Chapter 7

Relative r-noncommuting graphs of finite rings

In this chapter, we generalize the notion of r-noncommuting graph of a finite ring R. More precisely, we consider *relative r-noncommuting graph* of R relative to a subring S, which is denoted by $\Gamma^r_{S,R}$ and defined as a simple undirected graph with vertex set R and two vertices x and y are adjacent if $x \in S$ or $y \in S$ and $[x, y] \neq r, -r$. Clearly $\Gamma^r_{R,R}$ is the r-noncommuting graph of R. Further, if $r = 0$ then the induced subgraph of $\Gamma^r_{S,R}$ with vertex set $R \setminus C_R(S)$ is nothing but the relative non-commuting graph of R which has been studied in [20]. In Section 7.2, we derive formula for degree of any vertex in $\Gamma^r_{S,R}$ and characterize all finite rings such that $\Gamma^r_{S,R}$ is a star, lollipop or a regular graph. In Section 7.3, we show that $\Gamma^r_{S_1,R_1}$ is isomorphic to $\Gamma^{\psi(r)}_{S_2,R_2}$ if (ϕ, ψ) is an isoclinism between the pairs of finite rings (S_1, R_1), (S_2, R_2) and $|Z(S_1, R_1)| = |Z(S_2, R_2)|$. In Section 7.4, we obtain certain relations between the number of edges in $\Gamma^r_{S,R}$ and $\mathrm{Pr}_r(S, K)$. Finally, we conclude the chapter by deriving certain results on the induced subgraph of $\Gamma^r_{S,R}$ with vertex set $R \setminus Z(S, R)$. This chapter is based on our paper [87] submitted for publication.

7.1 Preliminary observations

We have the following observations regarding $\Gamma^r_{S,R}$ analogous to the observations in Section 3.1.

Observation 7.1.1. Let S be a subring of a finite ring R and $r \in R$. Then we have the following.

(a) If $r \notin K(S,R)$ then $\Gamma^r_{S,R} = K_{|S|} + \overline{K_{|R|-|S|}}$ and so

$$\deg(x) = \begin{cases} |R| - 1 & \text{if } x \in S \\ |S| & \text{if } x \in R \setminus S. \end{cases}$$

(b) If $K(S,R) = \{0\}$ and $r = 0$ then $\Gamma^r_{S,R} = \overline{K_{|R|}}$.

It follows that if $r \notin K(S,R)$ then

(i) $\Gamma^r_{S,R}$ is a tree if and only if $S = \{0\}$ or $|S| = |R| = 2$.

(ii) $\Gamma^r_{S,R}$ is a star graph if and only if $S = \{0\}$.

(iii) $\Gamma^r_{S,R}$ is a complete graph if and only if $S = R$.

Note that if R is commutative or $S = Z(S,R)$ then $K(S,R) = \{0\}$. Therefore, in view of Observation 7.1.1, we consider R to be non-commutative, S to be a subring of R such that $S \neq Z(S,R)$ and $r \in K(S,R)$ throughout this chapter.

7.2 Vertex degree and consequences

For any two given elements $x, r \in R$ we write $C^r_S(x)$ to denote the set $\{s \in S : [x,s] = r\}$. Note that $C^r_S(x)$ is the centralizer of x in R if $S = R$ and $r = 0$. The following theorem gives degree of any vertex in $\Gamma^r_{S,R}$ in terms of $C^r_S(x)$.

Theorem 7.2.1. *Let x be any vertex in $\Gamma^r_{S,R}$.*

(a) *If $r = 0$ then* $\deg(x) = \begin{cases} |R| - |C_R(x)|, & \text{if } x \in S \\ |S| - |C_S(x)|, & \text{if } x \in R \setminus S. \end{cases}$

(b) *If* $r \neq 0$ *and* $2r = 0$ *then* $\deg(x) = \begin{cases} |R| - |C_R^r(x)| - 1, & \text{if } x \in S \\ |S| - |C_S^r(x)|, & \text{if } x \in R \setminus S. \end{cases}$

(c) *If* $r \neq 0$ *and* $2r \neq 0$ *then* $\deg(x) = \begin{cases} |R| - 2|C_R^r(x)| - 1, & \text{if } x \in S \\ |S| - 2|C_S^r(x)|, & \text{if } x \in R \setminus S. \end{cases}$

Proof. (a) Let $r = 0$. If $x \in S$ then $\deg(x)$ is the number of $s \in R$ such that $sx \neq sx$. Hence, $\deg(x) = |R| - |C_R(x)|$. If $x \in R \setminus S$ then $\deg(x)$ is the number of $s \in S$ such that $sx \neq sx$. Hence, $\deg(x) = |S| - |C_S(x)|$.

(b) Let $r \neq 0$ and $2r = 0$. In this case, $r = -r$. If $x \in S$ then $s \in R$ is not adjacent to x if and only if $s = x$ or $s \in C_R^r(x)$. Hence, $\deg(x) = |R| - |C_R^r(x)| - 1$. If $x \in R \setminus S$ then $s \in S$ is not adjacent to x if and only if $s \in C_S^r(x)$. Hence, $\deg(x) = |S| - |C_S^r(x)|$.

(c) Let $r \neq 0$ and $2r \neq 0$. In this case, $r \neq -r$. Also, $C_S^r(x) \cap C_S^{-r}(x) = \emptyset$ and $s \in C_S^r(x)$ if and only if $-s \in C_S^{-r}(x)$. Therefore, $C_S^r(x)$ and $C_S^{-r}(x)$ have same cardinality. Further, if $x \in S$ then $s \in R$ is not adjacent to x if and only if $s = x$, $s \in C_R^r(x)$ or $s \in C_R^{-r}(x)$. Hence, $\deg(x) = |R| - |C_R^r(x)| - |C_R^{-r}(x)| - 1$. If $x \in R \setminus S$ then $s \in S$ is not adjacent to x if and only if $s \in C_S^r(x)$ or $s \in C_S^{-r}(x)$. Hence, $\deg(x) = |S| - |C_S^r(x)| - |C_S^{-r}(x)|$. Hence, the result follows. □

The next lemma shows that for all $x, r \in R$ the cardinality of $C_S^r(x)$ is either zero or $|C_S(x)|$.

Lemma 7.2.2. *If* $C_S^r(x)$ *is non-empty then* $|C_S^r(x)| = |C_S(x)|$ *for all* $x, r \in R$.

