Chapter 1

Introduction and preliminaries

The non-commuting graph of a finite group/ring F, represented by Γ_F, is a simple undirected graph connecting two distinct vertices x and y whenever $xy \neq yx$ with $F \setminus Z(F)$ as the vertex set, where $Z(F)$ is the center of F. It is the complement of commuting graph, the first graph that has been defined on finite groups and rings using commutativity. The purpose of studying these graphs associated to finite groups/rings is to characterize finite groups/rings through graph theoretic parameters and vice-versa. The origin of commuting graph of a finite group lies in a paper of Brauer and Fowler [12]. However, the study of non-commuting graph of a finite group gets its popularity after the works of Erdös and Neumann [76]. The study of non-commuting graph of a finite ring has been pioneered by Erfanian, Khashyarmanesh and Nafar [38]. The following is a list of some works that discuss the non-commuting and commuting graphs of finite groups [1, 3, 9, 10, 14, 19, 26, 28, 29, 31, 32, 36, 45, 46, 56, 64, 66, 84]. Similar studies for the non-commuting and commuting graphs of finite rings can be found in the articles [6, 8, 23, 38, 41, 77, 96, 97]. In these works, we observe that both theoretical as well as computational aspects of these graphs have been considered. Some mathematicians have taken the challenge to recognize finite groups/rings through the properties of non-commuting/commuting graphs defined on them while some have analyzed the spectral aspects of these graphs.

In this thesis, we consider some untouched spectral properties of non-commuting graphs of finite groups/rings. In particular, we compute Signless Laplacian spectrum and Signless Laplacian energy of non-commuting graphs for various families of finite non-abelian groups. We also compute spectrum, Laplacian spectrum, Signless Laplacian spectrum and their corresponding energies of non-commuting graphs for certain finite non-commutative rings. Further, we introduce and study a few fresh generalizations of non-commuting graphs of finite groups/rings and explore the interplay of algebraic and graph theoretic properties.

In this chapter, we recall certain definitions and results from Graph Theory, Group Theory and Ring Theory that are useful in the subsequent chapters. We record various probabilities defined on finite groups/rings such as commuting probability of a group (denoted by $\Pr(G)$ and introduced by Erdös and Turán [37]), relative commuting probability of a subgroup H of G (denoted by $\Pr(H, G)$ and introduced by Erfanian et al. [39]), g-commuting probability for an element $g \in G$ (denoted by $\Pr_g(G)$ and introduced by Pournaki and Sobhani [79]), relative g-commuting probability (denoted by $\Pr_g(H, K)$ and introduced by Das and Nath [16]) and the corresponding probabilities for finite rings which have been introduced in [63], [22], [35] and [34] respectively. We also review literature on non-commuting graph and its generalizations defined so far on finite groups and rings. In particular, we review non-commuting graph of a finite group G (denoted by Γ_G), relative non-commuting graph for a given subgroup H of G (denoted by $\Gamma_{H,G}$ and introduced by Tolue and Erfanian [93]), g-noncommuting graph for any given element g in G (denoted by Γ_G^g and introduced by Tolue et al. [94]), non-commuting graph of a finite ring R (denoted by Γ_R) and relative non-commuting graph for any finite non-commutative ring R with subring S (denoted by $\Gamma_{S,R}$ and introduced by Dutta and Basnet [20]).

In Chapter 2, we compute Signless Laplacian spectrum and Signless Laplacian energy of non-commuting graphs for several families of finite non-abelian groups and determine several groups such that Γ_G is Q-integral. We compare various energies of non-commuting graphs of the groups we consider and find that $E(\Gamma_G) \leq LE^+(\Gamma_G) \leq LE(\Gamma_G)$, where $E(\Gamma_G)$, $LE(\Gamma_G)$ and $LE^+(\Gamma_G)$ denote the energy, Laplacian energy and Signless Laplacian energy of Γ_G respectively. In addition, we look into the energetic hyper- and hypo-properties of Γ_G . We also assess whether the same graphs are L-hyperenergetic and

Q-hyperenergetic. In this journey, we produce a counter example for Conjecture 1.1.7 (see page 7) posed by Gutman [48]. The contents of Chapter 2 is based on our paper [88].

In Chapter 3, fusing the concepts of $\Gamma_{H,G}$ and Γ_G^g, we introduce relative g-noncommut -ing graph for a given subgroup H of G which is denoted by $\Gamma_{H,G}^g$. We obtain computing formula for degree of any vertex in $\Gamma_{H,G}^g$ and characterize whether $\Gamma_{H,G}^g$ is a tree, lollipop or a complete graph along with some other results. We also obtain relations between $\Gamma_{H,G}^g$ and $\mathrm{Pr}_g(H,G)$. The contents of Chapter 3 is based on our paper [85].

In Chapter 4, the induced subgraph of $\Gamma_{H,G}^g$ on $G \setminus Z(H,G)$ is considered and its properties, including connectivity and diameter with special attention to the dihedral groups, are investigated. The contents of Chapter 4 is based on our paper [89].

In Chapter 5, we compute spectrum, energy, Laplacian spectrum, Laplacian energy, Signless Laplacian spectrum and Signless Laplacian energy of Γ_R for certain classes of finite rings and check whether these graphs are integral/L-integral/Q-integral and hyperenergetic /Q-hyperenergetic/L-hyperenergetic. We also check whether the inequalities given in Conjecture 1.1.5 and Question 1.1.6 (see page 6) satisfy for Γ_R. The contents of Chapter 5 is based on our paper [86].

In Chapter 6, we introduce r-noncommuting graph of a finite ring R and an element $r \in R$, denoted by Γ_R^r, analogous to g-noncommuting graph of a finite group G. We compute degree of any vertex of Γ_R^r in terms of its centralizers and characterize R if Γ_R^r is a tree. Among other results we show that Γ_R^r is not a regular graph (for some $r \in R$) or a lollipop graph for any non-commutative ring R. We also consider the induced subgraph Δ_R^r of Γ_R^r induced by $R \setminus Z(R)$ and obtain results on diameter of Δ_R^r along with certain characterization of finite non-commutative rings such that Δ_R^r is n-regular for some positive integer n. The contents of Chapter 6 is based on our paper [74].

In Chapter 7, we generalize the graph Γ_R^r by introducing the relative r-noncommuting graph of R for a given subring S of R, denoted by $\Gamma_{S,R}^r$, analogous to $\Gamma_{H,G}^g$ studied in Chapters 3 and 4. We study various properties of $\Gamma_{S,R}^r$ including characterizations of finite rings through $\Gamma_{S,R}^r$. We obtain certain relations between the number of edges in $\Gamma_{S,R}^r$ and $\mathrm{Pr}_r(S,R)$. We also consider an induced subgraph of $\Gamma_{S,R}^r$ induced by $R \setminus Z(S,R)$. The contents of Chapter 7 is based on our paper [87].

In Chapter 8, we conclude the thesis by suggesting some problems for future research.

