Abstract

Assume that F is a finite group/ring with $Z(F)$ as its center. The non-commuting graph of F (denoted by Γ_F) is a simple undirected graph connecting two distinct non-commuting vertices, with $F \setminus Z(F)$ as the vertex set. Non-commuting graphs of finite groups G (denoted by Γ_G) have been considered by Neumann in 1976 while responding to a query posed by Erdös. However, in case of finite rings R, the study of non-commuting graphs (denoted by Γ_R) has been initiated by Erfanian, Khashyarmanesh and Nafar in 2015. Following them, many mathematicians have worked on non-commuting graphs of finite groups/rings and their generalizations over the years.

In this thesis, we consider some untouched spectral properties of non-commuting graphs of finite groups/rings. In particular, we compute Signless Laplacian spectrum and Signless Laplacian energy of non-commuting graphs for various families of finite non-abelian groups. We also compute spectrum, Laplacian spectrum, Signless Laplacian spectrum and their corresponding energies of non-commuting graphs for certain finite non-commutative rings. Further, we introduce and study a few fresh generalizations of non-commuting graphs of finite groups/rings and explore the interplay of algebraic and graph theoretic properties.

In Chapter 1 of this thesis, we recall several definitions, notations and results from Graph Theory, Group Theory and Ring Theory that are useful in the subsequent chapters. We record certain results on commuting probabilities of finite groups/rings and their generalizations. We also review literature on non-commuting graphs of finite groups/rings along with some of their generalizations.

In Chapter 2, we compute Signless Laplacian spectrum and Signless Laplacian energy of non-commuting graphs for various families of finite non-abelian groups. We obtain several conditions such that the non-commuting graphs of such groups are Q-integral and observe relations between energy, Signless Laplacian energy and Laplacian energy. In addition, we look into the energetic hyper- and hypo-properties of non-commuting graphs of these finite non-abelian groups. We also assess whether the same graphs are Q-hyperenergetic and L-hyperenergetic.

In Chapter 3, we introduce relative g-noncommuting graph of a finite group G. For a given element g in G and a given subgroup H of G, the relative g-noncommuting graph of G is a simple undirected graph whose vertex set is G and two vertices x and y are adjacent if $x \in H$ or $y \in H$ and $[x, y] \neq g, g^{-1}$, where $[x, y] = x^{-1}y^{-1}xy$. We denote this graph by $\Gamma^g_{H,G}$. We obtain computing formulae for degree of any vertex in $\Gamma^g_{H,G}$ and characterize whether $\Gamma^g_{H,G}$ is a tree, star graph, lollipop or a complete graph together with some properties of $\Gamma^g_{H,G}$ involving isomorphism of graphs. We also present certain relations between the number of edges in $\Gamma^g_{H,G}$ and certain generalized commuting probabilities of G which give some computing formulae for the number of edges in $\Gamma^g_{H,G}$. Finally, we conclude the chapter by deriving some bounds for the number of edges in $\Gamma^g_{H,G}$.

In Chapter 4, we consider the induced subgraph of $\Gamma^g_{H,G}$ on $G \setminus Z(H, G)$, where $Z(H, G) = \{x \in H : xy = yx, \forall y \in G\}$ for a given subgroup H of a finite non-abelian group G. We denote this graph by $\Delta^g_{H,G}$ and determine whether $\Delta^g_{H,G}$ is a tree among other results. We also discuss about its diameter and connectivity with special attention to the dihedral groups.

In Chapter 5, we compute spectrum, energy, Laplacian spectrum, Laplacian energy, Signless Laplacian spectrum and Signless Laplacian energy of non-commuting graphs of certain finite non-commutative rings. In particular, we consider finite non-commutative rings R such that $|R| = p^2, p^3, p^4$, p^5, p^2q and p^3q, where p and q are primes. Further, we consider n-centralizer finite rings for $n = 4, 5$ and $p + 2$; more generally, finite rings with central quotients isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Our computations reveal that non-commuting graphs of these rings are L-integral. Additionally, we identify the finite rings under consideration that yield integral, Q-integral, hyperenergetic, L-hyperenergetic and Q-hyperenergetic non-commuting graphs.

In Chapter 6, we generalize the notion of non-commuting graph of a finite ring R to r-noncommuting graph of R, denoted by Γ_R^r, for a given element $r \in R$. Γ_R^r is a simple undirected graph whose vertex set is R and two vertices x and y are adjacent if and only if $[x, y] \neq r$ and $[x, y] \neq -r$, where $[x, y]$ is the additive commutator $xy - yx$. We obtain expressions for vertex degrees and show that Γ_R^r is neither a regular graph nor a lollipop graph if R is non-commutative. We characterize finite non-commutative rings such that Γ_R^r is a tree, in particular a star graph. It is also shown that $\Gamma_{R_1}^r$ and $\Gamma_{R_2}^{\psi(r)}$ are isomorphic if R_1 and R_2 are two isoclinic rings with isoclinism (ϕ, ψ). Further, we consider the induced subgraph Δ_R^r of Γ_R^r (induced by the non-central elements of R) and obtain results on the bounds for clique number and diameter of Δ_R^r along with certain characterizations of finite non-commutative rings such that Δ_R^r is n-regular for some positive integer n. More precisely, we characterize certain finite non-commutative rings such that their non-commuting graphs are n-regular for $n \leq 6$.

In Chapter 7, we discuss relative r-noncommuting graph of a finite ring R relative to a subring S of R, denoted by $\Gamma_{S,R}^r$, which is a simple undirected graph whose vertex set is R and two vertices x and y are adjacent if and only if $x \in S$ or $y \in S$ and the additive commutator $[x, y] \neq r, -r$. We determine degree of any vertex in $\Gamma_{S,R}^r$ and characterize all finite rings such that $\Gamma_{S,R}^r$ is a star, lollipop or a regular graph. We derive connections between relative r-noncommuting graphs of two isoclinic pairs of rings. We also derive certain relations between the number of edges in $\Gamma_{S,R}^r$ and generalized commuting probabilities of R. Finally, we conclude the chapter by studying an induced subgraph of $\Gamma_{S,R}^r$.

In Chapter 8, we summarize the work completed for the thesis and conclude the thesis by suggesting some problems for future research.