Chapter 2

Various energies of non-commuting graphs of finite groups

Spectral aspects of non-commuting graphs of finite groups have grabbed attention of numerous mathematicians. In [46], Ghorbani et al. have calculated spectrum of Γ_G for certain groups. The energy of the same has later been explored by Ghorbani and Gharavi–Alkhansari in [45]. In [26], Dutta et al. have computed the Laplacian spectrum of Γ_G following which Dutta and Nath [32] have worked on Laplacian energy of the same. Signless Laplacian spectrum and energy of Γ_G have not yet computed. In [40] energy of Γ_G for several classes of finite groups has also been computed and has verified Conjecture 1.1.5 for the non-commuting graphs of those groups. At present, [3] is the only paper where Abdussakir et al. have investigated the Signless Laplacian spectrum of non-commuting graphs of dihedral groups. However, various spectra and energies (including Signless Laplacian spectrum and energy) of the complement of Γ_G, known as commuting graph of G, have already been computed in [19, 28, 29, 31, 36, 40, 84].

In this chapter, we compute Signless Laplacian spectrum and energy of Γ_G for various families of finite non-abelian groups and answer Questions 1.4.1–1.4.3 up to some extent. We compare various energies of Γ_G and show that Γ_G satisfies Conjecture 1.1.5 for almost all the groups considered in this chapter. Using energies of Γ_G for various classes of finite groups we determine whether they are hyperenergetic or hypoenergetic.

We disprove Conjecture 1.1.7 by considering non-commuting graphs of finite groups (see Theorem 2.5.3(b)). We also determine finite groups such that their non-commuting graphs are Q-hyperenergetic and L-hyperenergetic. This chapter is based on our paper [88] submitted for publication.

2.1 $\frac{G}{Z(G)}$ is isomorphic to D_{2m}

Here, we primarily compute the Signless Laplacian spectrum and Signless Laplacian energy of Γ_G, where G is isomorphic to D_{2m}, QD_{2^n}, M_{2rs}, Q_{4n} and U_{6n}. Further, we compare different energies of Γ_G and look into the hyper- and hypo-energetic properties of Γ_G for each of the above-mentioned groups. The energy and Laplacian energy of Γ_G for each of the aforementioned groups have already been determined and it is already noted in Chapter 1.

2.1.1 The dihedral groups, D_{2m}

We consider $D_{2m} := \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1}\rangle$, the dihedral groups of order $2m$ (where $m > 2$). Results regarding different energies of non-commuting graphs of D_{2m} are given below.

Theorem 2.1.1. *Let G be isomorphic to D_{2m}, where m is odd. Then*
$$\text{Q-spec}(\Gamma_{D_{2m}}) = \left\{(m)^{m-2}, (2m-3)^{m-1}, \left(\tfrac{4m-3+\sqrt{8m^2-16m+9}}{2}\right)^1, \left(\tfrac{4m-3-\sqrt{8m^2-16m+9}}{2}\right)^1\right\}$$

and
$$LE^+(\Gamma_{D_{2m}}) = \begin{cases} \frac{9}{5} + \sqrt{33}, & \text{if } m = 3 \\ \frac{2m^3-10m^2+12m-3}{2m-1} + \sqrt{8m^2-16m+9}, & \text{if } m \geq 5. \end{cases}$$

Proof. If $G \cong D_{2m}$ and m is odd then $|v(\Gamma_{D_{2m}})| = 2m-1$ and $\Gamma_{D_{2m}} = K_{m.1,1.(m-1)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_{D_{2m}}}(x) &= \prod_{i=1}^{2}(x-(2m-1)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(2m-1)+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(2m-1)+2p_i}\right) \\ &= (x-(2m-2))^0(x-m)^{m-2}(x-2m+3)^m(x-1)\left(1-\frac{m}{x-2m+3}-\frac{m-1}{x-1}\right) \end{aligned}$$

$$=(x-m)^{m-2}(x-(2m-3))^{m-1}(x^2-(4m-3)x+2m^2-2m).$$

Thus $\text{Q-spec}(\Gamma_{D_{2m}}) = \left\{(m)^{m-2}, (2m-3)^{m-1}, \left(\frac{4m-3+\sqrt{8m^2-16m+9}}{2}\right)^1, \left(\frac{4m-3-\sqrt{8m^2-16m+9}}{2}\right)^1\right\}$.

Number of edges in $\overline{\Gamma_{D_{2m}}}$ is $\frac{m^2-3m+2}{2}$. Thus, $|e(\Gamma_{D_{2m}})| = \frac{(2m-1)(2m-1-1)}{2} - \frac{m^2-3m+2}{2}$ $= \frac{3m(m-1)}{2}$. Now,

$$\left|m - \frac{2|e(\Gamma_{D_{2m}})|}{|v(\Gamma_{D_{2m}})|}\right| = \left|\frac{-m(m-2)}{2m-1}\right| = \frac{m(m-2)}{2m-1},$$

$$\left|2m-3 - \frac{2|e(\Gamma_{D_{2m}})|}{|v(\Gamma_{D_{2m}})|}\right| = \left|\frac{m^2-5m+3}{2m-1}\right| = \begin{cases} \frac{3}{5}, & \text{if } m = 3 \\ \frac{m^2-5m+3}{2m-1}, & \text{if } m \geq 5, \end{cases}$$

$$\begin{aligned}\left|\frac{\left(4m-3+\sqrt{8m^2-16m+9}\right)}{2} - \frac{2|e(\Gamma_{D_{2m}})|}{|v(\Gamma_{D_{2m}})|}\right| =& \left|\frac{\sqrt{8m^2-16m+9}}{2} + \frac{4m^2-8m+6}{2(4m-2)}\right| \\ =& \frac{\sqrt{8m^2-16m+9}}{2} + \frac{4m^2-8m+6}{2(4m-2)}\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\left(4m-3-\sqrt{8m^2-16m+9}\right)}{2} - \frac{2|e(\Gamma_{D_{2m}})|}{|v(\Gamma_{D_{2m}})|}\right| =& \left|\frac{-\sqrt{8m^2-16m+9}}{2} + \frac{4m^2-8m+6}{2(4m-2)}\right| \\ =& \frac{\sqrt{8m^2-16m+9}}{2} - \frac{4m^2-8m+6}{2(4m-2)}.\end{aligned}$$

Therefore, for $m = 3$ we have $LE^+(\Gamma_{D_{2m}}) = \frac{9}{5} + \sqrt{33}$. For $m \geq 5$ we have

$$\begin{aligned}LE^+(\Gamma_{D_{2m}}) =& (m-2) \times \frac{m(m-2)}{2m-1} + (m-1) \times \frac{m^2-5m+3}{2m-1} \\ &+ \frac{1}{2}\left(\sqrt{8m^2-16m+9} + m - \frac{3}{2} + \frac{3}{4m-2}\right) \\ &+ \frac{1}{2}\left(\sqrt{8m^2-16m+9} - m + \frac{3}{2} - \frac{3}{4m-2}\right)\end{aligned}$$

and the result follows on simplification. □

Theorem 2.1.2. *Let G be isomorphic to D_{2m}, where m is even. Then*

$$\text{Q-spec}(\Gamma_{D_{2m}}) = \Big\{(2m-4)^{\frac{m}{2}}, (m)^{m-3}, (2m-6)^{\frac{m}{2}-1}, \left(2m-3+\sqrt{2m^2-8m+9}\right)^1, \left(2m-3-\sqrt{2m^2-8m+9}\right)^1\Big\}$$

and

$$LE^+(\Gamma_{D_{2m}}) = \begin{cases} \frac{m^3-4m^2+12}{2m-2} + 2\sqrt{2m^2-8m+9}, & \text{if } 4 \le m \le 8 \\ \frac{m^3-8m^2+16m-6}{m-1} + 2\sqrt{2m^2-8m+9}, & \text{if } m \ge 10. \end{cases}$$

Proof. If $G \cong D_{2m}$ and m is even then $|v(\Gamma_{D_{2m}})| = 2m-2$ and $\Gamma_{D_{2m}} = K_{\frac{m}{2}.2, 1.(m-2)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_{D_{2m}}}(x) &= \prod_{i=1}^{2}(x-(2m-2)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(2m-2)+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(2m-2)+2p_i}\right) \\ &= (x-(2m-4))^{\frac{n}{2}}(x-m)^{m-3}(x-2m+6)^{\frac{m}{2}}(x-2)\left(1-\frac{m}{x-2m+6}-\frac{m-2}{x-2}\right) \\ &= (x-(2m-4))^{\frac{m}{2}}(x-m)^{m-3}(x-(2m-6))^{\frac{m}{2}-1}(x^2-(4m-6)x+2m^2-4m). \end{aligned}$$

Thus $\text{Q-spec}(\Gamma_{D_{2m}}) = \Big\{(2m-4)^{\frac{m}{2}}, (m)^{m-3}, (2m-6)^{\frac{m}{2}-1}, \left(2m-3+\sqrt{2m^2-8m+9}\right)^1, \left(2m-3-\sqrt{2m^2-8m+9}\right)^1\Big\}$.

Number of edges in $\overline{\Gamma_{D_{2m}}}$ is $\frac{m^2-4m+6}{2}$ and so $|e(\Gamma_{D_{2m}})| = \frac{(2m-2)(2m-2-1)}{2} - \frac{m^2-4m+6}{2} = \frac{3m(m-2)}{2}$. Now,

$$\left|2m-4-\frac{2|e(\Gamma_{D_{2m}})|}{|v(\Gamma_{D_{2m}})|}\right| = \left|\frac{(m-2)(m-4)}{2m-2}\right| = \frac{(m-2)(m-4)}{2m-2},$$

$$\left|m-\frac{2|e(\Gamma_{D_{2m}})|}{|v(\Gamma_{D_{2m}})|}\right| = \left|\frac{(-m^2+4m)}{2m-2}\right| = \frac{(m^2-4m)}{2m-2},$$

$$\left|2m-6-\frac{2|e(\Gamma_{D_{2m}})|}{|v(\Gamma_{D_{2m}})|}\right| = \left|\frac{(m^2-10m+12)}{2m-2}\right| = \begin{cases} \frac{(-m^2+10m-12)}{2m-2}, & \text{if } m \le 8 \\ \frac{(m^2-10m+12)}{2m-2}, & \text{if } m \ge 10, \end{cases}$$

$$\left|2m-3+\sqrt{2m^2-8m+9}-\frac{2|e(\Gamma_{D_{2m}})|}{|v(\Gamma_{D_{2m}})|}\right| = \left|\sqrt{2m^2-8m+9}+\frac{m^2-4m+6}{2m-2}\right|$$
$$=\sqrt{2m^2-8m+9}+\frac{m^2-4m+6}{2m-2}$$

and

$$\left|2m-3-\sqrt{2m^2-8m+9}-\frac{2|e(\Gamma_{D_{2m}})|}{|v(\Gamma_{D_{2m}})|}\right| = \left|-\sqrt{2m^2-8m+9}+\frac{m^2-4m+6}{2m-2}\right|$$
$$=\sqrt{2m^2-8m+9}-\frac{m^2-4m+6}{2m-2}.$$

Therefore, for $4 \leq m \leq 8$, we have

$$\begin{aligned} LE^+(\Gamma_{D_{2m}}) =& \frac{m}{2}\times\frac{(m-2)(m-4)}{2m-2}+(m-3)\times\frac{(m^2-4m)}{2m-2} \\ &+\left(\frac{m}{2}-1\right)\times\frac{-(m^2-10m+12)}{2m-2}+\sqrt{2m^2-8m+9}+\frac{m^2-4m+6}{2m-2} \\ &+\sqrt{2m^2-8m+9}-\frac{m^2-4m+6}{2m-2} \end{aligned}$$

and for $m \geq 10$, we have

$$\begin{aligned} LE^+(\Gamma_{D_{2m}}) =& \frac{m}{2}\times\frac{(m-2)(m-4)}{2m-2}+(m-3)\times\frac{(m^2-4m)}{2m-2} \\ &+\left(\frac{m}{2}-1\right)\times\frac{(m^2-10m+12)}{2m-2}+\sqrt{2m^2-8m+9}+\frac{m^2-4m+6}{2m-2} \\ &+\sqrt{2m^2-8m+9}-\frac{m^2-4m+6}{2m-2}. \end{aligned}$$

The required expressions for $LE^+(\Gamma_{D_{2m}})$ can be obtained on simplification. □

Theorem 2.1.3. *If G is isomorphic to D_{2m} then*

(a) $E(\Gamma_{D_{2m}}) \leq LE^+(\Gamma_{D_{2m}}) \leq LE(\Gamma_{D_{2m}})$, *equality holds if and only if $G \cong D_8$.*

(b) $\Gamma_{D_{2m}}$ *is non-hypoenergetic as well as non-hyperenergetic.*

(c) Γ_{D_6} *is L-hyperenergetic but not Q-hyperenergetic.* Γ_{D_8} *is not L-hyperenergetic and not Q-hyperenergetic. If $m \neq 3, 4$ then $\Gamma_{D_{2m}}$ is Q-hyperenergetic and L-hyperenergetic.*

Proof. (a) **Case 1:** m is odd

For $m = 3$, using Result 1.4.4 and Theorem 2.1.1, we have $E(\Gamma_{D_6}) = 2 + 2\sqrt{7}$, $LE(\Gamma_{D_6}) = \frac{42}{5}$ and $LE^+(\Gamma_{D_6}) = \frac{9}{5} + \sqrt{33}$. Clearly, $E(\Gamma_{D_6}) < LE^+(\Gamma_{D_6}) < LE(\Gamma_{D_6})$.

For $m \geq 5$, using Result 1.4.4 and Theorem 2.1.1, we have

$$LE(\Gamma_{D_{2m}}) - LE^+(\Gamma_{D_{2m}}) = \frac{8m^2 - 10m + 3}{2m - 1} - \sqrt{8m^2 - 16m + 9} \tag{2.1.1}$$

and

$$LE^+(\Gamma_{D_{2m}}) - E(\Gamma_{D_{2m}}) = \frac{2m^2(m-6) + 15m - 4}{2m - 1} + \sqrt{8m^2 - 16m + 9} - \sqrt{5m^2 - 6m + 1}. \tag{2.1.2}$$

Since $8m^2 - 10m + 3 > 0$, $(2m-1)\sqrt{8m^2 - 16m + 9} > 0$ and $(8m^2 - 10m + 3)^2 - \left(\sqrt{8m^2 - 16m + 9}\right)^2 (2m-1)^2 = 32m^3(m-2) + 8m(5m-1) > 0$ we have $8m^2 - 10m + 3 - (2m-1)\sqrt{8m^2 - 16m + 9} > 0$. Therefore, by equation (2.1.1), $(2m-1)(LE(\Gamma_{D_{2m}}) - LE^+(\Gamma_{D_{2m}})) > 0$. Hence, $LE(\Gamma_{D_{2m}}) > LE^+(\Gamma_{D_{2m}})$.

Again, we have $\sqrt{8m^2 - 16m + 9} > 0, \sqrt{5m^2 - 6m + 1} > 0$ and $\left(\sqrt{8m^2 - 16m + 9}\right)^2 - \left(\sqrt{5m^2 - 6m + 1}\right)^2 = m(3m - 10) + 8 > 0$. Thus, $\sqrt{8m^2 - 16m + 9} > \sqrt{5m^2 - 6m + 1}$. Since $2m^2(m-6) + 15m - 4 > 0$ we have $\frac{2m^2(m-6)+15m-4}{2m-1} + \sqrt{8m^2 - 16m + 9} - \sqrt{5m^2 - 6m + 1} > 0$. Therefore, by equation (2.1.2), $LE^+(\Gamma_{D_{2m}}) > E(\Gamma_{D_{2m}})$. Hence, $E(\Gamma_{D_{2m}}) < LE^+(\Gamma_{D_{2m}}) < LE(\Gamma_{D_{2m}})$.

Case 2: m is even

For $4 \leq m \leq 8$, using Result 1.4.4 and Theorem 2.1.2, we have

$$LE(\Gamma_{D_{2m}}) - LE^+(\Gamma_{D_{2m}}) = \frac{m^3 - 4m^2 + 12m - 12}{2m - 2} - 2\sqrt{2m^2 - 8m + 9} \tag{2.1.3}$$

and

$$LE^+(\Gamma_{D_{2m}}) - E(\Gamma_{D_{2m}}) = \frac{(m-4)(m^2 - 2m - 2)}{m - 1} + 2\sqrt{2m^2 - 8m + 9} - \sqrt{5m^2 - 12m + 4}. \tag{2.1.4}$$

Since $m^3 - 4m^2 + 12m - 12 > 0$, $2(2m-2)\sqrt{2m^2 - 8m + 9} > 0$ and $(m^3 - 4m^2 + 12m - 12)^2 - \left(2\sqrt{2m^2 - 8m + 9}\right)^2 (2m-2)^2 = m(m-4)^2(m-2)(m^2 + 2m - 4) \geq 0$

(equality holds if and only if $m = 4$). It follows that $m^3 - 4m^2 + 12m - 12 - 2(2m - 2)\sqrt{2m^2 - 8m + 9} \geq 0$. Therefore, by equation (2.1.3), $(2m-2)\left(LE(\Gamma_{D_{2m}}) - LE^{+}(\Gamma_{D_{2m}})\right) \geq 0$. Hence, $LE(\Gamma_{D_{2m}}) \geq LE^{+}(\Gamma_{D_{2m}})$ equality holds if and only if $G \cong D_8$.

Again, we have $2\sqrt{2m^2 - 8m + 9} > 0$, $\sqrt{5m^2 - 12m + 4} > 0$ and $\left(2\sqrt{2m^2 - 8m + 9}\right)^2 - \left(\sqrt{5m^2 - 12m + 4}\right)^2 = (m-4)(3m-8) \geq 0$. So, $2\sqrt{2m^2 - 8m + 9} \geq \sqrt{5m^2 - 12m + 4}$ (equality holds if and only if $m = 4$). Since $(m-4)(m^2 - 2m - 2) \geq 0$ we have $\frac{(m-4)(m^2-2m-2)}{m-1} + 2\sqrt{2m^2 - 8m + 9} - \sqrt{5m^2 - 12m + 4} \geq 0$ (equality holds if and only if $m = 4$). Therefore, by equation (2.1.4), $LE^{+}(\Gamma_{D_{2m}}) \geq E(\Gamma_{D_{2m}})$. Hence, $E(\Gamma_{D_{2m}}) \leq LE^{+}(\Gamma_{D_{2m}}) \leq LE(\Gamma_{D_{2m}})$ equality holds if and only if $G \cong D_8$.

For $m \geq 10$, using Result 1.4.4 and Theorem 2.1.2, we have

$$LE(\Gamma_{D_{2m}}) - LE^{+}(\Gamma_{D_{2m}}) = \frac{4m^2 - 10m + 6}{m-1} - 2\sqrt{2m^2 - 8m + 9} \tag{2.1.5}$$

and

$$LE^{+}(\Gamma_{D_{2m}}) - E(\Gamma_{D_{2m}}) = \frac{m^3 - 9m^2 + 19m - 8}{m-1} + 2\sqrt{2m^2 - 8m + 9} - \sqrt{5m^2 - 12m + 4}. \tag{2.1.6}$$

Since $4m^2 - 10m + 6 > 0$, $2(m-1)\sqrt{2m^2 - 8m + 9} > 0$ and $(4m^2 - 10m + 6)^2 - \left(2\sqrt{2m^2 - 8m + 9}\right)^2 (m-1)^2 = 8m^3(m-4) + 8m(5m-2) > 0$ we have $4m^2 - 10m + 6 - 2(m-1)\sqrt{2m^2 - 8m + 9} > 0$. Therefore, by equation (2.1.5), $(m-1)(LE(\Gamma_{D_{2m}}) - LE^{+}(\Gamma_{D_{2m}})) > 0$. Hence, $LE(\Gamma_{D_{2m}}) > LE^{+}(\Gamma_{D_{2m}})$.

Again, we have $2\sqrt{2m^2 - 8m + 9} > 0$, $\sqrt{5m^2 - 12m + 4} > 0$ and $\left(2\sqrt{2m^2 - 8m + 9}\right)^2 - \left(\sqrt{5m^2 - 12m + 4}\right)^2 = m(3m - 10) + 8 > 0$. So, $2\sqrt{2m^2 - 8m + 9} > \sqrt{5m^2 - 12m + 4}$. Since $m^3 - 9m^2 + 19m - 8 > 0$ thus $\frac{m^3 - 9m^2 + 19m - 8}{m-1} + 2\sqrt{2m^2 - 8m + 9} - \sqrt{5m^2 - 12m + 4} > 0$. Therefore, by equation (2.1.6), $LE^{+}(\Gamma_{D_{2m}}) > E(\Gamma_{D_{2m}})$. Hence, $E(\Gamma_{D_{2m}}) < LE^{+}(\Gamma_{D_{2m}}) < LE(\Gamma_{D_{2m}})$.

(b) **Case 1:** m is odd

Here, $|v(\Gamma_{D_{2m}})| = 2m - 1$ and $E(K_{|v(\Gamma_{D_{2m}})|}) = LE(K_{|v(\Gamma_{D_{2m}})|}) = LE^{+}(K_{|v(\Gamma_{D_{2m}})|}) = 4m - 4$. Using Result 1.4.4, we have

$$E(\Gamma_{D_{2m}}) - |v(\Gamma_{D_{2m}})| = \sqrt{(m-1)(5m-1)} - m \tag{2.1.7}$$

and

$$E(K_{|v(\Gamma_{D_{2m}})|}) - E(\Gamma_{D_{2m}}) = 3(m-1) - \sqrt{(m-1)(5m-1)}. \tag{2.1.8}$$

Since $\sqrt{(m-1)(5m-1)} > 0$, $m > 0$ and $\left(\sqrt{(m-1)(5m-1)}\right)^2 - m^2 = 4m^2 - 6m + 1 > 0$ we have $\sqrt{(m-1)(5m-1)} - m > 0$. Therefore, by equation (2.1.7), $E(\Gamma_{D_{2m}}) > |v(\Gamma_{D_{2m}})|$.

Again, $\sqrt{(m-1)(5m-1)} > 0$, $3(m-1) > 0$ and

$$(3(m-1))^2 - \left(\sqrt{(m-1)(5m-1)}\right)^2 = 4(m^2 - 3m + 2) > 0$$

and so $3(m-1) - \sqrt{(m-1)(5m-1)} > 0$. Therefore, by equation (2.1.8), $E(K_{|v(\Gamma_{D_{2m}})|}) > E(\Gamma_{D_{2m}})$.

Case 2: m is even

Here, $|v(\Gamma_{D_{2m}})| = 2m-2$ and $E(K_{|v(\Gamma_{D_{2m}})|}) = LE(K_{|v(\Gamma_{D_{2m}})|}) = LE^+(K_{|v(\Gamma_{D_{2m}})|}) = 4m - 6$. Using Result 1.4.4, we have

$$E(\Gamma_{D_{2m}}) - |v(\Gamma_{D_{2m}})| = \sqrt{(m-2)(5m-2)} - m \tag{2.1.9}$$

and

$$E(K_{|v(\Gamma_{D_{2m}})|}) - E(\Gamma_{D_{2m}}) = 3(m-2) + 2 - \sqrt{(m-2)(5m-2)}. \tag{2.1.10}$$

Since $\sqrt{(m-2)(5m-2)} > 0$, $m > 0$ and $\left(\sqrt{(m-2)(5m-2)}\right)^2 - m^2 = 4(m^2 - 3m + 1) > 0$ we have $\sqrt{(m-2)(5m-2)} - m > 0$. Therefore, by equation (2.1.9), $E(\Gamma_{D_{2m}}) > |v(\Gamma_{D_{2m}})|$.

Again, $\sqrt{(m-2)(5m-2)} > 0$, $3(m-2) + 2 > 0$ and

$$(3(m-2) + 2)^2 - \left(\sqrt{(m-2)(5m-2)}\right)^2 = 4(m^2 - 3m + 3) > 0$$

and so $3(m-2) + 2 - \sqrt{(m-2)(5m-2)} > 0$. Therefore, by equation (2.1.10), $E(K_{|v(\Gamma_{D_{2m}})|}) > E(\Gamma_{D_{2m}})$.

(c) **Case 1:** m is odd

For $m = 3$, using Result 1.4.4 and Theorem 2.1.1, $LE(\Gamma_{D_6}) = \frac{42}{5}$, $LE^+(\Gamma_{D_6}) = \frac{9}{5} + \sqrt{33}$ and $LE^+(K_{|v(\Gamma_{D_6})|}|) = LE(K_{|v(\Gamma_{D_6})|}) = 8$. Clearly,

$$LE^+(\Gamma_{D_6}) < LE^+(K_{|v(\Gamma_{D_6})|}) = LE(K_{|v(\Gamma_{D_6})|}) < LE(\Gamma_{D_6}).$$

For $m \geq 5$, using Theorem 2.1.1, we have

$$LE^+(\Gamma_{D_{2m}}) - LE^+(K_{|v(\Gamma_{D_{2m}})|}) = \frac{2m^2(m-9) + 24m - 7}{2m-1} + \sqrt{8m^2 - 16m + 9} > 0.$$

Therefore, $LE^+(\Gamma_{D_{2m}}) > LE^+(K_{|v(\Gamma_{D_{2m}})|})$ which implies $\Gamma_{D_{2m}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{D_{2m}}$ is L-hyperenergetic.

Case 2: m is even

For $m = 4$, using Result 1.4.4, we have $LE(\Gamma_{D_8}) = 8$ and $LE(K_{|v(\Gamma_{D_8})|}) = 10$. Clearly, $LE(\Gamma_{D_8}) < LE(K_{|v(\Gamma_{D_8})|})$. Therefore, Γ_{D_8} is not L-hyperenergetic and not Q-hyperenergetic.