1.1 Notations and results from Graph Theory

For all the standard notations and basic results of Graph Theory we refer to [100]. All the graphs we consider in our study are finite, simple and undirected. Let $\mathcal{G}$ be a graph of order n ($n \geq 1$) and let its vertex set be $v(\mathcal{G}) = \{v_1, ..., v_n\}$ and edge set be $e(\mathcal{G})$. The *degree* of a vertex $x \in v(\mathcal{G})$, denoted by $\deg(x)$, is defined to be the number of vertices adjacent to x. A subset S of $v(\mathcal{G})$ is called a *dominating set* if for every vertex $u \in v(\mathcal{G}) \setminus S$, there is a vertex $v \in S$ such that u is adjacent to v in $\mathcal{G}$. The *domination number* of $\mathcal{G}$, denoted by $\gamma(\mathcal{G})$, is the minimum cardinality of dominating sets of $\mathcal{G}$. For any subset S of $v(\mathcal{G})$, the *induced subgraph* of $\mathcal{G}$ with vertex set S is the graph whose edge set consists of all of the edges in $e(\mathcal{G})$ that have both endpoints in S. A subset of $v(\mathcal{G})$ is called a *clique* of $\mathcal{G}$ if it consists entirely of pairwise adjacent vertices. The least upper bound of the sizes of all the cliques of $\mathcal{G}$ is called the *clique number* of $\mathcal{G}$, and it is denoted by $\omega(\mathcal{G})$. The *girth* of $\mathcal{G}$ is the minimum of the lengths of all cycles in $\mathcal{G}$, and it is denoted by $\mathrm{girth}(\mathcal{G})$. The distance between two vertices u and v of $\mathcal{G}$ is denoted by $d(u, v)$. The *diameter* of a graph $\mathcal{G}$, denoted by $\mathrm{diam}(\mathcal{G})$, is the maximum distance between the pair of vertices. The adjacency matrix $A(\mathcal{G}) = [a_{ij}]_{n\times n}$ of $\mathcal{G}$ is a $n \times n$ matrix where $a_{ij} = 1$ if $v_iv_j \in e(\mathcal{G})$ (that is, v_i is adjacent to v_j) and $a_{ij} = 0$ otherwise. The degree matrix $D(\mathcal{G}) = [d_{ij}]_{n\times n}$ of $\mathcal{G}$ is a diagonal matrix where $d_{ii} = \deg(v_i)$ and $d_{ij} = 0$ for $i \neq j$. The Laplacian matrix $L(\mathcal{G})$ of $\mathcal{G}$ is defined by $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$ and the Signless Laplacian matrix $Q(\mathcal{G})$ of $\mathcal{G}$ is defined by $Q(\mathcal{G}) = D(\mathcal{G}) + A(\mathcal{G})$. The set $\{(\alpha_1)^{a_1}, (\alpha_2)^{a_2}, \dots, (\alpha_t)^{a_t}\}$, where α_i's are the eigenvalues of $A(\mathcal{G})$ with multiplicities a_i for $1 \leq i \leq t$, respectively is called the *spectrum* of $\mathcal{G}$ and is denoted by $\mathrm{Spec}(\mathcal{G})$. The set $\{(\beta_1)^{b_1}, (\beta_2)^{b_2}, \dots, (\beta_m)^{b_m}\}$, where β_j's are the eigenvalues of $L(\mathcal{G})$ with multiplicities b_j for $1 \leq j \leq m$, respectively is referred to as the *Laplacian spectrum* of $\mathcal{G}$ and is denoted by $\text{L-spec}(\mathcal{G})$. The set $\{(\gamma_1)^{c_1}, (\gamma_2)^{c_2}, \dots, (\gamma_l)^{c_l}\}$, where γ_k's are the eigenvalues of $Q(\mathcal{G})$ with multiplicities c_k for $1 \leq k \leq l$, respectively is known as the *Signless Laplacian spectrum* of $\mathcal{G}$ and is denoted by $\text{Q-spec}(\mathcal{G})$. A graph $\mathcal{G}$ is called integral, L-integral and Q-integral if eigenvalues of $A(\mathcal{G})$, $L(\mathcal{G})$ and $Q(\mathcal{G})$ are integers. The notion of integral graph and L-integral graph have been introduced by Harary and Schwenk [55] and Grone and Merris [47] respectively. Integral and L-integral graphs have been studied extensively over the years while Q-integral graphs have not studied much.

It is worth mentioning that the notion of Q-integral graph has been introduced by Simić and Stanić [90].

The following well-known result gives the spectrum of a strongly regular graph.

Result 1.1.1. Let $\mathcal{H}$ be a strongly regular graph with parameters (k, λ, μ) then

$$\mathrm{Spec}(\mathcal{H}) = \left\{ (k)^1, \left(\frac{\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}{2} \right)^{n_1}, \left(\frac{\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}{2} \right)^{n_2} \right\},$$

where $n_1 = \frac{1}{2}\left(|v(\mathcal{H})| - 1 + \frac{2k + (|v(\mathcal{H})|-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right)$ and $n_2 = \frac{1}{2}\left(|v(\mathcal{H})| - 1 - \frac{2k + (|v(\mathcal{H})|-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right)$.

Result 1.1.2. [67, Theorem 3.6] For any graph $\mathcal{H}$, if L-spec$(\mathcal{H}) = \{(\beta_1)^{b_1}, (\beta_2)^{b_2}, \ldots, (\beta_m)^{b_m}\}$ where $\beta_1 < \beta_2 < \cdots < \beta_m$ then L-spec$(\overline{\mathcal{H}}) = \{(0)^1, (|v(\mathcal{H})| - \beta_m)^{b_m}, (|v(\mathcal{H})| - \beta_{m-1})^{b_{m-1}}, (|v(\mathcal{H})| - \beta_{m-2})^{b_{m-2}}, \ldots, (|v(\mathcal{H})| - \beta_1)^{b_1 - 1}\}$.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two graphs such that $v(\mathcal{G}_1) \cap v(\mathcal{G}_2) = \emptyset$. Then $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ (called disjoint union of $\mathcal{G}_1$ and $\mathcal{G}_2$) is the graph with $v(\mathcal{G}) = v(\mathcal{G}_1) \cup v(\mathcal{G}_2)$ and $e(\mathcal{G}) = e(\mathcal{G}_1) \cup e(\mathcal{G}_2)$. Throughout this thesis we write $mK_n = \underbrace{K_n \cup \cdots \cup K_n}_{m\text{-times}}$, where K_n is the complete graph and $|v(K_n)| = n$. The following results will help us to compute spectrum, Laplacian spectrum and Signless Laplacian spectrum of complete r-partite graph in the succeeding chapters.

Result 1.1.3. [26, Corollary 2.3] Let $\mathcal{G}$ be a graph and $\mathcal{G} = l_1 K_{m_1} \cup l_2 K_{m_2} \cup \cdots \cup l_k K_{m_k}$, where $l_i K_{m_i}$ denotes the disjoint union of l_i copies of K_{m_i} for $1 \le i \le k$ and $m_1 < m_2 < \cdots < m_k$. Then

$$\text{L-spec}(\overline{\mathcal{G}}) = \left\{ (0)^1, \left(\sum_{i=1}^{k} l_i m_i - m_k\right)^{l_k(m_k - 1)}, \left(\sum_{i=1}^{k} l_i m_i - m_{k-1}\right)^{l_{k-1}(m_{k-1}-1)}, \right.$$
$$\left. \ldots, \left(\sum_{i=1}^{k} l_i m_i - m_1\right)^{l_1(m_1-1)}, \left(\sum_{i=1}^{k} l_i m_i\right)^{\sum_{i=1}^{k} l_i - 1} \right\}.$$

Result 1.1.4. ([99, Corollary 2.3] and [101, Corollary 2.2]) Let $\mathcal{G}$ be the complete r-partite graph $K_{\underbrace{p_1, \ldots, p_1}_{a_1\text{-times}}, \underbrace{p_2, \ldots, p_2}_{a_2\text{-times}}, \ldots, \underbrace{p_s, \ldots, p_s}_{a_s\text{-times}}} := K_{a_1.p_1, a_2.p_2, \ldots, a_s.p_s}$ on n vertices, where $r = a_1 + a_2 + \cdots + a_s$. Then

(a) the characteristic polynomial of $A(\mathcal{G})$ is

$$P_{\mathcal{G}}(x) := x^{n-r}\prod_{i=1}^{s}(x+p_i)^{a_i-1}\left(\prod_{i=1}^{s}(x+p_i) - \sum_{j=1}^{s}a_jp_j\prod_{i=1, i\neq j}^{s}(x+p_i)\right).$$

