Chapter 4

An induced subgraph of relative g-noncommuting graph of a finite group

Let H be a subgroup of a finite non-abelian group G and $g \in G$. In this chapter, we consider the *induced subgraph* of $\Gamma_{H,G}^{g}$ on $G \setminus Z(H,G)$, denoted by $\Delta_{H,G}^{g}$. Thus $\Delta_{H,G}^{g}$ is a simple undirected graph whose vertex set is $G \setminus Z(H,G)$ and two distinct vertices x and y are adjacent if $x \in H$ or $y \in H$ and $[x,y] \neq g, g^{-1}$. Clearly, $\Delta_{H,G}^{g} = \Delta_{H,G}^{g^{-1}}$. If $H = G$ and $g = 1$ then $\Delta_{H,G}^{g} = \Gamma_G$, the non-commuting graph of G. If $H = G$ then $\Delta_{H,G}^{g} = \Delta_{G}^{g}$, a generalization of Γ_G called an induced g-noncommuting graph of G on $G \setminus Z(G)$ which has been studied extensively in [69, 70, 71] by Erfanian and his collaborators. In Section 4.1, we determine whether $\Delta_{H,G}^{g}$ is a tree while determining the degree of a vertex in $\Delta_{H,G}^{g}$. In Section 4.2, we discuss the connectivity of $\Delta_{H,G}^{g}$ and conclude the chapter by investigating the diameter and connectivity of $\Delta_{H,G}^{g}$ with special attention to the dihedral groups. This chapter is based on our paper [89] published in *Mathematics*.

If $g \notin K(H,G) := \{[x,y] : x \in H \text{ and } y \in G\}$ then any pair of vertices (x,y) are adjacent in $\Delta_{H,G}^{g}$ trivially if $x, y \in H$ or one of x and y belongs to H. Therefore, we consider $g \in K(H,G)$. In addition, if $H = Z(H,G)$ then $K(H,G) = \{1\}$ and so $g = 1$.

Thus, throughout this chapter, we consider $H \neq Z(H,G)$ and $g \in K(H,G)$. We like to mention the following examples of $\Delta^g_{H,G}$, where $G = A_4 = \langle a, b : a^2 = b^3 = (ab)^3 = 1\rangle$ and the subgroup H is given by $H_1 = \{1, a\}$, $H_2 = \{1, bab^2\}$ or $H_3 = \{1, b^2ab\}$ (see Figures 4.1– 4.6).