Using Theorem 2.1.2, for $m = 6$ and 8, we have

$$LE^+(\Gamma_{D_{2m}}) - LE^+(K_{|v(\Gamma_{D_{2m}})|}) = \frac{m^2(m-12)+20m}{2m-2} + 2\sqrt{2m^2-8m+9} > 0$$

and for $m \geq 10$, we have

$$LE^+(\Gamma_{D_{2m}}) - LE^+(K_{|v(\Gamma_{D_{2m}})|}) = \frac{m^2(m-12)+26m-9}{m-1} + 2\sqrt{2m^2-8m+9} > 0.$$

Therefore, $LE^+(\Gamma_{D_{2m}}) > LE^+(K_{|v(\Gamma_{D_{2m}})|})$ which implies $\Gamma_{D_{2m}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{D_{2m}}$ is L-hyperenergetic. □

In Theorem 2.1.3, we compare $E(\Gamma_{D_{2m}})$, $LE(\Gamma_{D_{2m}})$ and $LE^+(\Gamma_{D_{2m}})$. However, in the following figures, we show how close are they.

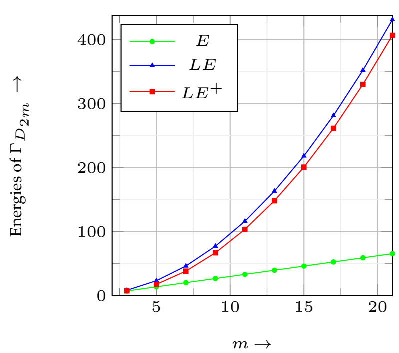

Figure 2.1: Energies of $\Gamma_{D_{2m}}$, m is odd

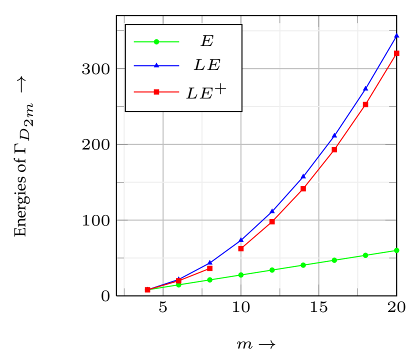

Figure 2.2: Energies of $\Gamma_{D_{2m}}$, m is even

2.1.2 The Quasidihedral groups, QD_{2^n}

We consider $QD_{2^n} := \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1}\rangle$, the quasidihedral groups of order 2^n (where $n \geq 4$). Results regarding different energies of non-commuting graphs of QD_{2^n} are given below.

Theorem 2.1.4. *Let G be isomorphic to QD_{2^n}. Then*

$$\text{Q-spec}(\Gamma_{QD_{2^n}}) = \Big\{(2^n-4)^{2^{n-2}}, (2^n-2^{n-1})^{2^{n-1}-3}, (2^n-6)^{2^{n-2}-1}, \left(2^n-3+\sqrt{2^{2n-1}-2^{n+2}+9}\right)^1, \left(2^n-3-\sqrt{2^{2n-1}-2^{n+2}+9}\right)^1\Big\}$$

and

$$LE^+(\Gamma_{QD_{2^n}}) = \begin{cases} \frac{134}{7}+2\sqrt{73}, & \textit{if } n=4 \\ \frac{2^{3n-2}+2^{n+4}-2^{2n+2}-12}{2^n-2}+2\sqrt{2^{2n-1}-2^{n+2}+9}, & \textit{if } n\geq 5. \end{cases}$$

Proof. If $G \cong QD_{2^n}$ then $|v(\Gamma_{QD_{2^n}})| = 2^n-2$ and $\Gamma_{QD_{2^n}} = K_{2^{n-2}.2,1.(2^{n-1}-2)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_{QD_{2^n}}}(x) =&\prod_{i=1}^{2}(x-(2^n-2)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(2^n-2)+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(2^n-2)+2p_i}\right) \\ =&(x-2^n+4)^{2^{n-2}}(x-2^n+2^{n-1})^{2^{n-1}-3}(x-2^n+6)^{2^{n-2}}(x-2) \\ &\times\left(1-\frac{2^{n-1}}{x-2^n+6}-\frac{2^{n-1}-2}{x-2}\right) \\ =&(x-(2^n-4))^{2^{n-2}}(x-(2^n-2^{n-1}))^{2^{n-1}-3}(x-(2^n-6))^{2^{n-2}-1} \\ &\times(x^2-(2^{n+1}-6)x+2^{2n-1}-2^{n+1}). \end{aligned}$$

Thus $$\text{Q-spec}(\Gamma_{QD_{2^n}}) = \Big\{(2^n-4)^{2^{n-2}}, (2^n-2^{n-1})^{2^{n-1}-3}, (2^n-6)^{2^{n-2}-1}, \left(2^n-3+\sqrt{2^{2n-1}-2^{n+2}+9}\right)^1, \left(2^n-3-\sqrt{2^{2n-1}-2^{n+2}+9}\right)^1\Big\}.$$

Number of edges of $\overline{\Gamma_{QD_{2^n}}}$ is $2^{2n-3}-2^n+3$. Thus, $|e(\Gamma_{QD_{2^n}})| = \frac{(2^n-2)(2^n-2-1)}{2} - (2^{2n-3}-2^n+3) = 3(2^{2n-3}-2^{n-1})$. Now,

$$\left|2^n-4-\frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| = \left|\frac{8+2^{2n-2}-3\times 2^n}{2^n-2}\right| = \frac{8+2^{2n-2}-3\times 2^n}{2^n-2},$$

$$\left|2^n-2^{n-1}-\frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| = \left|\frac{2^{n+1}-2^{2n-2}}{2^n-2}\right| = \frac{2^{2n-2}-2^{n+1}}{2^n-2},$$

$$\left|2^n - 6 - \frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| = \left|\frac{12 + 2^{2n-2} - 5 \times 2^n}{2^n - 2}\right| = \begin{cases} \frac{2}{7}, & \text{if } n = 4 \\ \frac{12+2^{2n-2}-5\times 2^n}{2^n-2}, & \text{if } n \geq 5, \end{cases}$$

$$\begin{aligned}\left|2^n - 3 + \sqrt{2^{2n-1} - 2^{n+2} + 9} - \frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| &= \left|\sqrt{2^{2n-1} - 2^{n+2} + 9} + \frac{2^{2n-2} - 2^{n+1} + 6}{2^n - 2}\right| \\ &= \sqrt{2^{2n-1} - 2^{n+2} + 9} + \frac{2^{2n-2} - 2^{n+1} + 6}{2^n - 2}\end{aligned}$$

and

$$\begin{aligned}\left|2^n - 3 - \sqrt{2^{2n-1} - 2^{n+2} + 9} - \frac{2|e(\Gamma_{QD_{2^n}})|}{|v(\Gamma_{QD_{2^n}})|}\right| &= \left|-\sqrt{2^{2n-1} - 2^{n+2}+9} + \frac{2^{2n-2} - 2^{n+1}+6}{2^n - 2}\right| \\ &= \sqrt{2^{2n-1} - 2^{n+2} + 9} - \frac{2^{2n-2} - 2^{n+1} + 6}{2^n - 2}.\end{aligned}$$

Therefore, for $n = 4$ we have $LE^+(\Gamma_{QD_{2^n}}) = \frac{134}{7} + 2\sqrt{73}$.

For $n \geq 5$ we have

$$\begin{aligned}LE^+(\Gamma_{QD_{2^n}}) =&(2^{n-2}) \times \frac{8 + 2^n(2^{n-2} - 3)}{2^n - 2} + (2^{n-1} - 3) \times \frac{2^{n+1}(2^{n-3} - 1)}{2^n - 2} + (2^{n-2} - 1) \\ &\times \frac{12 + 2^n(2^{n-2} - 5)}{2^n - 2} + \sqrt{2^{2n-1} - 2^{n+2} + 9} + \frac{2^{2n-2} - 2^{n+1} + 6}{2^n - 2} \\ &+ \sqrt{2^{2n-1} - 2^{n+2} + 9} - \frac{2^{2n-2} - 2^{n+1} + 6}{2^n - 2}\end{aligned}$$

and the result follows on simplification. □

Theorem 2.1.5. *If G is isomorphic to QD_{2^n} then*

(a) $E(\Gamma_{QD_{2^n}}) < LE^+(\Gamma_{QD_{2^n}}) < LE(\Gamma_{QD_{2^n}})$.

(b) $\Gamma_{QD_{2^n}}$ *is non-hypoenergetic as well as non-hyperenergetic.*

(c) $\Gamma_{QD_{2^n}}$ *is Q-hyperenergetic and L-hyperenergetic.*

Proof. (a) For $n = 4$, using Result 1.4.5 and Theorem 2.1.4, we have $E(\Gamma_{QD_{2^n}}) = 6+2\sqrt{57}$, $LE(\Gamma_{QD_{2^n}}) = \frac{304}{7}$ and $LE^+(\Gamma_{QD_{2^n}}) = \frac{134}{7} + 2\sqrt{73}$. Clearly, $E(\Gamma_{QD_{16}}) < LE^+(\Gamma_{QD_{16}}) < LE(\Gamma_{QD_{16}})$.

For $n \geq 5$, using Result 1.4.5 and Theorem 2.1.4, we have

$$LE(\Gamma_{QD_{2^n}}) - LE^+(\Gamma_{QD_{2^n}}) = \frac{12 + 2^{2n+1} - 5 \times 2^{n+1}}{2^n - 2} - 2\sqrt{2^{2n-1} - 2^{n+2} + 9} \quad (2.1.11)$$

and
$$LE^+(\Gamma_{QD_{2^n}}) - E(\Gamma_{QD_{2^n}}) = \frac{2^{2n-2}(2^n - 18) + 19 \times 2^n - 16}{2^n - 2} + 2\sqrt{2^{2n-1} - 2^{n+2} + 9} - 2\sqrt{5 \times 2^{2n-4} - 3 \times 2^{n-1} + 1}. \quad (2.1.12)$$

Since $12 + 2^{2n+1} - 5 \times 2^{n+1} > 0$, $2\sqrt{2^{2n-1} - 2^{n+2} + 9}(2^n - 2) > 0$ and

$$(12 + 2^{2n+1} - 5 \times 2^{n+1})^2 - \left(2\sqrt{2^{2n-1} - 2^{n+2} + 9}\right)^2 (2^n - 2)^2 = 2^{3n+1}(2^n - 8) + 2^{n+3}(5 \times 2^n - 4) > 0$$

we have $12 + 2^{2n+1} - 5 \times 2^{n+1} - 2(2^n - 2)\sqrt{2^{2n-1} - 2^{n+2} + 9} > 0$. Therefore, by equation (2.1.11), $(2^n - 2)(LE(\Gamma_{QD_{2^n}}) - LE^+(\Gamma_{QD_{2^n}})) > 0$. Hence, $LE(\Gamma_{QD_{2^n}}) > LE^+(\Gamma_{QD_{2^n}})$.

Again, $\sqrt{2^{2n-1} - 2^{n+2} + 9} > 0, \sqrt{5 \times 2^{2n-4} - 3 \times 2^{n-1} + 1} > 0$ and

$$\left(\sqrt{2^{2n-1} - 2^{n+2} + 9}\right)^2 - \left(\sqrt{5 \times 2^{2n-4} - 3 \times 2^{n-1} + 1}\right)^2 = 2^{n-4}(3 \times 2^n - 40) + 8 > 0.$$

Therefore, we have $\sqrt{2^{2n-1} - 2^{n+2} + 9} - \sqrt{5 \times 2^{2n-4} - 3 \times 2^{n-1} + 1} > 0$.
Since $2^{2n-2}(2^n - 18) + 19 \times 2^n - 16 > 0$ we have $\frac{2^{2n-2}(2^n-18)+19\times 2^n - 16}{2^n - 2} + 2\sqrt{2^{2n-1} - 2^{n+2} + 9} - 2\sqrt{5 \times 2^{2n-4} - 3 \times 2^{n-1} + 1} > 0$. Therefore, by equation (2.1.12), $LE^+(\Gamma_{QD_{2^n}}) \geq E(\Gamma_{QD_{2^n}})$. Hence, $E(\Gamma_{QD_{2^n}}) < LE^+(\Gamma_{QD_{2^n}}) < LE(\Gamma_{QD_{2^n}})$.

(b) Here, $|v(\Gamma_{QD_{2^n}})| = 2^n - 2$ and $E(K_{|v(\Gamma_{QD_{2^n}})|}) = LE(K_{|v(\Gamma_{QD_{2^n}})|}) = LE^+(K_{|v(\Gamma_{QD_{2^n}})|}) = 2^{n+1} - 6$. Using Result 1.4.5, we have

$$E(\Gamma_{QD_{2^n}}) - |v(\Gamma_{QD_{2^n}})| = 2\left(\sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)} - (2^{n-1} - 2^{n-2})\right) \quad (2.1.13)$$

and

$$E(K_{|v(\Gamma_{QD_{2^n}})|}) - E(\Gamma_{QD_{2^n}}) = 2\left(3 \times 2^{n-2} - 2 - \sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)}\right). \quad (2.1.14)$$

Since $\sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)} > 0$, $2^{n-1} - 2^{n-2} > 0$ and

$$\left(\sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)}\right)^2 - (2^{n-1} - 2^{n-2})^2 = 2^{n-2}(4 \times 2^{n-2} - 6) + 1 > 0$$

we have $\sqrt{(5 \times 2^{n-2} - 1)(2^{n-2} - 1)} - (2^{n-1} - 2^{n-2}) > 0$. Therefore, by equation (2.1.13), $E(\Gamma_{QD_{2^n}}) > |v(\Gamma_{QD_{2^n}})|$.

Again, $\sqrt{(5\times 2^{n-2}-1)(2^{n-2}-1)} > 0$, $3\times 2^{n-2}-2>0$ and

$$(3\times 2^{n-2}-2)^2-\left(\sqrt{(5\times 2^{n-2}-1)(2^{n-2}-1)}\right)^2=(2^{n-2}-3)(4\times 2^{n-2}+6)+21>0$$

and so $3\times 2^{n-2}-2-\sqrt{(5\times 2^{n-2}-1)(2^{n-2}-1)} > 0$. Therefore, by equation (2.1.14), $E(K_{|v(\Gamma_{QD_{2^n}})|}) > E(\Gamma_{QD_{2^n}})$.

(c) For $n=4$, using Theorem 2.1.4, $LE^+(\Gamma_{QD_{16}}) - LE^+(K_{|v(\Gamma_{QD_{16}})|}) = 2\sqrt{73}-\frac{48}{7} > 0$. Therefore, $LE^+(\Gamma_{QD_{16}}) > LE^+(K_{|v(\Gamma_{QD_{16}})|})$ which implies $\Gamma_{QD_{16}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{QD_{16}}$ is L-hyperenergetic.

For $n\geq 5$, using Theorem 2.1.4,

$$LE^+(\Gamma_{QD_{2^n}}) - LE^+(K_{|v(\Gamma_{QD_{2^n}})|}) = \frac{2^{2n-2}(2^n-24)+2(13\times 2^n-12)}{2^n-2}+2\sqrt{2^{2n-1}-2^{n+2}+9} > 0.$$

Therefore, $LE^+(\Gamma_{QD_{2^n}}) > LE^+(K_{|v(\Gamma_{QD_{2^n}})|})$ which implies $\Gamma_{QD_{2^n}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{QD_{2^n}}$ is L-hyperenergetic. □

In Theorem 2.1.5, we compare $E(\Gamma_{QD_{2^n}})$, $LE(\Gamma_{QD_{2^n}})$ and $LE^+(\Gamma_{QD_{2^n}})$. However, in the following figures, we show how close are they.

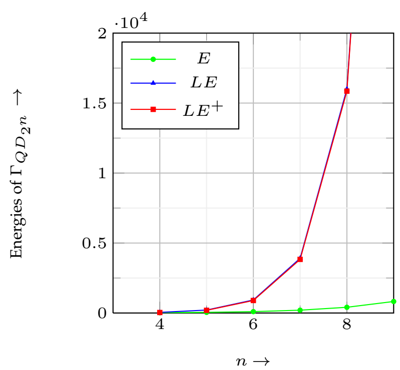

Figure 2.3: Energies of $\Gamma_{QD_{2^n}}$

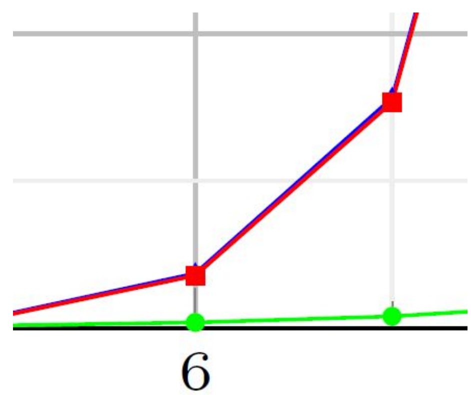

Figure 2.4: A close up view of Figure 3

2.1.3 The groups M_{2rs}

We consider the groups $M_{2rs} := \langle a, b : a^r = b^{2s} = 1, bab^{-1} = a^{-1}\rangle$, of order $2rs$ (where $r\geq 3$ and $s\geq 1$). Results regarding different energies of non-commuting graphs of M_{2rs} are given below.

Theorem 2.1.6. *Let G be isomorphic to M_{2rs}, where r is odd. Then*

$$\text{Q-spec}(\Gamma_{M_{2rs}}) = \left\{(2s(r-1))^{r(s-1)}, (rs)^{(r-1)s-1}, ((2r-3)s)^{r-1}, \left(\frac{s\left(4r-3+\sqrt{8r^2-16r+9}\right)}{2}\right)^1, \left(\frac{s\left(4r-3+\sqrt{8r^2-16r+9}\right)}{2}\right)^1\right\}$$

and

$$LE^+(\Gamma_{M_{2rs}}) = \begin{cases} \frac{3s(4s-1)}{5} + s\sqrt{33}, & \text{if } r = 3 \\ s\left(\frac{(2r(r-1)(r-2))s}{2r-1} - (2r-3) + \sqrt{8r^2-16r+9}\right), & \text{if } r \geq 5. \end{cases}$$

Proof. If $G \cong M_{2rs}$, where r is odd, then $|v(\Gamma_{M_{2rs}})| = (2r-1)s$ and $\Gamma_{M_{2rs}} = K_{r.s,1.((r-1)s)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_{M_{2rs}}}(x) =& \prod_{i=1}^{2}(x-(2rs-s)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(2rs-s)+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(2rs-s)+2p_i}\right) \\ =&(x-2s(r-1))^{r(s-1)}(x-rs)^{(r-1)s-1}(x-(2r-3)s)^r(x-s) \\ &\times\left(1-\frac{rs}{x-(2r-3)s}-\frac{(r-1)s}{x-s}\right) \\ =&(x-2s(r-1))^{r(s-1)}(x-rs)^{(r-1)s-1}(x-(2r-3)s)^{r-1} \\ &\times(x^2-(4r-3)sx+(2r^2-2r)s^2). \end{aligned}$$

Thus
$$\text{Q-spec}(\Gamma_{M_{2rs}}) = \left\{(2s(r-1))^{r(s-1)}, (rs)^{(r-1)s-1}, ((2r-3)s)^{r-1}, \left(\frac{s(4r-3+\sqrt{8r^2-16r+9})}{2}\right)^1, \left(\frac{s(4r-3-\sqrt{8r^2-16r+9})}{2}\right)^1\right\}.$$

Number of edges of $\overline{\Gamma_{M_{2rs}}}$ is $\frac{(r^2-r+1)s^2-(2r-1)s}{2}$. Thus, $|e(\Gamma_{M_{2rs}})| = \frac{(2r-1)^2s^2-(2r-1)s}{2} - \frac{(r^2-r+1)s^2-(2r-1)s}{2} = \frac{3r(r-1)s^2}{2}$. Now,

$$\left|(2r-2)s - \frac{2|e(\Gamma_{M_{2rs}})|}{|v(\Gamma_{M_{2rs}})|}\right| = \left|\frac{(r-1)(r-2)s}{2r-1}\right| = \frac{(r-1)(r-2)s}{2r-1},$$

$$\left|rs - \frac{2|e(\Gamma_{M_{2rs}})|}{|v(\Gamma_{M_{2rs}})|}\right| = \left|\frac{-r(r-2)s}{2r-1}\right| = \frac{r(r-2)s}{2r-1},$$

$$\left|(2r-3)s - \frac{2|e(\Gamma_{M_{2rs}})|}{|v(\Gamma_{M_{2rs}})|}\right| = \left|\frac{(r^2-5r+3)s}{2r-1}\right| = \begin{cases} \frac{3s}{5}, & \text{if } r = 3 \\ \frac{(r^2-5r+3)s}{2r-1}, & \text{if } r \geq 5, \end{cases}$$

$$\begin{aligned}\left|\frac{s}{2}\left(4r-3+\sqrt{8r^2-16r+9}\right) - \frac{2|e(\Gamma_{M_{2rs}})|}{|v(\Gamma_{M_{2rs}})|}\right| &= \left|\frac{s}{2}\left(\sqrt{8r^2-16r+9}+r-\frac{3}{2}+\frac{3}{4r-2}\right)\right| \\ &= \frac{s}{2}\left(\sqrt{8r^2-16r+9}+r-\frac{3}{2}+\frac{3}{4r-2}\right)\end{aligned}$$

and

$$\begin{aligned}\left|\frac{s}{2}\left(4r-3-\sqrt{8r^2-16r+9}\right) - \frac{2|e(\Gamma_{M_{2rs}})|}{|v(\Gamma_{M_{2rs}})|}\right| &= \left|\frac{s}{2}\left(-\sqrt{8r^2-16r+9}+r-\frac{3}{2}+\frac{3}{4r-2}\right)\right| \\ &= \frac{s}{2}\left(\sqrt{8r^2-16r+9}-r+\frac{3}{2}-\frac{3}{4r-2}\right).\end{aligned}$$

Therefore, for $r = 3$, we have $LE^+(\Gamma_{M_{2rs}}) = \frac{3s(4s-1)}{5} + s\sqrt{33}$. For $r \geq 5$, we have

$$\begin{aligned}LE^+(\Gamma_{M_{2rs}}) =& r(s-1) \times \frac{(r-1)(r-2)s}{2r-1} + ((r-1)s-1) \times \frac{r(r-2)s}{2r-1} \\ &+ (r-1) \times \frac{(r^2-5r+3)s}{2r-1} + \frac{s}{2}\left(\sqrt{8r^2-16r+9}+r-\frac{3}{2}+\frac{3}{4r-2}\right) \\ &+ \frac{s}{2}\left(\sqrt{8r^2-16r+9}-r+\frac{3}{2}-\frac{3}{4r-2}\right)\end{aligned}$$

and the result follows on simplification. □

Theorem 2.1.7. *Let G be isomorphic to M_{2rs}, where r is even. Then*

$$\begin{aligned}\text{Q-spec}(\Gamma_{M_{2rs}}) =& \Big\{ (2s(r-2))^{rs-\frac{r}{2}}, (rs)^{rs-2s-1}, (2s(r-3))^{\frac{r}{2}-1}, \\ & \left(4rs-6s+2s\sqrt{2r^2-8r+9}\right)^1, \left(4rs-6s-2s\sqrt{2r^2-8r+9}\right)^1 \Big\}\end{aligned}$$

$$\textit{and} \qquad LE^+(\Gamma_{M_{2rs}}) = \begin{cases} \frac{s(r^3s-6r^2s+8rs-\frac{r^3}{2}+4r^2-8r+6)}{r-1} + 2s\sqrt{2r^2-8r+9}, & \textit{if } 4 \leq r \leq 8 \\ \frac{s(r^3s-6r^2s+8rs-2r^2+8r-6)}{r-1} + 2s\sqrt{2r^2-8r+9}, & \textit{if } r \geq 10. \end{cases}$$

Proof. If $G \cong M_{2rs}$ and r is even then $|v(\Gamma_{M_{2rs}})| = 2s(r-1)$ and $\Gamma_{M_{2rs}} = K_{\frac{r}{2}.(2s),1.((\frac{r}{2}-1)2s)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_{M_{2rs}}}(x) =&\prod_{i=1}^{2}(x-2(rs-s)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-2(rs-s)+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-2(rs-s)+2p_i}\right)\\ =&(x-2s(r-2))^{rs-\frac{r}{2})}(x-rs)^{rs-2s-1}(x-2s(r-3))^{\frac{r}{2}}(x-2s)\\ &\times\left(1-\frac{rs}{x-2s(r-3)}-\frac{rs-2s}{x-2s}\right)\\ =&(x-2s(r-2))^{rs-\frac{r}{2}}(x-rs)^{rs-2s-1}(x-2s(r-3))^{\frac{r}{2}-1}\\ &\times(x^2-(4r-6)sx+rs^2(2r-4)). \end{aligned}$$

Thus $$\text{Q-spec}(\Gamma_{M_{2rs}}) = \Big\{(2s(r-2))^{rs-\frac{r}{2}}, (rs)^{rs-2s-1}, (2s(r-3))^{\frac{r}{2}-1}, \left(4rs-6s+2s\sqrt{2r^2-8r+9}\right)^1, \left(4rs-6s-2s\sqrt{2r^2-8r+9}\right)^1\Big\}.$$

Number of edges of $\overline{\Gamma_{M_{2rs}}}$ is $\frac{(r^2-2r+4)s^2-2(r-1)s}{2}$. Thus, $|e(\Gamma_{M_{2rs}})| = \frac{2(r-1)s(2(r-1)s-1)}{2} - \frac{(r^2-r+1)s^2-2(r-1)s}{2} = \frac{3r(r-2)s^2}{2}$. Now,

$$\left|2s(r-2)-\frac{2|e(\Gamma_{M_{2rs}})|}{|v(\Gamma_{M_{2rs}})|}\right| = \left|\frac{(r-2)(r-4)s}{2r-2}\right| = \frac{(r-2)(r-4)s}{2r-2},$$

$$\left|rs-\frac{2|e(\Gamma_{M_{2rs}})|}{|v(\Gamma_{M_{2rs}})|}\right| = \left|\frac{-r(r-4)s}{2r-2}\right| = \frac{r(r-4)s}{2r-2},$$

$$\left|(2r-3)s-\frac{2|e(\Gamma_{M_{2rs}})|}{|v(\Gamma_{M_{2rs}})|}\right| = \left|\frac{(r^2-10r+12)s}{2r-2}\right| = \begin{cases}\frac{(-r^2+10r-12)s}{2r-2}, & \text{if } 4\le r\le 8\\ \frac{(r^2-10r+12)s}{2r-2}, & \text{if } r\ge 10,\end{cases}$$

$$\begin{aligned}\left|4rs-6s+2s\sqrt{2r^2-8r+9}-\frac{2|e(\Gamma_{M_{2rs}})|}{|v(\Gamma_{M_{2rs}})|}\right| =&\left|\frac{rs}{2}-\frac{3s}{2}+\frac{3s}{2r-2}+s\sqrt{2r^2-8r+9}\right|\\ =&\frac{rs}{2}-\frac{3s}{2}+\frac{3s}{2r-2}+s\sqrt{2r^2-8r+9}\end{aligned}$$

and

$$\begin{aligned}\left|4rs-6s-2s\sqrt{2r^2-8r+9}-\frac{2|e(\Gamma_{M_{2rs}})|}{|v(\Gamma_{M_{2rs}})|}\right| &= \left|\frac{rs}{2}-\frac{3s}{2}+\frac{3s}{2r-2}-s\sqrt{2r^2-8r+9}\right| \\ &= -\frac{rs}{2}+\frac{3s}{2}-\frac{3s}{2r-2}+s\sqrt{2r^2-8r+9}.\end{aligned}$$

Therefore, for $4 \leq r \leq 8$, we have

$$\begin{aligned}LE^+(\Gamma_{M_{2rs}}) = &\left(rs-\frac{r}{2}\right)\times\frac{(r-2)(r-4)s}{2r-2}+(rs-2s-1)\times\frac{r(r-4)s}{2r-2} \\ &+\left(\frac{r}{2}-1\right)\times\frac{(-r^2+10r-12)s}{2r-2}+\frac{rs}{2}-\frac{3s}{2}+\frac{3s}{2r-2}+s\sqrt{2r^2-8r+9} \\ &-\frac{rs}{2}+\frac{3s}{2}-\frac{3s}{2r-2}+s\sqrt{2r^2-8r+9}\end{aligned}$$

and for $r \geq 10$, we have

$$\begin{aligned}LE^+(\Gamma_{M_{2rs}}) = &\left(rs-\frac{r}{2}\right)\times\frac{(r-2)(r-4)s}{2r-2}+(rs-2s-1)\times\frac{r(r-4)s}{2r-2} \\ &+\left(\frac{r}{2}-1\right)\times\frac{(r^2-10r+12)s}{2r-2}+\frac{rs}{2}-\frac{3s}{2}+\frac{3s}{2r-2}+s\sqrt{2r^2-8r+9} \\ &-\frac{rs}{2}+\frac{3s}{2}-\frac{3s}{2r-2}+s\sqrt{2r^2-8r+9}.\end{aligned}$$

Hence, the results follow on simplification. □

Theorem 2.1.8. *If G is isomorphic to M_{2rs} then*

(a) $E(\Gamma_{M_{2rs}}) \leq LE^+(\Gamma_{M_{2rs}}) \leq LE(\Gamma_{M_{2rs}})$*, equality holds if and only if* $G \cong M_{8s}$.

