Chapter 8

Conclusion and problems for future research

We have begun the thesis with an introductory chapter, in which we have introduced some notations and recalled certain useful results from Graph Theory, Group Theory and Ring Theory. We have also looked at the literature on non-commuting graph and some of its generalizations defined on finite groups/rings. In Chapter 2, we have computed Signless Laplacian spectrum and Signless Laplacian energy for the non-commuting graphs of the groups D_{2m}, QD_{2^n}, M_{2rs}, Q_{4n}, U_{6n}, SD_{8n}, V_{8n}, etc. along with the classes of finite groups such that their central quotient is isomorphic to D_{2m}, $\mathbb{Z}_p \times \mathbb{Z}_p$ or $Sz(2)$ and found conditions such that Γ_G is Q-integral. We have compared Signless Laplacian energy with energy and Laplacian energy of the non-commuting graphs of these groups, respectively and found that there exist groups such that their non-commuting graphs satify the inequalities viz. $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$ and $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G)$. Further, we have characterized certain groups such that their non-commuting graphs are hypoenergetic, hyperenergetic, L-hyperenergetic and Q-hyperenergetic.

In Chapter 3, we have introduced the relative g-noncommuting graph of a finite group G (denoted by $\Gamma^g_{H,G}$, where H is a subgroup of G), which is an amalgamation of relative non-commuting graph ($\Gamma_{H,G}$) and g-noncommuting graph (Γ^g_G) of finite groups. We have shown that $\Gamma^g_{H,G}$ is not a tree (whenever $|H| \neq 2$), lollipop (whenever $|H| \neq 2, 3$) or a complete graph when $1 \neq g \in K(H, G)$. We have also obtained the number of edges in $\Gamma^g_{H,G}$ in terms of $\Pr_g(H, G)$ and $\Pr_g(G)$ and consequently derived certain bounds for the same.

In Chapter 4, we have considered the induced subgraph (denoted by $\Delta^g_{H,G}$) of $\Gamma^g_{H,G}$ induced by $G \setminus Z(H, G)$ which is a generalization of g-noncommuting graph of G. We have generalized Results 1.4.25, 1.4.27 and 1.4.28 in Theorems 4.1.1, 4.2.1 and 4.2.3. In Result 1.4.24, it has been shown that $\Delta^g_{G,G}$ is not a tree. In Section 4.1, we have considered the question whether $\Delta^g_{H,G}$ is a tree or not and we have shown that $\Delta^g_{H,G}$ is not a tree in general. In [71], Nasiri et al. have shown that $\mathrm{diam}(\Delta^g_{G,G}) \leq 4$ if $\Delta^g_{G,G}$ is connected. Furthermore, they have conjectured that $\mathrm{diam}(\Delta^g_{G,G}) \leq 2$ if $\Delta^g_{G,G}$ is connected. In Section 4.2, we have shown that this is not true in case of the graph $\Delta^g_{H,G}$, where H is a proper subgroup of G. In particular, we have identified a subgroup H of D_{2n} in Theorem 4.2.8 such that $\mathrm{diam}(\Delta^g_{H,D_{2n}}) = 3$ while discussing connectivity and diameter of $\Delta^g_{H,D_{2n}}$.

In Chapter 5, we have considered non-commuting graphs of certain families of finite non-commutative rings and obtained $\mathrm{Spec}(\Gamma_R)$, Q-spec(Γ_R), L-spec(Γ_R) and their corresponding energies. As a consequence, we have determined many families of finite rings such that Γ_R is integral/L-integral/Q-integral and also resolved whether the non-commuting graphs of these rings are hyperenergetic, L-hyperenergetic and Q-hyperenergetic.

In Chapter 6, we have defined r-noncommuting graph of a finite ring R, denoted by Γ^r_R and its induced subgraph on $R \setminus Z(R)$, denoted by Δ^r_R. We have characterized R when Γ^r_R is a tree or a star graph. We have also proved that there does not exist any finite non-commutative ring R such that Γ^r_R is a regular graph (if $r \in K(R)$), a lollipop graph or a complete bipartite graph. We have shown that the commuting graph of R is a spanning subgraph of Δ^r_R and also determined the clique number and diameter of Δ^r_R. We have further characterized finite non-commutative rings R such that Δ^r_R is n-regular for some positive integer n.