Proof. Let $t \in C_S^r(x)$ and $p \in t + C_S(x)$. Then $p = t + m$ for some $m \in C_S(x)$. We have

$$[x, p] = [x, t + m] = x(t + m) - (t + m)x = [x, t] = r$$

and so $p \in C_S^r(x)$. Therefore, $t + C_S(x) \subseteq C_S^r(x)$. Again, if $y \in C_S^r(x)$ then $[x, t] = [x, y]$ which implies $(y - t)x = x(y - t)$. Therefore $(y - t) \in C_S(x)$ and so $y \in t + C_S(x)$. Thus $C_S^r(x) \subseteq t + C_S(x)$. Hence, $C_S^r(x) = t + C_S(x)$ and the result follows. □

By Lemma 7.2.2 and Theorem 7.2.1, we have the following two corollaries.

Corollary 7.2.3. *Let $x \in S$ be a vertex in $\Gamma^r_{S,R}$.*

(a) *If $r \neq 0$ and $2r = 0$ then* $\deg(x) = \begin{cases} |R| - |C_R(x)| - 1, & \text{if } C^r_R(x) \neq \emptyset \\ |R| - 1, & \text{otherwise.} \end{cases}$

(b) *If $r \neq 0$ and $2r \neq 0$ then* $\deg(x) = \begin{cases} |R| - 2|C_R(x)| - 1, & \text{if } C^r_R(x) \neq \emptyset \\ |R| - 1, & \text{otherwise.} \end{cases}$

Corollary 7.2.4. *Let $x \in R \setminus S$ be a vertex in $\Gamma^r_{S,R}$.*

(a) *If $r \neq 0$ and $2r = 0$ then* $\deg(x) = \begin{cases} |S| - |C_S(x)|, & \text{if } C^r_S(x) \neq \emptyset \\ |S|, & \text{otherwise.} \end{cases}$

(b) *If $r \neq 0$ and $2r \neq 0$ then* $\deg(x) = \begin{cases} |S| - 2|C_S(x)|, & \text{if } C^r_S(x) \neq \emptyset \\ |S|, & \text{otherwise.} \end{cases}$

In the next few results we discuss some properties of $\Gamma^r_{S,R}$. The following lemma shows that $\Gamma^r_{S,R}$ is a disconnected graph if $r = 0$.

Lemma 7.2.5. *If $x \in Z(S, R)$ then* $\deg(x) = \begin{cases} 0, & \text{if } r = 0 \\ |R| - 1, & \text{if } r \neq 0. \end{cases}$

Proof. The result follows from Theorem 7.2.1 noting that $x \in S$ and

$$C^r_R(x) = \begin{cases} C_R(x) = R, & \text{if } r = 0 \\ \emptyset, & \text{if } r \neq 0. \end{cases}$$

□

Lemma 7.2.6. *Let S be a subring of a non-commutative ring R with unity 1 and $r \neq 0$. If $1 \in S$ then $\deg(x) \geq 2$ for all $x \in R$.*

Proof. The result follows from the fact that $[x, 0] = [x, 1] \neq r$ and $-r$ for all $x \in R$. □

Theorem 7.2.7. *Let S be a subring of a non-commutative ring R and $r \in R$.*

(a) *If $r = 0$ then $\Gamma^r_{S,R}$ is not a tree, star graph, lollipop graph and complete graph.*

(b) *If $r \neq 0$ and R has unity $1 \in S$ then $\Gamma^r_{S,R}$ is not a tree and a star graph.*

Proof. The results follow from Lemma 7.2.5 and Lemma 7.2.6. □

Theorem 7.2.8. *Let S be a subring of a non-commutative ring R and $r \neq 0$. Then $\Gamma^r_{S,R}$ is a star if and only if $2r = 0$, $S \neq \{0\}$ and R is isomorphic to $E(4) = \langle a, b : 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$ or $F(4) = \langle x, y : 2x = 2y = 0, x^2 = x, y^2 = y, xy = y, yx = x\rangle$.*

Proof. If R is isomorphic to $E(4)$ or $F(4)$ then it is easy to see that $\Gamma^r_{S,R}$ is a star graph for any subring S.

Suppose that $\Gamma^r_{S,R}$ is a star graph. Clearly, $\deg(0) = |R| - 1$. Also, $\deg(x) = 1$ for all $0 \neq x \in R$. Since $r \neq 0$ and $r \in K(S, R)$ we have $S \neq \{0\}$. Let $0 \neq y \in R$. Then consider the following cases.

Case 1: $y \in S$.

Note that $\deg(y) \neq |R| - 1$. Therefore, if $2r = 0$ then, by Corollary 7.2.3(a), we have

$$1 = \deg(y) = |R| - |C_R(y)| - 1.$$

Therefore, $|R| - |C_R(y)| = 2$. We have $0, y \in C_R(y)$. Since $C_R(y)$ is a subring of R, $|C_R(y)|$ divides $|R| - |C_R(y)|$. Therefore, $|C_R(y)| = 2$ and hence $|R| = 4$.

If $2r \neq 0$ then, by Corollary 7.2.3(b), we have

$$1 = \deg(y) = |R| - 2|C_R(y)| - 1.$$

Therefore, $|C_R(y)| = 2$ and hence $|R| = 6$, a contradiction since R is non-commutative.

Hence, $2r = 0$ and $|R| = 4$.

Case 2: $y \in R \setminus S$.

Note that $\deg(y) \neq |S|$, otherwise $|S| = 1$; a contradiction. Therefore, if $2r = 0$ then, by Corollary 7.2.4(a), we have

$$1 = \deg(y) = |S| - |C_S(y)|.$$

We have $0 \in C_S(y)$. Since $C_S(y)$ is a subring of S, $|C_S(y)|$ divides $|S| - |C_S(y)|$. Therefore, $|C_S(y)| = 1$ and hence $|S| = 2$. Thus, S has a non-zero element and so, by Case 1, we have $|R| = 4$.

If $2r \neq 0$ then, by Corollary 7.2.4(b), we have

$$1 = \deg(y) = |S| - 2|C_S(y)|.$$

Therefore, $|C_S(y)| = 1$ and hence $|S| = 3$. Therefore, S has a non-zero element and so, by Case 1, we have $2r = 0$ and $|R| = 4$, a contradiction.

Hence, $2r = 0$ and R is isomorphic to $E(4)$ or $F(4)$. Hence, the result follows. □

Theorem 7.2.9. *Let S be a non-commutative subring of R. Then $\Gamma^r_{S,R}$ is not a lollipop graph.*

Proof. If $r = 0$ then the result follows from Theorem 7.2.7(a). Let $r \neq 0$ and $\Gamma^r_{S,R}$ be a lollipop graph. Then there exits an element $x \in R$ such that $\deg(x) = 1$.

