Chapter 3

Relative g-noncommuting graph of a finite group

Let H be a subgroup of a finite group G and $g \in G$. Fusing the concepts of relative non-commuting graph $\Gamma_{H,G}$ and g-noncommuting graph Γ_G^g, in this chapter, we introduce relative g-noncommuting graph of G. The *relative g-noncommuting graph* of G, denoted by $\Gamma_{H,G}^g$, is defined as the simple undirected graph whose vertex set is G and two vertices x and y ($x \neq y$) are adjacent if $x \in H$ or $y \in H$ and $[x, y] \neq g$ and g^{-1}. Note that if $g = 1$ then the induced subgraph of $\Gamma_{H,G}^g$ on $G \setminus Z(H, G)$ is the relative non-commuting graph $\Gamma_{H,G}$. Also, if $H = G$ then $\Gamma_{G,G}^g = \Gamma_G^g$. In Section 3.2, we obtain computing formula for degree of any vertex in $\Gamma_{H,G}^g$ and characterize whether $\Gamma_{H,G}^g$ is a tree, star graph, lollipop or a complete graph together with some properties of $\Gamma_{H,G}^g$ involving isomorphism of graphs. In Section 3.3, we obtain the number of edges in $\Gamma_{H,G}^g$ using $\Pr_g(H, G)$. We conclude this chapter with some bounds for the number of edges in $\Gamma_{H,G}^g$. This chapter is based on our paper [85] published in *Electronic Journal of Graph Theory and Applications*.

3.1 Preliminary observations

Let $\mathcal{G}_1 + \mathcal{G}_2$ be the join of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and let $\overline{\mathcal{G}}$ be the complement of $\mathcal{G}$. Then we have the following observations, where K_n is the complete graph on n vertices.

Observation 3.1.1. Let $H \leq G$ and $g \in G$.

(a) If $g \notin K(H, G)$ then $\Gamma_{H,G}^{g} = \overline{K_{|G|-|H|}} + K_{|H|}$ and so

$$\deg(x) = \begin{cases} |H|, & \text{if } x \in G \setminus H \\ |G| - 1, & \text{if } x \in H. \end{cases}$$

(b) If $g = 1$ and $K(H, G) = \{1\}$ then $\Gamma_{H,G}^{g} = \overline{K_{|G|}}$.

Observation 3.1.2. Let $H \leq G$ and $g \in G \setminus K(H, G)$. Then

(a) $\Gamma_{H,G}^{g}$ is a tree $\iff H = \{1\}$ and $|H| = |G| = 2$.

(b) $\Gamma_{H,G}^{g}$ is a star $\iff H = \{1\}$.

(c) $\Gamma_{H,G}^{g}$ is a complete graph $\iff H = G$.

Note that if $H = Z(H, G)$ or G is abelian then $K(H, G) = \{1\}$. Therefore, in view of Observation 3.1.1, we consider G to be non-abelian, $H \leq G$ such that $H \neq Z(H, G)$ and $g \in K(H, G)$ throughout this chapter.

3.2 Vertex degree and other properties

In this section we first obtain computing formula for $\deg(x)$ in terms of $|G|$, $|H|$ and the orders of the centralizers of x. We write $x \sim y$ if x is conjugate to y.

Theorem 3.2.1. *Let $x \in H$.*

(a) *For $g = 1$ we have $\deg(x) = |G| - |C_G(x)|$.*

(b) *For $g \neq 1$ and $g^2 \neq 1$ we have* $\deg(x) = \begin{cases} |G| - |C_G(x)| - 1, & \text{if } x \sim xg \text{ or } xg^{-1} \\ |G| - 2|C_G(x)| - 1, & \text{if } x \sim xg \text{ and } xg^{-1}. \end{cases}$

(c) *For $g \neq 1$ and $g^2 = 1$ we have $\deg(x) = |G| - |C_G(x)| - 1$, whenever $x \sim xg$.*

Proof. (a) Let $g = 1$. Then $\deg(x)$ is the number of $y \in G$ such that y does not commute with x. Hence, $\deg(x) = |G| - |C_G(x)|$.

(b) Let $g \neq 1$ and $g^2 \neq 1$. Then $g \neq g^{-1}$. Suppose that $x \sim xg$ or xg^{-1} but not to both. Without any loss we assume that x is conjugate to xg. Then there exits $y \in G$ such that $y^{-1}xy = xg$, that is $[x, y] = x^{-1}y^{-1}xy = g$. Therefore, the set $S_g := \{y \in G : y^{-1}xy = xg\}$ is non-empty. Also, for any $\alpha \in S_g$ we have $[x, \alpha] = g$ which gives that α is not adjacent to x. Thus, $\alpha \in G$ is not adjacent to x if and only if $\alpha = x$ or $\alpha \in S_g$. Therefore, the number of vertices not adjacent to x is equal to $|S_g| + 1$.

Let $y_1 \in S_g$ and $y_2 \in C_G(x)y_1$. Then $y_2 = uy_1$ for some $u \in C_G(x)$. We have

$$y_2^{-1}xy_2 = y_1^{-1}u^{-1}xuy_1 = y_1^{-1}xy_1 = xg.$$

Therefore, $y_2 \in S_g$ and so $C_G(x)y_1 \subseteq S_g$. Suppose that $y_3 \in S_g$. Then $y_1^{-1}xy_1 = y_3^{-1}xy_3$ which implies $y_3y_1^{-1} \in C_G(x)$. Therefore, $y_3 \in C_G(x)y_1$ and so $S_g \subseteq C_G(x)y_1$. Thus $S_g = C_G(x)y_1$ and so $|S_g| = |C_G(x)|$. Hence, the number of vertices not adjacent to x is equal to $|C_G(x)| + 1$ and so $\deg(x) = |G| - |C_G(x)| - 1$.

If x is conjugate to xg and xg^{-1} then $S_g \cap S_{g^{-1}} = \emptyset$, where $S_{g^{-1}} := \{y \in G : y^{-1}xy = xg^{-1}\}$ and $|S_{g^{-1}}| = |C_G(x)|$. In this case, $\alpha \in G$ is not adjacent to x if and only if $\alpha = x$ or $\alpha \in S_g \cup S_{g^{-1}}$. Therefore, the number of vertices not adjacent to x is equal to $|S_g| + |S_{g^{-1}}| + 1 = 2|C_G(x)| + 1$. Hence, $\deg(x) = |G| - 2|C_G(x)| - 1$.