(b) $\Gamma_{M_{2rs}}$ *is non-hypoenergetic as well as non-hyperenergetic.*

(c) Γ_{M_6} *is L-hyperenergetic but not Q-hyperenergetic.* $\Gamma_{M_{8s}}$ *is not L-hyperenergetic and not Q-hyperenergetic. If* $2rs \neq 6$ *and* $8s$ *then* $\Gamma_{M_{2rs}}$ *is Q-hyperenergetic and L-hyperenergetic.*

Proof. (a) **Case 1:** r is odd

For $r = 3$, using Result 1.4.6 and Theorem 2.1.6, we have $LE(\Gamma_{M_{2rs}}) - LE^+(\Gamma_{M_{2rs}}) = \frac{33s}{5} - s\sqrt{33} > 0$ and $LE^+(\Gamma_{M_{2rs}}) - E(\Gamma_{M_{2rs}}) = \frac{12s^2-13s}{5} + (\sqrt{33}-2\sqrt{7})s > 0$.
For $r \geq 5$, using Result 1.4.6 and Theorem 2.1.6, we have

$$LE(\Gamma_{M_{2rs}}) - LE^+(\Gamma_{M_{2rs}}) = \frac{s(8r^2-10r+3)}{2r-1} - s\sqrt{8r^2-16r+9} \tag{2.1.15}$$

$$\text{and} \qquad LE^+(\Gamma_{M_{2rs}}) - E(\Gamma_{M_{2rs}}) = \frac{(2r^3-6r^2+4r)s^2-6r^2s+11rs-4s}{2r-1} + s\sqrt{8r^2-16r+9} - s\sqrt{5r^2-6r+1}. \qquad (2.1.16)$$

Since $8r^2-10r+3>0$, $(2r-1)\sqrt{8r^2-16r+9}>0$ and

$$(8r^2-10r+3)^2-(2r-1)^2(8r^2-16r+9) = 32r^4-64r^3+40r^2-8r>0$$

we have $8r^2-10r+3-(2r-1)\sqrt{8r^2-16r+9} > 0$. Therefore, by equation (2.1.15), $(2r-1)(LE(\Gamma_{M_{2rs}})-LE^+(\Gamma_{M_{2rs}}))>0$. Hence, $LE(\Gamma_{M_{2rs}}) > LE^+(\Gamma_{M_{2rs}})$.

Again, we have $\sqrt{8r^2-16r+9}>0, \sqrt{5r^2-6r+1}>0$ and

$$(\sqrt{8r^2-16r+9})^2-(\sqrt{5r^2-6r+1})^2 = r(3r-10)+8>0.$$

Therefore, $\sqrt{8r^2-16r+9}-\sqrt{5r^2-6r+1}>0$. Since $2r^3-6r^2+4r>6r^2-11r+4$ we have $\frac{(2r^3-6r^2+4r)s-6r^2+11r-4}{2r-1}+\sqrt{8r^2-16r+9}-\sqrt{5r^2-6r+1}>0$. Therefore, by equation (2.1.16), $LE^+(\Gamma_{M_{2rs}}) > E(\Gamma_{M_{2rs}})$. Hence, $E(\Gamma_{M_{2rs}}) < LE^+(\Gamma_{M_{2rs}}) < LE(\Gamma_{M_{2rs}})$.

Case 2: r is even

For $4 \leq r \leq 8$, using Result 1.4.6 and Theorem 2.1.7, we have

$$LE(\Gamma_{M_{2rs}}) - LE^+(\Gamma_{M_{2rs}}) = \frac{s}{r-1}\left(\frac{r^3}{2}-2r^2+6r-6\right) - 2s\sqrt{2r^2-8r+9} \qquad (2.1.17)$$

and

$$LE^+(\Gamma_{M_{2rs}}) - E(\Gamma_{M_{2rs}}) = \frac{(r^3-6r^2+8r)s^2-\frac{r^3s}{2}+3r^2s-5rs+4s}{r-1} + 2s\sqrt{2r^2-8r+9} - s\sqrt{5r^2-12r+4}. \qquad (2.1.18)$$

Since $\frac{r^3}{2}-2r^2+6r-6>0$, $2(r-1)\sqrt{2r^2-8r+9}>0$ and

$$(\tfrac{r^3}{2}-2r^2+6r-6)^2-4(r-1)^2(2r^2-8r+9) = \tfrac{r^5(r-8)}{4}+2r^4+6r^2(3r-8)+32r \geq 0$$

(equality holds if and only if $r=4$) we have $\frac{r^3}{2}-2r^2+6r-6-2(r-1)\sqrt{2r^2-8r+9} \geq 0$. Therefore, by equation (2.1.17), $(r-1)(LE(\Gamma_{M_{2rs}})-LE^+(\Gamma_{M_{2rs}})) \geq 0$. Hence, $LE(\Gamma_{M_{2rs}}) \geq LE^+(\Gamma_{M_{2rs}})$ equality holds if and only if $G \cong M_{8s}$.

Again, we have $2\sqrt{2r^2-8r+9} > 0, \sqrt{5r^2-12r+4} > 0$ and $(2\sqrt{2r^2-8r+9})^2 - (\sqrt{5r^2-12r+4})^2 = (r-4)(3r-8) \geq 0$ (equality holds if and only if $r=4$). Therefore, $2\sqrt{2r^2-8r+9}-\sqrt{5r^2-12r+4} \geq 0$. Since $r^3-6r^2+8r \geq \frac{r^3}{2}-3r^2+5r-4$

we have $\frac{(r^3-6r^2+8r)s^2-\frac{r^3s}{2}+3r^2s-5rs+4s}{r-1}+2s\sqrt{2r^2-8r+9}-s\sqrt{5r^2-12r+4}\geq 0$. Therefore, by equation (2.1.18), $LE^+(\Gamma_{M_{2rs}})\geq E(\Gamma_{M_{2rs}})$. Hence, $E(\Gamma_{M_{2rs}})\leq LE^+(\Gamma_{M_{2rs}})$ $\leq LE(\Gamma_{M_{2rs}})$ equality holds if and only if $G\cong M_{8s}$.

For $r\geq 10$, using Result 1.4.6 and Theorem 2.1.7, we have

$$LE(\Gamma_{M_{2rs}})-LE^+(\Gamma_{M_{2rs}})=2s\left(\frac{2r^2-5r+3}{r-1}-\sqrt{2r^2-8r+9}\right) \tag{2.1.19}$$

and

$$\begin{aligned}LE^+(\Gamma_{M_{2rs}})-E(\Gamma_{M_{2rs}})=&\frac{(r^3-6r^2+8r)s^2-3r^2s+11rs-8s}{r-1}\\&+2s\sqrt{2r^2-8r+9}-s\sqrt{5r^2-12r+4}.\end{aligned} \tag{2.1.20}$$

Since $2r^2-5r+3>0$, $(r-1)\sqrt{2r^2-8r+9}>0$ and

$$(2r^2-5r+3)^2-(r-1)^2(2r^2-8r+9)=2r(r-2)(r-1)^2>0$$

so we have $2r^2-5r+3-(r-1)\sqrt{2r^2-8r+9}>0$. Therefore, by equation (2.1.19), $(r-1)(LE(\Gamma_{M_{2rs}})-LE^+(\Gamma_{M_{2rs}}))>0$. Hence, $LE(\Gamma_{M_{2rs}})>LE^+(\Gamma_{M_{2rs}})$.

Again, we have $2\sqrt{2r^2-8r+9}>0$, $\sqrt{5r^2-12r+4}>0$ and

$$(2\sqrt{2r^2-8r+9})^2-(\sqrt{5r^2-12r+4})^2=(r-4)(3r-8)>0$$

. Therefore, $2\sqrt{2r^2-8r+9}-\sqrt{5r^2-12r+4}>0$. Since $r^3-6r^2+8r>3r^2-11r+8$ we have $\frac{(r^3-6r^2+8r)s-3r^2+11r-8}{r-1}+2\sqrt{2r^2-8r+9}-\sqrt{5r^2-12r+4}>0$. Therefore, by equation (2.1.20), $LE^+(\Gamma_{M_{2rs}})>E(\Gamma_{M_{2rs}})$. Hence, $E(\Gamma_{M_{2rs}})<LE^+(\Gamma_{M_{2rs}})<LE(\Gamma_{M_{2rs}})$.

(b) **Case 1:** r is odd

Here, $|v(\Gamma_{M_{2rs}})|=2rs-s$ and $E(K_{|v(\Gamma_{M_{2rs}})|})=LE(K_{|v(\Gamma_{M_{2rs}})|})=LE^+(K_{|v(\Gamma_{M_{2rs}})|})=$ $4rs-2s-2$. Using Result 1.4.6, we have

$$E(\Gamma_{M_{2rs}})-|v(\Gamma_{M_{2rs}})|=s(\sqrt{(r-1)(5r-1)}-r) \tag{2.1.21}$$

and

$$E(K_{|v(\Gamma_{M_{2rs}})|})-E(\Gamma_{M_{2rs}})=3rs-s-2-s\sqrt{(r-1)(5r-1)}. \tag{2.1.22}$$

Since $\sqrt{(r-1)(5r-1)} > 0$, $r > 0$ and $\left(\sqrt{(r-1)(5r-1)}\right)^2 - (r)^2 = 2r(2r-3)+1 > 0$ we have $\sqrt{(r-1)(5r-1)} - r > 0$. Therefore, by equation (2.1.21), $E(\Gamma_{M_{2rs}}) > |v(\Gamma_{M_{2rs}})|$.

Again, we have $s\sqrt{(r-1)(5r-1)} > 0$, $3rs - s - 2 > 0$ and

$$(3rs - s - 2)^2 - \left(s\sqrt{(r-1)(5r-1)}\right)^2 = 4rs(rs-3) + 4(s+1) > 0$$

and so $3rs-s-2-s\sqrt{(r-1)(5r-1)} > 0$. Therefore, by equation (2.1.22), $E(K_{|v(\Gamma_{M_{2rs}})|}) > E(\Gamma_{M_{2rs}})$.

Case 2: r is even

Here, $|v(\Gamma_{M_{2rs}})| = 2rs-2s$ and $E(K_{|v(\Gamma_{M_{2rs}})|}) = LE(K_{|v(\Gamma_{M_{2rs}})|}) = LE^+(K_{|v(\Gamma_{M_{2rs}})|}) = 4rs - 4s - 2$. Using Result 1.4.6, we have

$$E(\Gamma_{M_{2rs}}) - |v(\Gamma_{M_{2rs}})| = s(\sqrt{(r-2)(5r-2)} - r) \tag{2.1.23}$$

and

$$E(K_{|v(\Gamma_{M_{2rs}})|}) - E(\Gamma_{M_{2rs}}) = 3rs - 2s - 2 - s\sqrt{(r-2)(5r-2)}. \tag{2.1.24}$$

Since $\sqrt{(r-2)(5r-2)} > 0$, $r > 0$ and $\left(\sqrt{(r-2)(5r-2)}\right)^2 - r^2 = 4(r(r-3)+1) > 0$ we have $\sqrt{(r-2)(5r-2)} - r > 0$. Therefore, by equation (2.1.23), $E(\Gamma_{M_{2rs}}) > |v(\Gamma_{M_{2rs}})|$.

Again, we have $s\sqrt{(r-2)(5r-2)} > 0$, $3rs - 2s - 2 > 0$ and

$$(3rs - 2s - 2)^2 - \left(s\sqrt{(r-2)(5r-2)}\right)^2 = 4rs(rs-3) + 4(2s+1) > 0$$

and so $3rs-2s-2-s\sqrt{(r-2)(5r-2)} > 0$. Therefore, by equation (2.1.24), $E(K_{|v(\Gamma_{M_{2rs}})|}) > E(\Gamma_{M_{2rs}})$.

(c) **Case 1:** r is odd

For $r = 3$, using Theorem 2.1.6, we have

$$LE^+(\Gamma_{M_{2rs}}) - LE^+(K_{|v(\Gamma_{M_{2rs}})|}) = \frac{12s^2 - 53s}{5} + 2 + s\sqrt{33} > 0 \text{ for all } s \neq 1.$$

Therefore, for $r = 3$ and $s \neq 1$, $LE^+(\Gamma_{M_{2rs}}) > LE^+(K_{|v(\Gamma_{M_{2rs}})|})$ which implies $\Gamma_{M_{2rs}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{M_{2rs}}$ is L-hyperenergetic. If $r = 3$ and $s = 1$, then $G \cong D_6$ so result follows from Theorem 2.1.3(c).

For $r \geq 5$, using Theorem 2.1.6, we have

$LE^+(\Gamma_{M_{2rs}}) - LE^+(K_{|v(\Gamma_{M_{2rs}})|}) = \frac{(2r^3-6r^2+4r)s^2}{2r-1} - \frac{(12r^2-16r+5)s}{2r-1} + s\sqrt{8r^2-16r+9} + 2 > 0.$

Therefore, $LE^+(\Gamma_{M_{2rs}}) > LE^+(K_{|v(\Gamma_{M_{2rs}})|})$ which implies $\Gamma_{M_{2rs}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{M_{2rs}}$ is L-hyperenergetic.

Case 2: r is even

For $r = 4$ and $s \neq 1$, using Result 1.4.6, we have $LE(K_{|v(\Gamma_{M_{2rs}})|}) - LE(\Gamma_{M_{2rs}}) = \frac{12s}{7} - 2 > 0$. Therefore, $\Gamma_{M_{8s}}$, is not L-hyperenergetic and consequently part (a) implies that it is not Q-hyperenergetic. If $r = 4$ and $s = 1$, then $G \cong D_8$ so result follows from Theorem 2.1.3(c).

Using Theorem 2.1.7, for $4 < r \leq 8$, we get

$$LE^+(\Gamma_{M_{2rs}}) - LE^+(K_{|v(\Gamma_{M_{2rs}})|}) = \frac{(r^3-6r^2+8r)s^2 - (\frac{r^3}{2}-2)s}{r-1} + 2s\sqrt{2r^2-8r+9} + 2 > 0.$$

Therefore, $LE^+(\Gamma_{M_{2rs}}) > LE^+(K_{|v(\Gamma_{M_{2rs}})|})$ which implies $\Gamma_{M_{2rs}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{M_{2rs}}$ is L-hyperenergetic.

Using Theorem 2.1.7, for $r \geq 10$, we get

$$LE^+(\Gamma_{M_{2rs}}) - LE^+(K_{|v(\Gamma_{M_{2rs}})|}) = \frac{(r^3-6r^2+8r)s^2 - (6r^2-16r+10)s}{r-1} + 2s\sqrt{2r^2-8r+9} + 2 > 0.$$

Therefore, $LE^+(\Gamma_{M_{2rs}}) > LE^+(K_{|v(\Gamma_{M_{2rs}})|})$ which implies $\Gamma_{M_{2rs}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{M_{2rs}}$ is L-hyperenergetic. □

In Theorem 2.1.8, we compare $E(\Gamma_{M_{2rs}})$, $LE(\Gamma_{M_{2rs}})$ and $LE^+(\Gamma_{M_{2rs}})$. However, in the following figures, we show how close are they.

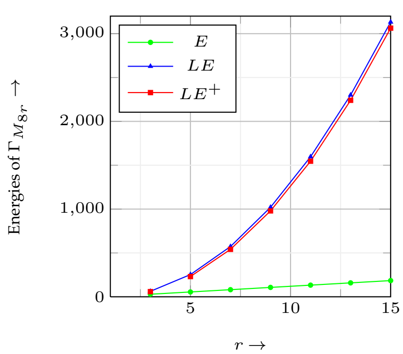

Figure 2.5: Energies of $\Gamma_{M_{8r}}$

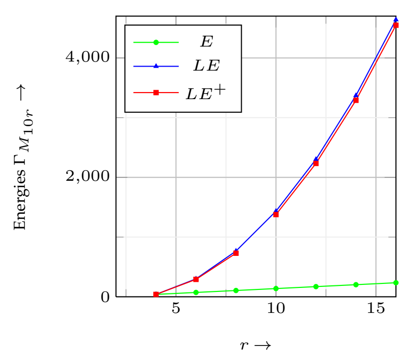

Figure 2.6: Energies of $\Gamma_{M_{10r}}$

2.1.4 The dicyclic groups, Q_{4n}

We consider $Q_{4n} := \langle x, y : x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1}\rangle$, the dicyclic groups of order $4n$ (where $n \geq 2$). Results regarding different energies of non-commuting graphs of Q_{4n} are given below.

Theorem 2.1.9. *Let G be isomorphic to Q_{4n}. Then*

$$\text{Q-spec}(\Gamma_{Q_{4n}}) = \Big\{(4n-4)^n, (2n)^{2n-3}, (4n-6)^{n-1}, (4n-3+\sqrt{8n^2-16n+9})^1, \left(4n-3-\sqrt{8n^2-16n+9}\right)^1\Big\}$$

and

$$LE^+(\Gamma_{Q_{4n}}) = \begin{cases} \frac{4n^3-8n^2+6}{2n-1} + 2\sqrt{8n^2-16n+9}, & \text{if } n \leq 4 \\ \frac{8n^3-32n^2+32n-6}{2n-1} + 2\sqrt{8n^2-16n+9}, & \text{if } n \geq 5. \end{cases}$$

Proof. If $G \cong Q_{4n}$ then $|v(\Gamma_{Q_{4n}})| = 4n-2$ and $\Gamma_{Q_{4n}} = K_{n.2,1.(2n-2)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_{Q_{4n}}}(x) =& \prod_{i=1}^{2}(x-(4n-2)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(4n-2)+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(4n-2)+2p_i}\right) \\ =&(x-(4n-4))^n(x-2n)^{2n-3}(x-4n+6)^n(x-2)\left(1-\frac{2n}{x-4n+6}-\frac{2n-2}{x-2}\right) \\ =&(x-(4n-4))^n(x-2n)^{2n-3}(x-(4n-6))^{n-1}(x^2-(8n-6)x+8n^2-8n). \end{aligned}$$

Thus
$$\text{Q-spec}(\Gamma_{Q_{4n}}) = \Big\{(4n-4)^n, (2n)^{2n-3}, (4n-6)^{n-1}, \left(4n-3+\sqrt{8n^2-16n+9}\right)^1, \left(4n-3-\sqrt{8n^2-16n+9}\right)^1\Big\}.$$

Number of edges of $\overline{\Gamma_{Q_{4n}}}$ is $2n^2-4n+3$. Thus, $|e(\Gamma_{Q_{4n}})| = \frac{(4n-2)(4n-2-1)}{2} - (2n^2 - 4n+3) = 6n(n-1)$. Now,

$$\left|4n-4-\frac{2|e(\Gamma_{Q_{4n}})|}{|v(\Gamma_{Q_{4n}})|}\right| = \left|\frac{(2n-4)(n-1)}{2n-1}\right| = \frac{(2n-4)(n-1)}{2n-1},$$

$$\left|2n-\frac{2|e(\Gamma_{Q_{4n}})|}{|v(\Gamma_{Q_{4n}})|}\right| = \left|\frac{-2n(n-2)}{2n-1}\right| = \frac{2n(n-2)}{2n-1},$$

$$\left|4n-6-\frac{2|e(\Gamma_{Q_{4n}})|}{|v(\Gamma_{Q_{4n}})|}\right| = \left|\frac{2(n^2-5n+3)}{2n-1}\right| = \begin{cases} \frac{-2(n^2-5n+3)}{2n-1}, & \text{if } n \leq 4 \\ \frac{2(n^2-5n+3)}{2n-1}, & \text{if } n \geq 5, \end{cases}$$

$$\begin{aligned}\left|4n-3+\sqrt{8n^2-16n+9}-\frac{2|e(\Gamma_{Q_{4n}})|}{|v(\Gamma_{Q_{4n}})|}\right| &= \left|\sqrt{8n^2-16n+9}+n-\frac{3}{2}+\frac{3}{4n-2}\right| \\ &= \sqrt{8n^2-16n+9}+n-\frac{3}{2}+\frac{3}{4n-2}\end{aligned}$$

and

$$\begin{aligned}\left|4n-3-\sqrt{8n^2-16n+9}-\frac{2|e(\Gamma_{Q_{4n}})|}{|v(\Gamma_{Q_{4n}})|}\right| &= \left|-\sqrt{8n^2-16n+9}+n-\frac{3}{2}+\frac{3}{4n-2}\right| \\ &= \sqrt{8n^2-16n+9}-n+\frac{3}{2}-\frac{3}{4n-2}.\end{aligned}$$

Therefore, for $n \leq 4$ we have

$$\begin{aligned}LE^+(\Gamma_{Q_{4n}}) =& n \times \frac{(2n-4)(n-1)}{2n-1} + (2n-3) \times \frac{2n(n-2)}{2n-1} + (n-1) \times \frac{-2(n^2-5n+3)}{2n-1} \\ &+ \sqrt{8n^2-16n+9}+n-\frac{3}{2}+\frac{3}{4n-2}+\sqrt{8n^2-16n+9}-n+\frac{3}{2}-\frac{3}{4n-2}\end{aligned}$$

and for $n \geq 5$ we have

$$\begin{aligned}LE^+(\Gamma_{Q_{4n}}) =& n \times \frac{(2n-4)(n-1)}{2n-1} + (2n-3) \times \frac{2n(n-2)}{2n-1} + (n-1) \times \frac{2(n^2-5n+3)}{2n-1} \\ &+ \sqrt{8n^2-16n+9}+n-\frac{3}{2}+\frac{3}{4n-2}+\sqrt{8n^2-16n+9}-n+\frac{3}{2}-\frac{3}{4n-2}\end{aligned}$$

Hence, the results follow on simplification. □

Theorem 2.1.10. *If G is isomorphic to Q_{4n} then*

(a) $E(\Gamma_{Q_{4n}}) \leq LE^+(\Gamma_{Q_{4n}}) \leq LE(\Gamma_{Q_{4n}})$, *equality holds if and only if $G \cong Q_8$.*

(b) $\Gamma_{Q_{4n}}$ *is non-hypoenergetic as well as non-hyperenergetic.*

(c) Γ_{Q_8} *is not L-hyperenergetic and not Q-hyperenergetic. If $n \neq 2$ then $\Gamma_{Q_{4n}}$ is Q-hyperenergetic and L-hyperenergetic.*

Proof. (a) For $n \leq 4$, using Result 1.4.7 and Theorem 2.1.9, we have

$$LE(\Gamma_{Q_{4n}}) - LE^+(\Gamma_{Q_{4n}}) = 2\left(\frac{2n^3 - 4n^2 + 6n - 3}{2n-1} - \sqrt{8n^2 - 16n + 9}\right) \tag{2.1.25}$$

and

$$LE^+(\Gamma_{Q_{4n}}) - E(\Gamma_{Q_{4n}}) = \frac{2(n-2)(2n^2 - 2n - 1)}{2n-1} + 2\sqrt{8n^2 - 16n + 9} - 2\sqrt{5n^2 - 6n + 1}. \tag{2.1.26}$$

Since $2n^3 - 4n^2 + 6n - 3 > 0$, $(2n-1)\sqrt{8n^2 - 16n + 9} > 0$ and $(2n^3 - 4n^2 + 6n - 3)^2 - \left(\sqrt{8n^2 - 16n + 9}\right)^2 (2n-1)^2 = 8n^5(n-4) + 16n^4 + 24n^2(3n-4) + 32n \geq 0$ (equality holds if and only if $n = 2$) we have $2n^3 - 4n^2 + 6n - 3 - (2n-1)\sqrt{8n^2 - 16n + 9} \geq 0$. Therefore, by equation (2.1.25), $(2n-1)(LE(\Gamma_{Q_{4n}}) - LE^+(\Gamma_{Q_{4n}})) \geq 0$. Hence, $LE(\Gamma_{Q_{4n}}) \geq LE^+(\Gamma_{Q_{4n}})$ equality holds if and only if $G \cong Q_8$.