(b) the characteristic polynomial of $Q(\mathcal{G})$ (also known as Q-polynomial) is

$$Q_{\mathcal{G}}(x) := \prod_{i=1}^{s}(x-n+p_i)^{a_i(p_i-1)}\prod_{i=1}^{s}(x-n+2p_i)^{a_i}\left(1-\sum_{i=1}^{s}\frac{a_ip_i}{x-n+2p_i}\right).$$

Depending on various spectra of a graph, there are various energies called energy, Laplacian energy and Signless Laplacian energy denoted by $E(\mathcal{G})$, $LE(\mathcal{G})$ and $LE^+(\mathcal{G})$ respectively. Let $\text{Spec}(\mathcal{G}) = \{(\alpha_1)^{a_1}, (\alpha_2)^{a_2}, \dots, (\alpha_t)^{a_t}\}$, L-spec$(\mathcal{G}) = \{(\beta_1)^{b_1}, (\beta_2)^{b_2}, \dots, (\beta_m)^{b_m}\}$ and Q-spec$(\mathcal{G}) = \{(\gamma_1)^{c_1}, (\gamma_2)^{c_2}, \dots, (\gamma_l)^{c_l}\}$. Then

$$E(\mathcal{G}) := \sum_{i=1}^{t} a_i|\alpha_i|, \tag{1.1.1}$$

$$LE(\mathcal{G}) := \sum_{j=1}^{m} b_j\left|\beta_j - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|}\right|, \tag{1.1.2}$$

$$LE^+(\mathcal{G}) := \sum_{k=1}^{l} c_k\left|\gamma_k - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|}\right|. \tag{1.1.3}$$

Note that $E(\mathcal{G})$, $LE(\mathcal{G})$ and $LE^+(\mathcal{G})$ have been defined by Gutman [48], Gutman and Zhou [53] and Abreua et al. [2] respectively. In 2008, Gutman et al. [51] have posed the following conjecture.

Conjecture 1.1.5. $E(\mathcal{G}) \leq LE(\mathcal{G})$, for any graph $\mathcal{G}$.

The above conjecture has been disproved in [62, 91], by providing some counter examples. Recently, Dutta et al. [19] have shown that Conjecture 1.1.5 holds for commuting graphs of several families of finite non-abelian groups and asked the following question.

Question 1.1.6. Is $LE(\mathcal{G}) \leq LE^+(\mathcal{G})$, for all graphs $\mathcal{G}$?

However, in the same paper, it has been observed that the inequality given in Question 1.1.6 does not hold for commuting graphs of finite non-abelian groups. Results on comparing various energies can be found in [11, 19, 51, 62, 91].
It is well-known that

$$E(K_n) = LE(K_n) = LE^+(K_n) = 2(n-1). \tag{1.1.4}$$

A graph $\mathcal{G}$ with n vertices is called *hyperenergetic*, *L-hyperenergetic* or *Q-hyperenergetic* according as $E(K_n) < E(\mathcal{G}), LE(K_n) < LE(\mathcal{G})$ or $LE^+(K_n) < LE^+(\mathcal{G})$. Also, $\mathcal{G}$ is called *hypoenergetic*, if $E(\mathcal{G}) < |v(\mathcal{G})|$. Gutman [49] and Walikar et al. [98] have pioneered the research of hyperenergetic graphs in 1999 while hypoenergetic graphs have been introduced by Gutman and Radenković [52] in 2007. L-hyperenergetic and Q-hyperenergetic graphs have been considered in [41]. We conclude this section with the following conjecture which has been posed by Gutman [48] but later has been disproved by different mathematicians providing counter examples (see [50]).

Conjecture 1.1.7. Any finite graph $\mathcal{G} \ncong K_{|v(\mathcal{G})|}$ is non-hyperenergetic.

1.2 Notations and results from Group Theory

In this section, we fix some notations and recall certain results from Group Theory which will be referred in the subsequent chapters. However, for all the standard notations and basic results we refer to [80, 81]. Throughout the thesis, G is a finite non-abelian group and $Z(G) = \{z \in G : zx = xz \,\forall\, x \in G\}$ is the *center* of G. The *centralizer* of an element x in a group G, denoted by $C_G(x)$, is defined as the set $\{y \in G : xy = yx\}$ which is a subgroup of G. Clearly, $Z(G) = \bigcap\limits_{x \in G} C_G(x)$. The conjugacy class of $x \in G$ is given by $\mathrm{Cl}_G(x) := \{gxg^{-1} : g \in G\}$. For any two elements x and y of G, $[x, y] = x^{-1}y^{-1}xy$ is called the commutator of x and y and the group generated by $K(G) = \{[x, y] : x, y \in G\}$ is called the commutator subgroup of G, denoted by G'. For any $H \leq G$ (H is a subgroup of G), we write $Z(H, G) = \{x \in H : xy = yx \,\forall\, y \in G\}$ and $Z(G, H) = \{x \in G : xy = yx \,\forall\, y \in H\}$, which implies $Z(G, G) = Z(G)$. For any element $x \in G$, we write $C_H(x) = \{y \in H : xy = yx\}$. Clearly, $Z(H, G) = \bigcap\limits_{x \in G} C_H(x)$. We write $K(H, G) = \{[x, y] : x \in H$ and $y \in G\}$ and

$[H,G] = \langle K(H,G)\rangle$. Therefore, $[G,G] = G'$. For any two subgroups H and K of G, the K-conjugacy class of $x \in H$ is given by $\mathrm{Cl}_K(x) := \{gxg^{-1} : g \in K\}$.

The concept of isoclinism between two groups has been introduced by Hall [54] in the year 1940. The isoclinism between two groups G_1 and G_2 is given in the following definition.

Definition 1.2.1. *Let $Z(G_1)$ and $Z(G_2)$ be the centers and let ${G_1}'$ and ${G_2}'$ be the commutator subgroups of the groups G_1 and G_2 respectively. If there exist an isomorphism ψ from the quotient group $\frac{G_1}{Z(G_1)}$ to $\frac{G_2}{Z(G_2)}$ and an isomorphism β from ${G_1}'$ to ${G_2}'$ then the pair (ψ,β) is said to be an isoclinism between G_1 and G_2 provided the following diagram commutes:*

$$\begin{array}{ccc}
\frac{G_1}{Z(G_1)} \times \frac{G_1}{Z(G_1)} & \xrightarrow{\psi\times\psi} & \frac{G_2}{Z(G_2)} \times \frac{G_2}{Z(G_2)} \\
\Big\downarrow a_{G_1} & & \Big\downarrow a_{G_2} \\
{G_1}' & \xrightarrow{\beta} & {G_2}'
\end{array}$$

here the maps $a_{G_1} : \frac{G_1}{Z(G_1)} \times \frac{G_1}{Z(G_1)} \to {G_1}'$ and $a_{G_2} : \frac{G_2}{Z(G_2)} \times \frac{G_2}{Z(G_2)} \to {G_2}'$ are given by $a_{G_1}(x_1Z(G_1), y_1Z(G_1)) = [x_1, y_1]$ and $a_{G_2}(x_2Z(G_2), y_2Z(G_2)) = [x_2, y_2]$ respectively. If there exists an isoclinism between G_1 and G_2 then we say that G_1 is isoclinic to G_2.