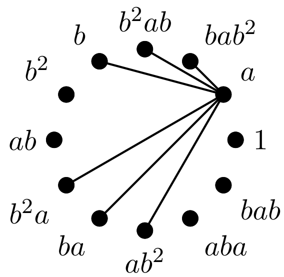

Figure 4.1: $\Delta^{bab^2}_{H_1,A_4}$

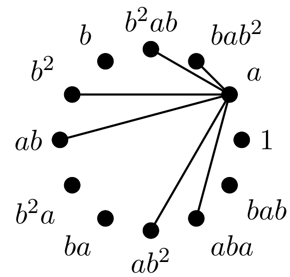

Figure 4.2: $\Delta^{b^2ab}_{H_1,A_4}$

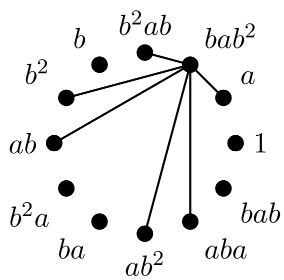

Figure 4.3: $\Delta^{a}_{H_2,A_4}$

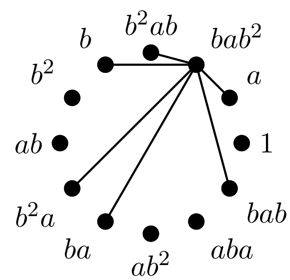

Figure 4.4: $\Delta^{b^2ab}_{H_2,A_4}$

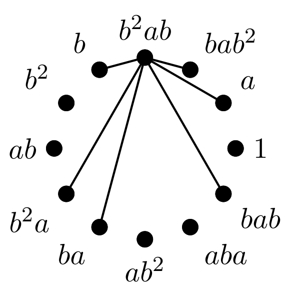

Figure 4.5: $\Delta^{a}_{H_3,A_4}$

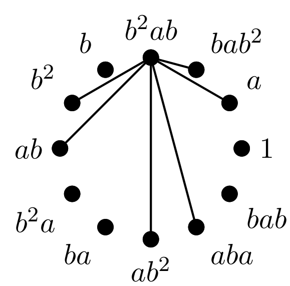

Figure 4.6: $\Delta^{bab^2}_{H_3,A_4}$

4.1 Vertex degree and a consequence

In this section we first determine $\deg(x)$, the degree of a vertex x in the graph $\Delta^g_{H,G}$. After that we determine whether $\Delta^g_{H,G}$ is a tree. Corresponding to Theorems 3.2.1 and 3.2.2, we have the following two results for $\Delta^g_{H,G}$.

Theorem 4.1.1. *Let $x \in H \setminus Z(H,G)$ be any vertex in $\Delta^g_{H,G}$.*

(a) *If $g = 1$ then $\deg(x) = |G| - |C_G(x)|$.*

(b) *If $g \neq 1$ and $g^2 \neq 1$ then*

$$\deg(x) = \begin{cases} |G| - |Z(H,G)| - |C_G(x)| - 1, & \text{if } x \sim xg \text{ or } xg^{-1} \\ |G| - |Z(H,G)| - 2|C_G(x)| - 1, & \text{if } x \sim xg \text{ and } xg^{-1}. \end{cases}$$

(c) *If $g \neq 1$ and $g^2 = 1$ then $\deg(x) = |G| - |Z(H,G)| - |C_G(x)| - 1$, whenever $x \sim xg$.*

Proof. (a) Let $g = 1$. Then $\deg(x)$ is the number of $y \in G \setminus Z(H,G)$ such that $xy \neq yx$. Hence,

$$\deg(x) = |G| - |Z(H,G)| - (|C_G(x)| - |Z(H,G)|) = |G| - |C_G(x)|.$$

Proceeding as the proof of Theorem 3.2.1(b) and (c), parts (b) and (c) follow noting that the vertex set of $\Delta^g_{H,G}$ is $G \setminus Z(H,G)$. □

Theorem 4.1.2. *Let $x \in G \setminus H$ be any vertex in $\Delta^g_{H,G}$.*

(a) *If $g = 1$ then $\deg(x) = |H| - |C_H(x)|$.*

(b) *If $g \neq 1$ and $g^2 \neq 1$ then*

$$\deg(x) = \begin{cases} |H| - |Z(H,G)| - |C_H(x)|, & \text{if } x \sim xg \text{ or } xg^{-1} \text{ for some element in } H \\ |H| - |Z(H,G)| - 2|C_H(x)|, & \text{if } x \sim xg \text{ and } xg^{-1} \text{ for some element in } H \end{cases}$$

(c) *If $g \neq 1$ and $g^2 = 1$ then $\deg(x) = |H| - |Z(H,G)| - |C_H(x)|$, whenever $x \sim xg$, for some element in H.*

Proof. (a) Let $g = 1$. Then $\deg(x)$ is the number of $y \in H \setminus Z(H, G)$ such that $xy \neq yx$. Hence,

$$\deg(x) = |H| - |Z(H, G)| - (|C_H(x)| - |Z(H, G)|) = |H| - |C_H(x)|.$$

Proceeding as the proof of Theorem 3.2.2(b) and (c), parts (b) and (c) follow noting that the vertex set of $\Delta^g_{H,G}$ is $G \setminus Z(H, G)$. □

As a consequence of the above results, we have the following:

Theorem 4.1.3. *If $|H| \neq 2, 3, 4, 6$ then $\Delta^g_{H,G}$ is not a tree.*

Proof. Suppose that $\Delta^g_{H,G}$ is a tree. Then there exists a vertex $x \in G \setminus Z(H, G)$ such that $\deg(x) = 1$. If $x \in H \setminus Z(H, G)$ then we have the following cases.