Again, we have $\sqrt{8n^2 - 16n + 9} > 0, \sqrt{5n^2 - 6n + 1} > 0$ and $\left(\sqrt{8n^2 - 16n + 9}\right)^2 - \left(\sqrt{5n^2 - 6n + 1}\right)^2 = n(3n - 10) + 8 \geq 0$ (equality holds if and only if $n = 2$). Therefore, $\sqrt{8n^2 - 16n + 9} - \sqrt{5n^2 - 6n + 1} \geq 0$. Since $(n-2)(2n^2 - 2n - 1) \geq 0$ we have $\frac{2(n-2)(2n^2-2n-1)}{2n-1} + 2\sqrt{8n^2 - 16n + 9} - 2\sqrt{5n^2 - 6n + 1} \geq 0$ (equality holds if and only if $n = 2$). Therefore, by equation (2.1.26), $LE^+(\Gamma_{Q_{4n}}) \geq E(\Gamma_{Q_{4n}})$. Hence, $E(\Gamma_{Q_{4n}}) \leq LE^+(\Gamma_{Q_{4n}}) \leq LE(\Gamma_{Q_{4n}})$ equality holds if and only if $G \cong Q_8$.

For $n \geq 5$, using Result 1.4.7 and Theorem 2.1.9, we have

$$LE(\Gamma_{Q_{4n}}) - LE^+(\Gamma_{Q_{4n}}) = 2\left(\frac{8n^2 - 10n + 3}{2n-1} - \sqrt{8n^2 - 16n + 9}\right) \tag{2.1.27}$$

and

$$LE^+(\Gamma_{Q_{4n}}) - E(\Gamma_{Q_{4n}}) = \frac{2(2n^2(4n-9) + 19n - 4)}{2n-1} + 2\sqrt{8n^2 - 16n + 9} - 2\sqrt{5n^2 - 6n + 1}. \tag{2.1.28}$$

Since $8n^2 - 10n + 3 > 0$, $(2n-1)\sqrt{8n^2 - 16n + 9} > 0$ and $(8n^2 - 10n + 3)^2 - \left(\sqrt{8n^2 - 16n + 9}\right)^2 (2n-1)^2 = 32n^3(n-2) + 8n(5n-1) > 0$ we have $8n^2 - 10n + 3 - (2n-1)\sqrt{8n^2 - 16n + 9} > 0$. Therefore, by equation (2.1.27), $(2n-1)(LE(\Gamma_{Q_{4n}}) - LE^+(\Gamma_{Q_{4n}})) > 0$. Hence, $LE(\Gamma_{Q_{4n}}) > LE^+(\Gamma_{Q_{4n}})$.

Again, we have $\sqrt{8n^2-16m+9}>0, \sqrt{5n^2-6n+1}>0$ and $\left(\sqrt{8n^2-16n+9}\right)^2 - \left(\sqrt{5n^2-6n+1}\right)^2 = n(3n-10)+8>0$. Therefore, $\sqrt{8n^2-16n+9}-\sqrt{5n^2-6n+1}>0$. Since $2n^2(4n-9)+19n-4>0$ we have $\frac{2(2n^2(4n-9)+19n-4)}{2n-1}+2\sqrt{8n^2-16n+9}-2\sqrt{5n^2-6n+1}>0$. Therefore, by equation (2.1.28), $LE^+(\Gamma_{Q_{4n}})>E(\Gamma_{Q_{4n}})$. Hence, $E(\Gamma_{Q_{4n}})<LE^+(\Gamma_{Q_{4n}})<LE(\Gamma_{Q_{4n}})$.

(b) Here, $|v(\Gamma_{Q_{4n}})|=4n-2$ and $E(K_{|v(\Gamma_{Q_{4n}})|})=LE(K_{|v(\Gamma_{Q_{4n}})|})=LE^+(K_{|v(\Gamma_{Q_{4n}})|})=8n-6$. Using Result 1.4.7,

$$E(\Gamma_{Q_{4n}})-|v(\Gamma_{Q_{4n}})|=2(\sqrt{(n-1)(5n-1)}-n) \tag{2.1.29}$$

and

$$E(K_{|v(\Gamma_{Q_{4n}})|})-E(\Gamma_{Q_{4n}})=2(3(n-1)+1-\sqrt{(n-1)(5n-1)}). \tag{2.1.30}$$

Since $\sqrt{(n-1)(5n-1)}>0$, $n>0$ and $\left(\sqrt{(n-1)(5n-1)}\right)^2-n^2=2n(2n-3)+1>0$ we have $\sqrt{(n-1)(5n-1)}-n>0$. Therefore, by equation (2.1.29), $E(\Gamma_{Q_{4n}})>|v(\Gamma_{Q_{4n}})|$.

Again, we have $\sqrt{(n-1)(5n-1)}>0$, $3(n-1)+1>0$ and $(3(n-1)+1)^2-\left(\sqrt{(n-1)(5n-1)}\right)^2=2n(2n-3)+3>0$ and so $3(n-1)+1-\sqrt{(n-1)(5n-1)}>0$. Therefore, by equation (2.1.30), $E(K_{|v(\Gamma_{Q_{4n}})|})>E(\Gamma_{Q_{4n}})$.

(c) For $n=2$, using Result 1.4.7, $LE(\Gamma_{Q_8})=8$ and $LE(K_{|v(\Gamma_{Q_8})|})=10$. Clearly, $LE(\Gamma_{Q_8})<LE(K_{|v(\Gamma_{Q_8})|})$. Therefore, Γ_{Q_8} is not L-hyperenergetic and consequently part (a) implies that Γ_{Q_8} is not Q-hyperenergetic.

Using Theorem 2.1.9, for $2<n\le 4$,

$$LE^+(\Gamma_{Q_{4n}})-LE^+(K_{|v(\Gamma_{Q_{4n}})|})=\frac{4n(n-1)(n-5)}{2n-1}+2\sqrt{8n^2-16n+9}>0.$$

Also, for $n\ge 5$, $LE^+(\Gamma_{Q_{4n}})-LE^+(K_{|v(\Gamma_{Q_{4n}})|})=\frac{8n^2(n-6)+52n-12}{2n-1}+2\sqrt{8n^2-16n+9}>0$. Therefore, $LE^+(\Gamma_{Q_{4n}})>LE^+(K_{|v(\Gamma_{Q_{4n}})|})$ which implies $\Gamma_{Q_{4n}}$ is Q-hyperenergetic and consequently part (a) implies that $\Gamma_{Q_{4n}}$ is L-hyperenergetic. Hence, the result holds. □

2.1.5 The groups U_{6n}

We consider the groups $U_{6n}:=\langle x,y : x^{2n}=y^3=1, x^{-1}yx=y^{-1}\rangle$, of order $6n$. Results regarding different energies of non-commuting graphs of U_{6n} are given below.

Theorem 2.1.11. *Let G be isomorphic to U_{6n}. Then*

$$\text{Q-spec}(\Gamma_{U_{6n}}) = \left\{(3n)^{2n+1}, (4n)^{3n-3}, \left(\frac{(9+\sqrt{33})n}{2}\right)^1, \left(\frac{(9-\sqrt{33})n}{2}\right)^1\right\}$$

$$\textit{and} \qquad LE^+(\Gamma_{U_{6n}}) = \frac{12n^2-3n}{5} + \sqrt{33}n.$$

Proof. If $G \cong U_{6n}$ then $|v(\Gamma_{U_{6n}})| = 5n$ and $\Gamma_{U_{6n}} = K_{1.2n,3.n}$. Using Result 1.1.4(b), we have

$$\begin{aligned}
Q_{\Gamma_{U_{6n}}}(x) &= \prod_{i=1}^{2}(x-5n+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-5n+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-5n+2p_i}\right)\\
&= (x-3n)^{2n-1}(x-4n)^{3n-3}(x-n)(x-3n)^3\left(1-\frac{2n}{x-n}-\frac{3n}{x-3n}\right)\\
&= (x-3n)^{2n+1}(x-4n)^{3n-3}(x^2-9nx+12n^2).
\end{aligned}$$

Thus, $\text{Q-spec}(\Gamma_{U_{6n}}) = \left\{(3n)^{2n+1}, (4n)^{3n-3}, \left(\frac{(9+\sqrt{33})n}{2}\right)^1, \left(\frac{(9-\sqrt{33})n}{2}\right)^1\right\}$.

Number of edges of $\overline{\Gamma_{U_{6n}}}$ is $\frac{7n^2-5n}{2}$. Thus, $|e(\Gamma_{U_{6n}})| = \frac{5n(5n-1)}{2} - \frac{7n^2-5n}{2} = \frac{18n^2}{2}$. Now,

$$\left|3n - \frac{2|e(\Gamma_{U_{6n}})|}{|v(\Gamma_{U_{6n}})|}\right| = \left|\frac{-3n}{5}\right| = \frac{3n}{5}, \quad \left|4n - \frac{2|e(\Gamma_{U_{6n}})|}{|v(\Gamma_{U_{6n}})|}\right| = \left|\frac{2n}{5}\right| = \frac{2n}{5},$$

$$\left|\frac{(9+\sqrt{33})n}{5} - \frac{2|e(\Gamma_{U_{6n}})|}{|v(\Gamma_{U_{6n}})|}\right| = \left|\frac{(9+5\sqrt{33})n}{10}\right| = \frac{(9+5\sqrt{33})n}{10}$$

and

$$\left|\frac{(9-\sqrt{33})n}{5} - \frac{2|e(\Gamma_{U_{6n}})|}{|v(\Gamma_{U_{6n}})|}\right| = \left|\frac{(9-5\sqrt{33})n}{10}\right| = \frac{(5\sqrt{33}-9)n}{10}.$$

Therefore, $LE^+(\Gamma_{U_{6n}}) = (2n+1)\times\frac{3n}{5} + (3n-3)\times\frac{2n}{5} + \frac{(9+5\sqrt{33})n}{10} + \frac{(5\sqrt{33}-9)n}{10}$ and the result follows on simplification. □

Theorem 2.1.12. *If G is isomorphic to U_{6n} then*

(a) $E(\Gamma_{U_{6n}}) < LE^+(\Gamma_{U_{6n}}) < LE(\Gamma_{U_{6n}})$.

(b) $\Gamma_{U_{6n}}$ *is non-hypoenergetic as well as non-hyperenergetic.*

(c) $\Gamma_{U_{6n}}$ *is Q-hyperenergetic and L-hyperenergetic.*

Proof. (a) Using Result 1.4.8 and Theorem 2.1.11, we have

$$LE(\Gamma_{U_{6n}}) - LE^+(\Gamma_{U_{6n}}) = \frac{33n}{5} - \sqrt{33}n > 0 \tag{2.1.31}$$

and

$$LE^+(\Gamma_{U_{6n}}) - E(\Gamma_{U_{6n}}) = \frac{12n^2 - 13n}{5} + (\sqrt{33} - 2\sqrt{7})n > 0. \tag{2.1.32}$$

Thus, the conclusion is drawn from equations (2.1.31) and (2.1.32).

(b) Here, $|v(\Gamma_{U_{6n}})| = 5n$ and $E(K_{|v(\Gamma_{U_{6n}})|}) = LE(K_{|v(\Gamma_{U_{6n}})|}) = LE^+(K_{|v(\Gamma_{U_{6n}})|}) = 10n-2$. Using Result 1.4.8, we have

$$E(\Gamma_{U_{6n}}) - |v(\Gamma_{U_{6n}})| = (2\sqrt{7} - 3)n > 0 \tag{2.1.33}$$

and

$$E(K_{|v(\Gamma_{U_{6n}})|}) - E(\Gamma_{U_{6n}}) = (8 - 2\sqrt{7})n - 2 > 0. \tag{2.1.34}$$

Thus, the conclusion is drawn from equations (2.1.33) and (2.1.34).

(c) Using Theorem 2.1.11, we have $LE^+(\Gamma_{U_{6n}}) - LE^+(K_{|v(\Gamma_{U_{6n}})|}) = \frac{12n^2-53n+10}{5} + n\sqrt{33} > 0$. Therefore, $LE^+(\Gamma_{U_{6n}}) > LE^+(K_{|v(\Gamma_{U_{6n}})|})$ which implies $\Gamma_{U_{6n}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{U_{6n}}$ is L-hyperenergetic. □

In Theorems 2.1.10 and 2.1.12, we compare $E(\Gamma_G)$, $LE(\Gamma_G)$ and $LE^+(\Gamma_G)$ if $G \cong Q_{4n}$ and U_{6n} respectively. However, in the following figures, we show how close are they for both the groups.

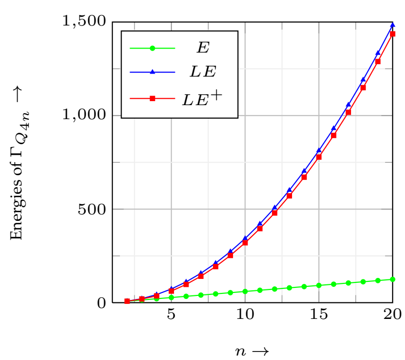

Figure 2.7: Energies of $\Gamma_{Q_{4n}}$

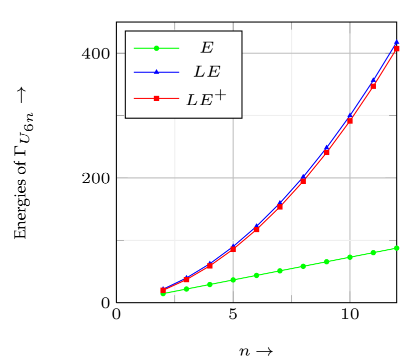

Figure 2.8: Energies of $\Gamma_{U_{6n}}$

It can be seen that if G is isomorphic to D_{2m}, QD_{2^n}, M_{2rs}, Q_{4n} or U_{6n}, then the central quotient of G is also isomorphic to some dihedral group. Therefore, we conclude this section with the following theorems for the non-commuting graphs of the groups G such that $\frac{G}{Z(G)} \cong D_{2m}$.

Theorem 2.1.13. *Let $\frac{G}{Z(G)}$ be isomorphic to D_{2m} $(m \geq 3)$ and $|Z(G)| = n$. Then*

$$\text{Q-spec}(\Gamma_G) = \left\{ ((2m-2)n)^{m(n-1)}, (mn)^{(m-1)n-1}, ((2m-3)n)^{m-1}, \left(\frac{n(4m-3+\sqrt{8m^2-16m+9})}{2}\right)^1, \left(\frac{n(4m-3-\sqrt{8m^2-16m+9})}{2}\right)^1 \right\}$$

and

$$LE^+(\Gamma_G) = \begin{cases} \frac{12n^2-3n}{5} + n\sqrt{33}, & \text{if } m = 3 \\ \frac{48n^2-29n}{7} + n\sqrt{73}, & \text{if } m = 4 \\ \frac{(2m^3-6m^2+4m)n^2-(4m^2-8m+3)n}{2m-1} + n\sqrt{8m^2-16m+9}, & \text{if } m \geq 5. \end{cases}$$

Proof. If $\frac{G}{Z(G)} \cong D_{2m}$ then $|v(\Gamma_G)| = (2m-1)n$ and $\Gamma_G = K_{m.n,1.((m-1)n)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_G}(x) =& \prod_{i=1}^{2}(x-(2m-1)n+p_i)^{a_i(p_i-1)} \prod_{i=1}^{2}(x-(2m-1)n+2p_i)^{a_i} \left(1 - \sum_{i=1}^{2} \frac{a_i p_i}{x-(2m-1)n+2p_i}\right) \\ =& (x-(2m-2)n)^{m(n-1)}(x-mn)^{(m-1)n-1}(x-(2m-3)n)^m(x-n) \\ & \times \left(1 - \frac{mn}{x-(2m-3)n} - \frac{(m-1)n}{x-n}\right) \\ =& (x-(2m-2)n)^{m(n-1)}(x-mn)^{(m-1)n-1}(x-(2m-3)n)^{m-1} \\ & \times (x^2-(4m-3)nx+(2m^2-2m)n^2). \end{aligned}$$

Thus
$$\text{Q-spec}(\Gamma_G) = \left\{ ((2m-2)n)^{m(n-1)}, (mn)^{(m-1)n-1}, ((2m-3)n)^{m-1}, \left(\tfrac{n(4m-3+\sqrt{8m^2-16m+9})}{2}\right)^1, \left(\tfrac{n(4m-3-\sqrt{8m^2-16m+9})}{2}\right)^1 \right\}.$$

Number of edges of $\overline{\Gamma_G}$ is $\frac{(m^2-m+1)n^2-(2m-1)n}{2}$. Therefore,

$$|e(\Gamma_G)| = \frac{(2m-1)^2n^2-(2m-1)n}{2} - \frac{(m^2-m+1)n^2-(2m-1)n}{2} = \frac{3m(m-1)n^2}{2}.$$

Now,

$$\left|(2m-2)n - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{(m-1)(m-2)n}{2m-1}\right| = \frac{(m-1)(m-2)n}{2m-1},$$

$$\left|mn - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{-m(m-2)n}{2m-1}\right| = \frac{m(m-2)n}{2m-1},$$

$$\left|(2m-3)n - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{(m^2-5m+3)n}{2m-1}\right| = \begin{cases} \frac{(-m^2+5m-3)n}{2m-1}, & \text{if } m \leq 4 \\ \frac{(m^2-5m+3)n}{2m-1}, & \text{if } m \geq 5, \end{cases}$$

$$\begin{aligned}
&\left|\frac{n}{2}\left(4m-3+\sqrt{8m^2-16m+9}\right) - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| \\
=&\left|\frac{n}{2}\left(\sqrt{8m^2-16m+9}+m-\frac{3}{2}+\frac{3}{4m-2}\right)\right| \\
=&\frac{n}{2}\left(\sqrt{8m^2-16m+9}+m-\frac{3}{2}+\frac{3}{4m-2}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{n}{2}\left(4m-3-\sqrt{8m^2-16m+9}\right) - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| \\
=&\left|\frac{n}{2}\left(-\sqrt{8m^2-16m+9}+m-\frac{3}{2}+\frac{3}{4m-2}\right)\right| \\
=&\frac{n}{2}\left(\sqrt{8m^2-16m+9}-m+\frac{3}{2}-\frac{3}{4m-2}\right).
\end{aligned}$$

Therefore, for $m \leq 4$ we have

$$\begin{aligned}
LE^+(\Gamma_G) =& m(n-1) \times \frac{(m-1)(m-2)n}{2m-1} + ((m-1)n-1) \times \frac{m(m-2)n}{2m-1} \\
&+ (m-1) \times \frac{(-m^2+5m-3)n}{2m-1} + \frac{n}{2}\left(\sqrt{8m^2-16m+9}+m-\frac{3}{2}+\frac{3}{4m-2}\right) \\
&+ \frac{n}{2}\left(\sqrt{8m^2-16m+9}-m+\frac{3}{2}-\frac{3}{4m-2}\right)
\end{aligned}$$

and for $m \geq 5$ we have

$$\begin{aligned} LE^{+}(\Gamma_G) =& m(n-1) \times \frac{(m-1)(m-2)n}{2m-1} + ((m-1)n-1) \times \frac{m(m-2)n}{2m-1} \\ &+ (m-1) \times \frac{(m^2-5m+3)n}{2m-1} + \frac{n}{2}\left(\sqrt{8m^2-16m+9} + m - \frac{3}{2} + \frac{3}{4m-2}\right) \\ &+ \frac{n}{2}\left(\sqrt{8m^2-16m+9} - m + \frac{3}{2} - \frac{3}{4m-2}\right). \end{aligned}$$

Hence, the results follow on simplification. □

Theorem 2.1.14. *If $\frac{G}{Z(G)}$ is isomorphic to D_{2m} $(m \geq 3)$ and $|Z(G)| = n$ then*

(a) $E(\Gamma_G) < LE^{+}(\Gamma_G) < LE(\Gamma_G)$.

(b) Γ_G *is non-hypoenergetic as well as non-hyperenergetic.*

(c) Γ_G *is L-hyperenergetic but not Q-hyperenergetic if $m = 3$ and $|Z(G)| = 1$. For $m = 3, 4$ and $|Z(G)| \neq 1$ or $m \geq 5$ and $|Z(G)| \geq 1$, Γ_G is Q-hyperenergetic and L-hyperenergetic.*

Proof. (a) For $m = 3$, using Result 1.4.9 and Theorem 2.1.13, we have $LE(\Gamma_G) - LE^{+}(\Gamma_G) = \frac{33n}{5} - n\sqrt{33} > 0$ and $LE^{+}(\Gamma_G) - E(\Gamma_G) = \frac{12n^2-13n}{5} + (\sqrt{33} - 2\sqrt{7})n > 0$.

For $m = 4$, using Result 1.4.9 and Theorem 2.1.13, we have $LE(\Gamma_G) - LE^{+}(\Gamma_G) = \frac{85n}{7} - n\sqrt{73} > 0$ and $LE^{+}(\Gamma_G) - E(\Gamma_G) = \frac{48n^2-50n}{5} + (\sqrt{73} - \sqrt{57})n > 0$.

For $m \geq 5$, using Result 1.4.9 and Theorem 2.1.13, we have

$$LE(\Gamma_G) - LE^{+}(\Gamma_G) = \frac{n(8m^2-10m+3)}{2m-1} - n\sqrt{8m^2-16m+9} \tag{2.1.35}$$

and

$$\begin{aligned} LE^{+}(\Gamma_G) - E(\Gamma_G) =& \frac{(2m^3-6m^2+4m)n^2 - 6m^2n + 11mn - 4n}{2m-1} \\ &+ n\sqrt{8m^2-16m+9} - n\sqrt{5m^2-6m+1}. \end{aligned} \tag{2.1.36}$$

Since $8m^2 - 10m + 3 > 0$, $(2m-1)\sqrt{8m^2-16m+9} > 0$ and

$$(8m^2-10m+3)^2 - (2m-1)^2(8m^2-16m+9) = 32m^4 - 64m^3 + 40m^2 - 8m > 0$$

we have $8m^2 - 10m + 3 - (2m-1)\sqrt{8m^2 - 16m + 9} > 0$. Therefore, by equation (2.1.35), $(2m-1)(LE(\Gamma_G) - LE^+(\Gamma_G)) > 0$. Hence, $LE(\Gamma_G) > LE^+(\Gamma_G)$.

Again, we have $\sqrt{8m^2 - 16m + 9} > 0, \sqrt{5m^2 - 6m + 1} > 0$ and

$$(\sqrt{8m^2 - 16m + 9})^2 - (\sqrt{5m^2 - 6m + 1})^2 = m(3m - 10) + 8 > 0.$$

Thus, $\sqrt{8m^2 - 16m + 9} - \sqrt{5m^2 - 6m + 1} > 0$. Since $2m^3 - 6m^2 + 4m > 6m^2 - 11m + 4$ we have $\frac{(2m^3-6m^2+4m)n-6m^2+11m-4}{2m-1} + \sqrt{8m^2 - 16m + 9} - \sqrt{5m^2 - 6m + 1} > 0$. Therefore, by equation (2.1.36), $LE^+(\Gamma_G) > E(\Gamma_G)$. Hence, $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G)$.

(b) Here, $|v(\Gamma_G)| = 2mn - n$ and $E(K_{|v(\Gamma_G)|}) = LE(K_{|v(\Gamma_G)|}) = LE^+(K_{|v(\Gamma_G)|}) = 4mn - 2n - 2$. Using Result 1.4.9, we have

$$E(\Gamma_G) - |v(\Gamma_G)| = n(\sqrt{(m-1)(5m-1)} - m) \tag{2.1.37}$$

and

$$E(K_{|v(\Gamma_G)|}) - E(\Gamma_G) = 3mn - n - 2 - n\sqrt{(m-1)(5m-1)}. \tag{2.1.38}$$

Since $\sqrt{(m-1)(5m-1)} > 0, m > 0$ and $(\sqrt{(m-1)(5m-1)})^2 - m^2 = 2m(2m-3)+1 > 0$ we have $\sqrt{(m-1)(5m-1)} - m > 0$. Therefore, by equation (2.1.37), $E(\Gamma_G) > |v(\Gamma_G)|$.

Again, we have $n\sqrt{(m-1)(5m-1)} > 0$, $3mn - n - 2 > 0$ and $(3mn - n - 2)^2 - \left(n\sqrt{(m-1)(5m-1)}\right)^2 = 4mn(mn-3) + 4(n+1) > 0$ and thus $3mn - n - 2 - n\sqrt{(m-1)(5m-1)} > 0$. Therefore, by equation (2.1.38), $E(K_{|v(\Gamma_G)|}) > E(\Gamma_G)$.