(c) Let $g \neq 1$ and $g^2 = 1$. Then $g = g^{-1}$ and so $xg = xg^{-1}$. Now, if x is conjugate to xg then, as shown in the proof of part (b), we have $\deg(x) = |G| - |C_G(x)| - 1$. □

Theorem 3.2.2. *Let $x \in G \setminus H$.*

(a) *For $g = 1$ we have* $\deg(x) = |H| - |C_H(x)|$.

(b) *For $g \neq 1$ and $g^2 \neq 1$ we have*

$$\deg(x) = \begin{cases} |H| - |C_H(x)|, & \text{if } x \sim xg \text{ or } xg^{-1} \text{ for some element in } X. \\ |H| - 2|C_H(x)|, & \text{if } x \sim xg \text{ and } xg^{-1} \text{ for some element in } H. \end{cases}$$

(c) *For $g \neq 1$ and $g^2 = 1$ we have* $\deg(x) = |H| - |C_H(x)|$, *whenever $x \sim xg$, for some element in H.*

Proof. The proof is analogous to the proof of Theorem 3.2.1. □

It is noteworthy that $g \notin K(H,G)$ if x is not conjugate to xg and xg^{-1}. Therefore, this case does not arise in Theorems 3.2.1 and 3.2.2. The degree of a vertex, in such case, is given by Observation 3.1.1.

Now, we present some properties of $\Gamma^g_{H,G}$. The following lemmas are useful in this regard.

Lemma 3.2.3. *If $g \neq 1$ and H has an element of order 3 then $\Gamma^g_{H,G}$ is not triangle free.*

Proof. Let $x \in H$ having order 3. Then the vertices $1, x$ and x^{-1} forms a triangle in $\Gamma^g_{H,G}$. Hence, the lemma follows. □

Lemma 3.2.4. *If $x \in Z(H,G)$ then* $\deg(x) = \begin{cases} 0, & \text{if } g = 1 \\ |G| - 1, & \text{if } g \neq 1. \end{cases}$

Proof. By definition of $Z(H,G)$, it follows that $x \in H$ and $[x,y] = 1$ for all $y \in G$ and so $C_G(x) = G$. Therefore, if $g = 1$ then by Theorem 3.2.1(a) we have $\deg(x) = 0$. If $g \neq 1$ then all the elements of G except x are adjacent to x. Therefore, $\deg(x) = |G| - 1$. □

As a consequence of Lemma 3.2.4, we have that $\gamma(\Gamma^g_{H,G}) = 1$ if $g \neq 1$ since $\{x\}$ is a dominating set for all $x \in Z(H,G)$, where $\gamma(\Gamma^g_{H,G})$ is the domination number of $\Gamma^g_{H,G}$. If $g \in H$ having even order then it can be seen that $\{g\}$ is also a dominating set in $\Gamma^g_{H,G}$. If $g = 1$ then $\gamma(\Gamma^g_{H,G}) \geq |Z(H,G)| + 1$. This lower bound is sharp because $\gamma(\Gamma^{(1)}_{H,S_3})$ is $2 = |Z(H,S_3)| + 1$, where H is any subgroup of S_3 of order 2. If $g = 1$ then, by Lemma 3.2.4, we also have that $\Gamma^g_{H,G}$ is disconnected. Hence, $\Gamma^1_{H,G}$ is not a tree and complete graph. Now we determine whether $\Gamma^g_{H,G}$ is a tree, star graph or complete graph if $g \neq 1$.

Theorem 3.2.5. *Let $H \leq G$ and $|H| \neq 2$. Then $\Gamma^{g\neq 1}_{H,G}$ is not a tree.*

Proof. Suppose for any $H \leq G$, $\Gamma^g_{H,G}$ is a tree, where $g \neq 1$. There exits a vertex x in $\Gamma^g_{H,G}$ of degree one.

Case 1: $x \in H$

By Theorem 3.2.1, $\deg(x) = |G| - |C_G(x)| - 1 = 1$ or $\deg(x) = |G| - 2|C_G(x)| - 1 = 1$. That is, $|G| - |C_G(x)| = 2$ or $|G| - 2|C_G(x)| = 2$. Therefore, $|C_G(x)| = 2$ and $|G| = 4, 6$.

Since G is non-abelian and $|H| \neq 1, 2$, we must have $G \cong S_3$ and $H = A_3$ or S_3. Therefore, by Lemma 3.2.3, $\Gamma^g_{H,G}$ has a triangle which is a contradiction.

Case 2: $x \in G \setminus H$

By Theorem 3.2.2, $\deg(x) = |H| - |C_H(x)| = 1$ or $\deg(x) = |H| - 2|C_H(x)| = 1$. Therefore, $|C_H(x)| = 1$ and $|H| = 2, 3$. However, $|H| \neq 2$ (by assumption). If $|H| = 3$ then, by Lemma 3.2.3, $\Gamma^g_{H,G}$ has a triangle which is a contradiction. Hence, the result follows. □

The proof of Theorem 3.2.5 also gives the following result.

Theorem 3.2.6. *Let $H \leq G$ and $|H| \neq 2, 3$. Then $\Gamma^{g \neq 1}_{H,G}$ is not a lollipop. Further, if $|H| \neq 2, 3, 6$ then $\Gamma^{g \neq 1}_{H,G}$ has no vertex of degree 1.*

As a consequence of Theorem 3.2.5 we have the following results.

Corollary 3.2.7. *Let $H \leq G$ and $|H| \neq 2$. Then $\Gamma^{g \neq 1}_{H,G}$ is not a star graph.*

Corollary 3.2.8. *If $g \neq 1$ and G is a group of odd order then $\Gamma^g_{H,G}$ is not a tree and hence not a star.*

Theorem 3.2.9. *If $g \neq 1$ then $\Gamma^g_{H,G}$ is a star $\Longleftrightarrow G \cong S_3$ and $|H| = 2$.*

Proof. By Lemma 3.2.4, $\deg(1) = |G| - 1$. Suppose that $\Gamma^g_{H,G}$ is a star graph. Then $\deg(x) = 1 \;\; \forall \;\; 1 \neq x \in G$. Since $g \in K(H, G)$ and $g \neq 1$ we have $H \neq \{1\}$. Suppose that $1 \neq y \in H$. If $g^2 = 1$, then by Theorem 3.2.1, we get $1 = \deg(y) = |G| - |C_G(y)| - 1$ which gives $|G| = 4$, a contradiction since G is non-abelian. If $g^2 \neq 1$, then by Theorem 3.2.1, we get $1 = \deg(y) = |G| - |C_G(y)| - 1$ or $|G| - 2|C_G(y)| - 1$ which gives $|G| = 6$. Therefore, $G \cong S_3$, $g = (123), (132)$ and $H = \{(1), (12)\}, \{(1), (13)\}, \{(1), (23)\}$ or $H = \{(1), (123), (132)\}$. If $|H| = 3$ then, by Lemma 3.2.3, Γ^g_{H,S_3} is not a star. If $|H| = 2$ then it is easy to see that Γ^g_{H,S_3} is a star. This completes the proof. □