Case 1: $x \in S$

By Corollary 7.2.3, we have $\deg(x) = |R| - 1 = 1$ or $\deg(x) = |R| - |C_R(x)| - 1 = 1$ or $\deg(x) = |R| - 2|C_R(x)| - 1 = 1$. Therefore $|R| - |C_R(x)| = 2$ or $|R| - 2|C_R(x)| = 2$ since $|R| \neq 2$. Thus $|C_R(x)| = 2$ and so $|R| = 4$ since $|R| \neq 6$. Hence, by Theorem 7.2.8, $\Gamma^r_{S,R}$ is a star graph; a contradiction.

Case 2: $x \in R \setminus S$

By Corollary 7.2.4, we have $\deg(x) = |S| - |C_S(x)| = 1$ or $\deg(x) = |S| - 2|C_S(x)| = 1$ since $|S| \neq 1$. Therefore $|C_S(x)| = 1$ and so $|S| = 2$ or 3. Hence, S is commutative, a contradiction. □

Note that Theorem 7.2.9 is a generalization of Theorem 6.1.4. We conclude this section with the following result.

Theorem 7.2.10. *Let S be a subring of a non-commutative ring R. Then $\Gamma^r_{S,R}$ is regular if and only if $K(S,R) = \{0\}$.*

Proof. If $K(S,R) = \{0\}$ then $r = 0$. Therefore, by Observation 7.1.1(b), it follows that $\Gamma^r_{S,R}$ is regular. Suppose that $\Gamma^r_{S,R}$ is regular. If $r = 0$ then, by Lemma 7.2.5, we have $\deg(0) = 0$. Therefore $\Gamma^r_{S,R} = \overline{K_{|R|}}$ and so $K(S,R) = \{0\}$. If $r \neq 0$ then, by Lemma 7.2.5, we have $\deg(0) = |R| - 1$. Therefore, $\Gamma^r_{S,R}$ is a complete graph and so $S = R$. That is, $\Gamma^r_{R,R}$ is regular; which is a contradiction by Theorem 6.1.5. $\square$

7.3 $\Gamma^r_{S,R}$ of isoclinic pairs

In this section, we mainly prove the following result.

Theorem 7.3.1. *Let R_1 and R_2 be two finite rings. Let S_1 and S_2 be two subrings of R_1 and R_2 respectively such that $|Z(S_1, R_1)| = |Z(S_2, R_2)|$. If $r \in [S_1, R_1]$ and (ϕ, ψ) is an isoclinism between the pairs (S_1, R_1) and (S_2, R_2) then $\Gamma^r_{S_1,R_1} \cong \Gamma^{\psi(r)}_{S_2,R_2}$.*

Proof. We have $\phi : \frac{R_1}{Z(S_1,R_1)} \to \frac{R_2}{Z(S_2,R_2)}$ is an isomorphism such that $\phi\left(\frac{S_1}{Z(S_1,R_1)}\right) = \frac{S_2}{Z(S_2,R_2)}$. Therefore, $|\frac{R_1}{Z(S_1,R_1)}| = |\frac{R_2}{Z(S_2,R_2)}|$ and $|\frac{S_1}{Z(S_1,R_1)}| = |\frac{S_2}{Z(S_2,R_2)}|$. Let $|\frac{S_1}{Z(S_1,R_1)}| = m$ and $|\frac{R_1}{Z(S_1,R_1)}| = n$. Let $\{s_1, s_2, \dots, s_m, r_{m+1}, \dots, r_n\}$ and $\{s'_1, s'_2, \dots, s'_m, r'_{m+1}, \dots, r'_n\}$ be two transversals of $\frac{R_1}{Z(S_1,R_1)}$ and $\frac{R_2}{Z(S_2,R_2)}$ respectively such that $\{s_1, s_2, \dots, s_m\}$ and $\{s'_1, s'_2, \dots, s'_m\}$ are transversals of $\frac{S_1}{Z(S_1,R_1)}$ and $\frac{S_2}{Z(S_2,R_2)}$ respectively.

Let ϕ be defined as $\phi(s_i+Z(S_1,R_1)) = s'_i+Z(S_2,R_2)$, $\phi(r_j+Z(S_1,R_1)) = r'_j+Z(S_2,R_2)$ for $1 \le i \le m$ and $m+1 \le j \le n$. Let $\theta : Z(S_1,R_1) \to Z(S_2,R_2)$ be a one-to-one correspondence. Let us define a map $\alpha : R_1 \to R_2$ such that $\alpha(s_i + z) = s'_i + \theta(z)$, $\alpha(r_j + z) = r'_j + \theta(z)$ for $z \in Z(S_1,R_1)$, $1 \le i \le m$ and $m+1 \le j \le n$. Then α is a bijection. Suppose u, v are adjacent in $\Gamma^r_{S_1,R_1}$. Then $u \in S_1$ or $v \in S_1$ and $[u,v] \ne r, -r$. Without any loss of generality, let us assume that $u \in S_1$. Then $u = s_i + z$ for $1 \le i \le m$ and $v = t + z_1$ where $z, z_1 \in Z(S_1,R_1)$, $t \in \{s_1, s_2, \dots, s_m, r_{m+1}, \dots, r_n\}$. Therefore, for some $t' \in \{s'_1, \dots, s'_m, r'_{m+1}, \dots, r'_n\}$, we have

$$\begin{aligned}
&[s_i + z, t + z_1] \ne r, -r \\
&\Rightarrow \psi([s_i + z, t + z_1]) \ne \psi(r), -\psi(r) \\
&\Rightarrow [s'_i + \theta(z), t' + \theta(z_1)] \ne \psi(r), -\psi(r) \\
&\Rightarrow [\alpha(s_i + z), \alpha(t + z_1)] \ne \psi(r), -\psi(r) \\
&\Rightarrow [\alpha(u), \alpha(v)] \ne \psi(r), -\psi(r).
\end{aligned}$$

This shows that $\alpha(u)$ and $\alpha(v)$ are adjacent in $\Gamma^{\psi(r)}_{S_2,R_2}$ noting that $\alpha(u) \in S_2$. Hence, α is an isomorphism between the graphs $\Gamma^r_{S_1,R_1}$ and $\Gamma^{\psi(r)}_{S_2,R_2}$. This completes the proof. $\square$

7.4 Connecting $\Gamma^r_{S,R}$ with $\Pr_r(S,R)$

In this section, we derive some connections between $\Gamma^r_{S,R}$ and $\Pr_r(S,R)$. Let $|e(\Gamma^r_{S,R})|$ denotes the number of edges in $\Gamma^r_{S,R}$. If $r \notin K(S,R)$ then it follows from Observation 7.1.1 that

$$|e(\Gamma^r_{S,R})| = |S||R| - \frac{|S|^2 + |S|}{2}.$$

The following theorem gives the number of edges in $\Gamma^r_{S,R}$ in terms of $\Pr_r(S,R)$ and $\Pr_r(S)$.