In [75, 82, 93] relative isoclinism between two pairs of groups (H_1, G_1) and (H_2, G_2), where $H_i \leq G_i$ for $i = 1, 2$ has been introduced. This coincides with the fascinating concept of isoclinism between two groups, if $H_i = G_i$ for $i = 1, 2$. A pair of isomorphisms (ϕ, ψ) is called a relative isoclinism between the pairs of groups (H_1, G_1) and (H_2, G_2), where $\phi : \frac{G_1}{Z(H_1,G_1)} \to \frac{G_2}{Z(H_2,G_2)}$ and $\psi : [H_1, G_1] \to [H_2, G_2]$, if

$$\phi\left(\frac{H_1}{Z(H_1, G_1)}\right) = \frac{H_2}{Z(H_2, G_2)} \text{ and } \psi \circ a_{(H_1,G_1)} = a_{(H_2,G_2)} \circ (\phi \times \phi),$$

where $a_{(H_i,G_i)} : \frac{H_i}{Z(H_i,G_i)} \times \frac{G_i}{Z(H_i,G_i)} \to [H_i, G_i]$ is given by

$$a_{(H_i,G_i)}((h_iZ(H_i, G_i), g_iZ(H_i, G_i))) = [h_i, g_i]$$

and $\phi \times \phi : \frac{H_1}{Z(H_1,G_1)} \times \frac{G_1}{Z(H_1,G_1)} \to \frac{H_2}{Z(H_2,G_2)} \times \frac{G_2}{Z(H_2,G_2)}$ is given by

$$(\phi \times \phi)((h_1Z(H_1, G_1), g_1Z(H_1, G_1))) = (\phi(h_1Z(H_1, G_1)), \phi(g_1Z(H_1, G_1))).$$

Thus for all $h_1 \in H_1$ and $g_1 \in G_1$ we must have $\psi([h_1, g_1]) = [h_2, g_2]$, where $g_2 \in \phi(g_1 Z(H_1, G_1))$ and $h_2 \in \phi(h_1 Z(H_1, G_1))$.

The pairs (H_1, G_1) and (H_2, G_2) are called relative isoclinic if there is a relative isoclinism between them.

1.2.1 Commuting probability of finite groups

The commuting probability of a finite group G, denoted by $\Pr(G)$, is given by

$$\Pr(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

Thus $\Pr(G)$ is the probability that a randomly chosen pair of elements of G commute. The study of $\Pr(G)$, has been initiated by Erdös and Turán [37] in the year 1968. $\Pr(G)$ has been widely studied in the last five decades. Many mathematicians have also worked on generalizations of this probability over the years. A survey on the recent generalizations of $\Pr(G)$ can be found in [17].

In [39], Erfanian et al. have considered the probability $\Pr(H, G)$ that a randomly chosen element of H commute with a randomly chosen element of G which is given by the following ratio:

$$\Pr(H, G) = \frac{|\{(x, y) \in H \times G : xy = yx\}|}{|H||G|}.$$

Clearly, if $H = G$ then $\Pr(H, G) = \Pr(G)$. Thus $\Pr(H, G)$ is a generalization of $\Pr(G)$. This probability $\Pr(H, G)$ is called the relative commuting probability of a subgroup H of G. In [79], Pournaki and Sobhani have generalized the notion of commuting probability of a finite group G by considering the following ratio:

$$\Pr_g(G) = \frac{|\{(x, y) \in G \times G : [x, y] = g\}|}{|G|^2},$$

where $g \in G$. Thus $\Pr_g(G)$ is the probability that the commutator of a randomly chosen pair of elements of G is equal to a given element g of G. Note that $\Pr_g(G) = \Pr(G)$ if $g = 1$. The probability $\Pr_g(G)$ is called g-commuting probability of G. Das and Nath [16] in 2010 have further generalized these notions and study the following ratio:

$$\Pr_g(H, K) = \frac{|\{(x, y) \in H \times K : [x, y] = g\}|}{|H||K|},$$

where $g \in G$ and H and K two subgroups of G. Thus $\Pr_g(H,K)$ is the probability that the commutator of a randomly chosen pair of elements, one from a subgroup H and another from a subgroup K, of G is equal to a given element $g \in G$. Clearly, $\Pr_g(H,K) = \Pr_g(G)$ if $H = K = G$. If $K = G$ then $\Pr_g(H,K) = \Pr_g(H,G)$ which is the relative commuting probability of a subgroup H of G when $g = 1$. We would like to mention the following useful results in this section.

Result 1.2.2. [16, Corollary 2.4] If H is normal in G then $\Pr(H,K) = \frac{k_K(H)}{|H|}$, where $k_K(H)$ is the number of K-conjugacy classes that constitute H.

Result 1.2.3. [16, Proposition 2.1] $\Pr_g(H,K) = \Pr_g^{-1}(K,H)$. However, if $g^2 = 1$, or if $g \in H \cup K$ (for example, when H or K is normal in G), we have $\Pr_g(H,K) = \Pr_g(K,H) = \Pr_g^{-1}(H,K)$.

Result 1.2.4. [16, Proposition 3.1] If $g \neq 1$ then

(a) $\Pr_g(H,K) \neq 0 \implies \Pr_g(H,K) \geq \frac{|C_H(K)||C_K(H)|}{|H||K|}$.

(b) $\Pr_g(H,K) \neq 0 \implies \Pr_g(H,K) \geq \frac{|H \cap Z(G)||C_G(H)|}{|H||G|}$.

(c) $\Pr_g(G) \neq 0 \implies \Pr_g(G) \geq \frac{3}{|G:Z(G)|^2}$.

Result 1.2.5. [16, Proposition 3.3] Let p be the smallest prime dividing $|G|$, and $g \neq 1$. Then

$$\Pr_g(H,K) \leq \frac{|H| - |C_H(K)|}{p|H|} < \frac{1}{p}.$$

Result 1.2.6. [16, equation (6)] Let $\mathrm{Irr}(G)$ be the set of all irreducible characters of G. If H is a normal subgroup of G then we have

$$\Pr_g(H,G) = \frac{1}{|G|} \sum_{\phi \in \mathrm{Irr}(G)} \langle \phi_H, \phi_H \rangle \frac{\phi(g)}{\phi(1)},$$

where ϕ_H is the restriction of $\phi \in \mathrm{Irr}(G)$ on H and $\langle \phi_H, \phi_H \rangle = \sum_{x \in H} \phi_H(x)\phi_H(x^{-1})$.

Result 1.2.7. [75, Theorem B] Let H be a subgroup of a finite nilpotent group G. If $|[G,H]| = p$, a prime (not necessarily the smallest one), and $g \in [G,H]$. Then

$$\Pr_g(H,G) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|H:Z(H,G)|}\right), & \text{if } g = 1 \\ \frac{1}{p}\left(1 - \frac{1}{|H:Z(H,G)|}\right), & \text{if } g \neq 1. \end{cases}$$

Result 1.2.8. [75, Lemma 3] Let H be a subgroup of a finite group G and p be the smallest prime dividing $|G|$. If $|[G,H]| = p$, and $g \in [G,H]$. Then

$$\Pr_g(H,G) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|H:Z(H,G)|}\right), & \text{if } g = 1 \\ \frac{1}{p}\left(1 - \frac{1}{|H:Z(H,G)|}\right), & \text{if } g \neq 1. \end{cases}$$

1.3 Notations and results from Ring Theory

Let R be a finite non-commutative ring and $Z(R) = \{z \in R : zr = rz, \forall r \in R\}$ be the *center* of R. For any element $x \in R$, the *centralizer* of x in R is a subring given by $C_R(x) := \{y \in R : xy = yx\}$. Clearly, $Z(R) = \underset{x \in R}{\cap} C_R(x)$. For any subring S of R we write $\frac{R}{S}$ to denote the additive quotient group.

Result 1.3.1. [63, Lemma 1] Let R be a non-commutative ring. Then $\frac{R}{Z(R)}$ is not cyclic.

Let $\mathrm{Cent}(R) = \{C_R(r) : r \in R\}$. Then $|\mathrm{Cent}(R)|$ gives the number of distinct centralizers in R. If $|\mathrm{Cent}(R)| = n$ then R is called n-centralizer ring. Recently, Dutta et al. (see [24, 25, 30]) have characterized n-centralizer finite rings for some n. Some of their results are given below.