Theorem 3.2.10. *If $g \neq 1$ then $\Gamma^g_{H,G}$ is not complete.*

Proof. Let $\Gamma^g_{H,G}$ be complete graph where $g \neq 1$. Then $\deg(x) = |G| - 1 \;\; \forall \;\; x \in G$. Since $g \in K(H, G)$ and $g \neq 1$ we have $H \neq \{1\}$. Suppose that $1 \neq y \in H$. Then by Theorem 3.2.1, we get $|G| - 1 = \deg(y) = |G| - |C_G(y)| - 1$ or $|G| - 1 = \deg(y) = |G| - 2|C_G(y)| - 1$. Therefore, $|C_G(y)| = 0$, a contradiction. Hence, $\Gamma^g_{H,G}$ is not complete. □

Theorem 3.2.11. *Let $H \trianglelefteq G$ and $g \sim h$. Then $\Gamma^g_{H,G} \cong \Gamma^h_{H,G}$.*

Proof. Let $h = g^x := x^{-1}gx$ for some $x \in G$. Then for any two elements $a_1, a_2 \in G$, we have

$$[a_1^x, a_2^x] = h \text{ or } h^{-1} \text{ if and only if } [a_1, a_2] = g \text{ or } g^{-1}. \tag{3.2.1}$$

Consider the bijection $\phi : v(\Gamma^g_{H,G}) \to v(\Gamma^h_{H,G})$ given by $\phi(a) = a^x$ for all $a \in G$. We show that ϕ preserves adjacency.

Suppose that $a_1, a_2 \in v(\Gamma^h_{H,G})$. If a_1 and a_2 are not adjacent in $\Gamma^g_{H,G}$ then $[a_1, a_2] = g$ or g^{-1}. Therefore, by equation (3.2.1), it follows that $\phi(a_1)$ and $\phi(a_2)$ are not adjacent in $\Gamma^h_{H,G}$. If a_1 and a_2 are adjacent then atleast one of a_1 and a_2 must belong to H and $[a_1, a_2] \neq g, g^{-1}$. Without any loss assume that $a_1 \in H$. Since $H \trianglelefteq G$ we have $\phi(a_1) \in H$. by equation (3.2.1), we have $[\phi(a_1), \phi(a_2)] \neq h, h^{-1}$. Thus $\phi(a_1)$ and $\phi(a_2)$ are adjacent in $\Gamma^h_{H,G}$. Hence, the result follows. □

In Result 1.4.21, it has been shown that Γ_{H_1,G_1} is isomorphic to Γ_{H_2,G_2} if (H_1, G_1) and (H_2, G_2) are relative isoclinic satisfying certain conditions. Tolue et al. in Result 1.4.22, also proved that $\Gamma^g_{G_1}$ is isomorphic to $\Gamma^{\psi(g)}_{G_2}$ if G_1 and G_2 are isoclinic such that $|Z(G_1)| = |Z(G_2)|$. We conclude Section 3.2 with Theorem 3.2.12 which generalizes Result 1.4.22.

Theorem 3.2.12. *Let (ϕ, ψ) be a relative isoclinism between the pairs of groups (H_1, G_1) and (H_2, G_2). If $|Z(H_1, G_1)| = |Z(H_2, G_2)|$ then $\Gamma^g_{H_1,G_1}$ is isomorphic to $\Gamma^{\psi(g)}_{H_2,G_2}$.*

Proof. Since $\phi : \frac{G_1}{Z(H_1,G_1)} \to \frac{G_2}{Z(H_2,G_2)}$ is an isomorphism such that $\phi\left(\frac{H_1}{Z(H_1,G_1)}\right) = \frac{H_2}{Z(H_2,G_2)}$. So we have $|\frac{H_1}{Z(H_1,G_1)}| = |\frac{H_2}{Z(H_2,G_2)}|$ and $|\frac{G_1}{Z(H_1,G_1)}| = |\frac{G_2}{Z(H_2,G_2)}|$. Let $|\frac{H_1}{Z(H_1,G_1)}| = |\frac{H_2}{Z(H_2,G_2)}| = m$ and $|\frac{G_1}{Z(H_1,G_1)}| = |\frac{G_2}{Z(H_2,G_2)}| = n$. Given $|Z(H_1, G_1)| = |Z(H_2, G_2)|$, so $\exists$ a bijection $\theta : Z(H_1, G_1) \to Z(H_2, G_2)$. Let $\{h_1, h_2, \dots, h_m, g_{m+1}, \dots, g_n\}$ and $\{h'_1, h'_2, \dots, h'_m, g'_{m+1}, \dots, g'_n\}$ be two trans-versals of $\frac{G_1}{Z(H_1,G_1)}$ and $\frac{G_2}{Z(H_2,G_2)}$ respectively where $\{h_1, h_2, \dots, h_m\}$ and $\{h'_1, h'_2, \dots, h'_m\}$ are transversals of $\frac{H_1}{Z(H_1,G_1)}$ and $\frac{H_2}{Z(H_2,G_2)}$ respectively. Let us define ϕ as $\phi(h_iZ(H_1, G_1)) = h'_iZ(H_2, G_2)$ and $\phi(g_jZ(H_1, G_1)) = g'_jZ(H_2, G_2)$ for $1 \leq i \leq m$ and $m + 1 \leq j \leq n$.