Theorem 7.4.1. *Let S be a subring of a finite ring R.*

(a) *If $r = 0$ then*

$$2|e(\Gamma^r_{S,R})| = 2|S||R|(1 - \Pr(S,R)) - |S|^2(1 - \Pr(S)).$$

(b) *If $r \neq 0$ and $2r = 0$ then*

$$2|e(\Gamma^r_{S,R})| = \begin{cases} 2|S||R|(1 - \Pr_r(S,R)) - |S|^2(1 - \Pr_r(S)) - |S|, & \text{if } r \in S \\ 2|S||R|(1 - \Pr_r(S,R)) - |S|^2 - |S|, & \text{if } r \in R \setminus S. \end{cases}$$

(c) *If $r \neq 0$ and $2r \neq 0$ then*

$$2|e(\Gamma^r_{S,R})| = \begin{cases} 2|S||R|(1 - \sum\limits_{u=r,-r} \Pr_u(S,R)) - \\ \qquad |S|^2(1 - \sum\limits_{u=r,-r} \Pr_u(S)) - |S|, & \text{if } r \in S \\ 2|S||R|(1 - \sum\limits_{u=r,-r} \Pr_u(S,R)) - |S|^2 - |S|, & \text{if } r \in R \setminus S. \end{cases}$$

Proof. Let $\mathbb{I} = \{(x,y) \in S \times R : x \neq y, [x,y] \neq r \text{ and } [x,y] \neq -r\}$ and $\mathbb{J} = \{(x,y) \in R \times S : x \neq y, [x,y] \neq r \text{ and } [x,y] \neq -r\}$. Then $\mathbb{I} \cap \mathbb{J} = \{(x,y) \in S \times S : x \neq y, [x,y] \neq r \text{ and } [x,y] \neq -r\}$.

It is easy to see that $(x, y) \mapsto (y, x)$ defines a bijective map from $\mathbb{I}$ to $\mathbb{J}$ and so $|\mathbb{I}| = |\mathbb{J}|$. Also, $2|e(\Gamma^r_{S,R})| = |\mathbb{I} \cup \mathbb{J}|$. Therefore,

$$2|e(\Gamma^r_{S,R})| = 2|\mathbb{I}| - |\mathbb{I} \cap \mathbb{J}|. \tag{7.4.1}$$

(a) If $r = 0$ then, by equation (1.3.2), we have

$$\begin{aligned}|\mathbb{I}| =& |\{(x, y) \in S \times R : [x, y] \neq 0\}|\\ =& |S||R| - |\{(x, y) \in S \times R : [x, y] = 0\}|\\ =& |S||R|(1 - \Pr(S, R))\end{aligned}$$

and

$$\begin{aligned}|\mathbb{I} \cap \mathbb{J}| =& |\{(x, y) \in S \times S : [x, y] \neq 0\}|\\ =& |S|^2 - |\{(x, y) \in S \times S : [x, y] = 0\}|\\ =& |S|^2(1 - \Pr(S)).\end{aligned}$$

Hence, the result follows from equation (7.4.1).
(b) If $r \neq 0$ and $2r = 0$ then $r = -r$. Therefore, by equation (1.3.2), we have

$$\begin{aligned}|\mathbb{I}| =& |\{(x, y) \in S \times R : x \neq y, [x, y] \neq r\}|\\ =& |S||R| - |\{(x, y) \in S \times R : [x, y] = r\}| - |\{(x, y) \in S \times S : x = y\}|\\ =& |S||R|(1 - \Pr_r(S, R)) - |S|.\end{aligned}$$

If $r \in S$ then, by equation (1.3.2), we have

$$\begin{aligned}|\mathbb{I} \cap \mathbb{J}| =& |\{(x, y) \in S \times S : x \neq y, [x, y] \neq r\}|\\ =& |S|^2 - |\{(x, y) \in S \times S : [x, y] = r\}| - |\{(x, y) \in S \times S : x = y\}|\\ =& |S|^2(1 - \Pr_r(S)) - |S|.\end{aligned}$$

If $r \in R \setminus S$ then we have

$$|\mathbb{I} \cap \mathbb{J}| = |S|^2 - |S|$$

noting that $\{(x, y) \in S \times S : [x, y] = r\}$ is empty. Therefore,

$$|\mathbb{I} \cap \mathbb{J}| = \begin{cases} |S|^2(1 - \Pr_r(S)) - |S|, & \text{if } r \in S \\ |S|^2 - |S|, & \text{if } r \in R \setminus S. \end{cases}$$

Hence, the result follows from equation (7.4.1).

(c) If r $\neq 0$ and $2r \neq 0$ then, by equation (1.3.2), we have

$$\begin{aligned} |\mathbb{I}| =& |\{(x, y) \in S \times R : x \neq y, [x, y] \neq r \text{ and } [x, y] \neq -r\}| \\ =& |S||R| - |\{(x, y) \in S \times R : [x, y] = r\}| - |\{(x, y) \in S \times R : [x, y] = -r\}| \\ & - |\{(x, y) \in S \times S : x = y\}| \\ =& |S||R|(1 - \sum_{u=r,-r} \Pr_u(S, R)) - |S|. \end{aligned}$$

If $r \in S$ then, by equation (1.3.2), we have

$$\begin{aligned} |\mathbb{I} \cap \mathbb{J}| =& |\{(x, y) \in S \times S : x \neq y, [x, y] \neq r \text{ and } [x, y] \neq -r\}| \\ =& |S|^2 - |\{(x, y) \in S \times S : [x, y] = r\}| - |\{(x, y) \in S \times S : [x, y] = -r\}| \\ & - |\{(x, y) \in S \times S : x = y\}| \\ =& |S|^2(1 - \sum_{u=r,-r} \Pr_u(S)) - |S|. \end{aligned}$$

If $r \in R \setminus S$ then we have

$$|\mathbb{I} \cap \mathbb{J}| = |S|^2 - |S|.$$

noting that $\{(x, y) \in S \times S : [x, y] = r\}$ and $\{(x, y) \in S \times S : [x, y] = -r\}$ are empty. Therefore,

$$|\mathbb{I} \cap \mathbb{J}| = \begin{cases} |S|^2(1 - \sum_{u=r,-r} \Pr_u(S)) - |S|, & \text{if } r \in S \\ |S|^2 - |S|, & \text{if } r \in R \setminus S. \end{cases}$$

Hence, the result follows from equation (7.4.1). □

As an application of Theorem 7.4.1, in the following two theorems, we compute the number of edges in $\Gamma^r_{S,R}$ if $[S, R]$ has prime order.