In Chapter 7, we have generalized the graph Γ^r_R and considered r-noncommuting graph of a finite ring R relative to a given subring S of R, denoted by $\Gamma^r_{S,R}$. We have determined degree of any vertex in $\Gamma^r_{S,R}$ and studied certain graph theoretical properties of $\Gamma^r_{S,R}$ using it. We have shown that $\Gamma^r_{S_1,R_1}$ is isomorphic to $\Gamma^{\psi(r)}_{S_2,R_2}$ if (ϕ, ψ) is an isoclinism between the pairs of finite rings (S_1, R_1), (S_2, R_2) and $|Z(S_1, R_1)| = |Z(S_2, R_2)|$. In addition, we have derived certain relations between the number of edges in $\Gamma^r_{S,R}$ and various generalized commuting probabilities of R. Finally, we have studied the induced subgraph of $\Gamma^r_{S,R}$ with vertex set $R \setminus Z(S, R)$.

8.1 Problems for future research

During our research on various properties of non-commuting graphs of finite groups and rings, as well as their generalizations, we have identified a number of problems for further investigation. In this section we list all those problems.

We have noticed that non-commuting graphs, for all the groups considered in Chapter 2 except S_4, are complete r-partite. Further, Γ_G is non-hypoenergetic as well as non-hyperenergetic except for S_4 and Γ_G satisfy the relation $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G)$ for all the groups considered in Chapter 2. Therefore, the following problem arises naturally.

Problem 8.1.1. Let $\mathcal{G}$ be a complete r-partite graph on n vertices. To prove/disprove that

(a) $\mathcal{G}$ is non-hypoenergetic as well as non-hyperenergetic.

(b) $E(\mathcal{G}) < LE^+(\mathcal{G}) < LE(\mathcal{G})$.

In Theorems 2.5.3 and 2.5.4 of Chapter 2, it has been proved that Γ_G is non-hypoene -rgetic if $\overline{\Gamma_G}$ is planar or toroidal respectively. In this regard the following problem comes naturally.

Problem 8.1.2. Let $\mathcal{G}$ be any graph. To prove/disprove that $\mathcal{G}$ is non-hypoenergetic if $\overline{\mathcal{G}}$ is planar or toroidal.

In Table 2.1 of Chapter 2 we have determined certain positive integers n such that $8n^2 - 16n + 9$, $2n^2 - 8n + 9$ and $32n^2 - 32n + 9$ are perfect squares. Such positive integers produce Q-integral graphs. For example, for the integers $n \geq 2$ listed in column 1 of Table 2.1 we have Q-integral graphs namely $\Gamma_{Q_{4n}}$ (see Theorem 2.5.1). At present we are able to list only a small number of such positive integers. The following problem may be interesting to consider.

Problem 8.1.3. Determine general terms of the sequences of positive integers given in the 1st, 3rd and 5th columns of the Table 2.1.

Spectral aspects of commuting and non-commuting graphs of finite groups have been well-studied. However, the same has not been carried for g-noncommuting and relative g-noncommuting graphs of finite groups. Therefore, it is worth studying spectral aspects of g-noncommuting and relative g-noncommuting graphs of finite groups.

In Chapter 4, we have discussed about diameter and connectivity of $\Delta^g_{H,G}$ with special attention to the dihedral groups. It will be interesting to consider other families of finite groups (e.g. semidihedral groups, quasidihedral groups and dicyclic groups) and find $\text{diam}(\Delta^g_{H,G})$.

In Chapter 6, we have obtained results on Γ^r_R and Δ^r_R analogous to certain results for g-noncommuting graphs of finite groups obtained in [70, 94]. Of course we have also obtained results not analogous to the results for g-noncommuting graphs of finite groups. However, it will be interesting to discover more properties of Γ^r_R and Δ^r_R different from the case of groups. Many of our results that characterize finite non-commutative rings such that the graph Δ^r_R is n-regular for $1 \leq n \leq 6$ involve conditions on $|R|$. Therefore, the question of recognizing rings with these graphs is still not clear for such cases. One may continue further research to remove those conditions on $|R|$ and recognize the rings clearly. Again regarding regularity of Δ^r_R, we have the following problem.

Problem 8.1.4. Determine all the positive integers n such that Δ^r_R is n-regular.

In Chapter 7, we have noticed that for a non-commutative subring S of a ring R with unity 1 such that $1 \in S$, $\Delta^r_{S,R}$ has no vertex of degree 1 if $|R| \neq 8$ and of degree 2 if $|R| \neq 12$ and $|S| \neq 8$. Therefore, the following challenge emerges in this regard.

Problem 8.1.5. To obtain conditions on $|R|$ and $|S|$ in terms of n such that $\Delta^r_{S,R}$ has no vertex having degree n, for $n \in \mathbb{N}$.