Let $\mu : G_1 \to G_2$ be a map such that $\mu(h_iz) = h'_i\theta(z)$, $\mu(g_jz) = g'_j\theta(z)$ for $z \in Z(H_1, G_1)$, $1 \leq i \leq m$ and $m + 1 \leq j \leq n$. Clearly μ is a bijection. Suppose two vertices

x and y in $\Gamma^g_{H_1,G_1}$ are adjacent. Then $x \in H_1$ or $y \in H_1$ and $[x,y] \neq g, g^{-1}$. Without any loss of generality, let us assume that $x \in H_1$. Then $x = h_i z_1$ for $1 \leq i \leq m$ and $y = kz_2$ where $z_1, z_2 \in Z(H_1, G_1)$, $k \in \{h_1, h_2, \dots, h_m, g_{m+1}, \dots, g_n\}$. Therefore, for some $k' \in \{h'_1, \dots, h'_m, g'_{m+1}, \dots, g'_n\}$, we have

$$\begin{aligned}\psi([h_i z_1, kz_2]) = \psi([h_i, k]) &= \psi \circ a_{(H_1,G_1)}\left((h_i Z(H_1, G_1), kZ(H_1, G_1))\right)\\ &= a_{(H_2,G_2)} \circ (\phi \times \phi)\left((h_i Z(H_1, G_1), kZ(H_1, G_1))\right)\\ &= a_{(H_2,G_2)}((h'_i Z(H_2, G_2), k' Z(H_2, G_2)))\\ &= [h'_i, k'] = [h'_i z'_1, k' z'_2], \end{aligned} \tag{3.2.2}$$

where $z'_1, z'_2 \in Z(H_2, G_2)$. Also,

$$\begin{aligned} & [h_i z_1, kz_2] \neq g, g^{-1}\\ \Rightarrow\; & \psi([h_i z_1, kz_2]) \neq \psi(g), \psi(g^{-1})\\ \Rightarrow\; & [h'_i z'_1, k' z'_2] \neq \psi(g), \psi^{-1}(g) \text{ (using equation (3.2.2))}\\ \Rightarrow\; & [h'_i \theta(z_1), k' \theta(z_2)] \neq \psi(g), \psi^{-1}(g)\\ \Rightarrow\; & [\mu(h_i z_1), \mu(kz_2)] \neq \psi(g), \psi^{-1}(g)\\ \Rightarrow\; & [\mu(x), \mu(y)] \neq \psi(g), \psi^{-1}(g). \end{aligned}$$

Thus $\mu(x)$ is adjacent to $\mu(y)$ in $\Gamma^{\psi(g)}_{H_2,G_2}$ since $\mu(x) \in H_2$. Hence, the graphs $\Gamma^g_{H_1,G_1}$ and $\Gamma^{\psi(g)}_{H_2,G_2}$ are isomorphic under the map μ. □

3.3 Relation between $\Gamma^g_{H,G}$ and $\Pr_g(H, G)$

In [93], Tolue and Erfanian have established some relations between $\Pr_1(H, G)$ and relative non-commuting graphs of finite groups. In [94], Tolue et al. have also established relations between Γ^g_G and $\Pr_g(G)$. Their results stimulate us to obtain relations between $\Gamma^g_{H,G}$ and $\Pr_g(H, G)$. We obtain the number of edges in $\Gamma^g_{H,G}$, denoted by $|e(\Gamma^g_{H,G})|$, in terms of $\Pr_g(H, G)$. Clearly, if $g \notin K(H, G)$ then from Observation 3.1.1, we get

$$2|e(\Gamma^g_{H,G})| = 2|H||G| - |H|^2 - |H|.$$

The following theorem expresses $|e(\Gamma_{H,G}^g)|$ in terms of $\Pr_g(H,G)$, where $g \in K(H,G)$.

Theorem 3.3.1. *Let* $\{1\} \neq H \leq G$ *and* $g \in K(H,G)$. *Let* $\Pr_{x\neq g}(H,G) := 1 - \Pr_g(H,G)$.

(a) *If* $g = 1$ *then* $2|e(\Gamma_{H,G}^g)| = 2|H||G|\Pr_{x\neq g}(H,G) - |H|^2(1 - \Pr_g(H))$.

(b) *If* $g \neq 1$ *and* $g^2 = 1$ *then*

$$2|e(\Gamma_{H,G}^g)| = \begin{cases} 2|H||G|\Pr_{x\neq g}(H,G) - |H|^2(1 - \Pr_g(H)) - |H|, & \text{if } g \in H \\ 2|H||G|\Pr_{x\neq g}(H,G) - |H|^2 - |H|, & \text{if } g \in G \setminus H. \end{cases}$$

(c) *If* $g \neq 1$ *and* $g^2 \neq 1$ *then*

$$2|e(\Gamma_{H,G}^g)| = \begin{cases} 2|H||G|(1 - \sum\limits_{u=g,g^{-1}} \Pr_u(H,G)) & \\ \qquad - |H|^2(1 - \sum\limits_{u=g,g^{-1}} \Pr_u(H)) - |H|, & \text{if } g \in H \\ 2|H||G|(1 - \sum\limits_{u=g,g^{-1}} \Pr_u(H,G)) - |H|^2 - |H|, & \text{if } g \in G \setminus H. \end{cases}$$

Proof. Let $E_1 = \{(z,w) \in H \times G : z \neq w, [z,w] \neq g \text{ and } [z,w] \neq g^{-1}\}$ and $E_2 = \{(z,w) \in G \times H : z \neq w, [z,w] \neq g \text{ and } [z,w] \neq g^{-1}\}$. Clearly we have a bijection from E_1 to E_2 defined by $(z,w) \mapsto (w,z)$. So $|E_1| = |E_2|$. It is easy to see that $|e(\Gamma_{H,G}^g)|$ is equal to half of $|E_1 \cup E_2|$. Therefore,

$$2|e(\Gamma_{H,G}^g)| = 2|E_1| - |E_1 \cap E_2|, \tag{3.3.1}$$

where $E_1 \cap E_2 = \{(z,w) \in H \times H : z \neq w, [z,w] \neq g \text{ and } [z,w] \neq g^{-1}\}$.
(a) If $g = 1$ then we have

$$\begin{aligned} |E_1| &= |\{(z,w) \in H \times G : [z,w] \neq 1\}| \\ &= |H||G| - |\{(z,w) \in H \times G : [z,w] = 1\}| = |H||G|(1 - \Pr_g(H,G)) \end{aligned}$$

and

$$\begin{aligned} |E_1 \cap E_2| &= |\{(z,w) \in H \times H : [z,w] \neq 1\}| \\ &= |H|^2 - |\{(z,w) \in H \times H : [z,w] = 1\}| = |H|^2(1 - \Pr_g(H)). \end{aligned}$$

Hence, the result follows from equation (3.3.1).