Theorem 7.4.2. *Let S be a commutative subring of a finite ring R such that $|[S,R]| = p$, a prime.*

(a) *If $r = 0$ then $|e(\Gamma^r_{S,R})| = \frac{(p-1)|R|(|S|-|Z(S,R)|)}{p}$.*

(b) *If $r \neq 0$ and $2r = 0$ then*

$$|e(\Gamma^r_{S,R})| = \frac{2|R|((p-1)|S| + |Z(S,R)|) - p|S|^2 - p|S|}{2p}.$$

(c) *If $r \neq 0$ and $2r \neq 0$ then*

$$|e(\Gamma^r_{S,R})| = \frac{2|R|((p-2)|S| + 2|Z(S,R)|) - p|S|^2 - p|S|}{2p}.$$

Proof. If $|[S,R]| = p$ then, by Result 1.3.12, we have

$$\Pr_r(S,R) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|S:Z(S,R)|}\right), & \text{if } r = 0 \\ \frac{1}{p}\left(1 - \frac{1}{|S:Z(S,R)|}\right), & \text{if } r \neq 0. \end{cases}$$

Since S is commutative, we have $[S,S] = \{0\}$. Therefore, by equation (1.3.2), we have

$$\Pr_r(S) = \begin{cases} 1, & \text{if } r = 0 \\ 0, & \text{if } r \neq 0. \end{cases}$$

Hence, the results follows from Theorem 7.4.1. □

Theorem 7.4.3. *Let S be a non-commutative subring of a finite ring R such that $|[S,R]| = p$, a prime.*

(a) *If $r = 0$ then*

$$|e(\Gamma^r_{S,R})| = \frac{(p-1)[2|R|(|S| - |Z(S,R)|) - |S|(|S| - |Z(S)|)]}{2p}.$$

(b) *If $r \neq 0$ and $2r = 0$ then*

$$|e(\Gamma^r_{S,R})| = \begin{cases} \frac{2|R|((p-1)|S|+|Z(S,R)|)-|S|((p-1)|S|+|Z(S)|)-p|S|}{2p}, & \text{if } r \in S \\ \frac{2|R|((p-1)|S|+|Z(S,R)|)-p|S|^2-p|S|}{2p}, & \text{if } r \in R \setminus S. \end{cases}$$

(c) *If $r \neq 0$ and $2r \neq 0$ then*

$$|e(\Gamma^r_{S,R})| = \begin{cases} \frac{2|R|((p-2)|S|+2|Z(S,R)|)-|S|((p-2)|S|+2|Z(S)|)-p|S|}{2p}, & \textit{if } r \in S \\ \frac{2|R|((p-2)|S|+2|Z(S,R)|)-p|S|^2-p|S|}{2p}, & \textit{if } r \in R \setminus S. \end{cases}$$

Proof. If $|[S, R]| = p$ then, by Result 1.3.12, we have

$$\Pr_r(S, R) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|S:Z(S,R)|}\right), & \text{if } r = 0 \\ \frac{1}{p}\left(1 - \frac{1}{|S:Z(S,R)|}\right), & \text{if } r \neq 0. \end{cases}$$

If S is non-commutative then $|[S, S]| = |[S, R]| = p$. Therefore, by Result 1.3.11, we have

$$\Pr_r(S) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|S:Z(S)|}\right), & \text{if } r = 0 \\ \frac{1}{p}\left(1 - \frac{1}{|S:Z(S)|}\right), & \text{if } r \neq 0. \end{cases}$$

Hence, the results follows from Theorem 7.4.1. □

Corollary 7.4.4. *Let $R = E(p^2) = \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$ for any prime p and S be a subring of R.*

(a) *If $|S| = p$ then*

$$|e(\Gamma^r_{S,R})| = \begin{cases} p(p-1)^2, & \textit{if } r = 0 \\ \frac{p(p-1)(2p-1)}{2}, & \textit{if } r \neq 0 \textit{ and } 2r = 0 \\ \frac{p(p-1)(2p-3)}{2}, & \textit{if } r \neq 0 \textit{ and } 2r \neq 0. \end{cases}$$

(b) *If $S = R$ then*

$$|e(\Gamma^r_{S,R})| = \begin{cases} \frac{p(p-1)^2(p+1)}{2}, & \textit{if } r = 0 \\ \frac{p(p-1)^2(p+1)}{2}, & \textit{if } r \neq 0 \textit{ and } 2r = 0 \\ \frac{p(p-1)(p-2)(p+1)}{2}, & \textit{if } r \neq 0 \textit{ and } 2r \neq 0. \end{cases}$$

Proof. We have $[S, R] = \{ma + (p - m)b : 1 \le m \le p\}$ and $Z(S, R) = \{0\}$. Therefore, $|[S, R]| = p$ and $|Z(S, R)| = 1$. Hence, the result follows from Theorems 7.4.2 and 7.4.3 noting that $|Z(S)| = 1$ if $S = R$. □

The following corollaries of Theorem 7.4.1 give certain lower bounds and upper bounds respectively for the number of edges in $\Gamma^r_{S,R}$, if $r \neq 0$.

