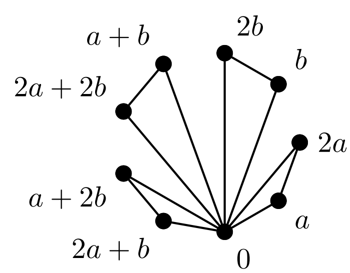

Figure 6.4: $\Gamma^r_{E(9)}$, where $r = a + 2b$ or $2a + b$

It is worth noting here that the graphs $\Gamma^0_{F(4)}, \Gamma^{x+y}_{F(4)}, \Gamma^0_{F(9)}$ and $\Gamma^{x+2y}_{F(9)}$ are isomorphic to $\Gamma^0_{E(4)}, \Gamma^{a+b}_{E(4)}, \Gamma^0_{E(9)}$ and $\Gamma^{a+2b}_{E(9)}$ respectively.

6.1 Some properties of Γ^r_R

In this section, we characterize R when Γ^r_R is a tree or a star graph. We also show the non-existence of finite non-commutative rings R whose r-noncommuting graph is a regular graph (if $r \in K(R)$), a lollipop graph or a complete bipartite graph. However, we first compute degree of any vertex in the graph Γ_R^r. For any two given elements x and r of R, we write $C_R^r(x)$ to denote the generalized centralizer $\{y \in R : [x, y] = r\}$ of x. The following theorem gives the degree of any vertex of Γ_R^r in terms of its generalized centralizers.

Theorem 6.1.1. *Let x be any vertex in Γ_R^r. Then*

(a) $\deg(x) = |R| - |C_R(x)|$ *if* $r = 0$.

(b) *if* $r \neq 0$ *then* $\deg(x) = \begin{cases} |R| - |C_R^r(x)| - 1, & \text{if } 2r = 0 \\ |R| - 2|C_R^r(x)| - 1, & \text{if } 2r \neq 0. \end{cases}$

Proof. (a) If $r = 0$ then $\deg(x)$ is the number of $y \in R$ such that $xy \neq yx$. Note that $|C_R(x)|$ gives the number of elements that commute with x. Hence, $\deg(x) = |R| - |C_R(x)|$.

(b) Consider the case when $r \neq 0$. If $2r = 0$ then $r = -r$. Note that $y \in R$ is not adjacent to x if and only if $y = x$ or $y \in C_R^r(x)$. Therefore, $\deg(x) = |R| - |C_R^r(x)| - 1$. If $2r \neq 0$ then $r \neq -r$. It is easy to see that $C_R^r(x) \cap C_R^{-r}(x) = \emptyset$ and $y \in C_R^r(x)$ if and only if $-y \in C_R^{-r}(x)$. Therefore, $|C_R^r(x)| = |C_R^{-r}(x)|$. Note that $y \in R$ is not adjacent to x if and only if $y = x$ or $y \in C_R^r(x)$ or $y \in C_R^{-r}(x)$. Therefore, $\deg(x) = |R| - |C_R^r(x)| - |C_R^{-r}(x)| - 1$. Hence the result follows. □

The following corollary gives degree of any vertex of Γ_R^r in terms of its centralizers.

Corollary 6.1.2. *Let x be any vertex in Γ_R^r.*

(a) *If* $r \neq 0$ *and* $2r = 0$ *then* $\deg(x) = \begin{cases} |R| - 1, & \text{if } C_R^r(x) = \emptyset \\ |R| - |C_R(x)| - 1, & \text{otherwise.} \end{cases}$

(b) *If* $r \neq 0$ *and* $2r \neq 0$ *then* $\deg(x) = \begin{cases} |R| - 1, & \text{if } C_R^r(x) = \emptyset \\ |R| - 2|C_R(x)| - 1, & \text{otherwise.} \end{cases}$

Proof. Notice that $C_R^r(x) \neq \emptyset$ if and only if $r \in [x, R]$. Suppose that $C_R^r(x) \neq \emptyset$. Let $t \in C_R^r(x)$ and $p \in t + C_R(x)$. Then $[x, p] = r$ and so $p \in C_R^r(x)$. Therefore, $t + C_R(x) \subseteq C_R^r(x)$. Again, if $y \in C_R^r(x)$ then $(y - t) \in C_R(x)$ and so $y \in t + C_R(x)$. Therefore, $C_R^r(x) \subseteq t + C_R(x)$. Thus $|C_R^r(x)| = |C_R(x)|$ if $C_R^r(x) \neq \emptyset$. Hence the result follows from Theorem 6.1.1. □

We now present some results regarding realization of the graph Γ_R^r and characterization of R through certain properties of Γ_R^r as applications of Theorem 6.1.1.

Theorem 6.1.3. *Let R be a ring with unity. The r-noncommuting graph Γ_R^r is a tree if and only if $|R| = 2$ and $r \neq 0$.*

Proof. If $r = 0$ then, by Theorem 6.1.1(a), we have $\deg(r) = 0$. Hence, Γ_R^r is not a tree. Suppose that $r \neq 0$. If R is commutative then $r \notin K(R)$. Hence, Γ_R^r is complete graph. Therefore Γ_R^r is a tree if and only if $|R| = 2$. If R is non-commutative then $[x, 0] \neq r, -r$ and $[x, 1] \neq r, -r$ for any $x \in R$. Therefore $\deg(x) \geq 2$ for all $x \in R$. Hence, Γ_R^r is not a tree. □

Theorem 6.1.4. *Let R be a non-commutative ring. If Γ_R^r has an end vertex then $r \neq 0$ and $\Gamma_R^{r\neq 0}$ is a star graph if and only if R is isomorphic to $E(4) = \langle a, b : 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$ or $F(4) = \langle a, b : 2a = 2b = 0, a^2 = a, b^2 = b, ab = b, ba = a\rangle$. Hence, Γ_R^r is not a lollipop graph.*

Proof. Let $x \in R$ be an end vertex in Γ_R^r. Then $\deg(x) = 1$. If $r = 0$ then $x \notin Z(R)$ and so $|C_R(x)| \leq \frac{|R|}{2}$. Also, by Theorem 6.1.1(a), we have $\deg(x) = |R| - |C_R(x)|$. These give $|R| - |C_R(x)| = 1$. Hence, $|R| \leq 2$, a contradiction. Therefore, $r \neq 0$. By Corollary 6.1.2, we have $\deg(x) = |R| - 1, |R| - |C_R(x)| - 1$ or $|R| - 2|C_R(x)| - 1$. These give $|R| - |C_R(x)| = 2$ or $|R| - 2|C_R(x)| = 2$. Clearly $x \notin Z(G)$ and so $|C_R(x)| \leq \frac{|R|}{2}$. Therefore, if $|R| - |C_R(x)| = 2$ then $|R| \leq 4$. If $|R| - 2|C_R(x)| = 2$ then $|R|$ is even and $|C_R(x)| \leq \frac{|R|}{2}$. Therefore, $|R| \leq 6$. Since R is non-commutative we have $|R| = 4$ and so R is isomorphic to either $E(4)$ or $F(4)$. In Figure 6.2, it is shown that $\Gamma_{E(4)}^r$ is a star graph if $r \neq 0$. Also, $\Gamma_{E(4)}^r$ is isomorphic to $\Gamma_{F(4)}^r$. Hence, the result follows. □

It follows that if R is non-commutative having more than four elements then there is no vertex of degree one in Γ_R^r.