(b) If $g \neq 1$ and $g^2 = 1$ then we have

$$\begin{aligned}|E_1| &= |\{(z,w) \in H \times G : z \neq w, [z,w] \neq g\}| \\ &= |H||G| - |\{(z,w) \in H \times G : [z,w] = g\}| - |\{(z,w) \in H \times G : z = w\}| \\ &= |H||G|(1 - \Pr_g(H,G)) - |H|.\end{aligned}$$

Now, if $g \in H$ then

$$\begin{aligned}|E_1 \cap E_2| &= |\{(z,w) \in H \times H : z \neq w, [z,w] \neq g\}| \\ &= |H|^2 - |\{(z,w) \in H \times H : [z,w] = g\}| - |\{(z,w) \in H \times H : z = w\}| \\ &= |H|^2(1 - \Pr_g(H)) - |H|.\end{aligned}$$

If $g \in G \setminus H$ then $|E_1 \cap E_2| = |H|^2 - |H|$. Hence, the result follows from equation (3.3.1).
(c) If $g \neq 1$ and $g^2 \neq 1$ then we have

$$\begin{aligned}|E_1| &= |\{(z,w) \in H \times G : z \neq w, [z,w] \neq g \text{ and } [z,w] \neq g^{-1}\}| \\ &= |H||G| - |\{(z,w) \in H \times G : [z,w] = g\}| \\ &\qquad - |\{(z,w) \in H \times G : [z,w] = g^{-1}\}| - |\{(z,w) \in H \times H : z = w\}| \\ &= |H||G|(1 - \sum_{u=g,g^{-1}} \Pr_u(H,G)) - |H|.\end{aligned}$$

Now, if $g \in H$ then

$$\begin{aligned}|E_1 \cap E_2| &= |\{(z,w) \in H \times H : z \neq w, [z,w] \neq g \text{ and } [z,w] \neq g^{-1}\}| \\ &= |H|^2 - |\{(z,w) \in H \times H : [z,w] = g\}| \\ &\qquad - |\{(z,w) \in H \times H : [z,w] = g^{-1}\}| - |\{(z,w) \in H \times H : z = w\}| \\ &= |H|^2(1 - \sum_{u=g,g^{-1}} \Pr_u(H)) - |H|.\end{aligned}$$

If $g \in G \setminus H$ then $|E_1 \cap E_2| = |H|^2 - |H|$. Hence, the result follows from equation (3.3.1). □

For an abelian group H we have

$$\Pr_g(H) = \begin{cases} 1, & \text{if } g = 1 \\ 0, & \text{if } g \neq 1. \end{cases}$$

Using these values in Theorem 3.3.1 we get Corollary 3.3.2.

Corollary 3.3.2. *Let $g \in K(H, G)$ where $\{1\} \neq H \leq G$ is abelian.*

(a) *If $g = 1$ then $|e(\Gamma^g_{H,G})| = |H||G|(1 - \Pr_g(H, G))$.*

(b) *If $g \neq 1$ and $g^2 = 1$ then*

$$2|e(\Gamma^g_{H,G})| = 2|H||G|(1 - \Pr_g(H, G)) - |H|^2 - |H|.$$

(c) *If $g \neq 1$ and $g^2 \neq 1$ then*

$$2|e(\Gamma^g_{H,G})| = 2|H||G|(1 - \sum_{u=g,g^{-1}} \Pr_u(H, G)) - |H|^2 - |H|.$$

Theorem 3.3.3. *Let $H \leq G$ and $g \in K(H, G)$. Let $|[H, G]| = p$, the smallest prime dividing $|G|$.*

(a) *If $g = 1$ then*

$$2p|e(\Gamma^g_{H,G})| = (p - 1)[2|G|(|H| - |Z(H, G)|) - |H|(|H| - |Z(H)|)].$$

(b) *If $g \neq 1$ and $g^2 = 1$ then*

$$2p|e(\Gamma^g_{H,G})| = \begin{cases} 2|G|((p-1)|H| + |Z(H,G)|) \\ \quad - |H|((p-1)|H| + |Z(H)| + p), & \text{if } g \in H \\ 2|G|((p-1)|H| + |Z(H,G)|) \\ \quad - p|H|(|H| + 1), & \text{if } g \in G \setminus H. \end{cases}$$

(c) *If $g \neq 1$ and $g^2 \neq 1$ then*

$$2p|e(\Gamma^g_{H,G})| = \begin{cases} 2|G|((p-2)|H| + 2|Z(H,G)|) \\ \quad - |H|((p-2)|H| + 2|Z(H)| + p), & \text{if } g \in H \\ 2|G|((p-2)|H| + 2|Z(H,G)|) \\ \quad - p|H|(|H| + 1), & \text{if } g \in G \setminus H. \end{cases}$$

Proof. By Result 1.2.8, we have

$$\Pr_g(H,G) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|H:Z(H,G)|}\right), & \text{if } g = 1 \\ \frac{1}{p}\left(1 - \frac{1}{|H:Z(H,G)|}\right), & \text{if } g \neq 1. \end{cases}$$

Hence, the result follows from Theorem 3.3.1. □

It is worth mentioning that, in view of Result 1.2.7, the conclusion of Theorem 3.3.3 also holds if G is nilpotent such that $|[H,G]| = p$, where p is not necessarily the smallest prime. We also have the following corollary.

Corollary 3.3.4. *Let $H \leq G$ where H is abelian and G is nilpotent. Let $|[H,G]| = p$ be any prime and $g \in K(H,G)$.*

(a) *If $g = 1$ then $p|e(\Gamma^g_{H,G})| = (p-1)|G|(|H| - |Z(H,G)|)$.*

(b) *If $g \neq 1$ and $g^2 = 1$ then*

$$2p|e(\Gamma^g_{H,G})| = 2|G|(|Z(H,G)| + (p-1)|H|) - p|H|(1 + |H|).$$

(c) *If $g \neq 1$ and $g^2 \neq 1$ then*

$$2p|e(\Gamma^g_{H,G})| = 2|G|(2|Z(H,G)| + (p-2)|H|) - p|H|(1 + |H|).$$

In Result 1.4.23, Toule et al. have obtained a relation between $|e(\Gamma^g_G)|$ and $\Pr_g(G)$. It is noteworthy that their result can also be obtained from the next theorem considering $H = G$, where $k(H)$ denotes the number of conjugacy classes in H.