Corollary 7.4.5. *Let p be the smallest prime dividing $|R|$ and $r \neq 0$. Then for a non-commutative subring S of R we have the following lower bounds for $|e(\Gamma^r_{S,R})|$.*

(a) *If $2r = 0$ then*

$$|e(\Gamma^r_{S,R})| \ge \begin{cases} \frac{2(p-1)|R||S|+2|R||Z(S,R)|-p|S|^2+6p|Z(S)|^2-p|S|)}{2p}, & \text{if } r \in S \\ \frac{2(p-1)|R||S|+2|R||Z(S,R)|-p|S|^2-p|S|}{2p}, & \text{if } r \in R \setminus S. \end{cases}$$

(b) *If $2r \neq 0$ then*

$$|e(\Gamma^r_{S,R})| \ge \begin{cases} \frac{2(p-2)|R||S|+4|R||Z(S,R)|-p|S|^2+12p|Z(S)|^2-p|S|)}{2p}, & \text{if } r \in S \\ \frac{2(p-2)|R||S|+4|R||Z(S,R)|-p|S|^2-p|S|}{2p}, & \text{if } r \in R \setminus S. \end{cases}$$

Proof. By Result 1.3.13 we have

$$1 - \Pr_r(S, R) \ge \frac{(p-1)|S| + |Z(S, R)|}{p|S|} \tag{7.4.2}$$

and

$$1 - \sum_{u=r,-r} \Pr_u(S, R) \ge \frac{(p-2)|S| + 2|Z(S, R)|}{p|S|}. \tag{7.4.3}$$

By Result 1.3.10 we have

$$\Pr_r(S) \ge \frac{6|Z(S)|^2}{|S|^2}. \tag{7.4.4}$$

(a) We have $2r = 0$. Therefore, if $r \in S$ then, using Theorem 7.4.1(b) and equations (7.4.2) and (7.4.4), we get

$$2|e(\Gamma^r_{S,R})| + |S|^2 + |S| \ge 2|S||R| \left(\frac{(p-1)|S| + |Z(S, R)|}{p|S|}\right) + |S|^2 \left(\frac{6|Z(S)|^2}{|S|^2}\right).$$

Hence the result follows.

If $r \in R \setminus S$ then, using Theorem 7.4.1(b) and equation (7.4.2), we get

$$2|e(\Gamma^r_{S,R})| + |S|^2 + |S| \geq 2|S||R|\left(\frac{(p-1)|S| + |Z(S,R)|}{p|S|}\right).$$

Hence the result follows.

(b) We have $2r \neq 0$. Therefore, if $r \in S$ then, using Theorem 7.4.1(c) and equations (7.4.3) and (7.4.4), we get

$$2|e(\Gamma^r_{S,R})| + |S|^2 + |S| \geq 2|S||R|\left(\frac{(p-2)|S| + 2|Z(S,R)|}{p|S|}\right) + |S|^2\left(\frac{12|Z(S)|^2}{|S|^2}\right).$$

Hence the result follows.

If $r \in R \setminus S$ then, using Theorem 7.4.1(c) and equation (7.4.3), we get

$$2|e(\Gamma^r_{S,R})| + |S|^2 + |S| \geq 2|S||R|\left(\frac{(p-2)|S| + 2|Z(S,R)|}{p|S|}\right).$$

Hence the result follows. □

Corollary 7.4.6. *Let p be the smallest prime dividing $|R|$ and $Z(R,S) = \{t \in R : ts = st$ for all $s \in S\}$ for any non-commutative subring S of R. If $r \neq 0$, then we have the following upper bounds for $|e(\Gamma^r_{S,R})|$.*

(a) *If $2r = 0$ then*

$$|e(\Gamma^r_{S,R})| \leq \begin{cases} \frac{2p|R||S| - 4p|Z(S,R)||Z(R,S)| - (p-1)|S|^2 - |S||Z(S)| - p|S|}{2p}, & \text{if } r \in S \\ \frac{2|R||S| - 4|Z(S,R)||Z(R,S)| - |S|^2 - |S|}{2p}, & \text{if } r \in R \setminus S. \end{cases}$$

(b) *If $2r \neq 0$ then*

$$|e(\Gamma^r_{S,R})| \leq \begin{cases} \frac{2p|R||S| - 8p|Z(S,R)||Z(R,S)| - (p-2)|S|^2 - 2|S||Z(S)| - p|S|}{2p}, & \text{if } r \in S \\ \frac{2|R||S| - 8|Z(S,R)||Z(R,S)| - |S|^2 - |S|}{2p}, & \text{if } r \in R \setminus S. \end{cases}$$

Proof. By Result 1.3.9 we have

$$\Pr_r(S) \leq \frac{|S| - |Z(S)|}{p|S|}. \tag{7.4.5}$$

By Result 1.3.14(b) we have

$$1 - \Pr_r(S, R) \leq \frac{|S||R| - 2|Z(S, R)||Z(R, S)|}{|S||R|} \tag{7.4.6}$$

and

$$1 - \sum_{u=r,-r} \Pr_u(S, R) \leq \frac{|S||R| - 4|Z(S, R)||Z(R, S)|}{|S||R|}. \tag{7.4.7}$$

(a) We have $2r = 0$. Therefore, if $r \in S$ then, using Theorem 7.4.1(b) and equations (7.4.5) and (7.4.6), we get

$$\begin{aligned} 2|e(\Gamma^r_{S,R})| + &|S|^2 + |S| \\ &\leq 2|S||R| \left(\frac{|S||R| - 2|Z(S, R)||Z(R, S)|}{|S||R|} \right) + |S|^2 \left(\frac{|S| - |Z(S)|}{p|S|} \right). \end{aligned}$$

Hence the result follows.

If $r \in R \setminus S$ then, using Theorem 7.4.1(b) and equation (7.4.6), we get

$$2|e(\Gamma^r_{S,R})| + |S|^2 + |S| \leq 2|S||R| \left(\frac{|S||R| - 2|Z(S, R)||Z(R, S)|}{|S||R|} \right).$$

Hence the result follows.

(b) We have $2r \neq 0$. Therefore, if $r \in S$ then, using Theorem 7.4.1(c) and equations (7.4.5) and (7.4.7), we get

$$\begin{aligned} 2|e(\Gamma^r_{S,R})| + &|S|^2 + |S| \\ &\leq 2|S||R| \left(\frac{|S||R| - 4|Z(S, R)||Z(R, S)|}{|S||R|} \right) + 2|S|^2 \left(\frac{|S| - |Z(S)|}{p|S|} \right). \end{aligned}$$

Hence the result follows.

If $r \in R \setminus S$ then, using Theorem 7.4.1(c) and equation (7.4.7), we get

$$2|e(\Gamma^r_{S,R})| + |S|^2 + |S| \leq 2|S||R| \left(\frac{|S||R| - 4|Z(S, R)||Z(R, S)|}{|S||R|} \right).$$

Hence the result follows. □

We conclude the section noting that if $r = 0$ then using Theorem 7.4.1(a) and various bounds for $\Pr(S, R)$ and $\Pr(S)$, obtained in [22], we may derive various bounds for $|e(\Gamma^r_{S,R})|$.