Result 1.3.2. [25, Theorem 3.1] Let R be a finite 4-centralizer ring. Then the additive quotient group $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Result 1.3.3. [25, Theorem 4.1] Let R be a finite 5-centralizer ring. Then the additive quotient group $\frac{R}{Z(R)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Result 1.3.4. [25, Theorem 2.6] Let R be a non-commutative ring whose order is a power of a prime p. If $|\mathrm{Cent}(R)| = p + 2$ then the additive quotient group $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

For any two elements x and y of R, $[x,y] := xy - yx$ is called the additive commutator of x and y. Let $K(R) = \{[x,y] : x, y \in R\}$ and $[R,R]$ and $[x,R]$ for $x \in R$ denote the additive subgroups of $(R,+)$ generated by the sets $K(R)$ and $\{[x,y] : y \in R\}$ respectively. The subgroup $[R,R]$ is called the commutator subgroup of R. Let S be a subring of a finite ring R and $Z(S,R) = \{z \in S : zx = xz, \forall x \in R\}$. For any element $x \in R$, we write $C_S(x)$ to denote the set $\{y \in S : xy = yx\}$. Clearly, $Z(S,R) = \underset{x \in R}{\cap} C_S(x)$. We have the following generalizations of Result 1.3.1.

Result 1.3.5. Let R be a non-commutative ring. Then $\frac{R}{Z(S,R)}$ is not cyclic.

Let $K(S,R) = \{[x,y] : x \in S \text{ and } y \in R\}$ and $[S,R]$ be the additive subgroup of $(R,+)$ generated by the set $K(S,R)$.

Following Hall [54], Buckley et al. [13] have introduced an analogous concept of isoclinism in Ring Theory which is called $\mathbb{Z}$-isoclinism between two rings. Two rings R_1 and R_2 are said to be $\mathbb{Z}$-isoclinic if there exist additive group isomorphisms $\phi : \frac{R_1}{Z(R_1)} \to \frac{R_2}{Z(R_2)}$ and $\psi : [R_1,R_1] \to [R_2,R_2]$ such that $\psi([u,v]) = [u',v']$ whenever $\phi(u + Z(R_1)) = u' + Z(R_2)$ and $\phi(v + Z(R_1)) = v' + Z(R_2)$.

Dutta et al. [23] have further generalized the notion of $\mathbb{Z}$-isoclinism of rings as given in the following definition.

Definition 1.3.6. [23, Definition 5.1] *Let R_1 and R_2 be two rings with subrings S_1 and S_2 respectively. Then the pair of rings (S_1,R_1) is said to be $\mathbb{Z}$-isoclinic to a pair of rings (S_2,R_2) if there exist additive group isomorphisms $\phi : \frac{R_1}{Z(S_1,R_1)} \to \frac{R_2}{Z(S_2,R_2)}$ such that $\phi\left(\frac{S_1}{Z(S_1,R_1)}\right) = \frac{S_2}{Z(S_2,R_2)}$; and $\psi : [S_1,R_1] \to [S_2,R_2]$ such that $\psi([u,v]) = [u',v']$ whenever $\phi(u+Z(S_1,R_1)) = u'+Z(S_2,R_2)$ and $\phi(v+Z(S_1,R_1)) = v'+Z(S_2,R_2)$. Such a pair of mappings (ϕ,ψ) is called a $\mathbb{Z}$-isoclinism between (S_1,R_1) and (S_2,R_2).*

1.3.1 Commuting probability of finite rings

In 1976, MacHale [63] has considered commuting probability of a finite ring R, denoted by $\Pr(R)$, which is analogous to the commuting probability of finite groups. Recall that

$$\Pr(R) = \frac{|\{(a,b) \in R \times R : ab = ba\}|}{|R|^2}.$$

For any details regarding $\Pr(R)$, one may conf. [13, 22, 33, 63]. The following results are useful in our study.

Result 1.3.7. [63, Theorem 1] Let R be a finite ring. Then $\Pr(R) \leq \frac{5}{8}$ with equality if and only if $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Result 1.3.8. [63, Theorem 3] Let R be a finite ring and p be the smallest prime divisor of $|R|$. Then $\Pr(R) \leq \frac{p^2+p-1}{p^3}$. The equality holds if and only if the additive quotient group $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

Let S be a subring of R. Dutta et al. [22] have generalized $\Pr(R)$ through the ratio:

$$\Pr(S, R) = \frac{|\{(s, r) \in S \times R : sr = rs\}|}{|S||R|}.$$

Note that $\Pr(S, R)$ is the relative commuting probability of a finite ring R relative to a subring S of R and $\Pr(R, R) = \Pr(R)$.

For a fixed element $r \in R$, Dutta and Nath [35] have generalized the notion of commuting probability of a finite non-commutative ring R by considering the following ratio:

$$\Pr_r(R) = \frac{|\{(x, y) \in R \times R : [x, y] = r\}|}{|R|^2},$$

which gives the probability that the commutator of a randomly chosen pair of elements of R equals r. This probability $\Pr_r(R)$ is called r-commuting probability of R. Notice that $\Pr_r(R) = \Pr(R)$ if $r = 0$. Results on $\Pr_r(R)$ that are useful in subsequent chapters are listed below.

Result 1.3.9. [35, Theorem 2.9] Let R be a finite non-commutative ring. If p is the smallest prime dividing $|R|$ and $r \neq 0$ then

$$\Pr_r(R) \leq \frac{|R| - |Z(R)|}{p|R|} < \frac{1}{p}.$$

Result 1.3.10. [35, Theorem 2.10] Let R be a finite non-commutative ring and $r \in K(R)$. If $r \neq 0$ then

$$\Pr_r(R) \geq \frac{6}{|R : Z(R)|^2}.$$

Result 1.3.11. [35, Lemma 3.2] Let R be a finite non-commutative ring and $|[R, R]| = p$ be any prime. Then

$$\Pr_r(R) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|R:Z(R)|}\right), & \text{if } r = 0 \\ \frac{1}{p}\left(1 - \frac{1}{|R:Z(R)|}\right), & \text{if } r \neq 0. \end{cases}$$

Dutta and Nath [34] have further generalized the notion of r-commuting probability and introduced relative r-commuting probability of R with respect to a subring S which is given by the ratio:

$$\Pr_r(S, R) = \frac{|\{(x, y) \in S \times R : [x, y] = r\}|}{|S||R|}. \tag{1.3.1}$$

Note that $\Pr_r(S, R) = \Pr_r(R)$ if $S = R$. If $r = 0$ then $\Pr_r(S, R) = \Pr(S, R)$. We use the following results on $\Pr_r(S, R)$ in the subsequent chapters.

Result 1.3.12. [34, Corollary 2] Let S be a subring of a finite non-commutative ring R. If $|[S, R]| = p$, a prime, then

$$\Pr_r(S, R) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|S:Z(S,R)|}\right), & \text{if } r = 0 \\ \frac{1}{p}\left(1 - \frac{1}{|S:Z(S,R)|}\right), & \text{if } r \neq 0. \end{cases}$$

Result 1.3.13. [34, Proposition 3] Let S be a subring of a finite ring R. If p is the smallest prime dividing $|R|$ and $r \neq 0$ then

$$\Pr_r(S, R) \leq \frac{|S| - |Z(S, R)|}{p|S|} < \frac{1}{p}.$$

One more generalization of commuting probability of a finite ring, which has been introduced by Dutta and Nath [34], is the generalized r-commuting probability of R with respect to the additive subgroups S and K of R which is defined as follows:

$$\Pr_r(S, K) := \frac{|\{(x, y) \in S \times K : [x, y] = r\}|}{|S||K|},$$

where $r \in R$. Thus $\Pr_r(S, K)$ is the probability that the commutator of a randomly chosen pair of elements $(x, y) \in S \times K$ equals a given element $r \in R$. We have

$$\Pr_r(S, K) = \begin{cases} \Pr(S, R), & \text{if } r = 0 \text{ and } K = R \\ \Pr_r(S, R), & \text{if } K = R \\ \Pr_r(S), & \text{if } S = K \\ \Pr(R), & \text{if } r = 0 \text{ and } S = K = R. \end{cases} \tag{1.3.2}$$

We conclude this section with the following result on $\Pr_r(S, K)$.