(c) Using Theorem 2.1.13, for $m = 3$, we have $LE^+(\Gamma_G) - LE^+(K_{|v(\Gamma_G)|}) = \frac{12n^2-53n}{7} + 2 + n\sqrt{33} > 0$ for all $n \neq 1$. Thus, for $m = 3$ and $n \neq 1$, $LE^+(\Gamma_G) > LE^+(K_{|v(\Gamma_G)|})$ which implies Γ_G is Q-hyperenergetic and consequently part (a) implies Γ_G is L-hyperenergetic. If $m = 3$ and $n = 1$, then $G \cong D_6$ so result follows from Theorem 2.1.3(c). Using Theorem 2.1.13, for $m = 4$, we have $LE^+(\Gamma_G) - LE^+(K_{|v(\Gamma_G)|}) = \frac{48n^2-127n}{7} + 2 + n\sqrt{73} > 0$ for all $n \neq 1$. Therefore, for $m = 4$ and $n \neq 1$, $LE^+(\Gamma_G) > LE^+(K_{|v(\Gamma_G)|})$ which implies Γ_G is Q-hyperenergetic and consequently part (a) implies Γ_G is L-hyperenergetic. The case $m = 4$ and $n = 1$ does not arise since $|Z(D_8)| = 2$.

For $m \geq 5$, using Theorem 2.1.13, we have $LE^+(\Gamma_G) - LE^+(K_{|v(\Gamma_G)|}) = \frac{(2m^3-6m^2+4m)n^2}{2m-1} - \frac{(12m^2-16m+5)n}{2m-1} + n\sqrt{8m^2 - 16m + 9} + 2 > 0$. Therefore, $LE^+(\Gamma_G) > LE^+(K_{|v(\Gamma_G)|})$ which implies Γ_G is Q-hyperenergetic, following which part (a) implies Γ_G is L-hyperenergetic. □

In Theorem 2.1.14, we compare $E(\Gamma_G)$, $LE(\Gamma_G)$ and $LE^+(\Gamma_G)$. However, in the following figures, we show how close are they.

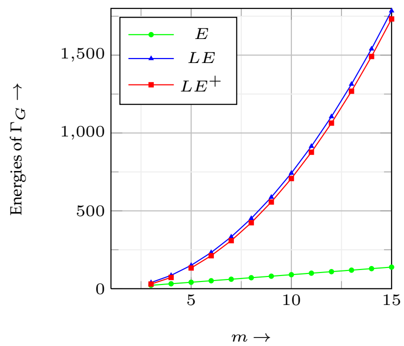

Figure 2.9: Energies of Γ_G, $\frac{G}{Z(G)} \cong D_{2m}$, $|Z(G)| = 3$

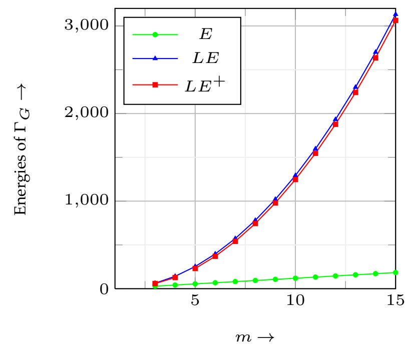

Figure 2.10: Energies of Γ_G, $\frac{G}{Z(G)} \cong D_{2m}$, $|Z(G)| = 4$

2.2 $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$

We compute the Signless Laplacian spectrum and Signless Laplacian energy of Γ_G considering the group G whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Further, we compare energy, Laplacian energy and Signless Laplacian energy of Γ_G and look into the hyper- and hypo-properties of Γ_G.

Theorem 2.2.1. *Let $\frac{G}{Z(G)}$ be isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Then*

$$\text{Q-spec}(\Gamma_G) = \left\{(pn(p-1))^{(p^2-1)n-(p+1)}, \left(n(p-1)^2\right)^p, (2pn(p-1))^1\right\}, \text{ where } |Z(G)| = n$$

and $LE^+(\Gamma_G) = 2p(p-1)|Z(G)|$. In particular, if G is non-abelian and $|G| = p^3$ then $LE^+(\Gamma_G) = 2p^2(p-1)$.

Proof. If $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then $|v(\Gamma_G)| = (p^2-1)n$ and $\Gamma_G = K_{(p+1).(p-1)n}$, where $|Z(G)| = n$. Using Result 1.1.4(b), we have

$$\begin{aligned}Q_{\Gamma_G}(x) =& (x - (p^2-1)n + (p-1)n)^{(p+1)((p-1)n-1)}(x - (p^2-1)n + 2(p-1)n)^{p+1} \\ & \times \left(1 - \frac{(p^2-1)n}{x - (p^2-1)n + 2(p-1)n}\right) \\ =& (x - pn(p-1))^{(p^2-1)n-(p+1)}(x - n(p-1)^2)^p(x - 2pn(p-1)).\end{aligned}$$

Thus, Q-spec$(\Gamma_G) = \left\{(pn(p-1))^{(p^2-1)n-(p+1)}, \left(n(p-1)^2\right)^p, (2pn(p-1))^1\right\}$.

Number of edges of $\overline{\Gamma_G}$ is $\frac{n(p^2-1)(pn-n-1)}{2}$. Therefore,

$$|e(\Gamma_G)| = \frac{n^2(p^2-1)^2 - n(p^2-1)}{2} - \frac{n(p^2-1)(pn-n-1)}{2} = \frac{(p^2-p)(p^2-1)n^2}{2}.$$

Now,

$$\left|pn(p-1) - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|pn(p-1) - (p^2-p)n\right| = 0,$$

$$\left|n(p-1)^2 - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = |n - pn| = pn - n$$

and

$$\left|2pn(p-1) - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|p^2 n - pn\right| = p^2 n - pn.$$

Therefore, $LE^+(\Gamma_G) = ((p^2-1)n - (p+1)) \times 0 + p \times (pn - n) + p^2 n - pn = 2pn(p-1)$. In particular, if G is non-abelian and $|G| = p^3$ then $n = p$. Therefore, $LE^+(\Gamma_G) = 2p^2(p-1)$. □

Theorem 2.2.2. *If* $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ *then*

(a) $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$.

(b) Γ_G *is non-hypoenergetic, non-hyperenergetic, not L-hyperenergetic as well as not Q-hyperenergetic.*

In particular, if G is non-abelian and $|G| = p^3$ then $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$ and Γ_G is non-hyperenergetic, non-hypoenergetic, not L-hyperenergetic as well as not Q-hyperenergetic.

Proof. (a) For $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, from Result 1.4.10 and Theorem 2.2.1, we have $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) = 2p^{2n}(p^n - 1)$.
(b) Here, $|v(\Gamma_G)| = (p^2-1)|Z(G)|$. Thus, $E(K_{|v(\Gamma_G)|}) = LE(K_{|v(\Gamma_G)|}) = LE^+(K_{|v(\Gamma_G)|}) = 2p(p-1)|Z(G)| + 2(p-1)|Z(G)| - 2$. Therefore, by Result 1.4.10 and Theorem 2.2.1, we have $E(K_{|v(\Gamma_G)|}) - E(\Gamma_G) = LE(K_{|v(\Gamma_G)|}) - LE(\Gamma_G) = LE^+(K_{|v(\Gamma_G)|}) - LE^+(\Gamma_G) = 2(p-1)|Z(G)| - 2 > 0$. Also, $E(\Gamma_G) - |v(\Gamma_G)| = (p-1)^2|Z(G)| > 0$. Hence, the results follow.

In particular, if G is non-abelian and $|G| = p^3$, then $|Z(G)| = p$ in the above cases so the results hold. □

2.3 $\frac{G}{Z(G)}$ is isomorphic to $Sz(2)$

We compute spectrum, energy, Signless Laplacian Spectrum and Signless Laplacian energy of Γ_G considering the group G whose central quotient is isomorphic to the Suzuki group of order 20 denoted by $Sz(2)$. Further, we compare energy, Laplacian energy and Signless Laplacian energy of Γ_G and look into the hyper- and hypo-properties of Γ_G.

Theorem 2.3.1. *Let* $\frac{G}{Z(G)} \cong Sz(2)$. *Then*

(a) $\text{Spec}(\Gamma_G) = \left\{(0)^{19n-6}, (-3n)^4, \left(2n\left(3+2\sqrt{6}\right)\right)^1, \left(2n\left(3-2\sqrt{6}\right)\right)^1\right\}$ *and* $E(\Gamma_G) = 4n(3+2\sqrt{6})$, *where* $n = |Z(G)|$.

(b) $\text{Q-spec}(\Gamma_G) = \left\{(16n)^{15n-5}, (15n)^{4n-1}, (13n)^4, \left(\frac{n(43+\sqrt{409})}{2}\right)^1, \left(\frac{n(43-\sqrt{409})}{2}\right)^1\right\}$ *and* $LE^+(\Gamma_G) = \frac{120n^2+177n}{19} + \sqrt{409}n$, *where* $n = |Z(G)|$.

Proof. If $\frac{G}{Z(G)} \cong Sz(2)$ and $|Z(G)| = n$ then $\Gamma_G = K_{5.3n,1.4n}$ and it is a complete 6-partite graph with $19n$ vertices.
(a) Using Result 1.1.4(a), the characteristic polynomial of Γ_G is

$$P_{\Gamma_G}(x) = x^{19n-6}(x+3n)^4(x^2-12nx-60n^2).$$

Therefore, $\text{Spec}(\Gamma_G) = \left\{(0)^{19n-6}, (-3n)^4, \left(2n\left(3+2\sqrt{6}\right)\right)^1, \left(2n\left(3-2\sqrt{6}\right)\right)^1\right\}$ and $E(\Gamma_G) = 4n\left(3+2\sqrt{6}\right)$.
(b) Using Result 1.1.4(b), we have

$$\begin{aligned}
Q_{\Gamma_G}(x) =& \prod_{i=1}^{2}(x-19n+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-19n+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-19n+2p_i}\right)\\
=& (x-19n+3n)^{5(3n-1)}(x-19n+4n)^{4n-1}(x-19n+2\times 3n)^5(x-19n+2\times 4n)^1\\
&\times\left(1-\frac{5\times 3n}{x-19n+2\times 3n}-\frac{4n}{x-19n+2\times 4n}\right)\\
=& (x-16n)^{15n-5}(x-15n)^{4n-1}(x-13n)^4(x^2-43nx+360n^2).
\end{aligned}$$

Thus, $\text{Q-spec}(\Gamma_G) = \left\{(16n)^{15n-5}, (15n)^{4n-1}, (13n)^4, \left(\frac{n(43+\sqrt{409})}{2}\right)^1, \left(\frac{n(43-\sqrt{409})}{2}\right)^1\right\}$.

Number of edges of $\overline{\Gamma_G}$ is $\frac{61n^2-19n}{2}$. Thus, $|e(\Gamma_G)| = \frac{19n(19n-1)}{2} - \frac{61n^2-19n}{2} = 150n^2$. Now,

$$\left|16n - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{4n}{19}\right| = \frac{4n}{19}, \quad \left|15n - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{-15n}{19}\right| = \frac{15n}{19},$$

$$\left|13n - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{-53n}{19}\right| = \frac{53n}{19},$$

$$\left|\frac{(43+\sqrt{409})n}{2} - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{(217+19\sqrt{409})n}{38}\right| = \frac{(217+19\sqrt{409})n}{38}$$

and

$$\left|\frac{(43-\sqrt{409})n}{2} - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{(217-19\sqrt{409})n}{38}\right| = \frac{(19\sqrt{409}-217)n}{38}.$$

Therefore, $LE^+(\Gamma_G) = (15n-5)\times\frac{4n}{19}+(4n-1)\times\frac{15n}{19}+4\times\frac{53n}{19}+\frac{(217+19\sqrt{409})n}{38}+\frac{(19\sqrt{409}-217)n}{38}$ and the result follows on simplification. □

Theorem 2.3.2. *If $\frac{G}{Z(G)}$ is isomorphic to $Sz(2)$ and $n = |Z(G)|$ then*

(a) $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G)$.

(b) Γ_G *is non-hypoenergetic as well as non-hyperenergetic.*

(c) $\Gamma_{Sz(2)}$ *is not Q-hyperenergetic but is L-hyperenergetic. If $G \ncong Sz(2)$ then Γ_G is Q-hyperenergetic and L-hyperenergetic.*

Proof. (a) Using Theorem 2.3.1(b) and Result 1.4.11, we have $LE(\Gamma_G) - LE^+(\Gamma_G) = \left(\frac{393}{19} - \sqrt{409}\right) n > 0$. Using Theorem 2.3.1, we have $LE^+(\Gamma_G) - E(\Gamma_G) = \frac{3n(40n-17)}{19} + \sqrt{409}n - 8n\sqrt{6} > 0$. Hence, the result follows.
(b) Here, $|v(\Gamma_G)| = 19n$ and $E(K_{|v(\Gamma_G)|}) = LE(K_{|v(\Gamma_G)|}) = LE^+(K_{|v(\Gamma_G)|}) = 38n - 2$. Using Theorem 2.3.1(a), we have $E(\Gamma_G) - |v(\Gamma_G)| = (8\sqrt{6} - 7)n > 0$ and also we have $E(K_{|v(\Gamma_G)|}) - E(\Gamma_G) = 2(13 - 4\sqrt{6})n - 2 > 0$. Hence, the result follows.
(c) For $n = 1$, by Result 1.4.17(c), we have $LE(\Gamma_{Sz(2)}) = \frac{690}{19} > 36 = LE(K_{|v(\Gamma_{Sz(2)})|})$ and by Theorem 2.3.1(b), we have $LE^+(\Gamma_{Sz(2)}) = \frac{297}{19} + \sqrt{409} < 36 = LE^+(K_{|v(\Gamma_{Sz(2)})|})$. Hence, for $n = 1$, Γ_G is L-hyperenergetic but not Q-hyperenergetic.

For $n > 1$, using Theorem 2.3.1(b), we have $LE^+(\Gamma_G) - LE^+(K_{|v(\Gamma_G)|}) = \frac{5n(24n-109)+38}{19} + \sqrt{409}n > 0$. Therefore, $LE^+(\Gamma_G) > LE^+(K_{|v(\Gamma_G)|})$ which implies Γ_G is Q-hyperenergetic and consequently part (a) implies Γ_G is L-hyperenergetic. □

For $\frac{G}{Z(G)} \cong Sz(2)$, the following figures also demonstrate that among the three energies, $E(\Gamma_G)$ is the least and the fact that although $LE^+(\Gamma_G) < LE(\Gamma_G)$ but these two energies are very close to each other.

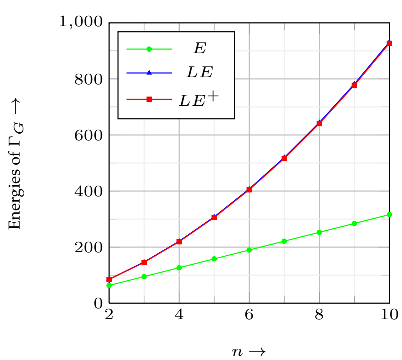

Figure 2.11: Energies of Γ_G where $\frac{G}{Z(G)} \cong Sz(2)$

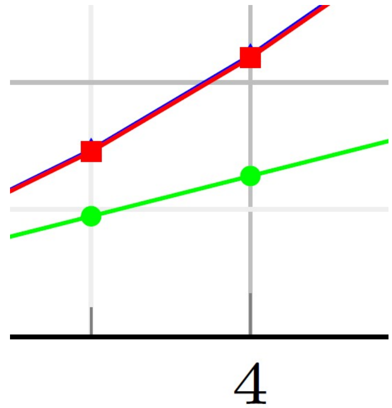

Figure 2.12: A close up view of Figure 11

2.4 Some more classes of groups

In this section we discuss results on energy, Laplacian energy and Signless Laplacian energy of non-commuting graph of certain well-known classes of finite groups.

2.4.1 The Hanaki groups

We consider the Hanaki groups

$$A(n,\mathcal{V}) = \left\{ U(a,b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \mathcal{V}(a) & 1 \end{bmatrix} : a,b \in GF(2^n) \right\} \quad (n \geq 2),$$

under matrix multiplication given by $U(a,b)U(a',b') = U(a+a', b+b'+a'\mathcal{V}(a))$ (here $\mathcal{V}$ be the Frobenius automorphism of $GF(2^n)$, i.e., $\mathcal{V}(x) = x^2$ for all $x \in GF(2^n)$) and

$$A(n,p) = \left\{ V(a,b,c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a,b,c \in GF(p^n) \right\} (p \text{ is any prime})$$

under matrix multiplication $V(a,b,c)V(a',b',c') = V(a+a', b+b'+ca', c+c')$. In this section, we compute Signless Laplacian spectrum and Signless Laplacian energy of non-commuting graph of the groups $A(n,\mathcal{V})$ and $A(n,p)$. Further, we compare Signless Laplacian energy of Γ_G with its predetermined energy, Laplacian energy and look into the hyper- and hypo-properties of Γ_G if G is isomorphic to $A(n,\mathcal{V})$ and $A(n,p)$.

Theorem 2.4.1. *If G is isomorphic to the Hanaki group $A(n,\mathcal{V})$ then* $\text{Q-spec}(\Gamma_G) = \left\{(2^{2n}-2^{n+1})^{2^{2n}-2^{n+1}+1}, (2^{2n}-3\times 2^n)^{2^n-2}, (2^{2n+1}-2^{n+2})^1\right\}$ *and* $LE^+(\Gamma_G) = 2^{2n+1}-2^{n+2}$.

Proof. If G is isomorphic to the Hanaki group $A(n,\mathcal{V})$ then $|v(\Gamma_G)| = 2^{2n}-2^n$ and $\Gamma_G = K_{(2^n-1).2^n}$. Using Result 1.1.4(b), we have

$$\begin{aligned}Q_{\Gamma_G}(x) =&(x-(2^{2n}-2^n)+2^n)^{(2^n-1)(2^n-1)}(x-(2^{2n}-2^n)+2\times 2^n)^{(2^n-1)}\\ &\times\left(1-\frac{(2^n-1).2^n}{x-(2^{2n}-2^n)+2\times 2^n}\right)\\ =&\,(x-(2^{2n}-2^{n+1}))^{(2^n-1)^2}(x-(2^{2n}-3\times 2^n))^{2^n-2}(x-(2^{2n+1}-2^{n+2})).\end{aligned}$$

Thus, $\text{Q-spec}(\Gamma_G) = \left\{(2^{2n}-2^{n+1})^{2^{2n}-2^{n+1}+1}, (2^{2n}-3\times 2^n)^{2^n-2}, (2^{2n+1}-2^{n+2})^1\right\}$.

Number of edges of $\overline{\Gamma_G}$ is $2^{n-2}(2^{2n}-2^{n+1}+1)$. Thus, $|e(\Gamma_G)| = \frac{(2^{2n}-2^n)(2^{2n}-2^n-1)}{2} - 2^{n-2}(2^{2n}-2^{n+1}+1) = 2^{4n-1}-3\times 2^{3n-1}+2^{2n}$. Now,

$$\left|2^{2n}-2^{n+1}-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{2^{3n+1}-2^{3n+1}}{2^{2n}-2^n}\right| = 0,$$

$$\left|2^{2n}-3\times 2^n-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{2^{2n}-2^{3n}}{2^{2n}-2^n}\right| = \frac{2^{3n}-2^{2n}}{2^{2n}-2^n}$$

and

$$\left|2^{2n+1}-2^{n+2}-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{2^{4n}-3\times 2^{3n}+2^{2n+1}}{2^{2n}-2^n}\right| = \frac{2^{4n}-3\times 2^{3n}+2^{2n+1}}{2^{2n}-2^n}.$$

Therefore, $LE^+(\Gamma_G) = (2^{2n}-2^{n+1}+1)\times 0+(2^n-2)\times\frac{2^{3n}-2^{2n}}{2^{2n}-2^n}+\frac{2^{4n}-3\times 2^{3n}+2^{2n+1}}{2^{2n}-2^n}$ and the result follows on simplification. □

Theorem 2.4.2. *If G is isomorphic to the Hanaki group $A(n,\mathcal{V})$ then*

(a) $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$.

(b) Γ_G *is non-hypoenergetic, non-hyperenergetic, not L-hyperenergetic and not Q-hyperenergetic.*

Proof. (a) Using Result 1.4.12 and Theorem 2.4.1, we have $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) = 2^{n+1}(2^n-2)$ and hence the result follows.

(b) Here, $|v(\Gamma_G)| = 2^n(2^n-1)$ and $E(K_{|v(\Gamma_G)|}) = LE(K_{|v(\Gamma_G)|}) = LE^+(K_{|v(\Gamma_G)|}) = 2^{n+1}(2^n-1)-2$. Therefore, by Result 1.4.12 and Theorem 2.4.1, we have $E(\Gamma_G)-|v(\Gamma_G)| = 2^n(2^n-3)>0$, for $n>1$ and $E(K_{|v(\Gamma_G)|})-E(\Gamma_G) = LE(K_{|v(\Gamma_G)|})-LE(\Gamma_G) = LE^+(K_{|v(\Gamma_G)|}) - LE^+(\Gamma_G) = 2(2^n-1)>0$. Hence, the results follow. □

Theorem 2.4.3. *If G is isomorphic to the Hanaki group $A(n,p)$ then*

$$\text{Q-spec}(\Gamma_G) = \left\{(p^{3n}-p^{2n})^{(p^n+1)(p^{2n}-p^n-1)}, (p^{3n}-2p^{2n}+p^n)^{p^n}, (2p^{3n}-2p^{2n})^1\right\}$$

and $$LE^+(\Gamma_G) = 2p^{2n}(p^n-1).$$

Proof. If G is isomorphic to the Hanaki group $A(n,p)$ then $|v(\Gamma_G)| = p^{3n}-p^n$ and $\Gamma_G = K_{(p^n+1).(p^{2n}-p^n)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_G}(x) =&(x-(p^{3n}-p^n)+(p^{2n}-p^n))^{(p^n+1)(p^{2n}-p^n-1)}(x-(p^{3n}-p^n)+2(p^{2n}-p^n))^{(p^n+1)} \\ &\times\left(1-\frac{p^{3n}-p^n}{x-(p^{3n}-p^n)+2(p^{2n}-p^n)}\right) \\ =&(x-(p^{3n}-p^{2n}))^{(p^n+1)(p^{2n}-p^n-1)}(x-(p^{3n}-2p^{2n}+p^n))^{p^n}(x-(2p^{3n}-2p^{2n})). \end{aligned}$$

Thus, $\text{Q-spec}(\Gamma_G) = \left\{(p^{3n}-p^{2n})^{(p^n+1)(p^{2n}-p^n-1)}, (p^{3n}-2p^{2n}+p^n)^{p^n}, (2p^{3n}-2p^{2n})^1\right\}$.

Number of edges of $\overline{\Gamma_G}$ is $\frac{(p^{3n}-p^n)(p^{2n}-p^n-1)}{2}$. Thus, $|e(\Gamma_G)| = \frac{(p^{3n}-p^n)(p^{3n}-p^n-1)}{2} - \frac{(p^{3n}-p^n)(p^{2n}-p^n-1)}{2} = \frac{p^{6n}-p^{5n}-p^{4n}+p^{3n}}{2}$. Now,

$$\left|p^{3n}-p^{2n}-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{0}{p^{3n}-p^n}\right| = 0,$$

$$\left|p^{3n}-2p^{2n}+p^n-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{-p^{5n}+p^{4n}+p^{3n}-p^{2n}}{p^{3n}-p^n}\right| = \frac{p^{5n}-p^{4n}-p^{3n}+p^{2n}}{p^{3n}-p^n}$$

and

$$\left|2p^{3n}-2p^{2n}-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{p^{6n}-p^{5n}-p^{4n}+p^{3n}}{p^{3n}-p^n}\right| = \frac{p^{6n}-p^{5n}-p^{4n}+p^{3n}}{p^{3n}-p^n}.$$

Therefore, $LE^+(\Gamma_G) = (p^n+1)(p^{2n}-p^n-1)\times 0 + p^n \times \frac{p^{5n}-p^{4n}-p^{3n}+p^{2n}}{p^{3n}-p^n} + \frac{p^{6n}-p^{5n}-p^{4n}+p^{3n}}{p^{3n}-p^n}$ and the result follows on simplification. □

Theorem 2.4.4. *If G is isomorphic to the Hanaki group $A(n,p)$ then*

(a) $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$.

(b) Γ_G *is non-hypoenergetic, non-hyperenergetic, not L-hyperenergetic and not Q-hyperenergetic.*

Proof. (a) Using Result 1.4.13 and Theorem 2.4.3 we have $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) = 2p^{2n}(p^n-1)$ and hence the result follows.

(b) Here, $|v(\Gamma_G)| = p^{3n}-p^n$ and $E(K_{|v(\Gamma_G)|}) = LE(K_{|v(\Gamma_G)|}) = LE^+(K_{|v(\Gamma_G)|}) = 2(p^{2n}(p^n-1)+p^{2n}-p^n-1)$. Therefore, by Result 1.4.13 and Theorem 2.4.3, we have $E(\Gamma_G)-|v(\Gamma_G)| = p^n(p^n-1)^2 > 0$ and $E(K_{|v(\Gamma_G)|}) - E(\Gamma_G) = LE(K_{|v(\Gamma_G)|}) - LE(\Gamma_G) = LE^+(K_{|v(\Gamma_G)|}) - LE^+(\Gamma_G) = 2(p^n(p^n-1)-1) > 0$. Hence, the results follow. □

2.4.2 The semi-dihedral groups, SD_{8n}

We consider $SD_{8n} := \langle a, b : a^{4n} = b^2 = 1, bab^{-1} = a^{2n-1}\rangle$, the semi-dihedral groups of order $8n$ (where $n > 1$). Results regarding different energies of non-commuting graph of SD_{8n} are given below.