Theorem 3.3.5. *Let $\{1\} \neq H \trianglelefteq G$ and $g \in K(H,G)$.*

(a) *If $g = 1$ then $2|e(\Gamma^g_{H,G})| = (2|G| - |H|)(|H| - k(H))$.*

(b) *If $g \neq 1$ and $g^2 = 1$ then*

$$2|e(\Gamma^g_{H,G})| = 2|H||G|(1 - \Pr_g(H,G)) - |H|^2(1 - \Pr_g(H)) - |H|.$$

(c) *If* $g \neq 1$ *and* $g^2 \neq 1$ *then*

$$2|e(\Gamma^g_{H,G})| = 2|H||G|(1 - 2\mathrm{Pr}_g(H,G)) \ - |H|^2(1 - 2\mathrm{Pr}_g(H)) - |H|.$$

Proof. If $g = 1$ then by Result 1.2.2 we have

$$\mathrm{Pr}_g(H,G) = \mathrm{Pr}_g(H) = \frac{k(H)}{|H|}.$$

Hence, part (a) follows from Theorem 3.3.1. Parts (b) and (c) also follow from Theorem 3.3.1 noting that the case $g \in G \setminus H$ does not arise (since $g \in H$ if H is normal) and $\mathrm{Pr}_g(H,G) = \mathrm{Pr}_{g^{-1}}(H,G)$ (as shown in Result 1.2.3). □

By the expression of Result 1.2.6 for $\mathrm{Pr}_g(H,G)$ and Theorem 3.3.5 we get the following character theoretic formula for $|e(\Gamma^g_{H,G})|$.

Corollary 3.3.6. *Let* $\{1\} \neq H \trianglelefteq G$ *and* $g \in K(H,G)$.

(a) *If* $g = 1$ *then* $2|e(\Gamma^g_{H,G})| = (|H| - |\operatorname{Irr}(H)|)(2|G| - |H|)$.

(b) *If* $g \neq 1$ *and* $g^2 = 1$ *then*

$$2|e(\Gamma^g_{H,G})| = 2|H| \left(|G| - \sum_{\phi \in \operatorname{Irr}(G)} \langle \phi_H, \phi_H \rangle \frac{\phi(g)}{\phi(1)}\right) - |H| \left(|H| - \sum_{\phi \in \operatorname{Irr}(H)} \frac{\phi(g)}{\phi(1)}\right) - |H|.$$

(c) *If* $g \neq 1$ *and* $g^2 \neq 1$ *then*

$$2|e(\Gamma^g_{H,G})| = 2|H| \left(|G| - 2\sum_{\phi \in \operatorname{Irr}(G)} \langle \phi_H, \phi_H \rangle \frac{\phi(g)}{\phi(1)}\right) - |H| \left(|H| - 2\sum_{\phi \in \operatorname{Irr}(H)} \frac{\phi(g)}{\phi(1)}\right) - |H|.$$

Corollary 3.3.7. *Let* $g \in K(G)$.

(a) *For* $g = 1$ *we have* $2|e(\Gamma^g_G)| = |G|(|G| - |\operatorname{Irr}(G)|)$.

(b) *If* $g \neq 1$ *then*

$$2|e(\Gamma^g_G)| = \begin{cases} |G| \left(|G| - 1 - \sum_{\phi \in \operatorname{Irr}(G)} \frac{\phi(g)}{\phi(1)}\right), & \text{if } g^2 = 1 \\ |G| \left(|G| - 1 - 2\sum_{\phi \in \operatorname{Irr}(G)} \frac{\phi(g)}{\phi(1)}\right), & \text{if } g^2 \neq 1. \end{cases}$$

3.4 Bounds for $|e(\Gamma_{H,G}^g)|$

In [93, Section 3], Tolue and Erfanian have obtained bounds for $|e(\Gamma_{H,G})|$. In this section some bounds for the number of edges in $\Gamma_{H,G}^g$ are obtained. By Theorem 3.3.1, we have

$$2|e(\Gamma_{H,G}^g)|+|H|^2+|H| = \begin{cases} |H|^2\Pr_g(H) + 2|H||G|(1-\Pr_g(H,G)), & \text{if } g \in H \\ 2|H||G|(1-\Pr_g(H,G)), & \text{if } g \in G \setminus H, \end{cases} \tag{3.4.1}$$

if $g \neq 1$ but $g^2 = 1$ and

$$2|e(\Gamma_{H,G}^g)|+|H|^2+|H| = \begin{cases} |H|^2 \sum\limits_{u=g,g^{-1}} \Pr_u(H) + 2|H||G|(1-\sum\limits_{u=g,g^{-1}} \Pr_u(H,G)), & \text{if } g \in H \\ 2|H||G|(1-\sum\limits_{u=g,g^{-1}} \Pr_u(H,G)), & \text{if } g \in G \setminus H, \end{cases} \tag{3.4.2}$$

if $g \neq 1$ and $g^2 \neq 1$.

Theorem 3.4.1. *Let $H \leq G$ and $g \neq 1$.*

(a) *If $g^2 = 1$ then*

$$|e(\Gamma_{H,G}^g)| \geq \begin{cases} \frac{|G||Z(H,G)|+|H|(|G|-1)+3|Z(H)|^2-|H|^2}{2}, & \text{if } g \in H \\ \frac{|G||Z(H,G)|+|H|(|G|-1)-|H|^2}{2}, & \text{if } g \in G \setminus H. \end{cases}$$

(b) *If $g^2 \neq 1$ then*

$$|e(\Gamma_{H,G}^g)| \geq \begin{cases} \frac{2|G||Z(H,G)|+6|Z(H)|^2-|H|^2-|H|}{2}, & \text{if } g \in H \\ \frac{2|G||Z(H,G)|-|H|^2-|H|}{2}, & \text{if } g \in G \setminus H. \end{cases}$$

Proof. By Result 1.2.5, we get

$$1-\Pr_g(H,G) \geq \frac{|H|+|Z(H,G)|}{2|H|} \text{ and } 1-\sum_{u=g,g^{-1}} \Pr_u(H,G) \geq \frac{|Z(H,G)|}{|H|}. \tag{3.4.3}$$

Again, by Result 1.2.4(c), we have

$$\Pr_g(H) \geq \frac{3|Z(H)|^2}{|H|^2}. \tag{3.4.4}$$

(a) We have $g^2 = 1$. Therefore, if $g \in H$ then, using equations (3.4.1), (3.4.3) and (3.4.4), we get

$$2|e(\Gamma_{H,G}^g)| + |H|^2 + |H| \geq |H|^2 \left(\frac{3|Z(H)|^2}{|H|^2}\right) + 2|H||G| \left(\frac{|H| + |Z(H,G)|}{2|H|}\right). \tag{3.4.5}$$

If $g \in G \setminus H$ then, using equations (3.4.1) and (3.4.3), we get

$$2|e(\Gamma_{H,G}^g)| + |H|^2 + |H| > 2|H||G| \left(\frac{|H| + |Z(H,G)|}{2|H|}\right). \tag{3.4.6}$$

Hence, the result follows from equations (3.4.5) and (3.4.6).