It is observed that Γ_R^r is $(|R| - 1)$-regular if $r \notin K(R)$. Also, if $r = 0$ and R is commutative then Γ_R^r is 0-regular. In the following theorem, we show that Γ_R^r is not regular if $r \in K(R)$.

Theorem 6.1.5. *Let R be a non-commutative ring and $r \in K(R)$. Then Γ_R^r is not regular.*

Proof. If $r = 0$ then, by Theorem 6.1.1(a), we have $\deg(r) = 0$. Let $x \in R$ be a non-central element. Then $|C_R(x)| \neq |R|$. Therefore, by Theorem 6.1.1(a), $\deg(x) \neq 0 = \deg(r)$. This shows that Γ_R^r is not regular. If $r \neq 0$ then $C_R^r(0) = \emptyset$. Therefore, by Corollary 6.1.2, we have $\deg(0) = |R| - 1$. Since $r \in K(R)$, there exists $0 \neq x \in R$ such that $C_R^r(x) \neq \emptyset$. Therefore, by Corollary 6.1.2, we have $\deg(x) = |R| - |C_R(x)| - 1$ or $|R| - 2|C_R(x)| - 1$. If Γ_R^r is regular then $\deg(x) = \deg(0)$. Therefore

$$|R| - |C_R(x)| - 1 = |R| - 2|C_R(x)| - 1 = |R| - 1$$

which gives $|C_R(x)| = 0$, a contradiction. Hence, Γ_R^r is not regular. This completes the proof. □

The following result shows that Γ_R^r is not complete bipartite if $|R| \geq 3$ and $|Z(R)| \geq 2$.

Theorem 6.1.6. *Let R be a finite ring.*

(a) *If $r = 0$ then Γ_R^r is not complete bipartite.*

(b) *If $r \neq 0$ then Γ_R^r is not complete bipartite for $|R| \geq 3$ with $|Z(R)| \geq 2$.*

Proof. Let Γ_R^r be complete bipartite. Then there exist subsets V_1 and V_2 of R such that $V_1 \cap V_2 = \emptyset, V_1 \cup V_2 = R$ and if $x \in V_1$ and $y \in V_2$ then x and y are adjacent.

(a) If $r = 0$ then for $x \in V_1$ and $y \in V_2$ we have $[x, y] \neq 0$. Therefore, $[x, x + y] \neq 0$ which implies $x+y \in V_2$. Again $[y, x+y] \neq 0$ which implies $x+y \in V_1$. Thus $x+y \in V_1 \cap V_2$, a contradiction. Hence Γ_R^r is not complete bipartite.

(b) If $r \neq 0, |R| \geq 3$ and $|Z(R)| \geq 2$ then for any $z_1, z_2 \in Z(R)$, z_1 and z_2 are adjacent. Let us take $z_1 \in V_1$ and $z_2 \in V_2$. Since $|R| \geq 3$ we have $x \in R$ such that $x \neq z_1$ and $x \neq z_2$. Also $[x, z_1] = 0 = [x, z_2]$. Therefore x is adjacent to both z_1 and z_2. Therefore $x \notin V_1 \cup V_2 = R$, a contradiction. Hence Γ_R^r is not complete bipartite. □

If R_1 and R_2 are two isomorphic rings and $\alpha : R_1 \to R_2$ is an isomorphism then it is easy to see that $\Gamma_{R_1}^r \cong \Gamma_{R_2}^{\alpha(r)}$. In the following theorem we show that $\Gamma_{R_1}^r \cong \Gamma_{R_2}^{\psi(r)}$ if R_1 and R_2 are two isoclinic rings with isoclinism (ϕ, ψ).

Theorem 6.1.7. *Let R_1 and R_2 be two finite rings such that $|Z(R_1)| = |Z(R_2)|$. If (ϕ, ψ) is an isoclinism between R_1 and R_2 then*

$$\Gamma_{R_1}^{r} \cong \Gamma_{R_2}^{\psi(r)}.$$

Proof. Since $\phi : \frac{R_1}{Z(R_1)} \to \frac{R_2}{Z(R_2)}$ is an isomorphism, $\frac{R_1}{Z(R_1)}$ and $\frac{R_2}{Z(R_2)}$ have same number of elements. Let $\left|\frac{R_1}{Z(R_1)}\right| = \left|\frac{R_2}{Z(R_2)}\right| = n$. Again since $|Z(R_1)| = |Z(R_2)|$, there exists a bijection $\theta : Z(R_1) \to Z(R_2)$. Let $\{r_i : 1 \leq i \leq n\}$ and $\{s_j : 1 \leq j \leq n\}$ be two transversals of $\frac{R_1}{Z(R_1)}$ and $\frac{R_2}{Z(R_2)}$ respectively. Let $\phi : \frac{R_1}{Z(R_1)} \to \frac{R_2}{Z(R_2)}$ and $\psi : [R_1, R_1] \to [R_2, R_2]$ be defined as $\phi(r_i + Z(R_1)) = s_i + Z(R_2)$ and $\psi([r_i + z_1, r_j + z_2]) = [s_i + z_1', s_j + z_2']$ for some $z_1, z_2 \in Z(R_1)$, $z_1', z_2' \in Z(R_2)$ and $1 \leq i, j \leq n$.

Let us define a map $\alpha : R_1 \to R_2$ such that $\alpha(r_i + z) = s_i + \theta(z)$ for $z \in Z(R)$. Clearly α is a bijection. We claim that α preserves adjacency. Let x and y be two elements of R_1 such that x and y are adjacent. Then $[x, y] \neq r, -r$. We have $x = r_i + z_i$ and $y = r_j + z_j$ where $z_i, z_j \in Z(R_1)$ and $1 \leq i, j \leq n$. Therefore

$$\begin{aligned}
&[r_i + z_i, r_j + z_j] \neq r, -r \\
\Rightarrow &\psi([r_i + z_i, r_j + z_j]) \neq \psi(r), -\psi(r) \\
\Rightarrow &[s_i + \theta(z_i), s_j + \theta(z_j)] \neq \psi(r), -\psi(r) \\
\Rightarrow &[\alpha(r_i + z_i), \alpha(r_j + z_j)] \neq \psi(r), -\psi(r) \\
\Rightarrow &[\alpha(x), \alpha(y)] \neq \psi(r), -\psi(r).
\end{aligned}$$

This shows that $\alpha(x)$ and $\alpha(y)$ are adjacent. Hence the result follows. □

6.2 An induced subgraph of r-noncommuting graph

We write Δ_R^r to denote the *induced subgraph* of Γ_R^r with vertex set $R \setminus Z(R)$. It is worth mentioning that Δ_R^0 is the non-commuting graph of R. If $r \neq 0$ then it is easy to see that the commuting graph of R is a spanning subgraph of Δ_R^r. The following result gives a condition such that Δ_R^r is the commuting graph of R.