Case 1: If $g = 1$, then by Theorem 4.1.1(a), we have $\deg(x) = |G| - |C_G(x)| = 1$. Therefore, $|C_G(x)| = 1$, contradiction.

Case 2: If $g \neq 1$ and $g^2 = 1$, then by Theorem 4.1.1(c), we have $\deg(x) = |G| - |Z(H, G)| - |C_G(x)| - 1 = 1$. That is,

$$|G| - |Z(H, G)| - |C_G(x)| = 2. \tag{4.1.1}$$

Therefore, $|Z(H, G)| = 1$ or 2. Thus equation (4.1.1) gives $|G| - |C_G(x)| = 3$ or 4. Therefore, $|G| = 6$ or 8. Since $|H| \neq 2, 3, 4, 6$ we must have $G \cong D_8$ or Q_8 and $H = G$ and hence, by Result 1.4.24, we get a contradiction.

Case 3: If $g \neq 1$ and $g^2 \neq 1$, then by Theorem 4.1.1(b), we have $\deg(x) = |G| - |Z(H, G)| - |C_G(x)| - 1 = 1$, which will lead to equation (4.1.1) (and eventually to a contradiction) or $\deg(x) = |G| - |Z(H, G)| - 2|C_G(x)| - 1 = 1$. That is,

$$\text{or } |G| - |Z(H, G)| - 2|C_G(x)| = 2. \tag{4.1.2}$$

Therefore, $|Z(H, G)| = 1$ or 2. Thus if $|Z(H, G)| = 1$ then equation (4.1.2) gives $|G| = 9$, which is a contradiction since G is non-abelian. Again if $|Z(H, G)| = 2$ then equation (4.1.2) gives $|C_G(x)| = 2$ or 4. Therefore, $|G| = 8$ or $|G| = 12$. If $|G| = 8$ then we get a contradiction as shown in Case 2 above. If $|G| = 12$ then $G \cong D_{12}$ or Q_{12}, since $|Z(H, G)| = 2$. In both of the cases, we must have $H = G$ and hence, by Result 1.4.24, we get a contradiction.

Now we assume that $x \in G \setminus H$ and consider the following cases.

Case 1: If $g = 1$, then by Theorem 4.1.2(a), we have $\deg(x) = |H| - |C_H(x)| = 1$. Therefore, $|H| = 2$, a contradiction.

Case 2: If $g \neq 1$ and $g^2 = 1$, then by Theorem 4.1.2(c), we have $\deg(x) = |H| - |Z(H,G)| - |C_H(x)| = 1$. That is,

$$|H| - |C_H(x)| = 2. \tag{4.1.3}$$

Therefore, $|H| = 3$ or 4, a contradiction.

Case 3: If $g \neq 1$ and $g^2 \neq 1$, then by Theorem 4.1.2(b), we have $\deg(x) = |H| - |Z(H,G)| - |C_H(x)| = 1$, which leads to equation (4.1.3) or $\deg(x) = |H| - |Z(H,G)| - 2|C_H(x)| = 1$. That is,

$$|H| - 2|C_H(x)| = 2. \tag{4.1.4}$$

Therefore, $|C_H(x)| = 1$ or 2. Thus if $|C_H(x)| = 1$ then equation (4.1.4) gives $|H| = 4$, a contradiction. If $|C_H(x)| = 2$ then equation (4.1.4) gives $|H| = 6$, a contradiction. □

The following theorems also show that the conditions on $|H|$ as mentioned in Theorem 4.1.3 can not be removed completely.