7.5 An induced subgraph of $\Gamma^r_{S,R}$

In this section we consider the *induced subgraph* $\Delta^r_{S,R}$ of $\Gamma^r_{S,R}$ with vertex set $R \backslash Z(S,R)$.

Theorem 7.5.1. *Let x be any vertex in $\Delta^r_{S,R}$.*

(a) *If $r = 0$ then* $\deg(x) = \begin{cases} |R| - |C_R(x)|, & \text{if } x \in S \setminus Z(S,R) \\ |S| - |C_S(x)|, & \text{if } x \in R \setminus S. \end{cases}$

(b) *If $r \neq 0$ and $2r = 0$ then*

$$\deg(x) = \begin{cases} |R| - |Z(S,R)| - |C^r_R(x)| - 1, & \text{if } x \in S \setminus Z(S,R) \\ |S| - |Z(S,R)| - |C^r_S(x)|, & \text{if } x \in R \setminus S. \end{cases}$$

(c) *If $r \neq 0$ and $2r \neq 0$ then*

$$\deg(x) = \begin{cases} |R| - |Z(S,R)| - 2|C^r_R(x)| - 1, & \text{if } x \in S \setminus Z(S,R) \\ |S| - |Z(S,R)| - 2|C^r_S(x)|, & \text{if } x \in R \setminus S. \end{cases}$$

Proof. Let x be a vertex in $\Delta^r_{S,R}$. If $x \in S \setminus Z(S,R)$ then $\deg(x)$ is the number of $y \in R \backslash Z(S,R)$ such that $xy \neq yx$. Hence, $\deg(x) = |R| - |Z(S,R)| - (|C_R(x)| - |Z(S,R)|) = |R| - |C_R(x)|$. If $x \in R \setminus S$ then $\deg(x)$ is the number of $s \in S \setminus Z(S,R)$ such that $sx \neq xs$. Hence, $\deg(x) = |S| - |Z(S,R)| - (|C_S(x)| - |Z(S,R)|) = |S| - |C_S(x)|$. Hence, part (a) follows.

The proofs of parts (b) and (c) follow from Theorem 7.2.1 (parts (b), (c)) noting that the vertex set of $\Delta^r_{S,R}$ is $R \setminus Z(S,R)$. □

By Lemma 7.2.2 and Theorem 7.5.1, we have the following two corollaries.

Corollary 7.5.2. *Let $x \in S$ be a vertex in $\Delta^r_{S,R}$.*

(a) *If $r \neq 0$ and $2r = 0$ then*

$$\deg(x) = \begin{cases} |R| - |Z(S,R)| - |C_R(x)| - 1, & \text{if } C^r_R(x) \neq \emptyset \\ |R| - |Z(S,R)| - 1, & \text{otherwise.} \end{cases}$$

(b) *If* $r \neq 0$ *and* $2r \neq 0$ *then*

$$\deg(x) = \begin{cases} |R| - |Z(S,R)| - 2|C_R(x)| - 1, & \text{if } C_R^r(x) \neq \emptyset \\ |R| - |Z(S,R)| - 1, & \text{otherwise.} \end{cases}$$

Corollary 7.5.3. *Let* $x \in R \setminus S$ *be a vertex in* $\Gamma_{S,R}^r$.

(a) *If* $r \neq 0$ *and* $2r = 0$ *then*

$$\deg(x) = \begin{cases} |S| - |Z(S,R)| - |C_S(x)|, & \text{if } C_S^r(x) \neq \emptyset \\ |S| - |Z(S,R)|, & \text{otherwise.} \end{cases}$$

(b) *If* $r \neq 0$ *and* $2r \neq 0$ *then*

$$\deg(x) = \begin{cases} |S| - |Z(S,R)| - 2|C_S(x)|, & \text{if } C_S^r(x) \neq \emptyset \\ |S| - |Z(S,R)|, & \text{otherwise.} \end{cases}$$

Theorem 7.5.4. *Let* S *be a subring of a non-commutative ring* R *with unity* 1 *such that* $|R| \neq 8$ *and* $1 \in S$. *Then* $\Delta_{S,R}^r$ *is not a tree.*

Proof. Suppose that $\Delta_{S,R}^r$ is a tree. Therefore, there exists $x \in R \backslash Z(S,R)$ such that $\deg(x) = 1$.

Case 1: $r = 0$

If $x \in S \setminus Z(S,R)$ then, by Theorem 7.5.1(a), we have $\deg(x) = |R| - |C_R(x)| = 1$. Therefore, $|C_R(x)| = 1$, contradiction. If $x \in R \setminus S$ then, by Theorem 7.5.1(a), we also have $\deg(x) = |S| - |C_S(x)| = 1$. Therefore, $|C_S(x)| = 1$, contradiction.

Case 2: $r \neq 0$ and $2r = 0$

Subcase 2.1: Let $x \in S \setminus Z(S,R)$. Then, by Corollary 7.5.2(a), we have $\deg(x) = |R| - |Z(S,R)| - 1 = 1$ or $\deg(x) = |R| - |Z(S,R)| - |C_R(x)| - 1 = 1$. That is,

$$|R| - |Z(S,R)| = 2 \text{ or} \tag{7.5.1}$$

$$|R| - |Z(S,R)| - |C_R(x)| = 2. \tag{7.5.2}$$

Note that $Z(S,R)$ is a subring of R as well as $C_R(x)$ containing 0 and 1. Therefore, $|Z(S,R)|$ divides the left hand sides of the equations (7.5.1) and (7.5.2). It follows that $|Z(S,R)| = 2$. Thus equation (7.5.1) gives $|R| = 4$, which is a contradiction since there is no non-commutative ring with unity having order 4. Again equation (7.5.2) gives $|R| - |C_R(x)| = 4$ and so $|C_R(x)| = 4$. Therefore, $|R| = 8$, which contradicts our assumption.

Subcase 2.2: Let $x \in R \setminus S$. Then, by Corollary 7.5.3(a), we have $\deg(x) = |S| - |Z(S,R)| = 1$ or $\deg(x) = |S| - |Z(S,R)| - |C_S(x)| = 1$. Note that $Z(S,R)$ is a subring of S as well as $C_S(x)$ containing 0 and 1. Therefore, $|Z(S,R)|$ divides $|S| - |Z(S,R)|$ and $|S| - |Z(S,R)| - |C_S(x)|$. It follows that $|Z(S,R)| = 1$, a contradiction.