Theorem 6.2.1. *Let R be a non-commutative ring and $r \neq 0$. If $K(R) = \{0, r, -r\}$ then Δ_R^r is the commuting graph of R.*

Proof. The result follows from the fact that two vertices x, y in Δ_R^r are adjacent if and only if $xy = yx$. □

Let $\omega(\Delta_R^r)$ be the clique number of Δ_R^r. The following result gives a lower bound for $\omega(\Delta_R^r)$.

Theorem 6.2.2. *Let R be a non-commutative ring and $r \neq 0$. If S is a commutative subring of R having maximal order then $\omega(\Delta_R^r) \geq |S| - |S \cap Z(R)|$.*

Proof. The result follows from the fact that the subset $S \setminus S \cap Z(R)$ of $R \setminus Z(R)$ is a clique of Δ_R^r. □

By Result 1.4.29, it follows that the diameter of Δ_R^0 is less than or equal to 2. The next result gives some information regarding diameter of Δ_R^r when $r \neq 0$. For any two vertices x and y, we write $x \leftrightarrow y$ to denote x and y are adjacent, otherwise $x \nleftrightarrow y$.

Theorem 6.2.3. *Let R be a non-commutative ring and $r \in R \setminus Z(R)$ such that $2r \neq 0$.*

(a) *If $3r \neq 0$ then* $\text{diam}(\Delta_R^r) \leq 3$.

(b) *If $|Z(R)| = 1$, $|C_R(r)| \neq 3$ and $3r = 0$ then* $\text{diam}(\Delta_R^r) \leq 3$.

Proof. (a) If $x \leftrightarrow r$ for all $x \in R \setminus Z(R)$ such that $x \neq r$ then, it is easy to see that $\text{diam}(\Delta_R^r) \leq 2$. Suppose there exists a vertex $x \in R \setminus Z(R)$ such that $x \nleftrightarrow r$. Then $[x, r] = r$ or $-r$. We have

$$[x, 2r] = 2[x, r] = \begin{cases} 2r, & \text{if } [x, r] = r \\ -2r, & \text{if } [x, r] = -r. \end{cases}$$

Since $2r \neq 0$ we have $[x, 2r] \neq 0$ and hence $2r \in R \setminus Z(R)$. Also, $2r \neq r, -r$. Therefore, $[x, 2r] \neq r, -r$ and so $x \leftrightarrow 2r$. Let $y \in R \setminus Z(R)$ such that $y \neq x$. If $y \leftrightarrow r$ then $d(x, y) \leq 3$ noting that $r \leftrightarrow 2r$. If $y \nleftrightarrow r$ then $y \leftrightarrow 2r$ (as shown above). In this case $d(x, y) \leq 2$. Hence, $\text{diam}(\Delta_R^r) \leq 3$.

(b) If $x \leftrightarrow r$ for all $x \in R\setminus Z(R)$ such that $x \neq r$ then, it is easy to see that $\operatorname{diam}(\Delta_R^r) \leq 2$. Suppose there exists a vertex $x \in R \setminus Z(R)$ such that $x \not\leftrightarrow r$. Let $y \in R \setminus Z(R)$ such that $y \neq x$. We consider the following two cases.

Case 1: $x \not\leftrightarrow r$ and $x \leftrightarrow 2r$.

If $y \leftrightarrow r$ then $d(x, y) \leq 3$ noting that $r \leftrightarrow 2r$. Therefore, $\operatorname{diam}(\Delta_R^r) \leq 3$. If $y \not\leftrightarrow r$ but $y \leftrightarrow 2r$ then $d(x, y) \leq 2$. Consider the case when $y \not\leftrightarrow r$ as well as $y \not\leftrightarrow 2r$. Therefore $[y, r] = r$ or $-r$. If $[y, r] = r$ then $[y, 2r] = 2[y, r] = 2r = -r$; otherwise $y \leftrightarrow 2r$, a contradiction. Let $a \in C_R(r)$ such that $a \neq 0, r, -r$ (such element exists, since $|C_R(r)| > 3$). Clearly $a \in R\setminus Z(R)$. Suppose $y \leftrightarrow a$. Then $x \leftrightarrow 2r \leftrightarrow a \leftrightarrow y$ and so $d(x, y) \leq 3$. Suppose $y \not\leftrightarrow a$. Then $[y, a] = r$ or $-r$. If $[y, a] = r$ then

$$[y, r - a] = [y, r] - [y, a] = r - r = 0.$$

Note that $r - a \in R \setminus Z(R)$; otherwise $a = r$, a contradiction. Therefore, $y \leftrightarrow r - a$. Also,

$$[r - a, 2r] = 2[r, a] = 0.$$

That is, $r - a \leftrightarrow 2r$. Thus $x \leftrightarrow 2r \leftrightarrow r - a \leftrightarrow y$. Therefore, $d(x, y) \leq 3$. If $[y, a] = -r$ then

$$[y, 2r - a] = [y, 2r] - [y, a] = -r - (-r) = 0.$$

Note that $2r-a \in R\setminus Z(R)$; otherwise $a = 2r = -r$, a contradiction. Therefore, $y \leftrightarrow 2r-a$. Also,

$$[2r - a, 2r] = 2[r, a] = 0.$$

That is, $2r - a \leftrightarrow 2r$. Thus $x \leftrightarrow 2r \leftrightarrow 2r - a \leftrightarrow y$. Therefore, $d(x, y) \leq 3$.