(b) We have $g^2 \neq 1$. Therefore, if $g \in H$ then, using equations (3.4.2), (3.4.3) and (3.4.4), we get

$$2|e(\Gamma_{H,G}^g)| + |H|^2 + |H| \geq 2|H||G| \left(\frac{|Z(H,G)|}{|H|}\right) + |H|^2 \left(\frac{6|Z(H)|^2}{|H|^2}\right). \tag{3.4.7}$$

If $g \in G \setminus H$ then, using equations (3.4.2) and (3.4.3), we have

$$2|e(\Gamma_{H,G}^g)| + |H|^2 + |H| \geq \left(\frac{2|Z(H,G)||H||G|}{|H|}\right). \tag{3.4.8}$$

Hence, the result follows from equations (3.4.7) and (3.4.8). □

Theorem 3.4.2. *Let $H \leq G$ and $g \neq 1$.*

(a) *If $g^2 = 1$ then*

$$|e(\Gamma_{H,G}^g)| \leq \begin{cases} \frac{4|H||G|-8|Z(H,G)||Z(G,H)|-|H|^2-|H|(|Z(H)|+2)}{4}, & \text{if } g \in H \\ \frac{2|H||G|-4|Z(H,G)||Z(G,H)|-|H|^2-|H|}{2}, & \text{if } g \in G \setminus H. \end{cases}$$

(b) *If $g^2 \neq 1$ then*

$$|e(\Gamma_{H,G}^g)| \leq \begin{cases} \frac{2|H||G|-8|Z(H,G)||Z(G,H)|-|H|(|Z(H)|+1)}{2}, & \text{if } g \in H \\ \frac{2|H||G|-8|Z(H,G)||Z(G,H)|-|H|^2-|H|}{2}, & \text{if } g \in G \setminus H. \end{cases}$$

Proof. By Result 1.2.4(b), we get

$$1 - \Pr_g(H, G) \leq \frac{|H||G| - 2|Z(H,G)||Z(G,H)|}{|H||G|} \tag{3.4.9}$$

and

$$1 - \sum_{u=g,g^{-1}} \Pr_u(H, G) \leq \frac{|H||G| - 4|Z(H,G)||Z(G,H)|}{|H||G|}. \tag{3.4.10}$$

Also, by Result 1.2.5, we get

$$\Pr_g(H) \leq \frac{|H| - |Z(H)|}{2|H|}. \tag{3.4.11}$$

(a) We have $g^2 = 1$. Therefore, if $g \in H$ then, using equations (3.4.1), (3.4.9) and (3.4.11), we get

$$\begin{aligned} 2|e(\Gamma^g_{H,G})| &+ |H|^2 + |H| \\ &\leq |H|^2 \left(\frac{|H| - |Z(H)|}{2|H|}\right) + 2|H||G| \left(\frac{|H||G| - 2|Z(H,G)||Z(G,H)|}{|H||G|}\right). \end{aligned} \tag{3.4.12}$$

If $g \in G \setminus H$ then, using equations (3.4.1) and (3.4.9), we get

$$2|e(\Gamma^g_{H,G})| + |H|^2 + |H| \leq 2|H||G| \left(\frac{|H||G| - 2|Z(H,G)||Z(G,H)|}{|H||G|}\right). \tag{3.4.13}$$

Hence, the result follows from equations (3.4.12) and (3.4.13).

(b) We have $g^2 \neq 1$. Therefore, if $g \in H$ then, using equations (3.4.2), (3.4.10) and (3.4.11), we get

$$\begin{aligned} 2|e(\Gamma^g_{H,G})| + |H|^2 + |H| \leq\, & 2|H||G| \left(\frac{|H||G| - 4|Z(H,G)||Z(G,H)|}{|H||G|}\right) \\ & + |H|^2 \left(\frac{|H| - |Z(H)|}{|H|}\right). \end{aligned} \tag{3.4.14}$$

If $g \in G \setminus H$ then, using equations (3.4.2) and (3.4.10), we get

$$2|e(\Gamma^g_{H,G})| + |H|^2 + |H| \leq 2|H||G| \left(\frac{|H||G| - 4|Z(H,G)||Z(G,H)|}{|H||G|}\right). \tag{3.4.15}$$

Hence, the result follows from equations (3.4.14) and (3.4.15). □

In the remaining results p stands for the smallest prime such that $p \mid |G|$ and $g \neq 1$.

Theorem 3.4.3. (a) *If $g^2 = 1$ then*

$$|e(\Gamma_{H,G}^g)| \geq \begin{cases} \frac{2(p-1)|H||G|+2|Z(H,G)||G|-p|H|^2+3p|Z(H)|^2-p|H|}{2p}, & \text{if } g \in H \\ \frac{2(p-1)|H||G|+2|Z(H,G)||G|-p|H|^2-p|H|}{2p}, & \text{if } g \in G \setminus H. \end{cases}$$

(b) *If $g^2 \neq 1$ then*

$$|e(\Gamma_{H,G}^g)| \geq \begin{cases} \frac{2(p-2)|H||G|+4|Z(H,G)||G|-p|H|^2+6p|Z(H)|^2-p|H|}{2p}, & \text{if } g \in H \\ \frac{2(p-2)|H||G|+4|Z(H,G)||G|-p|H|^2-p|H|}{2p}, & \text{if } g \in G \setminus H. \end{cases}$$

Proof. By Result 1.2.5, we get

$$1 - \Pr_g(H, G) \geq \frac{|Z(H,G)| + (p-1)|H|}{p|H|} \tag{3.4.16}$$

and

$$1 - \sum_{u=g,g^{-1}} \Pr_u(H, G) \geq \frac{2|Z(H,G)| + (p-2)|H|}{p|H|}. \tag{3.4.17}$$

(a) We have $g^2 = 1$. Therefore, if $g \in H$ then, using equations (3.4.1), (3.4.16) and (3.4.4), we get

$$2|e(\Gamma_{H,G}^g)| + |H|^2 + |H| \geq |H|^2 \left(\frac{3|Z(H)|^2}{|H|^2}\right) + 2|H||G| \left(\frac{|Z(H,G)| + (p-1)|H|}{p|H|}\right). \tag{3.4.18}$$

If $g \in G \setminus H$ then, using equations (3.4.1) and (3.4.16), we get

$$2|e(\Gamma_{H,G}^g)| + |H|^2 + |H| \geq 2|H||G| \left(\frac{|Z(H,G)| + (p-1)|H|}{p|H|}\right). \tag{3.4.19}$$

Hence, the result follows from equations (3.4.18) and (3.4.19).