Theorem 4.1.4. *If G is a non-abelian group of order ≤ 12 and $g = 1$ then $\Delta^g_{H,G}$ is a tree if and only if $G \cong D_6$ or D_{10} and $|H| = 2$.*

Proof. If H is the trivial subgroup of G then $\Delta^g_{H,G}$ is an empty graph. If $H = G$ then, by Result 1.4.24, we have $\Delta^g_{H,G}$ is not a tree. Thus, we examine only the proper subgroups of G, where $G \cong D_6, D_8, Q_8, D_{10}, D_{12}, Q_{12}$ or A_4. We consider the following cases:

Case 1: $G \cong D_6 = \langle a, b : a^3 = b^2 = 1, bab^{-1} = a^{-1} \rangle$. If $|H| = 2$ then $H = \langle x \rangle$, where $x = b, ab$ and a^2b. We have $[x, y] \neq 1$ for all $y \in G \setminus Z(H,G)$. Therefore, Δ^g_{H,D_6} is a star graph and hence, a tree. If $|H| = 3$ then $H = \{1, a, a^2\}$. In this case, the vertices a, ab, a^2 and b make a cycle since $[ab, a] = a^2 = [a^2, ab]$ and $[a, b] = a = [b, a^2]$.

Case 2: $G \cong D_8 = \langle a, b : a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$. If $|H| = 2$ then $H = Z(D_8)$ or $\langle a^r b \rangle$, where $r = 1, 2, 3, 4$. Clearly Δ^g_{H,D_8} is an empty graph if $H = Z(D_8)$. If $H = \langle a^r b \rangle$ then, in each case, a^2 is an isolated vertex in $G \setminus H$ (since $[a^2, a^r b] = 1$). Hence, Δ^g_{H,D_8} is disconnected. If $|H| = 4$ then $H = \{1, a, a^2, a^3\}$, $\{1, a^2, b, a^2b\}$ or $\{1, a^2, ab, a^3b\}$. If $H = \{1, a, a^2, a^3\}$ then the vertices ab, a, b and a^3 make a cycle; if $H = \{1, a^2, b, a^2b\}$ then the vertices ab, b, a^3 and a^2b make a cycle; and if $H = \{1, a^2, ab, a^3b\}$ then the vertices ab, a, a^3b and b make a cycle (since $[a, b] = [a^3, b] = [a^3, ab] = [a^3, a^2b] = [ab, a] = [a^2b, ab] = [a^3b, a] = [b, ab] = [b, a^3b] = a^2 \neq 1$).

Case 3: $G \cong Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1}\rangle$. If $|H| = 2$ then $H = Z(Q_8)$ and so Δ^g_{H,Q_8} is an empty graph. If $|H| = 4$ then $H = \{1, a, a^2, a^3\}$, $\{1, a^2, b, a^2b\}$ and $\{1, a^2, ab, a^3b\}$. Again, if $H = \{1, a, a^2, a^3\}$ then the vertices a, b, a^3 and ab make a cycle; if $H = \{1, a^2, b, a^2b\}$ then the vertices b, a^3b, a^2b and a^3 make a cycle; and if $H = \{1, a^2, ab, a^3b\}$ then the vertices ab, a, a^3b and a^2b make a cycle (since $[a, b] = [b, a^3] = [a^3, ab] = [ab, a] = [b, a^3b] = [a^3b, a^2b] = [a^2b, a^3] = [a, a^3b] = [a^2b, ab] = a^2 \neq 1$).

Case 4: $G \cong D_{10} = \langle a, b : a^5 = b^2 = 1, bab^{-1} = a^{-1}\rangle$. If $|H| = 2$ then $H = \langle a^rb\rangle$, for every integer r such that $1 \leq r \leq 5$. For each case of H, $\Delta^g_{H,D_{10}}$ is a star graph since $[a^rb, x] \neq g$ for all $x \in G \setminus H$. If $|H| = 5$ then $H = \{1, a, a^2, a^3, a^4\}$. In this case, the vertices a, ab, a^3 and a^3b make a cycle in $\Delta^g_{H,D_{10}}$ since $[a, ab] = a^3 \neq 1$, $[ab, a^3] = a \neq 1$, $[a^3, a^3b] = a^4 \neq 1$ and $[a^3b, a] = a^2 \neq 1$.