If $[y, r] = -r$ then $[y, 2r] = 2[y, r] = -2r = r$; otherwise $y \leftrightarrow 2r$, a contradiction. Let $a \in C_R(r)$ such that $a \neq 0, r, -r$. Suppose $y \leftrightarrow a$. Then $x \leftrightarrow 2r \leftrightarrow a \leftrightarrow y$ and so $d(x, y) \leq 3$. Suppose $y \not\leftrightarrow a$. Then $[y, a] = r$ or $-r$. If $[y, a] = r$ then

$$[y, r + a] = [y, r] + [y, a] = -r + r = 0.$$

Note that $r+a \in R\setminus Z(R)$; otherwise $a = -r$, a contradiction. Therefore, $y \leftrightarrow r+a$. Also,

$$[r + a, 2r] = 2[a, r] = 0.$$

That is, $r + a \leftrightarrow 2r$. Thus $x \leftrightarrow 2r \leftrightarrow r + a \leftrightarrow y$. Therefore, $d(x, y) \leq 3$. If $[y, a] = -r$ then

$$[y, 2r + a] = [y, 2r] + [y, a] = r + (-r) = 0.$$

Note that $2r+a \in R \setminus Z(R)$; otherwise $a = -2r = r$, a contradiction. Therefore, $y \leftrightarrow 2r+a$. Also,

$$[2r + a, 2r] = 2[a, r] = 0.$$

That is, $2r + a \leftrightarrow 2r$. Thus $x \leftrightarrow 2r \leftrightarrow 2r + a \leftrightarrow y$. Therefore, $d(x, y) \leq 3$ and hence $\text{diam}(\Delta_R^r) \leq 3$.

Case 2: $x \not\leftrightarrow r$ and $x \not\leftrightarrow 2r$.

Let $a \in C_R(r)$ such that $a \neq 0, r, -r$.

Subcase 2.1: $x \leftrightarrow a$

If $y \leftrightarrow r$ then $y \leftrightarrow r \leftrightarrow a \leftrightarrow x$. Therefore $d(x, y) \leq 3$. If $y \not\leftrightarrow r$ but $y \leftrightarrow 2r$ then $y \leftrightarrow 2r \leftrightarrow a \leftrightarrow x$. Therefore, $d(x, y) \leq 3$. Consider the case when $y \not\leftrightarrow r$ as well as $y \not\leftrightarrow 2r$. Therefore $[y, r] = r$ or $-r$. If $[y, r] = r$ then $[y, 2r] = 2[y, r] = 2r = -r$; otherwise $y \leftrightarrow 2r$, a contradiction. Suppose $y \leftrightarrow a$. Then $y \leftrightarrow a \leftrightarrow x$ and so $d(x, y) \leq 2$. Suppose $y \not\leftrightarrow a$. Then $[y, a] = r$ or $-r$. If $[y, a] = r$ then $[y, r - a] = 0$. Therefore, $y \leftrightarrow r - a \leftrightarrow a \leftrightarrow x$. Therefore, $d(x, y) \leq 3$. If $[y, a] = -r$ then $[y, 2r - a] = 0$. Therefore, $y \leftrightarrow 2r - a \leftrightarrow a \leftrightarrow x$ and so $d(x, y) \leq 3$.

If $[y, r] = -r$ then $[y, 2r] = 2[y, r] = -2r = r$; otherwise $y \leftrightarrow 2r$, a contradiction. Suppose $y \leftrightarrow a$. Then $y \leftrightarrow a \leftrightarrow x$ and so $d(x, y) \leq 2$. Suppose $y \not\leftrightarrow a$. Then $[y, a] = r$ or $-r$. If $[y, a] = r$ then $[y, r + a] = 0$. Therefore, $y \leftrightarrow r + a \leftrightarrow a \leftrightarrow x$. Therefore, $d(x, y) \leq 3$. If $[y, a] = -r$ then $[y, 2r + a] = 0$. Therefore, $y \leftrightarrow 2r + a \leftrightarrow a \leftrightarrow x$ and so $d(x, y) \leq 3$. Hence, $\text{diam}(\Delta_R^r) \leq 3$.

Subcase 2.2: $x \not\leftrightarrow a$

In this case we have $x \not\leftrightarrow r$ and $x \not\leftrightarrow 2r$. It can be seen that $[x, r] = r$ implies $[x, 2r] = -r$ and $[x, r] = -r$ implies $[x, 2r] = r$.

Suppose $[x, r] = r$ and $[x, a] = r$. Then $[x, r - a] = [x, r] - [x, a] = 0$. Hence, $x \leftrightarrow r - a$. Now, we have the following cases.

(i) $x \leftrightarrow r - a \leftrightarrow r \leftrightarrow y$ if $y \leftrightarrow r$.

(ii) $x \leftrightarrow r - a \leftrightarrow 2r \leftrightarrow y$ if $y \not\leftrightarrow r$ but $y \leftrightarrow 2r$.

Suppose $y \not\leftrightarrow r$ as well as $y \not\leftrightarrow 2r$. Then, proceeding as in Subcase 2.1, we get the following cases:

(iii) $x \leftrightarrow r - a \leftrightarrow a \leftrightarrow y$ if $y \not\leftrightarrow r$ and $2r$ but $y \leftrightarrow a$.

(iv) $y \leftrightarrow r - a \leftrightarrow x$ if $[y, r] = r$ and $[y, a] = r$.

(v) $y \leftrightarrow 2r - a \leftrightarrow r - a \leftrightarrow x$ if $[y, r] = r$ and $[y, a] = -r$.

(vi) $y \leftrightarrow r + a \leftrightarrow r - a \leftrightarrow x$ if $[y, r] = -r$ and $[y, a] = r$.

(vii) $y \leftrightarrow 2r + a \leftrightarrow r - a \leftrightarrow x$ if $[y, r] = -r$ and $[y, a] = -r$.

Therefore, $d(x, y) \leq 3$.

Suppose $[x, r] = r$ and $[x, a] = -r$. Then

$$[x, 2r - a] = [x, 2r] - [x, a] = -r - (-r) = 0.$$

Hence, $x \leftrightarrow 2r - a$. Now, proceeding as above we get the following cases:

(i) $x \leftrightarrow 2r - a \leftrightarrow r \leftrightarrow y$ if $y \leftrightarrow r$.

(ii) $x \leftrightarrow 2r - a \leftrightarrow 2r \leftrightarrow y$ if $y \not\leftrightarrow r$ but $y \leftrightarrow 2r$.

(iii) $x \leftrightarrow 2r - a \leftrightarrow a \leftrightarrow y$ if $y \not\leftrightarrow r$ and $2r$ but $y \leftrightarrow a$.

(iv) $y \leftrightarrow r - a \leftrightarrow 2r - a \leftrightarrow x$ if $[y, r] = r$ and $[y, a] = r$.

(v) $y \leftrightarrow 2r - a \leftrightarrow x$ if $[y, r] = r$ and $[y, a] = -r$.

(vi) $y \leftrightarrow r + a \leftrightarrow 2r - a \leftrightarrow x$ if $[y, r] = -r$ and $[y, a] = r$.

(vii) $y \leftrightarrow 2r + a \leftrightarrow 2r - a \leftrightarrow x$ if $[y, r] = -r$ and $[y, a] = -r$.

Therefore, $d(x, y) \leq 3$.

Suppose $[x, r] = -r$ and $[x, a] = r$. Then

$$[x, r + a] = [x, r] + [x, a] = -r + r = 0.$$

Hence, $x \leftrightarrow r + a$. Proceeding as above we get the following similar cases:

(i) $x \leftrightarrow r + a \leftrightarrow r \leftrightarrow y$ if $y \leftrightarrow r$.