(b) We have $g^2 \neq 1$. Therefore, if $g \in H$ then, using equations (3.4.2), (3.4.17) and (3.4.4), we get

$$2|e(\Gamma_{H,G}^g)| + |H|^2 + |H| \geq 2|H||G| \left(\frac{(p-2)|H| + 2|Z(H,G)|}{p|H|}\right) + |H|^2 \left(\frac{6|Z(H)|^2}{|H|^2}\right). \tag{3.4.20}$$

If $g \in G \setminus H$ then, using equations (3.4.2) and (3.4.17), we have

$$2|e(\Gamma_{H,G}^g)| + |H|^2 + |H| \geq 2|H||G| \left(\frac{(p-2)|H| + 2|Z(H,G)|}{p|H|}\right). \tag{3.4.21}$$

Hence, the result follows from equations (3.4.20) and (3.4.21). □

Theorem 3.4.4. (a) *If $g^2 = 1$ then*

$$|e(\Gamma^g_{H,G})| \leq \begin{cases} \frac{2p|H||G|-4p|Z(H,G)||Z(G,H)|-(p-1)|H|^2-|H||Z(H)|-p|H|}{2p}, & \text{if } g \in H \\ \frac{2|H||G|-4|Z(H,G)||Z(G,H)|-|H|^2-|H|}{2}, & \text{if } g \in G \setminus H. \end{cases}$$

(b) *If $g^2 \neq 1$ then*

$$|e(\Gamma^g_{H,G})| \leq \begin{cases} \frac{2p|H||G|-8p|Z(H,G)||Z(G,H)|-(p-2)|H|^2-2|H||Z(H)|-p|H|}{2p}, & \text{if } g \in H \\ \frac{2|H||G|-8|Z(H,G)||Z(G,H)|-|H|^2-|H|}{2}, & \text{if } g \in G \setminus H. \end{cases}$$

Proof. By Result 1.2.5, we get

$$\Pr_g(H) \leq \frac{|H| - |Z(H)|}{p|H|}. \tag{3.4.22}$$

(a) We have $g^2 = 1$. Therefore, if $g \in H$ then, using equations (3.4.1), (3.4.9) and (3.4.22), we get

$$2|e(\Gamma^g_{H,G})| + |H|^2 + |H| \leq |H|^2 \left(\frac{|H| - |Z(H)|}{p|H|}\right) + 2|H||G| \left(\frac{|H||G| - 2|Z(H,G)||Z(G,H)|}{|H||G|}\right). \tag{3.4.23}$$

If $g \in G \setminus H$ then, using equations (3.4.1) and (3.4.9), we get

$$2|e(\Gamma^g_{H,G})| + |H|^2 + |H| \leq 2|H||G| \left(\frac{|H||G| - 2|Z(H,G)||Z(G,H)|}{|H||G|}\right). \tag{3.4.24}$$

Hence, the result follows from equations (3.4.23) and (3.4.24).

(b) We have $g^2 \neq 1$. Therefore, if $g \in H$ then, using equations (3.4.2), (3.4.10) and (3.4.22), we get

$$2|e(\Gamma^g_{H,G})| + |H|^2 + |H| \leq 2|H|^2 \left(\frac{|H| - |Z(H)|}{p|H|}\right) + 2|H||G| \left(\frac{|H||G| - 4|Z(H,G)||Z(G,H)|}{|H||G|}\right). \tag{3.4.25}$$

If $g \in G \setminus H$ then, using equations (3.4.2) and (3.4.10), we get

$$2|e(\Gamma^g_{H,G})| + |H|^2 + |H| \leq 2|H||G| \left(\frac{|H||G| - 4|Z(H,G)||Z(G,H)|}{|H||G|}\right). \tag{3.4.26}$$

Hence, the result follows from equations (3.4.25) and (3.4.26). □

Note that several other bounds for $|e(\Gamma^g_{H,G})|$ can be obtained using different combinations of the bounds for $\Pr_g(H,G)$ and $\Pr_g(H)$. We conclude this chapter with certain bounds for $|e(\Gamma^g_G)|$ which are obtained by putting $H = G$ in the above theorems.

Corollary 3.4.5. (a) *If* $g^2 = 1$ *then*

$$\frac{3|G|^2 - 8|Z(G)|^2 - |G|(|Z(G)| + 2)}{4} \geq |e(\Gamma^g_G)| \geq \frac{|G||Z(G)| + 3|Z(G)|^2 - |G|}{2}.$$

(b) *If* $g^2 \neq 1$ *then*

$$\frac{2|G|^2 - 8|Z(G)|^2 - |G|(|Z(G)| + 1)}{2} \geq |e(\Gamma^g_G)| \geq \frac{2|G||Z(G)| + 6|Z(G)|^2 - |G|^2 - |G|}{2}$$

Corollary 3.4.6. (a) *If* $g^2 = 1$ *then*

$$\frac{(p+1)|G|^2 - 4p|Z(G)|^2 - |G||Z(G)| - p|G|}{2p} \geq |e(\Gamma^g_G)| \geq \frac{(p-2)|G|^2 + 2|Z(G)||G| + 3p|Z(G)|^2 - p|G|}{2p}.$$

(b) *If* $g^2 \neq 1$ *then*

$$\frac{(p+2)|G|^2 - 8p|Z(G)|^2 - 2|G||Z(G)| - p|G|}{2p} \geq |e(\Gamma^g_G)| \geq \frac{(p-4)|G|^2 + 4|Z(G)||G| + 6p|Z(G)|^2 - p|G|}{2p}.$$

3.5 Conclusion

In this chapter, we have introduced the concept of relative g-noncommuting graph ($\Gamma^g_{H,G}$), for any subgroup H of a finite group G. We have characterized finite groups such that $\Gamma^{g\neq 1}_{H,G}$ is a star. We have also shown that $\Gamma^g_{H,G}$ is not a tree (whenever $|H| \neq 2$), lollipop (whenever $|H| \neq 2, 3$) or a complete graph when $1 \neq g \in K(H,G)$ along with certain other results. Further, we have derived an expression for the number of edges in $\Gamma^g_{H,G}$ in terms of $\Pr_g(H,G)$ and $\Pr_g(G)$, and subsequently established certain bounds for this quantity. As a consequence of our research we obtain several new results on Γ^g_G (for example, see Corollaries 3.3.7, 3.4.5 and 3.4.6). Some of our results also generalize some existing results on Γ^g_G (for example, Theorem 3.2.12 generalizes Results 1.4.21 and 1.4.22; Theorem 3.3.5 generalizes Result 1.4.23).