Theorem 2.4.5. *Let G be isomorphic to SD_{8n}, where n is odd. Then*

$$\text{Q-spec}(\Gamma_{SD_{8n}}) = \Big\{(8n-8)^{3n}, (4n)^{4n-5}, (8n-12)^{n-1}, \Big(8n-6+2\sqrt{8n^2-16n+9}\Big)^1, \Big(8n-6-2\sqrt{8n^2-16n+9}\Big)^1\Big\}$$

and

$$LE^+(\Gamma_{SD_{8n}}) = \begin{cases} 36+4\sqrt{33}, & \text{if } n=3 \\ \frac{32n^3-112n^2+96n-12}{2n-1} + 4\sqrt{8n^2-16n+9}, & \text{if } n \geq 5. \end{cases}$$

Proof. If $G \cong SD_{8n}$ and n is odd then $|v(\Gamma_{SD_{8n}})| = 8n-4$ and $\Gamma_{SD_{8n}} = K_{n.4,1.(4n-4)}$. Using Result 1.1.4(b), we have

$$\begin{aligned}Q_{\Gamma_{SD_{8n}}}(x) =& \prod_{i=1}^{2}(x-(8n-4)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(8n-4)+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(8n-4)+2p_i}\right)\\ =&(x-8n+8)^{3n}(x-4n)^{4n-5}(x-8n+12)^{n}(x-4)\left(1-\frac{4n}{x-8n+12}-\frac{4n-4}{x-4}\right)\\ =&(x-(8n-8))^{3n}(x-4n)^{4n-5}(x-(8n-12))^{n-1}(x^2-(16n-12)x+32n^2-32n).\end{aligned}$$

Thus
$$\text{Q-spec}(\Gamma_{SD_{8n}}) = \left\{(8n-8)^{3n}, (4n)^{4n-5}, (8n-12)^{n-1}, \left(8n-6+2\sqrt{8n^2-16n+9}\right)^1, \left(8n-6-2\sqrt{8n^2-16n+9}\right)^1\right\}.$$

Number of edges of $\overline{\Gamma_{SD_{8n}}}$ is $8n^2-12n+10$. Therefore,

$$|e(\Gamma_{SD_{8n}})| = \frac{(8n-4)(8n-4-1)}{2} - (8n^2-12n+10) = 24n(n-1).$$

Now,

$$\left|8n-8-\frac{2|e(\Gamma_{SD_{8n}})|}{|v(\Gamma_{SD_{8n}})|}\right| = \left|\frac{4(n-1)(n-2)}{2n-1}\right| = \frac{4(n-1)(n-2)}{2n-1},$$

$$\left|4n-\frac{2|e(\Gamma_{SD_{8n}})|}{|v(\Gamma_{SD_{8n}})|}\right| = \left|\frac{4n(2-n)}{2n-1}\right| = \frac{4n(n-2)}{2n-1},$$

$$\left|8n-12-\frac{2|e(\Gamma_{SD_{8n}})|}{|v(\Gamma_{SD_{8n}})|}\right| = \left|\frac{4(n^2-5n+3)}{2n-1}\right| = \begin{cases}\frac{4(-n^2+5n-3)}{2n-1}, & \text{if } n \leq 3\\ \frac{4(n^2-5n+3)}{2n-1}, & \text{if } n \geq 5,\end{cases}$$

$$\begin{aligned}\left|8n-6+2\sqrt{8n^2-16n+9}-\frac{2|e(\Gamma_{SD_{8n}})|}{|v(\Gamma_{SD_{8n}})|}\right| =& \left|2\sqrt{8n^2-16n+9}+(2n-3)+\frac{3}{2n-1}\right|\\ =&2\sqrt{8n^2-16n+9}+(2n-3)+\frac{3}{2n-1}\end{aligned}$$

and

$$\begin{aligned}\left|8n-6-2\sqrt{8n^2-16n+9}-\frac{2|e(\Gamma_{SD_{8n}})|}{|v(\Gamma_{SD_{8n}})|}\right| =& \left|-2\sqrt{8n^2-16n+9}+(2n-3)+\frac{3}{2n-1}\right|\\ =&2\sqrt{8n^2-16n+9}-(2n-3)-\frac{3}{2n-1}.\end{aligned}$$

Therefore, for $n = 3$ we have $LE^+(\Gamma_{SD_{8n}}) = 36 + 4\sqrt{33}$.

For $n \geq 5$ we have

$$LE^+(\Gamma_{SD_{8n}}) = 3n \times \frac{4(n-1)(n-2)}{2n-1} + (4n-5) \times \frac{4n(n-2)}{2n-1} + (n-1) \times \frac{4(n^2-5n+3)}{2n-1} + 2\sqrt{8n^2-16n+9} - 2n - 3 + \frac{3}{2n-1} + 2\sqrt{8n^2-16n+9} + 2n + 3 - \frac{3}{2n-1}$$

and the result follows on simplification. □

Theorem 2.4.6. *Let G be isomorphic to SD_{8n}, where n is even. Then*

$$\text{Q-spec}(\Gamma_{SD_{8n}}) = \left\{(8n-4)^{2n}, (4n)^{4n-3}, (8n-6)^{2n-1}, \left(8n-3+\sqrt{32n^2-32n+9}\right)^1, \left(8n-3-\sqrt{32n^2-32n+9}\right)^1\right\}$$

and

$$LE^+(\Gamma_{SD_{8n}}) = \begin{cases} \frac{134}{7} + 2\sqrt{73}, & \text{if } n = 2 \\ \frac{64n^3-128n^2+64n-6}{4n-1} + 2\sqrt{32n^2-32n+9}, & \text{if } n \geq 4. \end{cases}$$

Proof. If $G \cong SD_{8n}$ and n is even then $|v(\Gamma_{SD_{8n}})| = 8n-2$ and $\Gamma_{SD_{8n}} = K_{2n.2,1.(4n-2)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_{SD_{8n}}}(x) &= \prod_{i=1}^{2}(x-(8n-2)+p_i)^{a_i(p_i-1)} \prod_{i=1}^{2}(x-(8n-2)+2p_i)^{a_i} \left(1 - \sum_{i=1}^{2} \frac{a_i p_i}{x-(8n-2)+2p_i}\right) \\ &= (x-8n+4)^{2n}(x-4n)^{4n-3}(x-8n+6)^{2n}(x-2)\left(1 - \frac{4n}{x-8n+6} - \frac{4n-2}{x-2}\right) \\ &= (x-(8n-4))^{2n}(x-4n)^{4n-3}(x-(8n-6))^{2n-1}(x^2-(16n-6)x+32n^2-16n). \end{aligned}$$

Thus $\text{Q-spec}(\Gamma_{SD_{8n}}) = \left\{(8n-4)^{2n}, (4n)^{4n-3}, (8n-6)^{2n-1}, \left(8n-3+\sqrt{32n^2-32n+9}\right)^1, \left(8n-3-\sqrt{32n^2-32n+9}\right)^1\right\}.$

Number of edges of $\overline{\Gamma_{SD_{8n}}}$ is $8n^2-8n+3$. Therefore, $|e(\Gamma_{SD_{8n}})| = \frac{(8n-2)(8n-2-1)}{2} - (8n^2-8n+3) = 12n(2n-1)$. Now,

$$\left|8n - 4 - \frac{2|e(\Gamma_{SD_{8n}})|}{|v(\Gamma_{SD_{8n}})|}\right| = \left|\frac{(8n-4)(n-1)}{4n-1}\right| = \frac{(8n-4)(n-1)}{4n-1},$$

$$\left|4n - \frac{2|e(\Gamma_{SD_{8n}})|}{|v(\Gamma_{SD_{8n}})|}\right| = \left|\frac{-8n(n-1)}{4n-1}\right| = \frac{8n(n-1)}{4n-1},$$

$$\left|8n-6 - \frac{2|e(\Gamma_{SD_{8n}})|}{|v(\Gamma_{SD_{8n}})|}\right| = \left|\frac{(8n^2-20n+6)}{4n-1}\right| = \begin{cases} \frac{2}{7}, & \text{if } n = 2 \\ \frac{(8n^2-20n+6)}{4n-1}, & \text{if } n \geq 4, \end{cases}$$

$$\begin{aligned}\left|8n-3+\sqrt{32n^2-32n+9} - \frac{2|e(\Gamma_{SD_{8n}})|}{|v(\Gamma_{SD_{8n}})|}\right| &= \left|\sqrt{32n^2-32n+9}+2n-\frac{3}{2}+\frac{3}{8n-2}\right| \\ &= \sqrt{32n^2-32n+9}+2n-\frac{3}{2}+\frac{3}{8n-2}\end{aligned}$$

and

$$\begin{aligned}\left|8n-3-\sqrt{32n^2-32n+9} - \frac{2|e(\Gamma_{SD_{8n}})|}{|v(\Gamma_{SD_{8n}})|}\right| &= \left|-\sqrt{32n^2-32n+9}+2n-\frac{3}{2}+\frac{3}{8n-2}\right| \\ &= \sqrt{32n^2-32n+9}-2n+\frac{3}{2}-\frac{3}{8n-2}.\end{aligned}$$

Therefore, for $n = 2$, we have $LE^+(\Gamma_{SD_{8n}}) = \frac{134}{7} + 2\sqrt{73}$. For $n \geq 4$, we have

$$\begin{aligned}LE^+(\Gamma_{SD_{8n}}) =& 2n \times \frac{(8n-4)(n-1)}{4n-1} + (4n-3) \times \frac{8n(n-1)}{4n-1} + (2n-1) \times \frac{(8n^2-20n+6)}{4n-1} \\ &+ \sqrt{32n^2-32n+9}+2n-\frac{3}{2}+\frac{3}{8n-2}+\sqrt{32n^2-32n+9}-2n+\frac{3}{2}-\frac{3}{8n-2}\end{aligned}$$

and the result follows on simplification. □

Theorem 2.4.7. *If G is isomorphic to SD_{8n} then*

(a) $E(\Gamma_{SD_{8n}}) < LE^+(\Gamma_{SD_{8n}}) < LE(\Gamma_{SD_{8n}})$.

(b) $\Gamma_{SD_{8n}}$ *is non-hypoenergetic as well as non-hyperenergetic.*

(c) $\Gamma_{SD_{8n}}$ *is Q-hyperenergetic and L-hyperenergetic.*

Proof. (a) **Case 1:** n is odd

For $n = 3$, using Result 1.4.14 and Theorem 2.4.5, we have $E(\Gamma_{SD_{24}}) = 8 + 8\sqrt{7}$, $LE(\Gamma_{SD_{24}}) = \frac{312}{5}$ and $LE^+(\Gamma_{SD_{24}}) = 36 + 4\sqrt{33}$. Clearly, $E(\Gamma_{SD_{24}}) < LE^+(\Gamma_{SD_{24}}) < LE(\Gamma_{SD_{24}})$.

For $n \geq 5$, using Result 1.4.14 and Theorem 2.4.5, we have

$$LE(\Gamma_{SD_{8n}}) - LE^+(\Gamma_{SD_{8n}}) = \frac{32n^2 - 40n + 12}{2n-1} - 4\sqrt{8n^2 - 16n + 9} \tag{2.4.1}$$

and

$$LE^+(\Gamma_{SD_{8n}}) - E(\Gamma_{SD_{8n}}) = \frac{32n^3 - 116n^2 + 102n - 14}{2n-1} + 4\sqrt{8n^2 - 16n + 9} - 4\sqrt{5n^2 - 6n + 1}. \tag{2.4.2}$$

Since $32n^2 - 40n + 12 > 0$, $4(2n-1)\sqrt{8n^2 - 16n + 9} > 0$ and $(32n^2 - 40n + 12)^2 - \left(4\sqrt{8n^2 - 16n + 9}\right)^2 (2n-1)^2 = 512n^3(n-2) + 128n(5n-1) > 0$ we have

$$32n^2 - 40n + 12 - 4(2n-1)\sqrt{8n^2 - 16n + 9} > 0.$$

Therefore, by equation (2.4.1), $(2n-1)(LE(\Gamma_{SD_{8n}}) - LE^+(\Gamma_{SD_{8n}})) > 0$. Hence, $LE(\Gamma_{SD_{8n}}) > LE^+(\Gamma_{SD_{8n}})$.

Again, we have $\sqrt{8n^2 - 16n + 9} > 0$, $\sqrt{5n^2 - 6n + 1} > 0$ and $\left(\sqrt{8n^2 - 16n + 9}\right)^2 - \left(\sqrt{5n^2 - 6n + 1}\right)^2 = n(3n-10) + 8 > 0$. Therefore, $\sqrt{8n^2 - 16n + 9} - \sqrt{5n^2 - 6n + 1} > 0$. Since $32n^3 - 116n^2 + 102n - 14 > 0$ we have $\frac{32n^3 - 116n^2 + 102n - 14}{2n-1} + 4\sqrt{8n^2 - 16n + 9} - 4\sqrt{5n^2 - 6n + 1} > 0$. Therefore, by equation (2.4.2), $LE^+(\Gamma_{SD_{8n}}) > E(\Gamma_{SD_{8n}})$. Hence, $E(\Gamma_{SD_{8n}}) < LE^+(\Gamma_{SD_{8n}}) < LE(\Gamma_{SD_{8n}})$.

Case 2: n is even

For $n = 2$, using Result 1.4.14 and Theorem 2.4.6, we have $E(\Gamma_{SD_{16}}) = 6 + 2\sqrt{57}$, $LE(\Gamma_{SD_{16}}) = \frac{304}{7}$ and $LE^+(\Gamma_{SD_{16}}) = \frac{134}{7} + 2\sqrt{73}$. Clearly, $E(\Gamma_{SD_{16}}) < LE^+(\Gamma_{SD_{16}}) < LE(\Gamma_{SD_{16}})$.

For $n \geq 4$, using Result 1.4.14 and Theorem 2.4.6, we have

$$LE(\Gamma_{SD_{8n}}) - LE^+(\Gamma_{SD_{8n}}) = \frac{64n^2 - 40n + 6}{4n-1} - 2\sqrt{32n^2 - 32n + 9} \tag{2.4.3}$$

and

$$LE^+(\Gamma_{SD_{8n}}) - E(\Gamma_{SD_{8n}}) = \frac{16n^2(4n-9) + 76n - 8}{4n-1} + 2\sqrt{32n^2 - 32n + 9} - 2\sqrt{20n^2 - 12n + 1}. \tag{2.4.4}$$

Since $64n^2 - 40n + 6 > 0$, $2(4n-1)\sqrt{32n^2 - 32n + 9} > 0$ and

$$(64n^2 - 40n + 6)^2 - (2\sqrt{32n^2 - 32n + 9})^2(4n-1)^2 = 2048n^3(n-1) + 64n(10n-1) > 0$$

we have $64n^2 - 40n + 6 - 2(4n-1)\sqrt{32n^2 - 32n + 9} > 0$. Therefore, by equation (2.4.3), $(4n-1)(LE^+(\Gamma_{SD_{8n}}) - LE(\Gamma_{SD_{8n}})) > 0$. Hence, $LE(\Gamma_{SD_{8n}}) > LE^+(\Gamma_{SD_{8n}})$.

Again, we have $\sqrt{32n^2 - 32n + 9} > 0, \sqrt{20n^2 - 12n + 1} > 0$ and $\left(\sqrt{32n^2 - 32n + 9}\right)^2 - \left(\sqrt{20n^2 - 12n + 1}\right)^2 = 4n(3n-5)+8 > 0$. Thus, $\sqrt{32n^2 - 32n + 9} - \sqrt{20n^2 - 12n + 1} > 0$. Since $16n^2(4n-9) + 76n - 8 > 0$ we have $\frac{16n^2(4n-9)+76n-8}{2n-1} + 2\sqrt{32n^2 - 32n + 9} - 2\sqrt{20n^2 - 12n + 1} > 0$. Therefore, by equation (2.4.4), $LE^+(\Gamma_{SD_{8n}}) > E(\Gamma_{SD_{8n}})$. Hence, $E(\Gamma_{SD_{8n}}) < LE^+(\Gamma_{SD_{8n}}) < LE(\Gamma_{SD_{8n}})$.

(b) **Case 1:** n is odd

Here $|v(\Gamma_{SD_{8n}})| = 8n - 4$ and $E(K_{|v(\Gamma_{SD_{8n}})|}) = LE(K_{|v(\Gamma_{SD_{8n}})|}) = LE^+(K_{|v(\Gamma_{SD_{8n}})|}) = 16n - 10$. Using Result 1.4.14, we have

$$E(\Gamma_{SD_{8n}}) - |v(\Gamma_{SD_{8n}})| = 4\sqrt{(n-1)(5n-1)} - 4n \tag{2.4.5}$$

and

$$E(K_{|v(\Gamma_{SD_{8n}})|}) - E(\Gamma_{SD_{8n}}) = 12n - 6 - 4\sqrt{(n-1)(5n-1)}. \tag{2.4.6}$$

Since $4\sqrt{(n-1)(5n-1)} > 0$, $4n > 0$ and $\left(4\sqrt{(n-1)(5n-1)}\right)^2 - (4n)^2 = 16(4n^2 - 6n + 1) > 0$ we have $4\sqrt{(n-1)(5n-1)} - 4n > 0$. Therefore, by equation (2.4.5), $E(\Gamma_{SD_{8n}}) > |v(\Gamma_{SD_{8n}})|$.

Again, $4\sqrt{(n-1)(5n-1)} > 0$, $12n - 6 > 0$ and $(12n-6)^2 - \left(4\sqrt{(n-1)(5n-1)}\right)^2 = 4(16n^2 - 12n + 5) > 0$ and so $12n - 6 - 4\sqrt{(n-1)(5n-1)} > 0$. Therefore, by equation (2.4.6), $E(K_{|v(\Gamma_{SD_{8n}})|}) > E(\Gamma_{SD_{8n}})$.

Case 2: n is even

Here $|v(\Gamma_{SD_{8n}})| = 8n - 2$ and $E(K_{|v(\Gamma_{SD_{8n}})|}) = LE(K_{|v(\Gamma_{SD_{8n}})|}) = LE^+(K_{|v(\Gamma_{SD_{8n}})|}) = 16n - 6$. Using Result 1.4.14, we have

$$E(\Gamma_{SD_{8n}}) - |v(\Gamma_{SD_{8n}})| = 2\left(\sqrt{(2n-1)(10n-1)} - 2n\right) \tag{2.4.7}$$

and

$$E(K_{|v(\Gamma_{SD_{8n}})|}) - E(\Gamma_{SD_{8n}}) = 2\left(3(2n-1)+1-\sqrt{(2n-1)(10n-1)}\right). \quad (2.4.8)$$

Since $\sqrt{(2n-1)(10n-1)} > 0$, $2n > 0$ and $\left(\sqrt{(2n-1)(10n-1)}\right)^2 - (2n)^2 = 4n(4n-3)+1 > 0$ we have $\sqrt{(2n-1)(10n-1)} - 2n > 0$. Therefore, by equation (2.4.7), $E(\Gamma_{SD_{8n}}) > |v(\Gamma_{SD_{8n}})|$.

Again, we have $\sqrt{(2n-1)(10n-1)} > 0$, $3(2n-1)+1 > 0$ and $(3(2n-1)+1)^2 - \left(\sqrt{(2n-1)(10n-1)}\right)^2 = 4n(4n-3)+3 > 0$ and so $3(2n-1)+1-\sqrt{(2n-1)(10n-1)} > 0$. Therefore, by equation (2.4.8), $E(K_{|v(\Gamma_{SD_{8n}})|}) > E(\Gamma_{SD_{8n}})$.

(c) **Case 1:** n is odd

Using Theorem 2.4.5, for $n = 3$ we have $LE^+(\Gamma_{SD_{24}}) = 36 + 4\sqrt{33}$ and $LE^+(K_{|v(\Gamma_{SD_{24}})|}) = 38$. Also, for $n \geq 5$ we have

$$LE^+(\Gamma_{SD_{8n}}) - LE^+(K_{|v(\Gamma_{SD_{8n}})|}) = \frac{2(8n^2(2n-9)+66n-11)}{2n-1} + 4\sqrt{8n^2-14n+9} > 0.$$

Therefore, $LE^+(\Gamma_{SD_{8n}}) > LE^+(K_{|v(\Gamma_{SD_{8n}})|})$ which implies $\Gamma_{SD_{8n}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{SD_{8n}}$ is L-hyperenergetic.

Case 2: n is even

For $n = 2$ we have $LE^+(K_{|v(\Gamma_{SD_{16}})|}) = 16$ and using Theorem 2.4.6, $LE^+(\Gamma_{SD_{16}}) = \frac{134}{7} + 2\sqrt{73}$. Therefore, $\Gamma_{SD_{16}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{SD_{16}}$ is L-hyperenergetic.

For $n \geq 4$, using Theorem 2.4.6, we have

$$LE^+(\Gamma_{SD_{8n}}) - LE^+(K_{|v(\Gamma_{SD_{8n}})|}) = \frac{64n^2(n-3)+144n-22}{4n-1} + 2\sqrt{32n^2-32n+9} > 0.$$

Therefore, $LE^+(\Gamma_{SD_{8n}}) > LE^+(K_{|v(\Gamma_{SD_{8n}})|})$ which implies $\Gamma_{SD_{8n}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{SD_{8n}}$ is L-hyperenergetic. □

In Theorem 2.4.7, we compare $E(\Gamma_{SD_{8n}})$, $LE(\Gamma_{SD_{8n}})$ and $LE^+(\Gamma_{SD_{8n}})$. However, in the following figures, we show how close are they.

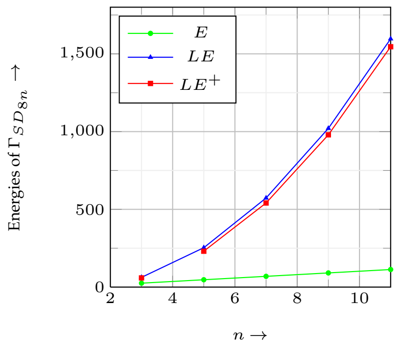

Figure 2.13: Energies of $\Gamma_{SD_{8n}}$, where n is odd

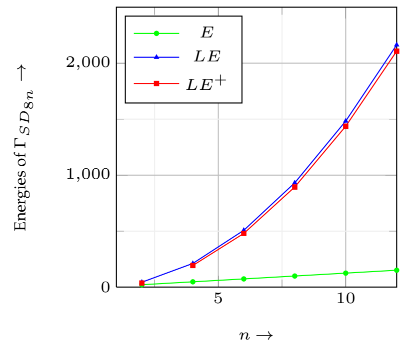

Figure 2.14: Energies of $\Gamma_{SD_{8n}}$, where n is even

2.4.3 The groups, V_{8n}

We consider the groups $V_{8n} := \langle a, b : a^{2n} = b^4 = 1, b^{-1}ab^{-1} = bab = a^{-1}\rangle$, of order $8n$ (where $n > 1$). We compute Signless Laplacian spectrum, Signless Laplacian energy, Laplacian spectrum, Laplacian energy and spectrum and energy of $\Gamma_{V_{8n}}$ (n is even). Energy and Laplacian energy of $\Gamma_{V_{8n}}$ (n is odd) are already known (see Result 1.4.15), the following theorem gives its Signless Laplacian spectrum and Signless Laplacian energy.