(ii) $x \leftrightarrow r + a \leftrightarrow 2r \leftrightarrow y$ if $y \not\leftrightarrow r$ but $y \leftrightarrow 2r$.

(iii) $x \leftrightarrow r + a \leftrightarrow a \leftrightarrow y$ if $y \not\leftrightarrow r$ and $2r$ but $y \leftrightarrow a$.

(iv) $y \leftrightarrow r - a \leftrightarrow r + a \leftrightarrow x$ if $[y, r] = r$ and $[y, a] = r$.

(v) $y \leftrightarrow 2r - a \leftrightarrow r + a \leftrightarrow x$ if $[y, r] = r$ and $[y, a] = -r$.

(vi) $y \leftrightarrow r + a \leftrightarrow x$ if $[y, r] = -r$ and $[y, a] = r$.

(vii) $y \leftrightarrow 2r + a \leftrightarrow r + a \leftrightarrow x$ if $[y, r] = -r$ and $[y, a] = -r$.

Therefore, $d(x, y) \leq 3$.

Suppose $[x, r] = -r$ and $[x, a] = -r$. Then

$$[x, 2r + a] = [x, 2r] + [x, a] = r + (-r) = 0.$$

Hence, $x \leftrightarrow 2r + a$ and so we get the the following similar cases:

(i) $x \leftrightarrow 2r + a \leftrightarrow r \leftrightarrow y$ if $y \leftrightarrow r$.

(ii) $x \leftrightarrow 2r + a \leftrightarrow 2r \leftrightarrow y$ if $y \not\leftrightarrow r$ but $y \leftrightarrow 2r$.

(iii) $x \leftrightarrow 2r + a \leftrightarrow a \leftrightarrow y$ if $y \not\leftrightarrow r$ and $2r$ but $y \leftrightarrow a$.

(iv) $y \leftrightarrow r - a \leftrightarrow 2r + a \leftrightarrow x$ if $[y, r] = r$ and $[y, a] = r$.

(v) $y \leftrightarrow 2r - a \leftrightarrow 2r + a \leftrightarrow x$ if $[y, r] = r$ and $[y, a] = -r$.

(vi) $y \leftrightarrow r + a \leftrightarrow 2r + a \leftrightarrow x$ if $[y, r] = -r$ and $[y, a] = r$.

(vii) $y \leftrightarrow 2r + a \leftrightarrow x$ if $[y, r] = -r$ and $[y, a] = -r$.

Therefore, $d(x, y) \leq 3$. Hence, in all the cases $\text{diam}(\Delta_R^r) \leq 3$. This completes the proof. □

As a consequence of Theorem 6.1.1(a) and Corollary 6.1.2 we get the following result.

Corollary 6.2.4. *Let x be any vertex in Δ_R^r.*

(a) *If $r = 0$ then* $\deg(x) = |R| - |C_R(x)|$.

(b) *If $r \neq 0$ and $2r = 0$ then*

$$\deg(x) = \begin{cases} |R| - |Z(R)| - 1, & \text{if } C_R^r(x) = \emptyset \\ |R| - |Z(R)| - |C_R(x)| - 1, & \text{otherwise.} \end{cases}$$

(c) *If $r \neq 0$ and $2r \neq 0$ then*

$$\deg(x) = \begin{cases} |R| - |Z(R)| - 1, & \text{if } C_R^r(x) = \emptyset \\ |R| - |Z(R)| - 2|C_R(x)| - 1, & \text{otherwise.} \end{cases}$$

Some applications of Corollary 6.2.4 are given below.

Theorem 6.2.5. *Let R be a non-commutative ring such that $|R| \neq 8$ and let K_n be the complete graph on n-vertices. If Δ_R^r has an end vertex then $r \neq 0$ and $\Delta_R^{r \neq 0} = 4K_2$ if and only if R is isomorphic to $E(9)$ or $F(9)$. Hence, Γ_R^r is neither a tree nor a lollipop graph.*

Proof. Let $x \in R \setminus Z(R)$ be an end vertex in Δ_R^r. Then $\deg(x) = 1$. If $r = 0$ then, by Corollary 6.2.4(a), we have $\deg(x) = |R| - |C_R(x)|$. Therefore, $|R| - |C_R(x)| = 1$ and hence $|C_R(x)| = 1$, a contradiction. Therefore, $r \neq 0$. Now we consider the following cases.

Case 1: $r \neq 0$ and $2r = 0$.

By Corollary 6.2.4(b), we have $\deg(x) = |R| - |Z(R)| - 1$ or $|R| - |Z(R)| - |C_R(x)| - 1$. Hence $|R| - |Z(R)| - 1 = 1$ or $|R| - |Z(R)| - |C_R(x)| - 1 = 1$.

Subcase 1.1: $|R| - |Z(R)| = 2$.

In this case we have $|Z(R)| = 1$ or 2. If $|Z(R)| = 1$ then $|R| = 3$, a contradiction. If $|Z(R)| = 2$ then $|R| = 4$. Therefore, the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction.

Subcase 1.2: $|R| - |Z(R)| - |C_R(x)| = 2$.

In this case, $|Z(R)| = 1$ or 2. If $|Z(R)| = 1$ then $|R| - |C_R(x)| = 3$. Therefore, $|C_R(x)| = 3$ and hence $|R| = 6$. Therefore, R is commutative; a contradiction. If $|Z(R)| = 2$ then $|R| - |C_R(x)| = 4$. Therefore, $|C_R(x)| = 4$ and so $|R| = 8$, a contradiction.

Case 2: $r \neq 0$ and $2r \neq 0$.

By Corollary 6.2.4(c), we have $\deg(x) = |R| - |Z(R)| - 1$ or $|R| - |Z(R)| - 2|C_R(x)| - 1$. Hence, $|R| - |Z(R)| - 1 = 1$ or $|R| - |Z(R)| - 2|C_R(x)| - 1 = 1$. If $|R| - |Z(R)| = 2$ then as shown in Subcase 1.1 we get a contradiction. If $|R| - |Z(R)| - 2|C_R(x)| = 2$ then $|Z(R)| = 1$ or 2.

Subcase 2.1: $|Z(R)| = 1$.

In this case, $|R| - 2|C_R(x)| = 3$. Therefore, $|C_R(x)| = 3$ and so $|R| = 9$. Hence, R is isomorphic to either $E(9)$ or $F(9)$. It follows from Figure 6.4 that $\Delta_R^r = 4K_2$ noting that $\Delta_{E(9)}^r$ and $\Delta_{F(9)}^r$ are isomorphic.

Subcase 2.2: $|Z(R)| = 2$.