Case 5: $G \cong D_{12} = \langle a, b : a^6 = b^2 = 1, bab^{-1} = a^{-1}\rangle$. If $|H| = 2$ then $H = Z(D_{12})$ or $\langle a^rb\rangle$, for every integer r such that $1 \leq r \leq 6$. If $|H| = 3$ then $H = \{1, a^2, a^4\}$. If $|H| = 4$ then $H = \{1, a^3, b, a^3b\}$, $\{1, a^3, ab, a^4b\}$ or $\{1, a^3, a^2b, a^5b\}$. If $|H| = 6$ then $H = \{1, a, a^2, a^3, a^4, a^5\}$, $\{1, a^2, a^4, b, a^2b, a^4b\}$ or $\{1, a^2, a^4, ab, a^3b, a^5b\}$. Note that $\Delta^g_{H,D_{12}}$ is an empty graph if $H = Z(D_{12})$. If $H = \langle a^rb\rangle$ (for $1 \leq r \leq 6$), $\{1, a^2, a^4\}$, $\{1, a^2, a^4, b, a^2b, a^4b\}$ or $\{1, a^2, a^4, ab, a^3b, a^5b\}$ then in each case the vertex a^3 is an isolated vertex in $G \setminus H$ (since $a^3 \in Z(D_{12})$) and hence $\Delta^g_{H,D_{12}}$ is disconnected. We have $[a, b] = [b, a^5] = [a, ab] = [a^4, a^4b] = [a^5b, a^2] = [b, a^2] = [a^2b, a^5] = [a^3b, a^2] = [a^3b, a^5] = a^4 \neq 1$ and $[a^5, a^5b] = [a^5b, a] = [ab, a^4] = [a^4b, a] = [a^2, a^2b] = [a^2b, a] = a^2 \neq 1$. Therefore, if $H = \{1, a^3, b, a^3b\}$ then the vertices b, a^2, a^3b and a^5 make a cycle; if $H = \{1, a^3, ab, a^4b\}$ then the vertices a, ab, a^4 and a^4b make a cycle; if $H = \{1, a^3, a^2b, a^5b\}$ then the vertices a^2, a^2b, a^5 and a^5b make a cycle; and if $H = \{1, a, a^2, a^3, a^4, a^5\}$ then the vertices a, b, a^2 and a^2b make a cycle.

Case 6: $G \cong A_4 = \langle a, b : a^2 = b^3 = (ab)^3 = 1\rangle$. If $|H| = 2$ then $H = \langle a\rangle$, $\langle bab^2\rangle$ or $\langle b^2ab\rangle$. Since the elements a, bab^2 and b^2ab commute among themselves, in each case the remaining two elements in $G \setminus H$ remain isolated and hence Δ^g_{H,A_4} is disconnected. If $|H| = 3$ then $H = \langle x\rangle$, where $x = b$, ab, ba, aba. In each case, the vertices x, a, x^{-1} and bab^2 make a cycle. If $|H| = 4$ then $H = \{1, a, bab^2, b^2ab\}$. In this case, the vertices a, b, bab^2 and ab make a cycle.

