Dedicated to my parents

Shri Jagat Chandra Sharma

&

Smt. Purnima Sharma

DECLARATION BY THE CANDIDATE

I, Ms. Monalisha Sharma, hereby declare that the subject matter in this thesis entitled "Non-commuting and generalized non-commuting graphs of finite groups and rings", is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

Date:

Place: Tezpur Signature of the Candidate

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CERTIFICATE OF THE SUPERVISOR

This is to certify that the thesis entitled "*Non-commuting and generalized non-commuting graphs of finite groups and rings*" submitted to the School of Sciences of Tezpur University in partial fulfillment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by Ms. Monalisha Sharma under my supervision and guidance.

All help received by her from various sources have been duly acknowledged.

No part of this thesis have been submitted elsewhere for award of any other degree.

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List of symbols

$\cup$	union of sets/graphs
$\cap$	intersection of sets
$\rtimes$	semidirect product
$*$	central product
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathcal{G}$	finite, simple and undirected graph
$\overline{\mathcal{G}}$	complement of $\mathcal{G}$
$v(\mathcal{G})$	vertex set of $\mathcal{G}$
$e(\mathcal{G})$	edge set of $\mathcal{G}$
$x \leftrightarrow y$	vertex x is adjacent to vertex y in the graph $\mathcal{G}$
$x \nleftrightarrow y$	vertex x is not adjacent to vertex y in the graph $\mathcal{G}$
$\deg(x)$	degree of a vertex x in $\mathcal{G}$
$\gamma(\mathcal{G})$	domination number of $\mathcal{G}$
$\omega(\mathcal{G})$	clique number of $\mathcal{G}$
$\mathrm{girth}(\mathcal{G})$	girth of $\mathcal{G}$
$d(u, v)$	distance between two vertices u and v in $\mathcal{G}$
$\mathrm{diam}(\mathcal{G})$	diameter of $\mathcal{G}$
K_n	complete graph on n vertices
$K_{a_1.p_1,a_2.p_2,...,a_s.p_s}$	complete r-partite graph $K_{\underbrace{p_1,\ldots,p_1}_{a_1\text{-times}},\underbrace{p_2,\ldots,p_2}_{a_2\text{-times}},\ldots,\underbrace{p_s,\ldots,p_s}_{a_s\text{-times}}}$ on n vertices, where $r = a_1 + a_2 + \cdots + a_s$
$A(\mathcal{G})$	adjacency matrix of $\mathcal{G}$
$D(\mathcal{G})$	degree matrix of $\mathcal{G}$

$L(\mathcal{G})$	Laplacian matrix of $\mathcal{G}$
$Q(\mathcal{G})$	Signless Laplacian matrix of $\mathcal{G}$
$P_{\mathcal{G}}(x)$	characteristic polynomial of $A(\mathcal{G})$
$Q_{\mathcal{G}}(x)$	characteristic polynomial of $Q(\mathcal{G})$
$\mathrm{Spec}(\mathcal{G})$	spectrum of $\mathcal{G}$
$\mathrm{L\text{-}spec}(\mathcal{G})$	Laplacian spectrum of $\mathcal{G}$
$\mathrm{Q\text{-}spec}(\mathcal{G})$	Signless Laplacian spectrum of $\mathcal{G}$
$E(\mathcal{G})$	energy of $\mathcal{G}$
$LE(\mathcal{G})$	Laplacian energy of $\mathcal{G}$
$LE^{+}(\mathcal{G})$	Signless Laplacian energy of $\mathcal{G}$
G	any finite group
H	any subgroup of G
$Z(G)$	center of G
$C_G(x)$	centralizer of x in G
$K(G)$	$\{[x,y] : x,y \in G\}$ (here $[x,y] = x^{-1}y^{-1}xy$)
$Z(H,G)$	$\{x \in H : xy = yx \,\forall\, y \in G\}$
$Z(G,H)$	$\{x \in G : xy = yx \,\forall\, y \in H\}$
$K(H,G)$	$\{[x,y] : x \in H$ and $y \in G\}$ (here $[x,y] = x^{-1}y^{-1}xy$)
x^g	gxg^{-1} for some $g \in G$
$x \sim y$	x is conjugate to y for some $x, y \in G$
$x \nsim y$	x is not conjugate to y for some $x, y \in G$
$o(x)$	order of x
$\Pr(G)$	commuting probability of G
$\Pr(H,G)$	relative commuting probability of a subgroup H of G
$\Pr_g(G)$	g-commuting probability of G for any given element $g \in G$
$\Pr_g(H,K)$	relative g-commuting probability of two subgroups H and K of G for any given element $g \in G$