Case 3: $r \neq 0$ and $2r \neq 0$

Subcase 3.1: Let $x \in S \setminus Z(S,R)$. Then, by Corollary 7.5.2(b), we have $\deg(x) = |R| - |Z(S,R)| - 1 = 1$ or $\deg(x) = |R| - |Z(S,R)| - 2|C_R(x)| - 1 = 1$. That is,

$$|R| - |Z(S,R)| = 2 \text{ or} \tag{7.5.3}$$

$$|R| - |Z(S,R)| - 2|C_R(x)| = 2. \tag{7.5.4}$$

Therefore, $|Z(S,R)| = 2$. Thus equation (7.5.3) leads to the same contradiction that we get in the first part of Subcase 2.1. By equation (7.5.4) we have $|R| - 2|C_R(x)| = 4$ and so $|C_R(x)| = 4$. Therefore, $|R| = 12$ and so $\frac{R}{Z(S,R)}$ is cyclic. Hence, by Result 1.3.5, R is commutative; a contradiction.

Subcase 3.2: Let $x \in R \setminus S$. Then, by Corollary 7.5.3(b), we have $\deg(x) = |S| - |Z(S,R)| = 1$ or $\deg(x) = |S| - |Z(S,R)| - 2|C_S(x)| = 1$. Therefore, $|Z(S,R)| = 1$, a contradiction. □

The proof of Theorem 7.5.4 also tells that there is no vertex in the graph $\Delta^r_{S,R}$ having degree 1 if R is a non-commutative ring with unity 1 such that $|R| \neq 8$ and S is any subring of R with the same unity. We conclude this chapter by obtaining conditions such that $\Delta^r_{S,R}$ has no vertex having degree 2.

Theorem 7.5.5. *Let S be a non-commutative subring of a ring R with unity 1 such that $1 \in S$. Then $\Delta^r_{S,R}$ has no vertex having degree 2 if $|R| \neq 12$ and $|S| \neq 8$.*

Proof. Suppose that $\Delta^r_{S,R}$ has a vertex x such that $\deg(x) = 2$.

Case 1: $r = 0$

If $x \in S \setminus Z(S, R)$ then, by Theorem 7.5.1(a), we have $\deg(x) = |R| - |C_R(x)| = 2$. Therefore, $|C_R(x)| = 2$, contradiction. If $x \in R \setminus S$ then, by Theorem 7.5.1(a), we also have $\deg(x) = |S| - |C_S(x)| = 2$. Therefore, $|C_S(x)| = 2$ and so $|S| = 4$, a contradiction since there is no non-commutative ring with unity having order 4.

Case 2: $r \neq 0$ and $2r = 0$

Subcase 2.1: Let $x \in S \setminus Z(S, R)$. Then, by Corollary 7.5.2(a), we have $\deg(x) = |R| - |Z(S, R)| - 1 = 2$ or $\deg(x) = |R| - |Z(S, R)| - |C_R(x)| - 1 = 2$. That is,

$$|R| - |Z(S, R)| = 3 \text{ or} \tag{7.5.5}$$

$$|R| - |Z(S, R)| - |C_R(x)| = 3. \tag{7.5.6}$$

Therefore, $|Z(S, R)| = 3$. Thus equation (7.5.5) gives $|R| = 6$, which is a contradiction since R is non-commutative. Again equation (7.5.6) gives $|R| - |C_R(x)| = 6$ and so $|C_R(x)| = 6$ since $|C_R(x)| \neq 3$. Therefore $|R| = 12$, which contradicts our assumption.

Subcase 2.2: Let $x \in R \setminus S$. Then, by Corollary 7.5.3(a), we have $\deg(x) = |S| - |Z(S, R)| = 2$ or $\deg(x) = |S| - |Z(S, R)| - |C_S(x)| = 2$. Therefore, $|Z(S, R)| = 2$ and so $|S| - |C_S(x)| = 4$ since $|S| \neq 4$. We have $|C_S(x)| = 4$ since $|C_S(x)| \neq 2$. Therefore $|S| = 8$ which contradicts our assumption.

Case 3: $r \neq 0$ and $2r \neq 0$

Subcase 3.1: Let $x \in S \setminus Z(S, R)$. Then, by Corollary 7.5.2(b), we have $\deg(x) = |R| - |Z(S, R)| - 1 = 2$ or $\deg(x) = |R| - |Z(S, R)| - 2|C_R(x)| - 1 = 2$. That is,

$$|R| - |Z(S, R)| = 3 \text{ or} \tag{7.5.7}$$

$$|R| - |Z(S, R)| - 2|C_R(x)| = 3. \tag{7.5.8}$$

Therefore, $|Z(S, R)| = 3$. Thus equation (7.5.7) leads to the same contradiction that we get in the first part of Subcase 2.1. By equation (7.5.8) we have $|R| - 2|C_R(x)| = 6$ and so $|C_R(x)| = 6$ since $|C_R(x)| \neq 3$. Therefore $|R| = 18$ and so $\frac{R}{Z(S,R)}$ is cyclic. Hence, by Result 1.3.5, R is commutative; a contradiction.

Subcase 3.2: Let $x \in R \setminus S$. Then, by Corollary 7.5.3(b), we have $\deg(x) = |S| - |Z(S, R)| = 2$ or $\deg(x) = |S| - |Z(S, R)| - 2|C_S(x)| = 2$. Therefore, $|Z(S, R)| = 2$ and so

$|S| - 2|C_S(x)| = 4$ since $|S| \neq 4$. Therefore, $|C_S(x)| = 4$. Therefore $|S| = 12$ and so $\frac{S}{Z(S,R)}$ is cyclic. Hence, by Result 1.3.5, S is commutative; a contradiction. $\square$

7.6 Conclusion

In this chapter, we have introduced relative r-noncommuting graph of a finite ring R relative to a subring S of R and obtained results analogous to the results obtained in Chapter 3. We have determined degree of any vertex in $\Gamma^r_{S,R}$ and studied certain graph theoretical properties of $\Gamma^r_{S,R}$. We have shown that $\Gamma^r_{S_1,R_1}$ is isomorphic to $\Gamma^{\psi(r)}_{S_2,R_2}$ if (ϕ, ψ) is an isoclinism between the pairs of finite rings (S_1, R_1), (S_2, R_2) and $|Z(S_1, R_1)| = |Z(S_2, R_2)|$. Furthermore, we have established connections between the number of edges in $\Gamma^r_{S,R}$ and various generalized commuting probabilities of R. Finally, we have studied a subgraph of $\Gamma^r_{S,R}$ induced on $R \setminus Z(S, R)$.