Case 7: $G \cong Q_{12} = \langle a, b : a^6 = 1, b^2 = a^3, bab^{-1} = a^{-1}\rangle$. If $|H| = 2$ then $H = Z(Q_{12})$ and so $\Delta^g_{H,Q_{12}}$ is an empty graph. If $|H| = 3$ then $H = \{1, a^2, a^4\}$. In this case, a^3 is an isolated vertex in $G \setminus H$ (since $a^3 \in Z(Q_{12})$) and so $\Delta^g_{H,Q_{12}}$ is disconnected. If $|H| = 4$ then $H = \{1, a^3, b, a^3b\}$, $\{1, a^3, ab, a^4b\}$ or $\{1, a^3, a^2b, a^5b\}$. If $|H| = 6$ then $H = \{1, a, a^2, a^3, a^4, a^5\}$. We have $[a, b] = [a, ab] = [a^4, a^4b] = [a^5b, a^2] = [b, a^2] = [b, a^5] = [a^2b, a^5] = [a^3b, a^2] = [a^3b, a^5] = a^4 \neq 1$ and $[a^5, a^5b] = [a^5b, a] = [ab, a^4] = [a^4b, a] = [a^2, a^2b] = [a^2b, a] = a^2 \neq 1$. Therefore, if $H = \{1, a^3, b, a^3b\}$ then the vertices a^2, b, a^5 and a^3b make a cycle; if $H = \{1, a^3, ab, a^4b\}$ then the vertices a, ab, a^4 and a^4b make a cycle; if $H = \{1, a^3, a^2b, a^5b\}$ then the vertices a^2, a^2b, a^5 and a^5b make a cycle; and if $H = \{1, a, a^2, a^3, a^4, a^5\}$ then the vertices a, b, a^2 and a^2b make a cycle. This completes the proof. □

Theorem 4.1.5. *If G is a non-abelian group of order ≤ 12 and $g \neq 1$ then $\Delta^g_{H,G}$ is a tree if and only if $g^2 = 1$, $G \cong A_4$ and $|H| = 2$ such that $H = \langle g\rangle$.*

Proof. If H is the trivial subgroup of G then $\Delta^g_{H,G}$ is an empty graph. If $H = G$ then, by Result 1.4.24, we have $\Delta^g_{H,G}$ is not a tree. Thus, we examine only the proper subgroups of G, where $G \cong D_6, D_8, Q_8, D_{10}, D_{12}, Q_{12}$ or A_4. We consider the following two cases.

Case 1: $g^2 = 1$

In this case $G \cong D_8$, Q_8 or A_4. If $G \cong D_8 = \langle a, b : a^4 = b^2 = 1, bab^{-1} = a^{-1}\rangle$ then $g = a^2$ and $|H| = 2, 4$. If $|H| = 2$ then $H = Z(D_8)$ or $\langle a^rb\rangle$, for every integer r such that $1 \leq r \leq 4$. For $H = Z(D_8)$, Δ^g_{H,D_8} is an empty graph. For $H = \langle a^rb\rangle$, in each case a is an isolated vertex in $G \setminus H$ (since $[a, a^rb] = a^2$) and hence, Δ^g_{H,D_8} is disconnected. If $|H| = 4$ then $H = \{1, a, a^2, a^3\}$, $\{1, a^2, b, a^2b\}$ or $\{1, a^2, ab, a^3b\}$. For $H = \{1, a, a^2, a^3\}$, b is an isolated vertex in $G \setminus H$ (since $[a, b] = a^2 = [a^3, b]$) and hence, Δ^g_{H,D_8} is disconnected. If $H = \{1, a^2, b, a^2b\}$ or $\{1, a^2, ab, a^3b\}$ then a is an isolated vertex in $G \setminus H$ (since $[a, a^rb] = a^2$ for every integer r such that $1 \leq r \leq 4$) and hence, Δ^g_{H,D_8} is disconnected.

If $G \cong Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1}\rangle$ then $g = a^2$ and $|H| = 2, 4$. If $|H| = 2$ then $H = Z(Q_8)$ and hence Δ^g_{H,Q_8} is an empty graph. If $|H| = 4$ then $H = \{1, a, a^2, a^3\}$, $\{1, a^2, b, a^2b\}$ or $\{1, a^2, ab, a^3b\}$. In each case, vertices of $H \setminus Z(H, G)$ commute with each other and commutator of these vertices and those of $G \setminus H$ equals a^2. Hence, the vertices in $G \setminus H$ remain isolated and so Δ^g_{H,Q_8} is disconnected.