Result 1.3.14. [33, Proposition 3.1] Let S and K be two additive subgroups of R. If $r \neq 0$, then

(a) $\Pr_r(S, K) \geq \frac{|Z(S,K)||Z(K,S)|}{|S||K|}$.

(b) If $S \subseteq K$ then $\Pr_r(S, K) \geq \frac{2|Z(S,K)||Z(K,S)|}{|S||K|}$.

1.4 Non-commuting graph and its generalizations

In this section, we list certain results on non-commuting graphs of finite groups and rings and their generalizations.

1.4.1 Non-commuting graph of a finite group

The *non-commuting graph* of a finite non-abelian group G, denoted by Γ_G, is a simple undirected graph with $G \setminus Z(G)$ as the vertex set and two distinct vertices x and y are adjacent whenever $xy \neq yx$. After the works of Erdös and Neumann [76], non-commuting graph and its generalizations have been studied extensively by many mathematicians (see [1, 4, 9, 15, 43, 66, 69, 70, 71, 92, 93, 94]). For instance, characterization of finite non-abelian groups with isomorphic non-commuting graphs has been discussed in [1, 14, 58, 66], Spectrum, Laplacian spectrum and their corresponding energies of Γ_G have been computed in [26, 32, 40, 44, 45, 46], expressions for various topological indices of Γ_G have been obtained in [59, 61] and a characterization of finite groups through domination number of Γ_G can be found in [95]. Graph theoretic invariants such as clique number, vertex chromatic number, independent number etc. for non-commuting graphs of dihedral groups have been investigated in [92]. People have computed various spectrum and energies of Γ_G in order to answer the following questions.

Question 1.4.1. Which finite non-abelian groups give integral, L-integral and Q-integral non-commuting graphs?

Question 1.4.2. Are there any finite non-abelian group G such that Γ_G is hypoenergetic, hyperenergetic, L-hyperenergetic and Q-hyperenergetic?

Question 1.4.3. Which finite non-abelian groups satisfy the following inequalities?

(a) $E(\Gamma_G) \leq LE(\Gamma_G)$.

(b) $LE(\Gamma_G) \leq LE^+(\Gamma_G)$.

The following results on Γ_G are useful in our study.

Result 1.4.4. [40, Corollary 4.1.7 and equation (4.3.e)] Consider the dihedral group $D_{2m} := \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1}\rangle$.

(a) If m is odd then

$$E(\Gamma_{D_{2m}}) = (m-1) + \sqrt{(m-1)(5m-1)} \text{ and}$$

$$LE(\Gamma_{D_{2m}}) = \frac{2m(m-1)(m-2) + 2m(2m-1)}{2m-1}.$$

(b) If m is even then

$$E(\Gamma_{D_{2m}}) = (m-2) + \sqrt{(m-2)(5m-2)} \text{ and}$$

$$LE(\Gamma_{D_{2m}}) = \frac{m(m-2)(m-4) + 2m(m-1)}{m-1}.$$

Result 1.4.5. ([40, Result 1.2.16(d)] and [32, Proposition 3.2]) Consider the quasidihedral group $QD_{2^n} := \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1}\rangle$. Then

$$E(\Gamma_{QD_{2^n}}) = (2^{n-1} - 2) + 2\sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)} \text{ and}$$

$$LE(\Gamma_{QD_{2^n}}) = \frac{2^{3n-3} - 2^{2n} + 3 \times 2^n}{2^{n-1} - 1}.$$

Result 1.4.6. [40, Corollary 4.1.6 and equation (4.3.d)] Consider the group $M_{2rs} := \langle a, b : a^r = b^{2s} = 1, bab^{-1} = a^{-1}\rangle$.

(a) If m is odd then

$$E(\Gamma_{M_{2rs}}) = s(r-1) + s\sqrt{(r-1)(5r-1)} \text{ and}$$

$$LE(\Gamma_{M_{2rs}}) = \frac{s}{2r-1}\left(2r^3s - 6r^2s + 4rs + 4r^2 - 2r\right).$$

(b) If m is even then

$$E(\Gamma_{M_{2rs}}) = s(r-2) + s\sqrt{(r-2)(5r-2)} \text{ and}$$

$$LE(\Gamma_{M_{2rs}}) = \frac{s}{r-1}\left(r^3s - 6r^2s + 8rs + 2r^2 - 2r\right).$$

Result 1.4.7. [40, Corollary 4.1.8 and equation (4.3.f)] Consider the dicyclic group $Q_{4n} := \langle x, y : x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1}\rangle$. Then

$$E(\Gamma_{Q_{4n}}) = 2\left((n-1) + \sqrt{(n-1)(5n-1)}\right) \text{ and}$$

$$LE(\Gamma_{Q_{4n}}) = \frac{8n(n-1)(n-2) + 4n(2n-1)}{2n-1}.$$

Result 1.4.8. [40, Corollary 4.1.9 and equation (4.3.c)] Consider the group $U_{6n} := \langle x, y : x^{2n} = y^3 = 1, x^{-1}yx = y^{-1}\rangle$. Then $E(\Gamma_{U_{6n}})) = 2n(1+\sqrt{7})$ and $LE(\Gamma_{U_{6n}})) = \frac{12n^2+30n}{5}$.

Result 1.4.9. [40, Theorem 4.1.5] Let G be a finite group such that $\frac{G}{Z(G)}$ is isomorphic to the dihedral group D_{2m} $(m \geq 3)$ and $|Z(G)| = n$. Then

$$E(\Gamma_G) = n\left((m-1) + \sqrt{(m-1)(5m-1)}\right) \text{ and}$$

$$LE(\Gamma_G) = \frac{n}{2m-1}\left((2m^3 - 6m^2 + 4m)n + 4m^2 - 2m\right).$$

Result 1.4.10. ([32, Theorem 2.2] and [40, Theorem 4.1.1(c)]) Let G be a finite group such that $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then $E(\Gamma_G) = LE(\Gamma_G) = 2p(p-1)|Z(G)|$. In particular, if G is a non-abelian group of order p^3 then $E(\Gamma_G) = LE(\Gamma_G) = 2p^2(p-1)$.

Result 1.4.11. [32, Theorem 2.1] Let G be a finite group such that $\frac{G}{Z(G)}$ is isomorphic to the Suzuki group $Sz(2) := \langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2\rangle$. Then $LE(\Gamma_G) = \left(\frac{120}{19}n + 30\right)n$, where $|Z(G)| = n$.

Result 1.4.12. ([32, Proposition 3.5] and [40, Theorem 4.1.3]) If G denotes the Hanaki group $A(n, \mathcal{V})$ (see page 59) then

$$E(\Gamma_G) = LE(\Gamma_G) = 2^{2n+1} - 2^{n+2}.$$

Result 1.4.13. ([32, Proposition 3.6] and [40, Theorem 4.1.4]) If G denotes the Hanaki group $A(n, p)$ (see page 59) then

$$E(\Gamma_G) = LE(\Gamma_G) = 2(p^{3n} - p^{2n}).$$

Result 1.4.14. [40, Theorem 4.1.1(b) and Theorem 4.3.2] Consider the group $SD_{8n} := \langle a, b : a^{4n} = b^2 = 1, bab^{-1} = a^{2n-1}\rangle$.