Theorem 2.4.8. *Let G be isomorphic to V_{8n}, where n is odd. Then*

$$\text{Q-spec}(\Gamma_{V_{8n}}) = \left\{(8n-4)^{2n}, (4n)^{4n-3}, (8n-6)^{2n-1}, \left(8n-3+\sqrt{32n^2-32n+9}\right)^1, \left(8n-3-\sqrt{32n^2-32n+9}\right)^1\right\}$$

and $$LE^+(\Gamma_{V_{8n}}) = \tfrac{64n^3-128n^2+64n-6}{4n-1} + 2\sqrt{32n^2-32n+9}.$$

Proof. If $G \cong V_{8n}$ and n is odd then $|v(\Gamma_{V_{8n}})| = 8n-2$ and $\Gamma_{V_{8n}} = K_{2n.2,1.(4n-2)}$. Using Result 1.1.4(b), we have

$$\begin{aligned}
Q_{\Gamma_{V_{8n}}}(x) =& \prod_{i=1}^{2}(x-(8n-2)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(8n-2)+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(8n-2)+2p_i}\right)\\
=&(x-8n+4)^{2n}(x-4n)^{4n-3}(x-8n+6)^{2n}(x-2)\left(1-\frac{4n}{x-8n+6}-\frac{4n-2}{x-2}\right)\\
=&(x-(8n-4))^{2n}(x-4n)^{4n-3}(x-(8n-6))^{2n-1}(x^2-(16n-6)x+32n^2-16n).
\end{aligned}$$

Thus, $$\text{Q-spec}(\Gamma_{V_{8n}}) = \left\{(8n-4)^{2n}, (4n)^{4n-3}, (8n-6)^{2n-1}, \left(8n-3+\sqrt{32n^2-32n+9}\right)^1, \left(8n-3-\sqrt{32n^2-32n+9}\right)^1\right\}.$$

Number of edges of $\overline{\Gamma_{V_{8n}}}$ is $8n^2-8n+3$. Therefore, $|e(\Gamma_{V_{8n}})| = \frac{(8n-2)(8n-2-1)}{2} - (8n^2-8n+3) = 12n(2n-1)$. Now,

$$\left|8n-4-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right| = \left|\frac{(8n-4)(n-1)}{4n-1}\right| = \frac{(8n-4)(n-1)}{4n-1},$$

$$\left|4n-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right| = \left|\frac{-8n(n-1)}{4n-1}\right| = \frac{8n(n-1)}{4n-1},$$

$$\left|8n-6-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right| = \left|\frac{(8n^2-20n+6)}{4n-1}\right| = \frac{8n^2-20n+6}{4n-1},$$

$$\begin{aligned}\left|8n-3+\sqrt{32n^2-32n+9}-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right| &= \left|\sqrt{32n^2-32n+9}+2n-\frac{3}{2}+\frac{3}{8n-2}\right| \\ &= \sqrt{32n^2-32n+9}+2n-\frac{3}{2}+\frac{3}{8n-2}\end{aligned}$$

and

$$\begin{aligned}\left|8n-3-\sqrt{32n^2-32n+9}-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right| &= \left|-\sqrt{32n^2-32n+9}+2n-\frac{3}{2}+\frac{3}{8n-2}\right| \\ &= \sqrt{32n^2-32n+9}-2n+\frac{3}{2}-\frac{3}{8n-2}.\end{aligned}$$

Therefore, for $n \geq 3$, we have

$$\begin{aligned}LE^+(\Gamma_{V_{8n}}) =& 2n \times \frac{(8n-4)(n-1)}{4n-1} + (4n-3) \times \frac{8n(n-1)}{4n-1} + (2n-1) \times \frac{8n^2-20n+6}{4n-1} + \\ & \sqrt{32n^2-32n+9}+2n-\frac{3}{2}+\frac{3}{8n-2}+\sqrt{32n^2-32n+9}-2n+\frac{3}{2}-\frac{3}{8n-2}\end{aligned}$$

and the result follows on simplification. □

Theorem 2.4.9. *Let G be isomorphic to V_{8n}, where n is even. Then*

$$\text{Q-spec}(\Gamma_{V_{8n}}) = \left\{(8n-8)^{3n}, (4n)^{4n-5}, (8n-12)^{n-1}, \left(8n-6+2\sqrt{8n^2-16n+9}\right)^1, \left(8n-6-2\sqrt{8n^2-16n+9}\right)^1\right\},$$

$$\text{L-spec}(\Gamma_{V_{8n}}) = \left\{0, (4n)^{4n-5}, (8n-8)^{3n}, (8n-4)^{n}\right\} \textit{ and}$$

$$\text{Spec}(\Gamma_{V_{8n}}) = \Big\{0^{7n-5}, (-4)^{n-1}, \left(2(n-1) + 2\sqrt{(n-1)(5n-1)}\right)^{1}, \left(2(n-1) + 2\sqrt{(n-1)(5n-1)}\right)^{1}\Big\}.$$

Further

$$LE^{+}(\Gamma_{V_{8n}}) = \begin{cases} \frac{24n^3-64n^2+32n+12}{2n-1} + 4\sqrt{8n^2-16n+9}, & \textit{if } n \leq 4 \\ \frac{32n^3-112n^2+96n-12}{2n-1} + 4\sqrt{8n^2-16n+9}, & \textit{if } n \geq 6, \end{cases}$$

$$LE(\Gamma_{V_{8n}}) = \frac{8n(4n^2-10n+7)}{2n-1} \textit{ and } E(\Gamma_{V_{8n}}) = 4(n-1) + 4\sqrt{(n-1)(5n-1)}.$$

Proof. If $G \cong V_{8n}$ and n is even then $|v(\Gamma_{V_{8n}})| = 8n-4$ and $\Gamma_{V_{8n}} = K_{n.4,1.(4n-4)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_{V_{8n}}}(x) =& \prod_{i=1}^{2}(x-(8n-4)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(8n-4)+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(8n-4)+2p_i}\right) \\ =&(x-8n+8)^{3n}(x-4n)^{4n-5}(x-8n+12)^{n}(x-4)\left(1-\frac{4n}{x-8n+12}-\frac{4n-4}{x-4}\right) \\ =&(x-(8n-8))^{3n}(x-4n)^{4n-5}(x-(8n-12))^{n-1}(x^2-(16n-12)x+32n^2-32n). \end{aligned}$$

Thus $\text{Q-spec}(\Gamma_{V_{8n}}) = \Big\{(8n-8)^{3n}, (4n)^{4n-5}, (8n-12)^{n-1}, \left(8n-6+2\sqrt{8n^2-16n+9}\right)^{1}, \left(8n-6-2\sqrt{8n^2-16n+9}\right)^{1}\Big\}.$

Number of edges of $\overline{\Gamma_{V_{8n}}}$ is $8n^2-12n+10$ and so $|e(\Gamma_{V_{8n}})| = \frac{(8n-4)(8n-4-1)}{2} - (8n^2-12n+10) = 24n(n-1)$. Now,

$$\left|8n-8-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right| = \left|\frac{4(n-1)(n-2)}{2n-1}\right| = \frac{4(n-1)(n-2)}{2n-1},$$

$$\left|4n-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right| = \left|\frac{4n(2-n)}{2n-1}\right| = \frac{4n(n-2)}{2n-1},$$

$$\left|8n-12-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right| = \left|\frac{4(n^2-5n+3)}{2n-1}\right| = \begin{cases} \frac{4(-n^2+5n-3)}{2n-1}, & \text{if } n \leq 4 \\ \frac{4(n^2-5n+3)}{2n-1}, & \text{if } n \geq 6, \end{cases}$$

$$\begin{aligned}\left|8n-6+2\sqrt{8n^2-16n+9}-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right| &= \left|2\sqrt{8n^2-16n+9}+(2n-3)+\frac{3}{2n-1}\right|\\ &=2\sqrt{8n^2-16n+9}+(2n-3)+\frac{3}{2n-1}\end{aligned}$$

and

$$\begin{aligned}\left|8n-6-2\sqrt{8n^2-16n+9}-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right| &= \left|-2\sqrt{8n^2-16n+9}+(2n-3)+\frac{3}{2n-1}\right|\\ &=2\sqrt{8n^2-16n+9}-(2n-3)-\frac{3}{2n-1}.\end{aligned}$$

Therefore, for $n \leq 4$, we have

$$\begin{aligned}LE^+(\Gamma_{V_{8n}}) =& 3n \times \frac{4(n-1)(n-2)}{2n-1} + (4n-5) \times \frac{4n(n-2)}{2n-1} + (n-1) \times \frac{4(-n^2+5n-3)}{2n-1} + \\ & 2\sqrt{8n^2-16n+9} - 2n - 3 + \frac{3}{2n-1} + 2\sqrt{8n^2-16n+9} + 2n + 3 - \frac{3}{2n-1}.\end{aligned}$$

For $n \geq 6$, we have

$$\begin{aligned}LE^+(\Gamma_{V_{8n}}) =& 3n \times \frac{4(n-1)(n-2)}{2n-1} + (4n-5) \times \frac{4n(n-2)}{2n-1} + (n-1) \times \frac{4(n^2-5n+3)}{2n-1} + \\ & 2\sqrt{8n^2-16n+9} - 2n - 3 + \frac{3}{2n-1} + 2\sqrt{8n^2-16n+9} + 2n + 3 - \frac{3}{2n-1}.\end{aligned}$$

Thus we get the required expressions for $LE^+(\Gamma_{V_{8n}})$ on simplification.

Since $\overline{\Gamma_{V_{8n}}} = nK_4 \cup K_{4n-4}$, using Result 1.1.3, we have

$$\text{L-spec}(\Gamma_{V_{8n}}) = \left\{(0)^1, \left(\sum_{i=1}^{2} l_i m_i - m_2\right)^{l_2(m_2-1)}, \left(\sum_{i=1}^{2} l_i m_i - m_1\right)^{l_1(m_1-1)}, \left(\sum_{i=1}^{2} l_i m_i\right)^{\sum_{i=1}^{2} l_i - 1}\right\},$$

where $l_1 = n, l_2 = 1, m_1 = 4$ and $m_2 = 4n-4$. Therefore

$$\begin{aligned}\text{L-spec}(\Gamma_{V_{8n}}) =& \Big\{(0)^1, (n.4+4n-4-4n+4)^{1(4n-4-1)}, (n.4+4n-4-4)^{n(4-1)},\\ & (n.4+4n-4)^{n+1-1}\Big\}\\ =& \Big\{(0)^1, (4n)^{4n-5}, (8n-8)^{3n}, (8n-4)^n\Big\}.\end{aligned}$$

Now,

$$\left|0-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right|=\left|\frac{-12n(n-1)}{2n-1}\right|=\frac{12n(n-1)}{2n-1},$$

$$\left|4n-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right|=\left|\frac{4n(2-n)}{2n-1}\right|=\frac{4n(n-2)}{2n-1},$$

$$\left|8n-8-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right|=\left|\frac{4(n-1)(n-2)}{2n-1}\right|=\frac{4(n-1)(n-2)}{2n-1}$$

and

$$\left|8n-4-\frac{2|e(\Gamma_{V_{8n}})|}{|v(\Gamma_{V_{8n}})|}\right|=\left|\frac{4(n^2-n+1)}{2n-1}\right|=\frac{4(n^2-n+1)}{2n-1}.$$

Therefore, $LE(\Gamma_{V_{8n}})=1\times\frac{12n(n-1)}{2n-1}+(4n-5)\times\frac{4n(n-2)}{2n-1}+3n\times\frac{4(n-1)(n-2)}{2n-1}+n\times\frac{4(n^2-n+1)}{2n-1}$ and we get the required expression for $LE(\Gamma_{V_{8n}})$ on simplification.

Since $\Gamma_{V_{8n}}$ is a complete $(n+1)$-partite graph with $8n-4$ vertices and $\Gamma_{V_{8n}}=K_{n.4,1.(4n-4)}$. Therefore, using Result 1.1.4(a), the characteristic polynomial of $\Gamma_{V_{8n}}$ is

$$\begin{aligned}P_{\Gamma_{V_{8n}}}(x)&=x^{(8n-4)-(n+1)}(x+4)^{n-1}(x+4n-4)^{1-1}(x^2-(4n-4)x-16n^2-16n)\\&=x^{7n-5}(x+4)^{n-1}(x^2-(4n-4)x-16n^2-16n).\end{aligned}$$

$$\text{Thus}\,\mathrm{Spec}(\Gamma_{V_{8n}})=\left\{(0)^{7n-5},(-4)^{n-1},\left(2(n-1)+2\sqrt{(n-1)(5n-1)}\right)^1,\left(2(n-1)+2\sqrt{(n-1)(5n-1)}\right)^1\right\}.$$

Therefore, $E(\Gamma_{V_{8n}})=(7n-5)\times|0|+(n-1)\times|-4|+|2(n-1)+2\sqrt{(n-1)(5n-1)}|+|2(n-1)-2\sqrt{(n-1)(5n-1)}|$ and we get the required expression for $E(\Gamma_{V_{8n}})$ on simplification. □

Theorem 2.4.10. *If G is isomorphic to V_{8n} then*

(a) $E(\Gamma_{V_{8n}})\leq LE^+(\Gamma_{V_{8n}})\leq LE(\Gamma_{V_{8n}})$*; equality holds if and only if* $G\cong V_{16}$.

(b) $\Gamma_{V_{8n}}$ *is non-hypoenergetic as well as non-hyperenergetic.*

(c) $\Gamma_{V_{16}}$ *is not L-hyperenergetic and not Q-hyperenergetic. If $n\neq 2$ then $\Gamma_{V_{8n}}$ is Q-hyperenergetic and L-hyperenergetic.*

Proof. (a) **Case 1:** n is odd

Using Result 1.4.15 and Theorem 2.4.8, we have

$$LE(\Gamma_{V_{8n}}) - LE^+(\Gamma_{V_{8n}}) = \frac{64n^2 - 40n + 6}{4n-1} - 2\sqrt{32n^2 - 32n + 9} \qquad (2.4.9)$$

and

$$LE^+(\Gamma_{V_{8n}}) - E(\Gamma_{V_{8n}}) = \frac{16n^2(4n-9) + 76n - 8}{4n-1} + 2\sqrt{32n^2 - 32n + 9} - 2\sqrt{20n^2 - 12n + 1}. \qquad (2.4.10)$$

Since $64n^2 - 40n + 6 > 0$, $2(4n-1)\sqrt{32n^2 - 32n + 9} > 0$ and

$$(64n^2 - 40n + 6)^2 - (2\sqrt{32n^2 - 32n + 9})^2(4n-1)^2 = 2048n^3(n-1) + 64n(10n-1) > 0$$

we have $64n^2 - 40n + 6 - 2(4n-1)\sqrt{32n^2 - 32n + 9} > 0$. Therefore, by equation (2.4.9), $(4n-1)(LE^+(\Gamma_{V_{8n}}) - LE(\Gamma_{V_{8n}})) > 0$. Hence, $LE(\Gamma_{V_{8n}}) > LE^+(\Gamma_{V_{8n}})$.

Again, we have $\sqrt{32n^2 - 32n + 9} > 0, \sqrt{20n^2 - 12n + 1} > 0$ and $\left(\sqrt{32n^2 - 32n + 9}\right)^2 - \left(\sqrt{20n^2 - 12n + 1}\right)^2 = 4n(3n-5) + 8 > 0$. Thus, $\sqrt{32n^2 - 32n + 9} - \sqrt{20n^2 - 12n + 1} > 0$. Since $16n^2(4n-9) + 76n - 8 > 0$ we have $\frac{16n^2(4n-9)+76n-8}{2n-1} + 2\sqrt{32n^2 - 32n + 9} - 2\sqrt{20n^2 - 12n + 1} > 0$. Therefore, by equation (2.4.10), $LE^+(\Gamma_{V_{8n}}) > E(\Gamma_{V_{8n}})$. Hence, $E(\Gamma_{V_{8n}}) < LE^+(\Gamma_{V_{8n}}) < LE(\Gamma_{V_{8n}})$.

Case 2: n is even

Using Result 1.4.15 and Theorem 2.4.9, for $n \leq 4$, we have

$$LE(\Gamma_{V_{8n}}) - LE^+(\Gamma_{V_{8n}}) = \frac{8n^3 - 16n^2 + 24n - 12}{2n-1} - 4\sqrt{8n^2 - 16n + 9} \qquad (2.4.11)$$

and

$$LE^+(\Gamma_{V_{8n}}) - E(\Gamma_{V_{8n}}) = \frac{4(n-2)(6n^2 - 6n - 1)}{2n-1} + 4\sqrt{8n^2 - 16n + 9} - 4\sqrt{5n^2 - 6n + 1}. \qquad (2.4.12)$$

Since $8n^3 - 16n^2 + 24n - 12 > 0$, $4(2n-1)\sqrt{8n^2 - 16n + 9} > 0$ and
$(8n^3 - 16n^2 + 24n - 12)^2 - \left(4\sqrt{8n^2 - 16n + 9}\right)^2(2n-1)^2 = 64n(n-2)^2(n-1)(n^2+n-1) \geq 0$

(equality holds if and only if $n = 2$) we have $8n^3-16n^2+24n-12-4(2n-1)\sqrt{8n^2-16n+9} \geq 0$. Therefore, by equation (2.4.11), $(2n-1)(LE(\Gamma_{V_{8n}})-LE^+(\Gamma_{V_{8n}})) \geq 0$. Hence, $LE(\Gamma_{V_{8n}}) \geq LE^+(\Gamma_{V_{8n}})$ equality holds if and only if $G \cong V_{16}$.

Again, we have $\sqrt{8n^2-16n+9} > 0, \sqrt{5n^2-6n+1} > 0$ and $\left(\sqrt{8n^2-16n+9}\right)^2 - \left(\sqrt{5n^2-6n+1}\right)^2 = n(3n-10)+8 \geq 0$ (equality holds if and only if $n = 2$). Therefore, $\sqrt{8n^2-16n+9} - \sqrt{5n^2-6n+1} \geq 0$. Since $4(n-2)(6n^2-6n-1) \geq 0$ we have $\frac{4(n-2)(6n^2-6n-1)}{2n-1}+4\sqrt{8n^2-16n+9}-4\sqrt{5n^2-6n+1} \geq 0$. Therefore, by equation (2.4.12), $LE^+(\Gamma_{V_{8n}}) \geq E(\Gamma_{V_{8n}})$. Hence, $E(\Gamma_{V_{8n}}) \leq LE^+(\Gamma_{V_{8n}}) \leq LE(\Gamma_{V_{8n}})$ equality holds if and only if $G \cong V_{16}$.

Using Result 1.4.15 and Theorem 2.4.9, for $n \geq 6$, we have

$$LE(\Gamma_{V_{8n}}) - LE^+(\Gamma_{V_{8n}}) = \frac{32n^2-40n+12}{2n-1} - 4\sqrt{8n^2-16n+9} \tag{2.4.13}$$

and

$$LE^+(\Gamma_{V_{8n}}) - E(\Gamma_{V_{8n}}) = \frac{32n^3-116n^2+102n-14}{2n-1} + 4\sqrt{8n^2-16n+9} - 4\sqrt{5n^2-6n+1}. \tag{2.4.14}$$

Since $32n^2-40n+12 > 0$, $4(2n-1)\sqrt{8n^2-16n+9} > 0$ and $(32n^2-40n+12)^2 - \left(4\sqrt{8n^2-16n+9}\right)^2(2n-1)^2 = 512n^3(n-2)+128n(5n-1) > 0$ we have $32n^2-40n+12-4(2n-1)\sqrt{8n^2-16n+9} > 0$. Therefore, by equation (2.4.13), $(2n-1)(LE(\Gamma_{V_{8n}}) - LE^+(\Gamma_{V_{8n}})) > 0$. Hence, $LE(\Gamma_{V_{8n}}) > LE^+(\Gamma_{V_{8n}})$.

Again, we have $\sqrt{8n^2-16n+9} > 0, \sqrt{5n^2-6n+1} > 0$ and $\left(\sqrt{8n^2-16n+9}\right)^2 - \left(\sqrt{5n^2-6n+1}\right)^2 = n(3n-10)+8 > 0$. Therefore, $\sqrt{8n^2-16n+9} - \sqrt{5n^2-6n+1} > 0$. Since $32n^3-116n^2+102n-14 > 0$ we have $\frac{32n^3-116n^2+102n-14}{2n-1} + 4\sqrt{8n^2-16n+9} - 4\sqrt{5n^2-6n+1} > 0$. Therefore, by equation (2.4.14), $LE^+(\Gamma_{V_{8n}}) > E(\Gamma_{V_{8n}})$. Hence, $E(\Gamma_{V_{8n}}) < LE^+(\Gamma_{V_{8n}}) < LE(\Gamma_{V_{8n}})$.

(b) **Case 1:** n is odd

Here $|v(\Gamma_{V_{8n}})| = 8n-2$ and $E(K_{|v(\Gamma_{V_{8n}})|}) = LE(K_{|v(\Gamma_{V_{8n}})|}) = LE^+(K_{|v(\Gamma_{V_{8n}})|}) = 16n-6$.

Using Result 1.4.15, we have

$$E(\Gamma_{V_{8n}}) - |v(\Gamma_{V_{8n}})| = 2\left(\sqrt{(2n-1)(10n-1)} - 2n\right) \tag{2.4.15}$$

and

$$E(K_{|v(\Gamma_{V_{8n}})|}) - E(\Gamma_{V_{8n}}) = 2\left(3(2n-1)+1-\sqrt{(2n-1)(10n-1)}\right). \tag{2.4.16}$$

Since $\sqrt{(2n-1)(10n-1)} > 0$, $2n > 0$ and $\left(\sqrt{(2n-1)(10n-1)}\right)^2 - (2n)^2 = 4n(4n-3) + 1 > 0$ we have $\sqrt{(2n-1)(10n-1)} - 2n > 0$. Therefore, by equation (2.4.15), $E(\Gamma_{V_{8n}}) > |v(\Gamma_{V_{8n}})|$.

Again, we have $\sqrt{(2n-1)(10n-1)} > 0$, $3(2n-1)+1 > 0$ and $(3(2n-1)+1)^2 - \left(\sqrt{(2n-1)(10n-1)}\right)^2 = 4n(4n-3)+3 > 0$ and so $3(2n-1)+1-\sqrt{(2n-1)(10n-1)} > 0$. Therefore, by equation (2.4.16), $E(K_{|v(\Gamma_{V_{8n}})|}) > E(\Gamma_{V_{8n}})$.

Case 2: n is even

Here $|v(\Gamma_{V_{8n}})| = 8n-4$ and $E(K_{|v(\Gamma_{V_{8n}})|}) = LE(K_{|v(\Gamma_{V_{8n}})|}) = LE^+(K_{|v(\Gamma_{V_{8n}})|}) = 16n-10$. Using Result 1.4.15, we have

$$E(\Gamma_{V_{8n}}) - |v(\Gamma_{V_{8n}})| = 4\sqrt{(n-1)(5n-1)} - 4n \tag{2.4.17}$$

and

$$E(K_{|v(\Gamma_{V_{8n}})|}) - E(\Gamma_{V_{8n}}) = 12n-6-4\sqrt{(n-1)(5n-1)}. \tag{2.4.18}$$

Since $4\sqrt{(n-1)(5n-1)} > 0$, $4n > 0$ and

$$\left(4\sqrt{(n-1)(5n-1)}\right)^2 - (4n)^2 = 16(4n^2-6n+1) > 0$$

we have $4\sqrt{(n-1)(5n-1)} - 4n > 0$. Therefore, by equation (2.4.17), $E(\Gamma_{V_{8n}}) > |v(\Gamma_{V_{8n}})|$.

Again, $4\sqrt{(n-1)(5n-1)} > 0$, $12n-6 > 0$ and

$$(12n-6)^2 - \left(4\sqrt{(n-1)(5n-1)}\right)^2 = 4(16n^2-12n+5) > 0$$

we have $12n-6-4\sqrt{(n-1)(5n-1)} > 0$. Therefore, by equation (2.4.18), $E(K_{|v(\Gamma_{V_{8n}})|}) > E(\Gamma_{V_{8n}})$.

(c) **Case 1:** n is odd

Using Theorem 2.4.8 we have

$$LE^+(\Gamma_{V_{8n}}) - LE^+(K_{|v(\Gamma_{V_{8n}})|}) = \frac{64n^2(n-3)+144n-22}{4n-1} + 2\sqrt{32n^2-32n+9} > 0.$$

Therefore, $LE^+(\Gamma_{V_{8n}}) > LE^+(K_{|v(\Gamma_{V_{8n}})|})$ which implies $\Gamma_{V_{8n}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{V_{8n}}$ is L-hyperenergetic.

Case 2: n is even

Using Theorem 2.4.9, for $n = 2$, we have $LE(\Gamma_{V_{8n}}) = 16$ and $LE(K_{|v(\Gamma_{V_{8n}})|}) = 22$. Clearly, $LE(\Gamma_{V_{8n}}) < LE(K_{|v(\Gamma_{V_{8n}})|})$. For $n \leq 4$,

$$LE^+(\Gamma_{V_{8n}}) - LE^+(K_{|v(\Gamma_{V_{8n}})|}) = \frac{4(6n^2(n-4) + 20n - 1)}{2n-1} + 4\sqrt{8n^2 - 16n + 9} > 0$$

for all $n \neq 2$. Therefore, for all $n \neq 2$, $LE^+(\Gamma_{V_{8n}}) > LE^+(K_{|v(\Gamma_{V_{8n}})|})$ which implies $\Gamma_{V_{8n}}$ is Q-hyperenergetic and consequently part (a) implies $\Gamma_{V_{8n}}$ is L-hyperenergetic. For $n \geq 6$,

$$LE^+(\Gamma_{V_{8n}}) - LE^+(K_{|v(\Gamma_{V_{8n}})|}) = \frac{2(8n^2(2n-9) + 66n - 11)}{2n-1} + 4\sqrt{8n^2 - 16n + 9} > 0.$$

Hence, the result holds. □

In Theorem 2.4.10, we compare $E(\Gamma_{V_{8n}})$, $LE(\Gamma_{V_{8n}})$ and $LE^+(\Gamma_{V_{8n}})$. However, in the following figures, we show how close are they.

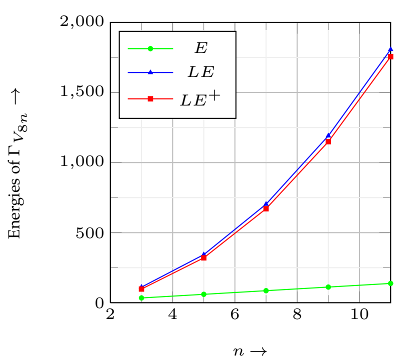

Figure 2.15: Energies of $\Gamma_{V_{8n}}$, where n is odd

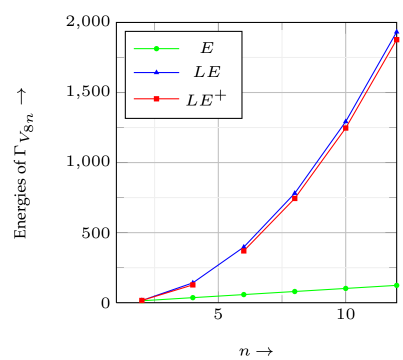

Figure 2.16: Energies of $\Gamma_{V_{8n}}$, where n is even

2.4.4 The Frobenious groups of order pq

We consider $F_{p,q} := \langle a, b : a^p = b^q = 1, b^{-1}ab = a^u\rangle$, the Frobenious groups of order pq, where p and q are two primes such that $q|(p-1)$ and u is an integer such that $\bar{u} \in \mathbb{Z}_p \setminus \{\bar{0}\}$ having order q. The following theorem gives the Signless Laplacian spectrum and Signless Laplacian energy of non-commuting graph of the group $F_{p,q}$.

Theorem 2.4.11. *Let G be isomorphic to $F_{p,q}$. Then*

$$\text{Q-spec}(\Gamma_G) = \left\{(pq-p)^{p-2}, (pq-q)^{pq-2p}, (pq-2q+1)^{p-1}, \left(\frac{A}{2}\right)^1, \left(\frac{B}{2}\right)^1\right\},$$

where $A = 3pq - 2p - 2q + 1 + \sqrt{pq(pq-2) + 4(p-q)(pq-p-q+1)+1}$ *and*
$B = 3pq - 2p - 2q + 1 - \sqrt{pq(pq-2) + 4(p-q)(pq-p-q+1)+1}$ *and*

$$\begin{aligned} LE^+(\Gamma_G) =& \frac{2p^3q - p^2q^2 - 2pq^2 - 6pq - 4p^3 + 6p^2 + 2q - 1}{pq-1} \\ &+ \sqrt{pq(pq-2) + 4(p-q)(pq-p-q+1)+1}. \end{aligned}$$

Proof. If $G \cong F_{p,q}$ then $|v(\Gamma_G)| = pq - 1$ and $\Gamma_G = K_{1.(p-1),p.(q-1)}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_G}(x) =& \prod_{i=1}^{2}(x-(pq-1)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(pq-1)+2p_i)^{a_i}\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(pq-1)+2p_i}\right) \\ =&(x-pq+p)^{p-2}(x-pq+q)^{pq-2p}(x-pq+2p-1)(x-pq+2q-1)^p \\ &\times\left(1-\frac{p-1}{x-pq+2p-1}-\frac{pq-p}{x-pq+2q-1}\right) \\ =&(x-(pq-p))^{p-2}(x-(pq-q))^{pq-2p}(x-(pq-2q+1))^p \\ &\times(x^2-(3pq-2p-2q+1)x+2p^2q^2-4p^2q+2pq-2pq^2+2p^2-2p+2q-1). \end{aligned}$$

Thus, $\text{Q-spec}(\Gamma_G) = \left\{(pq-p)^{p-2}, (pq-q)^{pq-2p}, (pq-2q+1)^{p-1}, \left(\frac{A}{2}\right)^1, \left(\frac{B}{2}\right)^1\right\}$.