If $G \cong A_4 = \langle a, b : a^2 = b^3 = (ab)^3 = 1\rangle$ then $g \in \{a, bab^2, b^2ab\}$ and $|H| = 2, 3, 4$. If $|H| = 2$ then $H = \langle a\rangle$, $\langle bab^2\rangle$ or $\langle b^2ab\rangle$. If $H = \langle g\rangle$ then Δ^g_{H,A_4} is a star graph because $[g, x] \neq g$ for all $x \in G \setminus H$ and hence a tree; otherwise Δ^g_{H,A_4} is not a tree as shown in Figures 1–6. If $|H| = 3$ then $H = \langle x\rangle$, where $x = b, ab, ba, aba$ or their inverses. We have $[x, x^{-1}] = 1$, $[x, g] \neq g$ and $[x^{-1}, g] \neq g$. Therefore, x, x^{-1} and g make a triangle for each such subgroup in the graph Δ^g_{H,A_4}. If $|H| = 4$ then $H = \{1, a, bab^2, b^2ab\}$. Since H is abelian, the vertices a, bab^2 and b^2ab make a triangle in the graph Δ^g_{H,A_4}.

Case 2: $g^2 \neq 1$

In this case $G \cong D_6$, D_{10}, D_{12} or Q_{12}.

If $G \cong D_6 = \langle a, b : a^3 = b^2 = 1, bab^{-1} = a^{-1}\rangle$ then $g \in \{a, a^2\}$ and $|H| = 2, 3$. We have $\Delta^a_{H,D_6} = \Delta^{a^2}_{H,D_6}$ since $a^{-1} = a^2$. If $|H| = 2$ then $H = \langle x\rangle$, where $x = b, ab$ and a^2b. We have $[x, y] \in \{g, g^{-1}\}$ for all $y \in G \setminus H$ and so Δ^g_{H,D_6} is an empty graph. If $|H| = 3$ then $H = \{1, a, a^2\}$. In this case, the vertices of $G \setminus H$ remain isolated since for $y \in G \setminus H$ we have $[a, y], [a^2, y] \in \{g, g^{-1}\}$.

If $G \cong D_{10} = \langle a, b : a^5 = b^2 = 1, bab^{-1} = a^{-1}\rangle$ then $g \in \{a, a^2, a^3, a^4\}$ and $|H| = 2, 5$. We have $\Delta^a_{H,D_{10}} = \Delta^{a^4}_{H,D_{10}}$ and $\Delta^{a^2}_{H,D_{10}} = \Delta^{a^3}_{H,D_{10}}$ since $a^{-1} = a^4$ and $(a^2)^{-1} = a^3$. Suppose that $|H| = 2$. Then $H = \langle a^r b\rangle$, for every integer r such that $1 \leq r \leq 5$. If $g = a$ then for each subgroup H, a^2 is an isolated vertex in $\Delta^g_{H,D_{10}}$ (since $[a^2, a^r b] = a^4$ for every integer r such that $1 \leq r \leq 5$). If $g = a^2$ then for each subgroup H, a is an isolated vertex in $\Delta^g_{H,D_{10}}$ (since $[a, a^r b] = a^2$ for every integer r such that $1 \leq r \leq 5$). Hence, $\Delta^g_{H,D_{10}}$ is disconnected for each g and each subgroup H of order 2. Now suppose that $|H| = 5$. Then we have $H = \{1, a, a^2, a^3, a^4\}$. In this case, the vertices a, a^2, a^3 and a^4 make a cycle in $\Delta^g_{H,D_{10}}$ for each g as they commute among themselves.