(a) If n is odd then

$$E(\Gamma_{SD_{8n}}) = 4(n-1) + 4\sqrt{(n-1)(5n-1)} \text{ and } LE(\Gamma_{SD_{8n}}) = \frac{8n(4n^2-10n+7)}{2n-1}.$$

(b) If n is even then

$$E(\Gamma_{SD_{8n}}) = 2(2n-1) + 2\sqrt{(2n-1)(10n-1)} \text{ and } LE(\Gamma_{SD_{8n}}) = \frac{8n(8n^2-8n+3)}{4n-1}.$$

Result 1.4.15. [40, Theorem 4.1.1(a) and Theorem 4.3.1] Consider the group $V_{8n} := \langle a, b : a^{2n} = b^4 = 1, b^{-1}ab^{-1} = bab = a^{-1}\rangle$, where n is odd. Then

$$E(\Gamma_{V_{8n}}) = 2(2n-1) + 2\sqrt{(2n-1)(10n-1)} \text{ and } LE(\Gamma_{V_{8n}}) = \frac{8n(8n^2-8n+3)}{4n-1}.$$

Result 1.4.16. [40, equation (4.1.e) and equation (4.3.b)] Consider the Frobenious group $F_{p,q} := \langle a, b : a^p = b^q = 1, b^{-1}ab = a^u\rangle$ of order pq, where p and q are two primes such that $q|(p-1)$ and u is an integer such that $\bar{u} \in \mathbb{Z}_p \setminus \{\bar{0}\}$ having order q. Then

$$E(\Gamma_G) = \alpha + \sqrt{\alpha^2 + 4p\alpha} \text{ and } LE(\Gamma_G) = \frac{2p^2\alpha + 2p(q-1)^2}{pq-1},$$

where $\alpha = (p-1)(q-1)$.

Result 1.4.17. [40, pages 75–81] Let G be a finite group.

(a) If $G \cong A_4$ then $\Gamma_{A_4} = K_{4.2,1.3}$, $E(\Gamma_{A_4}) = 6 + 2\sqrt{33}$ and $LE(\Gamma_{A_4}) = \frac{224}{11}$.

(b) If $G \cong A_5$ then $E(\Gamma_{A_5}) \approx 111.89$ and $LE(\Gamma_{A_5}) = \frac{8580}{59}$.

(c) If $G \cong Sz(2)$ then $LE(\Gamma_{Sz(2)}) = \frac{690}{19}$.

(d) If $G \cong SL(2,3)$ then $\Gamma_{SL(2,3)} = K_{3.2,4.4}$, $E(\Gamma_{SL(2,3)}) = 16 + 8\sqrt{7}$ and $LE(\Gamma_{SL(2,3)})) = \frac{552}{11}$.

(e) If $G \cong S_4$ then $E(\Gamma_{S_4}) \approx 35.866 + 4\sqrt{5}$ and $LE(\Gamma_{S_4}) = \frac{1072}{23} + 4\sqrt{13}$.

(f) If $G \cong D_6 \times \mathbb{Z}_3$ then $\Gamma_G = K_{3.3,1.6}$, $E(\Gamma_G) = 6 + 6\sqrt{7}$ and $LE(\Gamma_G) = \frac{594}{15}$.

(g) If $G \cong A_4 \times \mathbb{Z}_2$ then $\Gamma_G = K_{4.4,1.6}$, $E(\Gamma_G) = 12 + 4\sqrt{33}$ and $LE(\Gamma_G) = \frac{544}{11}$.

Result 1.4.18. [1, Proposition 2.3] The non-commuting graph of a finite group G is planar if and only if $G \cong D_6, D_8, Q_8$.

Note that the complement of Γ_G, known as the commuting graph of G, has also been well-studied (see [7, 10, 56, 57, 68, 78]). It is worth mentioning that Afkhami et al. [4] and Das et al. [28] have classified all finite non-abelian groups whose commuting graphs are planar or toroidal in the years 2015 and 2016 respectively. We conclude this section by recalling those results.

Result 1.4.19. [4, Theorem 2.2] The commuting graph of a finite group G is planar if and only if $G \cong D_6, D_8, D_{10}, D_{12}, Q_8, Q_{12}, \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, A_4, A_5, S_4, SL(2,3), Sz(2)$, $\mathcal{M}_{16} := \langle a, b : a^8 = b^2 = 1, bab = a^5 \rangle$, $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 := \langle a, b : a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$, $D_8 * \mathbb{Z}_4 := \langle a, x, y : a^4 = y^4 = x^2 = 1, a^2 = y^2, xax = a^{-1}, ay = ya, xy = yx \rangle$, $SG(16,3) := \langle a, b : a^4 = b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1} \rangle$.

Result 1.4.20. [28, Theorem 3.3] The commuting graph of a finite group G is toroidal if and only if $G \cong D_{14}, D_{16}, Q_{16}, QD_{16}$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3 := \langle a, b : a^7 = b^3 = 1, bab^{-1} = a^2 \rangle$, $D_6 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$.

1.4.2 Relative non-commuting graph and g-noncommuting graph of a finite group

In 2013, Tolue and Erfanian [93] have generalized the non-commuting graph of G by considering *relative non-commuting graph* for a given subgroup H of G. In particular, they have considered a simple undirected graph, denoted by $\Gamma_{H,G}$, whose vertex set is $G \setminus Z(H,G)$ and two distinct vertices a and b are adjacent if $a \in H$ or $b \in H$ and $ab \neq ba$. The following result is useful in our study.

Result 1.4.21. [93, Theorem 4.5] Suppose that $H_i \leq G_i$ for $i = 1, 2$ and the pairs (H_1, G_1) and (H_2, G_2) are relative isoclinic. If $|Z(H_1, G_1)| = |Z(H_2, G_2)|$ and $|Z(G_1)| = |Z(G_2)|$ then $\Gamma_{H_1,G_1} \cong \Gamma_{H_2,G_2}$.

In 2014, Tolue et al. [94] have introduced another generalization of non-commuting graph called the *g-noncommuting graph*, denoted by Γ_G^g, for any given element $g \in G$. Recall that for any non-abelian group G and a given element g in G, the g-noncommuting graph of G is a simple undirected graph with vertex set G and two distinct vertices x and y are adjacent if $[x, y] \neq g$ and g^{-1}. For the last five years, Γ_G^g of a finite group G for any given element $g \in G$ has been studied (see [69, 70, 71, 94]). Some results on Γ_G^g that we use in subsequent chapters are listed below.

Result 1.4.22. [94, Theorem 2.16] Let G and H be two finite isoclinic groups with $|Z(G)| = |Z(H)|$. If (ϕ, ψ) is an isoclinism between G and H then $\Gamma_G^g \cong \Gamma_H^{\psi(g)}$.

Result 1.4.23. [94, Proposition 2.14] Let G be a finite non-abelian group. Then

(a) for a non-identity element $g \in G'$ such that $g^2 \neq 1$ we have

$$|e(\Gamma_G^g)| = \frac{|G|^2 - |G| - 2|G|^2 \operatorname{Pr}_g(G)}{2}.$$

(b) for a non-identity element $g \in G'$ such that $g^2 = 1$ we have

$$|e(\Gamma_G^g)| = \frac{|G|^2 - |G| - |G|^2 \operatorname{Pr}_g(G)}{2}.$$

(c) for $g \notin G'$ we have $|e(\Gamma_G^g)| = \frac{|G|^2 - |G|}{2}$.

The *induced subgraph* of Γ_G^g with vertex set $G \setminus Z(G)$ is denoted by Δ_G^g . Note that Δ_G^g coincides with the non-commuting graph of G if $g = 1$. Also, if $g \notin K(G)$ then Δ_G^g is a complete graph. Therefore while studying Δ_G^g it is considered that $1 \neq g \in K(G)$. In the last five years, Δ_G^g has also been studied. Some results on Δ_G^g that are useful in the subsequent chapters are listed below.

Result 1.4.24. [94, Theorem 2.5] If G is a finite non-abelian group then Δ_G^g is not a tree.

Result 1.4.25. [70, Lemma 2.4] Consider the graph Δ_G^g. Let $x \in G \setminus Z(G)$.

(a) If $g^2 \neq 1$, then $\deg(x) = |G| - |Z(G)| - \epsilon|C_G(x)| - 1$, where $\epsilon = 1$ if x is conjugate to xg or xg^{-1}, but not to both, and $\epsilon = 2$ if x is conjugate to xg and xg^{-1}.