In this case, $|R| - 2|C_R(x)| = 4$. Therefore, $|C_R(x)| = 4$ and so $|R| = 12$. It follows that the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction. Hence, the result follows. □

We have the following corollary to Theorem 6.2.5.

Corollary 6.2.6. *Let R be a non-commutative ring such that $|R| \neq 8$. Then*

(a) Δ_R^r *is 1-regular if and only if $r \neq 0$ and R is isomorphic to $E(9)$ or $F(9)$.*

(b) *The non-commuting graph of R does not have any end vertex. In particular, non-commuting graph of such ring is neither a tree nor a lollipop graph.*

Proof. The results follow from Theorem 6.2.5 noting the facts that any 1-regular graph has end vertices and non-commuting graph of R is the graph Δ_R^0. □

Theorem 6.2.7. *Let R be a non-commutative ring such that $|R| \neq 8, 12$. If Δ_R^r has a vertex of degree 2 then $r = 0$ and Δ_R^0 is a triangle if and only if R is isomorphic to $E(4)$ or $F(4)$.*

Proof. Suppose Δ_R^r has a vertex x of degree 2. Consider the following cases.

Case 1: $r = 0$.

By Corollary 6.2.4(a), we have $\deg(x) = |R| - |C_R(x)|$. Therefore, $|R| - |C_R(x)| = 2$ and hence $|C_R(x)| = 2$. Therefore, $|R| = 4$ and so R is isomorphic to $E(4)$ or $F(4)$. Hence, Δ_R^r is a triangle (as shown in Figure 6.1 noting that $\Delta_{E(4)}^r$ and $\Delta_{F(4)}^r$ are isomorphic).

Case 2: $r \neq 0$ and $2r = 0$.

By Corollary 6.2.4(b), we have $\deg(x) = |R| - |Z(R)| - 1$ or $\deg(x) = |R| - |Z(R)| - |C_R(x)| - 1$. Therefore $|R| - |Z(R)| - 1 = 2$ or $|R| - |Z(R)| - |C_R(x)| - 1 = 2$.

Subcase 2.1: $|R| - |Z(R)| = 3$.

In this case we have $|Z(R)| = 1$ or 3. If $|Z(R)| = 1$ then $|R| = 4$. As shown in Figure 6.2, Δ_R^r is a null graph on three vertices. Therefore, it has no vertex of degree 2, which is a contradiction. If $|Z(R)| = 3$ then $|R| = 6$. Therefore, R is commutative; a contradiction.

Subcase 2.2: $|R| - |Z(R)| - |C_R(x)| = 3$.

In this case, $|Z(R)| = 1$ or 3. If $|Z(R)| = 1$ then $|R| - |C_R(x)| = 4$. Therefore, $|C_R(x)| = 2$ or 4 and hence $|R| = 6$ or 8; a contradiction. If $|Z(R)| = 3$ then $|R| - |C_R(x)| = 6$. Therefore, $|C_R(x)| = 6$ and so $|R| = 12$, which contradicts our assumption.

Case 3: $r \neq 0$ and $2r \neq 0$.

By Corollary 6.2.4(c), we have $\deg(x) = |R| - |Z(R)| - 1$ or $|R| - |Z(R)| - 2|C_R(x)| - 1$. Hence, $|R| - |Z(R)| - 1 = 2$ or $|R| - |Z(R)| - 2|C_R(x)| - 1 = 2$.

If $|R| - |Z(R)| = 3$ then as shown in Subcase 2.1 we get a contradiction. If $|R| - |Z(R)| - 2|C_R(x)| = 3$ then $|Z(R)| = 1$ or 3.

Subcase 3.1: $|Z(R)| = 1$.

In this case, $|R| - 2|C_R(x)| = 4$. Therefore, $|C_R(x)| = 2$ or 4 and hence $|R| = 8$ or 12 which is a contradiction.

Subcase 3.2: $|Z(R)| = 3$.

In this case, $|R| - 2|C_R(x)| = 6$. Therefore, $|C_R(x)| = 6$ and so $|R| = 18$. It follows that the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction. Hence, the result follows. □

We have the following corollary to Theorem 6.2.7.

Corollary 6.2.8. *Let R be a non-commutative ring such that $|R| \neq 8, 12$. Then*

(a) *Δ_R^r is 2-regular if and only if $r = 0$ and R is isomorphic to $E(4)$ or $F(4)$.*

(b) *The non-commuting graph of R is 2-regular if and only if R is isomorphic to $E(4)$ or $F(4)$.*

Proof. The results follow from Theorem 6.2.7 noting the facts that any 2-regular graph has vertices of degree 2 and non-commuting graph of R is the graph Δ_R^0. □

Theorem 6.2.9. *Let R be a non-commutative ring such that $|R| \neq 16, 18$. Then the graph Δ_R^r has no vertex of degree* 3.

Proof. Suppose Δ_R^r has a vertex x of degree 3.

Case 1: $r = 0$.

By Corollary 6.2.4(a), we have $\deg(x) = |R| - |C_R(x)|$. Therefore, $|R| - |C_R(x)| = 3$ and hence $|C_R(x)| = 3$. Therefore, $|R| = 6$ and hence R is commutative; a contradiction.

Case 2: $r \neq 0$ and $2r = 0$.

By Corollary 6.2.4(b), we have $\deg(x) = |R| - |Z(R)| - 1$ or $\deg(x) = |R| - |Z(R)| - |C_R(x)| - 1$. Therefore $|R| - |Z(R)| - 1 = 3$ or $|R| - |Z(R)| - |C_R(x)| - 1 = 3$.

Subcase 2.1: $|R| - |Z(R)| = 4$.

In this case we have $|Z(R)| = 1$ or 2 or 4. If $|Z(R)| = 1$ or 2 then $|R| = 5$ or 6 and hence R is commutative; a contradiction. If $|Z(R)| = 4$ then $|R| = 8$. Therefore, the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction.

Subcase 2.2: $|R| - |Z(R)| - |C_R(x)| = 4$.

In this case, $|Z(R)| = 1$ or 2 or 4. If $|Z(R)| = 1$ then $|R| - |C_R(x)| = 5$. Therefore, $|C_R(x)| = 5$ and hence $|R| = 10$. Therefore R is commutative; a contradiction. If $|Z(R)| = 2$ then $|R| - |C_R(x)| = 6$. Therefore, $|C_R(x)| = 6$ and so $|R| = 12$. It follows that the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction. If $|Z(R)| = 4$ then $|R| - |C_R(x)| = 8$. Therefore, $|C_R(x)| = 8$ and so $|R| = 16$; a contradiction.

Case 3: $r \neq 0$ and $2r \neq 0$.

By Corollary 6.2.4(c), we have $\deg(x) = |R| - |Z(R)| - 1$ or $|R| - |Z(R)| - 2|C_R(x)| - 1$. Hence, $|R| - |Z(R)| - 1 = 3$ or $|R| - |Z(R)| - 2|C_R(x)| - 1 = 3$.