$\mathbb{Z}_n$	cyclic group of order n
S_n	symmetric group of degree n
A_n	alternating group of degree n
D_{2m}	$\langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle$,
	the dihedral group of order $2m$
QD_{2^n}	$\langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1} \rangle$,
	the quasidihedral group of order 2^n
M_{2rs}	$\langle a, b : a^r = b^{2s} = 1, bab^{-1} = a^{-1} \rangle$
Q_{4n}	$\langle x, y : x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1} \rangle$,
	the dicyclic group of order $4n$
U_{6n}	$\langle x, y : x^{2n} = y^3 = 1, x^{-1}yx = y^{-1} \rangle$
$Sz(2)$	$\langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle$,
	the suzuki group of order 20
SD_{8n}	$\langle a, b : a^{4n} = b^2 = 1, bab = a^{2n-1} \rangle$,
	the semidihedral group of order $8n$
V_{8n}	$\langle a, b : a^{2n} = b^4 = 1, b^{-1}ab^{-1} = bab = a^{-1} \rangle$
$\mathcal{M}_{16}$	$\langle a, b : a^8 = b^2 = 1, bab = a^5 \rangle$
$SG(16, 3)$	$\langle a, b : a^4 = b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1} \rangle$
$SL(2, 3)$	special linear group of order 24
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	$\langle a, b : a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$
$D_8 * \mathbb{Z}_4$	$\langle a, x, y : a^4 = y^4 = x^2 = 1, a^2 = y^2, xax = a^{-1}, ay = ya, xy = yx \rangle$
Γ_G	non-commuting graph of G
$\Gamma_{H,G}$	relative non-commuting graph of G
Γ_G^g	g-noncommuting graph of G
Δ_G^g	induced subgraph of Γ_G^g on $G \setminus Z(G)$
$\Gamma_{H,G}^g$	relative g-noncommuting graph of G
$\Delta_{H,G}^g$	induced subgraph of $\Gamma_{H,G}^g$ on $G \setminus Z(H, G)$

R	any finite ring
S	any subring of R
$Z(R)$	center of R
$C_R(x)$	centralizer of x in R
$K(R)$	$\{[x,y] : x, y \in R\}$ (here $[x,y] = xy - yx$)
$Z(S,R)$	$\{z \in S : zx = xz, \forall x \in R\}$
$K(S,R)$	$\{[x,y] : x \in S$ and $y \in R\}$ (here $[x,y] = xy - yx$)
$\Pr(R)$	commuting probability of R
$\Pr(S,R)$	relative commuting probability of a subring S of R
$\Pr_r(R)$	r-commuting probability of R for any given element $r \in R$
$\Pr_r(S,R)$	relative r-commuting probability of R with respect to a subring S for any given element $r \in R$
Γ_R	non-commuting graph of R
$\Gamma_{S,R}$	relative non-commuting graph of R
Γ_R^r	r-noncommuting graph of R
Δ_R^r	induced subgraph of Γ_R^r on $R \setminus Z(R)$
$\Gamma_{S,R}^r$	relative r-noncommuting graph of R
$\Delta_{S,R}^r$	induced subgraph of $\Gamma_{S,R}^r$ on $R \setminus Z(S,R)$