Number of edges of $\overline{\Gamma_G}$ is $\frac{p^2-3p+2+pq^2-3pq+2p}{2}$. Therefore,

$$|e(\Gamma_G)| = \frac{(pq-1)(pq-2)}{2} - \frac{p^2-3p+2+pq^2-3pq+2p}{2} = \frac{(p^2-p)(q^1-1)}{2}.$$

Now,

$$\left|pq-p-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{(q-1)(pq-p^2)}{pq-1}\right| = \frac{(q-1)(p^2-pq)}{pq-1},$$

$$\left|pq-q-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{(p-q)(p-1)}{pq-1}\right| = \frac{(p-q)(p-1)}{pq-1},$$

$$\left|pq-2q+1-\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{(p^2+2q)-(pq^2+p+1)}{pq-1}\right| = \frac{-(p^2+2q)+(pq^2+p+1)}{pq-1},$$

$$\left|A - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{p^2q^2 + 2p^2 + p + 2q - 2p^2q - 2pq - 1}{2(pq-1)} + \frac{C}{2}\right|$$
$$= \frac{p^2q^2 + 2p^2 + p + 2q - 2p^2q - 2pq - 1}{2(pq-1)} + \frac{C}{2}$$

and

$$\left|B - \frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|}\right| = \left|\frac{p^2q^2 + 2p^2 + p + 2q - 2p^2q - 2pq - 1}{2(pq-1)} - \frac{C}{2}\right|$$
$$= \frac{-(p^2q^2 + 2p^2 + p + 2q - 2p^2q - 2pq - 1)}{2(pq-1)} + \frac{C}{2},$$

where $C = \sqrt{pq(pq-2) + 4(p-q)(pq-p-q+1) + 1}$. Therefore,

$$LE^+(\Gamma_G) = (p-2) \times \frac{(q-1)(p^2-pq)}{pq-1} + (pq-2p) \times \frac{(p-q)(p-1)}{pq-1}$$
$$+ (p-1) \times \frac{(pq^2+p+1) - (p^2+2q)}{pq-1} + \frac{p^2q^2 + 2p^2 + p + 2q - 2p^2q - 2pq - 1}{2(pq-1)}$$
$$+ \frac{C}{2} - \left(\frac{p^2q^2 + 2p^2 + p + 2q - 2p^2q - 2pq - 1}{2(pq-1)}\right) + \frac{C}{2}$$

and the result follows on simplification. □

2.5 Some implications of the preceding findings

It has been shown in [26] that the non-commuting graphs of the groups considered in this chapter are L-integral. Also, in [40, Chapter 4], several conditions have been obtained such that the non-commuting graphs of these groups are integral. In view of Theorems 2.1.11 and 2.3.1, it follows that Γ_G is not Q-integral if $G \cong U_{6n}$ or $\frac{G}{Z(G)} \cong Sz(2)$. However, Γ_G is Q-integral if $G \cong A(n, \mathcal{V})$, $A(n,p)$ or $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ (follows from Theorems 2.2.1, 2.4.1 and 2.4.3). As a consequence of our results we also have the following theorem related to Question 1.4.1.

Theorem 2.5.1. *Γ_G is Q-integral if*

(a) *$G \cong D_{2r}, M_{2rs}$ or SD_{8r}, r is odd and $8r^2 - 16r + 9$ is a perfect square.*

(b) *$G \cong V_{8n}$, n is even and $8n^2 - 16n + 9$ is a perfect square.*

(c) $G \cong Q_{4n}$ *or* $\frac{G}{Z(G)} \cong D_{2n}$ *and* $8n^2 - 16n + 9$ *is a perfect square.*

(d) $G \cong D_{2r}$ *or* M_{2rs}*,* r *is even and* $2r^2 - 8r + 9$ *is a perfect square.*

(e) $G \cong V_{8n}$*,* n *is odd and* $32n^2 - 32n + 9$ *is a perfect square.*

(f) $G \cong SD_{8n}$*,* n *is even and* $32n^2 - 32n + 9$ *is a perfect square.*

(g) $G \cong QD_{2^n}$ *and* $2^{2n-1} - 2^{n+2} + 9$ *is a perfect square.*

In the following table we give some positive integers n such that $8n^2 - 16n + 9$, $2n^2 - 8n + 9$ and $32n^2 - 32n + 9$ are perfect squares. It may be interesting to obtain general terms of such sequences of positive integers.

n	$\sqrt{8n^2 - 16n + 9}$	n	$\sqrt{2n^2 - 8n + 9}$	n	$\sqrt{32n^2 - 32n + 9}$
1	1	2	1	1	3
2	3	4	3	18	99
7	17	14	17	595	3363
36	99	72	99	20196	114243
205	577	410	577	686053	3880899
1190	3363	2380	3363	23305590	131836323
6931	19601	13862	19601		
40392	114243	80784	114243		
235417	665857	470834	665857		
1372106	3880899	2744212	3880899		
7997215	22619537	15994430	22619537		
46611180	131836323	93222360	131836323		
271669861	768398401	543339722	768398401		

Table 2.1

As consequences of our results we also have the following theorems related to Questions 1.4.2 and 1.4.3.

Theorem 2.5.2. *Let G be a finite non-abelian group. Then*

(a) $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$ *if* $G \cong D_8, Q_8, M_{8s}, A(n, \mathcal{V}), A(n, p), V_{16}$. *Also, if* $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, *where p is a prime, then* $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$.

(b) $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G)$ *if* $G \cong D_{2m}(m \neq 4)$, QD_{2^n}, $M_{2rs}(r \neq 4)$, $Q_{4n}(n \neq 2)$, U_{6n}, SD_{8n} *and* V_{8n} $(n \neq 2)$. *Also, if* $\frac{G}{Z(G)} \cong D_{2m}(m \geq 3)$ *and* $Sz(2)$ *then* $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G)$.

(c) Γ_G *is non-hypoenergetic as well as non-hyperenergetic if* $G \cong D_{2m}, QD_{2^n}, M_{2rs}, Q_{4n}, U_{6n}, A(n, \mathcal{V})$, $A(n, p), SD_{8n}$ *and* V_{8n}. *Also, if* $\frac{G}{Z(G)} \cong D_{2m}, \mathbb{Z}_p \times \mathbb{Z}_p$ *and* $Sz(2)$, *where* $m \geq 3$ *and p is a prime, then* Γ_G *is non-hypoenergetic as well as non-hyperenergetic.*

(d) Γ_G *is L-hyperenergetic but not Q-hyperenergetic if* $G \cong D_6, M_6$ *and* $Sz(2)$.

(e) Γ_G *is neither L-hyperenergetic nor Q-hyperenergetic if* $G \cong D_8, M_{8s}, Q_8, A(n, \mathcal{V})$, $A(n, p), V_{16}$. *Also, if* $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ *then* Γ_G *is neither L-hyperenergetic nor Q-hyperenergetic.*

(f) Γ_G *is L-hyperenergetic as well as Q-hyperenergetic if* $G \cong D_{2m}(m \neq 3, 4)$, QD_{2^n}, $M_{2rs}(2rs \neq 6, 8s)$, $Q_{4n}(n \neq 2)$, U_{6n}, SD_{8n} *and* $V_{8n}(n \neq 2)$. *Also, if* $\frac{G}{Z(G)} \cong Sz(2)$ $(G \ncong Sz(2))$ *and* D_{2m} $(m = 3, 4$ *and* $|Z(G)| \neq 1$ *or* $m \geq 5$ *and* $|Z(G)| \geq 1)$ *then* Γ_G *is L-hyperenergetic as well as Q-hyperenergetic.*

In the following theorem we get an example of a graph (non-commuting graph of the symmetric group of degree 4) disproving Conjecture 1.1.7.

Theorem 2.5.3. *Let the commuting graph of a finite group G be planar. Then*

(a) Γ_G *is Q-integral if* $G \cong D_8, Q_8, \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, \mathcal{M}_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 * \mathbb{Z}_4$ *or* $SG(16, 3)$, *otherwise not Q-integral.*

(b) Γ_G *is non-hypoenergetic.*

(c) Γ_G *is hyperenergetic if* $G \cong S_4$*, otherwise non-hyperenergetic.*

(d) Γ_G *is Q-hyperenergetic if* $G \cong D_{10}, D_{12}, Q_{12}, A_4, A_5, S_4$ *or* $SL(2,3)$*, otherwise not Q-hyperenergetic.*

(e) Γ_G *is L-hyperenergetic if* $G \cong D_6, D_{10}, D_{12}, Q_{12}, A_4, A_5, S_4, SL(2,3)$ *or* $Sz(2)$*, otherwise not L-hyperenergetic.*

Proof. If the commuting graph of G is planar then, by Result 1.4.19, $G \cong D_6, D_8, D_{10}, D_{12}, Q_8, Q_{12}, \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, \mathcal{M}_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 * \mathbb{Z}_4, SG(16,3), A_4, A_5, S_4, SL(2,3), Sz(2)$.

By Theorems 2.1.1, 2.1.2 and 2.1.9, Γ_G is Q-integral if $G \cong D_8, Q_8$ and not Q-integral if $G \cong D_6, D_{10}, D_{12}, Q_{12}$. If $G \cong D_6$, then from Theorem 2.1.3, Γ_G is non-hypoenergetic, non-hyperenergetic, not Q-hyperenergetic but is L-hyperenergetic. If $G \cong D_{10}, D_{12}$, then from Theorem 2.1.3, Γ_G is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic and L-hyperenergetic. If $G \cong D_8$, then from Theorem 2.1.3, Γ_G is non-hypoenergetic, non-hyperenergetic, not Q-hyperenergetic and not L-hyperenergetic.

If $G \cong Q_8$, then from Theorem 2.1.10, Γ_G is non-hypoenergetic, non-hyperenergetic, not Q-hyperenergetic and not L-hyperenergetic. If $G \cong Q_{12}$, then from Theorem 2.1.10, Γ_G is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic and L-hyperenergetic.

If $G \cong \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, \mathcal{M}_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 * \mathbb{Z}_4, SG(16,3)$, then $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Using Theorems 2.2.1 and 2.2.2, for $p = 2$, we get Γ_G is Q-integral but not hypoenergetic, hyperenergetic, Q-hyperenergetic as well as L-hyperenergetic.

If $G \cong A_4$, then by Result 1.4.17(a), we have $E(\Gamma_{A_4}) = 6 + 2\sqrt{33}$ and $LE(\Gamma_{A_4}) = \frac{224}{11}$. Now, $|v(\Gamma_{A_4})| = 11$ so $E(K_{|v(\Gamma_{A_4})|}) = LE^+(K_{|v(\Gamma_{A_4})|}) = LE(K_{|v(\Gamma_{A_4})|}) = 20$. Here, $\Gamma_{A_4} = K_{4.2,1.3}$ so using Result 1.1.4(b) we get

$$\text{Q-spec}(\Gamma_{A_4}) = \left\{ (9)^4, (8)^2, (7)^3, \left(\frac{23+\sqrt{145}}{2}\right)^1, \left(\frac{23-\sqrt{145}}{2}\right)^1 \right\}.$$

It follows that Γ_{A_4} is not Q-integral. We have $|e(\Gamma_{A_4})| = 48$ and so $\frac{2|e(\Gamma_{A_4})|}{|v(\Gamma_{A_4})|} = \frac{96}{11}$. Therefore, $\left|9 - \frac{96}{11}\right| = \frac{3}{11}$, $\left|8 - \frac{96}{11}\right| = \frac{8}{11}$, $\left|7 - \frac{96}{11}\right| = \frac{19}{11}$, $\left|\frac{23+\sqrt{145}}{2} - \frac{96}{11}\right| = \frac{61}{22} + \frac{\sqrt{145}}{2}$ and $\left|\frac{23-\sqrt{145}}{2} - \frac{96}{11}\right| = -\frac{61}{22} + \frac{\sqrt{145}}{2}$. Thus,

$$LE^+(\Gamma_{A_4}) = 4 \times \frac{3}{11} + 2 \times \frac{8}{11} + 3 \times \frac{19}{11} + \frac{61}{22} + \frac{\sqrt{145}}{2} - \frac{61}{22} + \frac{\sqrt{145}}{2} = \frac{85}{11} + \sqrt{145} > 20.$$

Hence, Γ_{A_4} is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic as well as L-hyperenergetic.

If $G \cong A_5$, then by Result 1.4.17(b), we have $E(\Gamma_{A_5}) \approx 111.89$ and $LE(\Gamma_{A_5}) = \frac{8580}{59}$. Now, $|v(\Gamma_{A_5})| = 59$ so $E(K_{|v(\Gamma_{A_5})|}) = LE^+(K_{|v(\Gamma_{A_5})|}) = LE(K_{|v(\Gamma_{A_5})|}) = 116$. Here, $\Gamma_{A_5} = K_{5.3,10.2,6.4}$ so using Result 1.1.4(b) we get

$$\text{Q-spec}(\Gamma_{A_5}) = \{(56)^{10}, (57)^{10}, (55)^{27}, (53)^4, (51)^5, (x_1)^1, (x_2)^1, (x_3)^1\},$$

where $x_1 \approx 52.03252, x_2 \approx 54.05266$ and $x_3 \approx 111.91482$ are the roots of the polynomial $x^3-218x^2+14685x-314760$. It follows that Γ_{A_5} is not Q-integral. We have $|e(\Gamma_{A_5})| = 1650$ and so $\frac{2|e(\Gamma_{A_5})|}{|v(\Gamma_{A_5})|} = \frac{3300}{59}$. Therefore, $\left|57 - \frac{3300}{59}\right| = \frac{63}{59}$, $\left|56 - \frac{3300}{59}\right| = \frac{4}{59}$, $\left|55 - \frac{3300}{59}\right| = \frac{55}{59}$, $\left|53 - \frac{3300}{59}\right| = \frac{173}{59}$, $\left|51 - \frac{3300}{59}\right| = -\frac{291}{59} + \frac{\sqrt{145}}{2}$, $\left|x_1 - \frac{3300}{59}\right| = -(x_1 - \frac{3300}{59})$, $\left|x_2 - \frac{3300}{59}\right| = -(x_2 - \frac{3300}{59})$ and $\left|x_3 - \frac{3300}{59}\right| = x_3 - \frac{3300}{59}$. Thus,

$$LE^+(\Gamma_{A_5}) = \frac{7602}{59} - x_1 - x_2 + x_3 > 116.$$

Hence, Γ_{A_5} is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic as well as L-hyperenergetic.

If $G \cong S_4$, then by Result 1.4.17(e), we have $E(\Gamma_{S_4}) \approx 35.866 + 4\sqrt{5}$ and $LE(\Gamma_{S_4}) = \frac{1072}{23} + 4\sqrt{13}$. Now, $|v(\Gamma_{S_4})| = 23$ so $E(K_{|v(\Gamma_{S_4})|}) = LE^+(K_{|v(\Gamma_{S_4})|}) = LE(K_{|v(\Gamma_{S_4})|}) = 44$. Using GAP [102], the characteristic polynomial of $Q(\Gamma_{S_4})$ is

$$Q_{\Gamma_{S_4}}(x) = x(x+20)^4(x+21)^7(x+23)^7(x^2+40x+394)^2$$

and so

$$\text{Q-spec}(\Gamma_{S_4}) = \left\{(0)^1, (-20)^4, (-21)^7, (-23)^7, \left(-20+\sqrt{6}\right)^2, \left(-20-\sqrt{6}\right)^2\right\}.$$

It follows that Γ_{S_4} is not Q-integral. We have $|e(\Gamma_{S_4})| = 228$ and so $\frac{2|e(\Gamma_{S_4})|}{|v(\Gamma_{S_4})|} = \frac{456}{23}$. Therefore, $\left|0 - \frac{456}{23}\right| = \frac{456}{23}$, $\left|20 - \frac{456}{23}\right| = \frac{4}{23}$, $\left|21 - \frac{456}{23}\right| = \frac{27}{23}$, $\left|23 - \frac{456}{23}\right| = \frac{73}{23}$, $\left|-20 + \sqrt{6} - \frac{456}{23}\right| = \frac{916}{23} - \sqrt{6}$ and $\left|-20 - \sqrt{6} - \frac{456}{23}\right| = \frac{916}{23} + \sqrt{6}$. Thus,

$$LE^+(\Gamma_{S_4}) = \frac{456}{23} + 4\times\frac{4}{23} + 7\times\frac{27}{23} + 7\times\frac{73}{23} + 2\times\left(\frac{916}{23} - \sqrt{6}\right) + 2\times\left(\frac{916}{23} + \sqrt{6}\right) = \frac{4836}{23}.$$

Hence, Γ_{S_4} is hyperenergetic, Q-hyperenergetic as well as L-hyperenergetic but is non-hypoenergetic.

If $G \cong SL(2,3)$, then by Result 1.4.17(d), we have $E(\Gamma_{SL(2,3)}) = 16 + 8\sqrt{7}$ and $LE(\Gamma_{SL(2,3)})) = \frac{552}{11}$. Now, $|v(\Gamma_{SL(2,3)}))| = 22$ so $E(K_{|v(\Gamma_{SL(2,3)}))|}) = LE^{+}(K_{|v(\Gamma_{SL(2,3)})|}) = LE(K_{|v(\Gamma_{SL(2,3)})|}) = 42$. Here, $\Gamma_{SL(2,3)} = K_{3.2,4.4}$ so using Result 1.1.4(b) we get

$$\text{Q-spec}(\Gamma_{SL(2,3)}) = \left\{(20)^3, (18)^{14}, (14)^3, \left(\frac{54+\sqrt{420}}{2}\right)^1, \left(\frac{54-\sqrt{420}}{2}\right)^1\right\}.$$

It follows that $\Gamma_{SL(2,3)}$ is not Q-integral. We have $|e(\Gamma_{SL(2,3)})| = 204$ and so $\frac{2|e(\Gamma_{SL(2,3)})|}{|v(\Gamma_{SL(2,3)})|} = \frac{204}{11}$. Therefore, $\left|20 - \frac{204}{11}\right| = \frac{16}{11}$, $\left|18 - \frac{204}{11}\right| = \frac{6}{11}$, $\left|14 - \frac{204}{11}\right| = \frac{50}{11}$, $\left|\frac{54+\sqrt{420}}{2} - \frac{204}{11}\right| = \frac{93}{11} + \frac{\sqrt{420}}{2}$ and $\left|\frac{54-\sqrt{420}}{2} - \frac{204}{11}\right| = -\frac{93}{22} + \frac{\sqrt{420}}{2}$. Thus,

$$LE^{+}(\Gamma_{SL(2,3)}) = 3 \times \frac{16}{11} + 14 \times \frac{6}{11} + 3 \times \frac{50}{11} + \frac{93}{11} + \frac{\sqrt{420}}{2} - \frac{93}{22} + \frac{\sqrt{420}}{2} = \frac{282}{11} + \sqrt{420}.$$

Hence, $\Gamma_{SL(2,3)}$ is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic as well as L-hyperenergetic.

If $G \cong Sz(2)$ then, by Theorem 2.3.1, we have Γ_G is not Q-integral. Also, Theorem 2.3.2 gives that Γ_G is non-hypoenergetic, non-hyperenergetic, not Q-hyperenergetic but is L-hyperenergetic. □

Theorem 2.5.4. *Let G be a finite group and the commuting graph of G is toroidal. Then*

(a) *Γ_G is Q-integral if $G \cong D_{14}$ or $A_4 \times \mathbb{Z}_2$, otherwise not Q-integral.*

(b) *Γ_G is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic and L-hyperenergetic.*

Proof. If commuting graph of G is toroidal then, by Result 1.4.20, $G \cong D_{14}, D_{16}, Q_{16}, QD_{16}, \mathbb{Z}_7 \rtimes \mathbb{Z}_3, D_6 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$. By Theorems 2.1.1, 2.1.4 and 2.1.9, Γ_G is Q-integral if $G \cong D_{14}$ and not Q-integral if $G \cong D_{16}, Q_{16}$ or QD_{16}. If $G \cong D_{14}, D_{16}$, then from Theorem 2.1.3, Γ_G is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic and L-hyperenergetic. If $G \cong Q_{16}$, then from Theorem 2.1.10, Γ_G is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic and L-hyperenergetic. If $G \cong QD_{16}$, then from Theorem 2.1.5, Γ_G is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic and L-hyperenergetic.

If $G \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ then, by Theorem 2.4.11, we have

$$\text{Q-spec}(\Gamma_G) = \left\{(14)^5, (18)^7, (16)^6, \left(22 + 2\sqrt{37}\right)^1, \left(22 - 2\sqrt{37}\right)^1\right\}.$$

Thus Γ_G is not Q-integral. By Result 1.4.16, we also have $E(\Gamma_G) = 12 + 4\sqrt{30}$ and $LE(\Gamma_G) = \frac{308}{5}$ and from Theorem 2.4.11, we have $LE^+(\Gamma_G) = \frac{292}{20} + 4\sqrt{37}$. Now, $|v(\Gamma_G)| = 20$ so $E(K_{|v(\Gamma_G)|}) = LE^+(K_{|v(\Gamma_G)|}) = LE(K_{|v(\Gamma_G)|}) = 38$. Hence, Γ_G is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic and L-hyperenergetic.

If $G \cong D_6 \times \mathbb{Z}_3$, then by Result 1.4.17(f), we have $E(\Gamma_G) = 6 + 6\sqrt{7}$ and $LE(\Gamma_G) = \frac{594}{15}$. Now, $|v(\Gamma_G)| = 15$ so $E(K_{|v(\Gamma_G)|}) = LE^+(K_{|v(\Gamma_G)|}) = LE(K_{|v(\Gamma_G)|}) = 28$. Here, $\Gamma_G = K_{3.3,1.6}$ so using Result 1.1.4(b) we get

$$\text{Q-spec}(\Gamma_G) = \left\{(12)^6, (9)^7, \left(\frac{27 + \sqrt{297}}{2}\right)^1, \left(\frac{27 - \sqrt{297}}{2}\right)^1\right\}.$$

It follows that Γ_G is not Q-integral. We have $|e(\Gamma_G)| = 81$ and so $\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{162}{15}$. Therefore, $\left|12 - \frac{162}{15}\right| = \frac{18}{15}$, $\left|9 - \frac{162}{15}\right| = \frac{27}{15}$, $\left|\frac{27+\sqrt{297}}{2} - \frac{162}{15}\right| = \frac{81}{30} + \frac{\sqrt{297}}{2}$ and $\left|\frac{27-\sqrt{145}}{2} - \frac{162}{15}\right| = -\frac{81}{30} + \frac{\sqrt{297}}{2}$. Thus,

$$LE^+(\Gamma_G) = 6 \times \frac{18}{15} + 7 \times \frac{189}{11} + \frac{81}{30} + \frac{\sqrt{297}}{2} - \frac{81}{30} + \frac{\sqrt{297}}{2} = \frac{99}{5} + 3\sqrt{33}.$$

Hence, Γ_G is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic as well as L-hyperenergetic.

If $G \cong A_4 \times \mathbb{Z}_2$, then by Result 1.4.17(g), we have $E(\Gamma_G) = 12 + 4\sqrt{33}$ and $LE(\Gamma_G) = \frac{544}{11}$. Now, $|v(\Gamma_G)| = 22$ so $E(K_{|v(\Gamma_G)|}) = LE^+(K_{|v(\Gamma_G)|}) = LE(K_{|v(\Gamma_G)|}) = 42$. Here, $\Gamma_G = K_{4.4,1.6}$ so using Result 1.1.4(b) we get

$$\text{Q-spec}(\Gamma_G) = \{(18)^{12}, (16)^5, (14)^3, (36)^1, (10)^1\}.$$

Clearly, Γ_G is Q-integral. We have $|e(\Gamma_G)| = 192$ and so $\frac{2|e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{192}{11}$. Therefore, $\left|18 - \frac{192}{11}\right| = \frac{6}{11}$, $\left|16 - \frac{192}{11}\right| = \frac{16}{11}$, $\left|14 - \frac{192}{11}\right| = \frac{38}{11}$, $\left|36 - \frac{192}{11}\right| = \frac{204}{11}$ and $\left|10 - \frac{192}{11}\right| = \frac{82}{11}$. Thus,

$$LE^+(\Gamma_G) = 12 \times \frac{6}{11} + 5 \times \frac{16}{11} + 3 \times \frac{38}{11} + \frac{204}{11} + \frac{82}{11} = \frac{552}{11}.$$

Hence, Γ_G is non-hypoenergetic, non-hyperenergetic but is Q-hyperenergetic as well as L-hyperenergetic. □

If non-commuting graph of G is planar then, by Result 1.4.18, $G \cong D_6, D_8, Q_8$. Therefore, we have the following theorem to conclude this chapter.

Theorem 2.5.5. *Let G be a finite group whose non-commuting graph is planar. Then*

(a) Γ_G *is non-hypoenergetic, non-hyperenergetic and not Q-hyperenergetic.*

(b) Γ_G *is not L-hyperenergetic but Q-integral if* $G \ncong D_6$.

2.6 Conclusion

In this chapter, we have computed Signless Laplacian spectrum and Signless Laplacian energy of non-commuting graphs of dihedral group, quasidihedral group, dicyclic group, semidihedral group along with several other classes of groups and obtained conditions such that the non-commuting graphs of these groups are Q-integral. In addition, we have characterized certain finite non-abelian groups such that their non-commuting graphs are hypoenergetic, hyperenergetic, L-hyperenergetic and Q-hyperenergetic. In this process, we have produced a new counter example for Conjecture 1.1.7 posed by Gutman [48]. Further, we have compared Signless Laplacian energy with energy and Laplacian energy of non-commuting graphs of these groups and found that $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$ or $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G)$.