If $|R| - |Z(R)| = 4$ then as shown in Subcase 2.1 we get a contradiction. If $|R| - |Z(R)| - 2|C_R(x)| = 4$ then $|Z(R)| = 1$ or 2 or 4.

Subcase 3.1: $|Z(R)| = 1$.

In this case, $|R| - 2|C_R(x)| = 5$. Therefore, $|C_R(x)| = 5$ then $|R| = 15$. Therefore R is commutative; a contradiction.

Subcase 3.2: $|Z(R)| = 2$.

In this case, $|R| - 2|C_R(x)| = 6$. Therefore, $|C_R(x)| = 6$ and so $|R| = 18$; a contradiction.

Subcase 3.3: $|Z(R)| = 4$.

In this case, $|R| - 2|C_R(x)| = 8$. Therefore, $|C_R(x)| = 8$ and so $|R| = 24$. It follows that the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction. This completes the proof. □

We have the following corollary to Theorem 6.2.9.

Corollary 6.2.10. *Let R be a non-commutative ring such that $|R| \neq 16, 18$. Then Δ_R^r is not 3-regular. In particular, the non-commuting graph of such R is not 3-regular.*

Theorem 6.2.11. *Let R be a non-commutative ring such that $|R| \neq 8, 12, 18, 20$. Then Δ_R^r has no vertex of degree 4.*

Proof. Suppose Δ_R^r has a vertex x of degree 4.

Case 1: $r = 0$.

By Corollary 6.2.4(a), we have $\deg(x) = |R| - |C_R(x)|$. Therefore, $|R| - |C_R(x)| = 4$ and hence $|C_R(x)| = 2$ or 4. If $|C_R(x)| = 2$ then $|R| = 6$ and hence R is commutative; a contradiction. If $|C_R(x)| = 4$ then $|R| = 8$; a contradiction.

Case 2: $r \neq 0$ and $2r = 0$.

By Corollary 6.2.4(b), we have $\deg(x) = |R| - |Z(R)| - 1$ or $\deg(x) = |R| - |Z(R)| - |C_R(x)| - 1$. Therefore $|R| - |Z(R)| - 1 = 4$ or $|R| - |Z(R)| - |C_R(x)| - 1 = 4$.

Subcase 2.1: $|R| - |Z(R)| = 5$.

In this case we have $|Z(R)| = 1$ or 5. Then $|R| = 6$ or 10 and hence R is commutative; a contradiction.

Subcase 2.2: $|R| - |Z(R)| - |C_R(x)| = 5$.

In this case, $|Z(R)| = 1$ or 5. If $|Z(R)| = 1$ then $|R| - |C_R(x)| = 6$. Therefore, $|C_R(x)| = 2$ or 3 or 6. If $|C_R(x)| = 2$ then $|R| = 8$; a contradiction. If $|C_R(x)| = 3$ then $|R| = 9$. It follows from Figure 6.4 that $\Delta_R^r = 4K_2$ which is a contradiction. If $|C_R(x)| = 6$ then $|R| = 12$; a contradiction. If $|Z(R)| = 5$ then $|R| - |C_R(x)| = 10$. Therefore, $|C_R(x)| = 10$ and so $|R| = 20$; a contradiction.

Case 3: $r \neq 0$ and $2r \neq 0$.

By Corollary 6.2.4(c), we have $\deg(x) = |R| - |Z(R)| - 1$ or $|R| - |Z(R)| - 2|C_R(x)| - 1$. Hence, $|R| - |Z(R)| - 1 = 4$ or $|R| - |Z(R)| - 2|C_R(x)| - 1 = 4$.

If $|R| - |Z(R)| = 5$ then as shown in Subcase 2.1 we get a contradiction. If $|R| - |Z(R)| - 2|C_R(x)| = 5$ then $|Z(R)| = 1$ or 5.

Subcase 3.1: $|Z(R)| = 1$.

In this case, $|R| - 2|C_R(x)| = 6$. Therefore, $|C_R(x)| = 2$ or 3 or 6. If $|C_R(x)| = 2$ then $|R| = 10$. Therefore R is commutative; a contradiction. If $|C_R(x)| = 3$ or 6 then $|R| = 12$ or 18; a contradiction.

Subcase 3.2: $|Z(R)| = 5$.

In this case, $|R| - 2|C_R(x)| = 10$. Therefore, $|C_R(x)| = 10$ and so $|R| = 30$. It follows that the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction. This completes the proof. □

We have the following corollary to Theorem 6.2.11.

Corollary 6.2.12. *Let R be a non-commutative ring such that $|R| \neq 8, 12, 18, 20$. Then Δ_R^r is not 4-regular. In particular, the non-commuting graph of such R is not 4-regular.*

Theorem 6.2.13. *Let R be a non-commutative ring such that $|R| \neq 8, 16, 24, 27$. Then Δ_R^r has no vertex of degree 5.*

Proof. Suppose Δ_R^r has a vertex x of degree 5.

Case 1: $r = 0$.

By Corollary 6.2.4(a), we have $\deg(x) = |R| - |C_R(x)|$. Therefore, $|R| - |C_R(x)| = 5$ and hence $|C_R(x)| = 5$. Then $|R| = 10$ and hence R is commutative; a contradiction.

Case 2: $r \neq 0$ and $2r = 0$.

By Corollary 6.2.4(b), we have $\deg(x) = |R| - |Z(R)| - 1$ or $\deg(x) = |R| - |Z(R)| - |C_R(x)| - 1$. Therefore $|R| - |Z(R)| - 1 = 5$ or $|R| - |Z(R)| - |C_R(x)| - 1 = 5$.

Subcase 2.1: $|R| - |Z(R)| = 6$.

In this case we have $|Z(R)| = 1$ or 2 or 3 or 6. If $|Z(R)| = 1$ then $|R| = 7$ and hence R is commutative; a contradiction. If $|Z(R)| = 2$ then $|R| = 8$; a contradiction. If $|Z(R)| = 3$ then $|R| = 9$. It follows from Figure 6.4 that $\Delta_R^r = 4K_2$ which is a contradiction.

If $|Z(R)| = 6$ then $|R| = 12$. Therefore, the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction.

Subcase 2.2: $|R| - |Z(R)| - |C_R(x)| = 6$.

In this case, $|Z(R)| = 1$ or 2 or 3 or 6. If $|Z(R)| = 1$ then $|R| - |C_R(x)| = 7$. Therefore, $|C_R(x)| = 7$ then $|R| = 14$ and hence R is commutative; a contradiction. If $|Z(R)| = 2$ then $|R| - |C_R(x)| = 8$. Therefore, $|C_R(x)| = 4$ or 8. If $|C_R(x)| = 4$ then $|R| = 12$. Therefore, the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction. If $|C_R(x)| = 8$ then $|R| = 16$; a contradiction. If $|Z(R)| = 3$ then $|R| - |C_R(x)| = 9$. Therefore, $|C_R(x)| = 9$. and so $|R| = 18$. It follows that the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction. If $|Z(R)| = 6$ then $|R| - |C_R(x)| = 12$. Therefore, $|C_R(x)| = 12$ and so $|R| = 24$; a contradiction.