(b) If $g^2 = 1$ and $g \neq 1$ then $\deg(x) = |G| - |Z(G)| - |C_G(x)| - 1$, whenever xg is conjugate to x.

(c) If xg and xg^{-1} are not conjugate to x then $\deg(x) = |G| - |Z(G)| - 1$.

Result 1.4.26. [69, Theorem 4] $\Delta^g_{D_{2n}}$ is connected if and only if $n \neq 3$, 4 and 6.

Result 1.4.27. [70, Lemma 3.1] Let g be a non-central element of G.

(a) If $g^2 = 1$ then $\mathrm{diam}(\Delta^g_G) = 2$.

(b) If $g^2 \neq 1$ and $g^3 \neq 1$ then $\mathrm{diam}(\Delta^g_G) \leq 3$.

Result 1.4.28. [71, Theorem 2.1] Let g be a non-central element of a finite non-abelian group G such that $o(g) \neq 3$. Then $\mathrm{diam}(\Delta^g_G) = 2$.

1.4.3 Non-commuting graph of a finite ring

The *non-commuting graph* Γ_R of a finite non-commutative ring R is defined as a graph with vertex set $R \setminus Z(R)$ and two distinct vertices x and y are adjacent whenever $xy \neq yx$. Erfanian, Khashyarmanesh and Nafar [38] have begun the study of non-commuting graphs of finite rings. Properties of this graph have been studied in [21, 38].

Result 1.4.29. [38, Theorem 2.1] Let R be a non-commutative ring. Then $\mathrm{diam}(\Gamma_R) \leq 2$ and $\mathrm{girth}(\Gamma_R) = 3$.

The *relative non-commuting graph of a ring R relative to a subring S*, denoted by $\Gamma_{S,R}$, is a simple undirected graph whose vertex set is $R \setminus Z(S, R)$ and two distinct vertices a and b are adjacent if $a \in S$ or $b \in S$ and $ab \neq ba$. This graph has been introduced by Dutta et al. and studied in [20, 23].

The complement of non-commuting graph, known as commuting graph, of a finite non-commutative ring has been considered in [18, 27, 41, 77]. Results on $\overline{\Gamma_R}$ which will be used in subsequent chapters are listed below.

Result 1.4.30. [40, Result 1.3.8] Let R be a finite ring and let p, q be primes.

(a) If $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then $\overline{\Gamma_R} = (p+1)K_{(p-1)\eta}$, where $\eta = |Z(R)|$.

(b) Let $|R| = p^2q$ and $Z(R) = \{0\}$.

(i) If "$t \in \{p, q, p^2, pq\}$ and $(t-1)$ divides (p^2q-1)" then $\overline{\Gamma_R} = \frac{p^2q-1}{t-1}K_{t-1}$ and L-spec$(\overline{\Gamma_R}) = \left\{(0)^{\frac{p^2q-1}{t-1}}, (t-1)^{\frac{(p^2q-1)(t-2)}{t-1}}\right\}$.

(ii) If $(p-1)l_1 + (q-1)l_2 + (p^2-1)l_3 + (pq-1)l_4 = p^2q - 1$ then $\overline{\Gamma_R} = l_4K_{pq-1} \cup l_3K_{p^2-1} \cup l_2K_{q-1} \cup l_1K_{p-1}$ and L-spec$(\overline{\Gamma_R}) = \{(0)^{l_1+l_2+l_3+l_4}, (p-1)^{(p-2)l_1}, (q-1)^{(q-2)l_2}, (p^2-1)^{(p^2-2)l_3}, (pq-1)^{(pq-2)l_4}\}$.

Result 1.4.31. ([96, Theorem 2.5], [40, Result 1.3.7] and [97, Theorem 2.12]) Suppose that R is a ring with unity.

(a) Let $|R| = p^4$.

(i) If $Z(R)$ has p elements then $\overline{\Gamma_R} = (1+p+p^2)K_{p(p-1)}$ or $l_1K_{p(p-1)} \cup l_2K_{p(p^2-1)}$ and L-spec$(\overline{\Gamma_R}) = \left\{(0)^{p^2+p+1}, (p^2-p)^{p^4-p^2-2p-1}\right\}$ or $\left\{(0)^{l_1+l_2}, (p^2-p)^{(p^2-p-1)l_1}, (p^3-p)^{(p^3-p-1)l_2}\right\}$ respectively, where $p^2+p+1 = l_1 + (p+1)l_2$.

(ii) If $Z(R)$ has p^2 elements then $\overline{\Gamma_R} = (p+1)K_{(p^3-p^2)}$.

(b) Let $|R| = p^5$ and $Z(R)$ is not a field.

(i) If $Z(R)$ has p^2 elements then $\overline{\Gamma_R} = (1+p+p^2)K_{p^2(p-1)}$ or $l_1K_{p^2(p-1)} \cup l_2K_{p^2(p^2-1)}$ and L-spec$(\overline{\Gamma_R}) = \left\{(0)^{p^2+p+1}, (p^3-p^2)^{p^5-2p^2-p-1}\right\}$ or $\left\{(0)^{l_1+l_2}, (p^3-p^2)^{(p^3-p^2-1)l_1}, \ (p^4-p^2)^{(p^4-p^2-1)l_2}\right\}$ respectively, where $p^2+p+1 = l_1 + (p+1)l_2$.

(ii) If $Z(R)$ has p^3 elements then $\overline{\Gamma_R} = (p+1)K_{(p^4-p^3)}$.

(c) If $|R| = p^3q$ and $Z(R)$ has pq elements then $\overline{\Gamma_R} = (p+1)K_{p^2q-pq}$ and L-spec$(\overline{\Gamma_R}) = \left\{(0)^{p+1}, (p^2q-pq)^{(p+1)(p^2q-pq-1)}\right\}$.

(d) Let $|R| = p^3q$ and $Z(R)$ has p^2 elements. Then

(i) $\overline{\Gamma_R} = \frac{pq-1}{p-1}K_{p^3-p^2}$ and L-spec$(\overline{\Gamma_R}) = \left\{(0)^{\frac{pq-1}{p-1}}, (p^3-p^2)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}\right\}$ whenever $(p-1)$ divides $(pq-1)$.

(ii) $\overline{\Gamma_R} = \frac{pq-1}{q-1} K_{p^2q-p^2}$ and L-spec$(\overline{\Gamma_R}) = \left\{(0)^{\frac{pq-1}{q-1}}, (p^2q-p^2)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}\right\}$ whenever $(q-1)$ divides $(pq-1)$.

(iii) $\overline{\Gamma_R} = l_1 K_{p^3-p^2} \cup l_2 K_{p^2q-p^2}$ and L-spec$(\overline{\Gamma_R}) = \left\{(0)^{l_1+l_2}, (p^3-p^2)^{l_1(p^3-p^2-1)}, (p^2q-p^2)^{l_2(p^2q-p^2-1)}\right\}$ whenever $pq-1 = (p-1)l_1 + (q-1)l_2$.

A class of non-commutative rings, which has been introduced by Erfanian et al. in [38], is referred to as CC-ring if the centralizers $C_R(y)$ are commutative whenever $y \in R \setminus Z(R)$. We conclude this section with the following result.

Result 1.4.32. [27, page 3] If $S_1, S_2, \ldots, S_n$ are the non-identical centralizers of $s \in R \setminus Z(R)$, where R is a finite CC-ring and $|Z(R)| = \eta$, then $\overline{\Gamma_R} = \bigcup\limits_{i=1}^{n} K_{|S_i|-\eta}$ and L-spec$(\overline{\Gamma_R})$ $= \{(0)^n, (|S_1|-\eta)^{|S_1|-\eta-1}, \ldots, (|S_n|-\eta)^{|S_n|-\eta-1}\}$.