Case 3: $r \neq 0$ and $2r \neq 0$.

By Corollary 6.2.4(c), we have $\deg(x) = |R| - |Z(R)| - 1$ or $|R| - |Z(R)| - 2|C_R(x)| - 1$. Hence, $|R| - |Z(R)| - 1 = 5$ or $|R| - |Z(R)| - 2|C_R(x)| - 1 = 5$.

If $|R| - |Z(R)| = 6$ then as shown in Subcase 2.1 we get a contradiction. If $|R| - |Z(R)| - 2|C_R(x)| = 6$ then $|Z(R)| = 1$ or 2 or 3 or 6.

Subcase 3.1: $|Z(R)| = 1$.

Here we have, $|R| - 2|C_R(x)| = 7$. Therefore, $|C_R(x)| = 7$ then $|R| = 21$ and hence R is commutative; a contradiction.

Subcase 3.2: $|Z(R)| = 2$.

In this case, $|R| - 2|C_R(x)| = 8$. Therefore, $|C_R(x)| = 4$ or 8. If $|C_R(x)| = 4$ or 8 then $|R| = 16$ or 24; a contradiction.

Subcase 3.3: $|Z(R)| = 3$.

In this case, $|R| - 2|C_R(x)| = 9$. Therefore, $|C_R(x)| = 9$ and so $|R| = 27$; a contradiction.

Subcase 3.4: $|Z(R)| = 6$.

In this case, $|R| - 2|C_R(x)| = 12$. Therefore, $|C_R(x)| = 12$ and so $|R| = 36$. It follows that the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction. This completes the proof. □

We have the following corollary to Theorem 6.2.13.

Corollary 6.2.14. *Let R be a non-commutative ring such that $|R| \neq 8, 16, 24, 27$. Then Δ_R^r is not 5-regular. In particular, the non-commuting graph of such R is not 5-regular.*

We conclude this chapter with the following characterization of R.

Theorem 6.2.15. *Let R be a non-commutative ring such that $|R| \neq 8, 12, 16, 24, 28$. Then Δ_R^r has a vertex of degree 6 if and only if $r = 0$ and R is isomorphic to $E(9)$ or $F(9)$.*

Proof. Suppose Δ_R^r has a vertex x of degree 6.

Case 1: $r = 0$.

By Corollary 6.2.4(a), we have $\deg(x) = |R| - |C_R(x)|$. Therefore, $|R| - |C_R(x)| = 6$ and hence $|C_R(x)| = 2$ or 3 or 6. If $|C_R(x)| = 2$ then $|R| = 8$; a contradiction. If $|C_R(x)| = 3$ then $|R| = 9$. Therefore, Δ_R^r is a 6-regular graph (as shown in Figure 6.3). If $|C_R(x)| = 6$ then $|R| = 12$; a contradiction.

Case 2: $r \neq 0$ and $2r = 0$.

By Corollary 6.2.4(b), we have $\deg(x) = |R| - |Z(R)| - 1$ or $\deg(x) = |R| - |Z(R)| - |C_R(x)| - 1$. Therefore $|R| - |Z(R)| - 1 = 6$ or $|R| - |Z(R)| - |C_R(x)| - 1 = 6$.

Subcase 2.1: $|R| - |Z(R)| = 7$.

In this case we have $|Z(R)| = 1$ or 7. If $|Z(R)| = 1$ then $|R| = 8$; a contradiction. If $|Z(R)| = 7$ then $|R| = 14$ and hence R is commutative; a contradiction.

Subcase 2.2: $|R| - |Z(R)| - |C_R(x)| = 7$.

In this case, $|Z(R)| = 1$ or 7. If $|Z(R)| = 1$ then $|R| - |C_R(x)| = 8$. Therefore, $|C_R(x)| = 2$ or 4 or 8. If $|C_R(x)| = 2$ then $|R| = 10$. Thus R is commutative; a contradiction. If $|C_R(x)| = 4$ or 8 then $|R| = 12$ or 16; which are contradictions. If $|Z(R)| = 7$ then $|R| - |C_R(x)| = 14$. Therefore, $|C_R(x)| = 14$ and so $|R| = 28$; a contradiction.

Case 3: $r \neq 0$ and $2r \neq 0$.

By Corollary 6.2.4(c), we have $\deg(x) = |R| - |Z(R)| - 1$ or $|R| - |Z(R)| - 2|C_R(x)| - 1$. Hence, $|R| - |Z(R)| - 1 = 6$ or $|R| - |Z(R)| - 2|C_R(x)| - 1 = 6$.

If $|R| - |Z(R)| = 7$ then as shown in Subcase 2.1 we get a contradiction. If $|R| - |Z(R)| - 2|C_R(x)| = 7$ then $|Z(R)| = 1$ or 7.

Subcase 3.1: $|Z(R)| = 1$.

In this case, $|R| - 2|C_R(x)| = 8$. Therefore, $|C_R(x)| = 2$ or 4 or 8 then $|R| = 12$ or 16 or 24; all are contradictions to the order of R.

Subcase 3.2: $|Z(R)| = 7$.

In this case, $|R| - 2|C_R(x)| = 14$. Therefore, $|C_R(x)| = 14$ and so $|R| = 42$. It follows that the additive quotient group $\frac{R}{Z(R)}$ is cyclic. Hence, by Result 1.3.1, R is commutative; a contradiction. This completes the proof. □

We have the following corollary to Theorem 6.2.15.

Corollary 6.2.16. *Let R be a non-commutative ring such that $|R| \neq 8, 12, 16, 24, 28$. Then Δ_R^r is 6-regular if and only if $r = 0$ and R is isomorphic to $E(9)$ or $F(9)$. In particular, the non-commuting graph of such R is 6-regular if and only if R is isomorphic to $E(9)$ or $F(9)$.*

6.3 Conclusion

In this chapter, the study of non-commuting graphs of finite rings (Γ_R) has been extended by introducing r-noncommuting graph of R (Γ_R^r) for a given element $r \in R$. Expressions for vertex degrees have been derived and showed that Γ_R^r is neither a regular graph nor a lollipop graph nor a complete bipartite graph if R is non-commutative. Among other results, finite non-commutative rings have been characterized such that Γ_R^r is a tree. As a consequence of our results, characterizations of certain finite non-commutative rings such that their non-commuting graphs are n-regular for $n \leq 6$ have been obtained.