If $G \cong D_{12} = \langle a, b : a^6 = b^2 = 1, bab^{-1} = a^{-1}\rangle$ then $g \in \{a^2, a^4\}$ and $|H| = 2, 3, 4, 6$. We have $\Delta^{a^2}_{H,D_{12}} = \Delta^{a^4}_{H,D_{12}}$ since $(a^2)^{-1} = a^4$. Suppose that $|H| = 2$ then $H = Z(D_{12})$ or $\langle a^r b\rangle$, for every integer r such that $1 \leq r \leq 6$. For $H = Z(D_{12})$, $\Delta^g_{H,D_{12}}$ is an empty graph. For $H = \langle a^r b\rangle$, in each case a is an isolated vertex in $G \setminus H$ (since $[a, a^r b] = a^2$ for every integer r such that $1 \leq r \leq 6$) and hence, $\Delta^g_{H,D_{12}}$ is disconnected. If $|H| = 3$ then $H = \{1, a^2, a^4\}$. In this case, the vertices a, a^2 and a^4 make a triangle in $\Delta^g_{H,D_{12}}$ since they commute among themselves. If $|H| = 4$ then $H = \{1, a^3, b, a^3b\}$, $\{1, a^3, ab, a^4b\}$ or $\{1, a^3, a^2b, a^5b\}$. For all these H, a is an isolated vertex in $G \setminus H$ (since $[a, a^rb] = a^2$ for every integer r such that $1 \leq r \leq 6$) and hence, $\Delta^g_{H,D_{12}}$ is disconnected. If $|H| = 6$ then $H = \{1, a, a^2, a^3, a^4, a^5\}$, $\{1, a^2, a^4, b, a^2b, a^4b\}$ or $\{1, a^2, a^4, ab, a^3b, a^5b\}$. For all these H the vertices a, a^2, a^4 and a^5 make a cycle in $\Delta^g_{H,D_{12}}$ since they commute among themselves.

If $G \cong Q_{12} = \langle a, b \, : \, a^6 = 1, b^2 = a^3, bab^{-1} = a^{-1}\rangle$ then $g \in \{a^2, a^4\}$ and $|H| = 2, 3, 4, 6$. We have $\Delta^{a^2}_{H,Q_{12}} = \Delta^{a^4}_{H,Q_{12}}$ since $(a^2)^{-1} = a^4$. If $|H| = 2$ then $H = Z(Q_{12})$ and so $\Delta^g_{H,Q_{12}}$ is an empty graph. If $|H| = 3$ then $H = \{1, a^2, a^4\}$. In this case, the vertices a, a^2 and a^4 make a triangle in $\Delta^g_{H,Q_{12}}$ since they commute among themselves. If $|H| = 4$ then $H = \{1, a^3, b, a^3b\}$, $\{1, a^3, ab, a^4b\}$ or $\{1, a^3, a^2b, a^5b\}$. For all these H, a is an isolated vertex in $G \setminus H$ (since $[a, a^rb] = a^2$ for every integer r such that $1 \leq r \leq 6$) and hence, $\Delta^g_{H,Q_{12}}$ is disconnected. If $|H| = 6$ then $H = \{1, a, a^2, a^3, a^4, a^5\}$. In this case, the vertices a, a^2, a^4 and a^5 make a cycle in $\Delta^g_{H,Q_{12}}$ since they commute among themselves. □

4.2 Connectivity and diameter

Connectivity of Δ^g_G has been studied in [69, 70, 71]. It has been conjectured that the diameter of Δ^g_G is equal to 2 if Δ^g_G is connected. In this section we discuss the connectivity of $\Delta^g_{H,G}$. In general, $\Delta^g_{H,G}$ is not connected. For any two vertices x and y, we write $x \leftrightarrow y$ and $x \nleftrightarrow y$ respectively to mean that they are adjacent or not.

Theorem 4.2.1. *If $g \in H \setminus Z(G)$ and $g^2 = 1$ then* $\operatorname{diam}(\Delta^g_{H,G}) = 2$.

Proof. Let $x \neq g$ be any vertex of $\Delta^g_{H,G}$. Then $[x, g] \neq g$ which implies $[x, g] \neq g^{-1}$ since $g^2 = 1$. Since $g \in H$, if follows that $x \leftrightarrow g$. Therefore, $d(x, g) = 2$ and hence $\operatorname{diam}(\Delta^g_{H,G}) = 2$. □

Lemma 4.2.2. *Let $g \in H \setminus Z(H, G)$ such that $g^2 \neq 1$ and $o(g) \neq 3$, where $o(g)$ is the order of g. If $x \in G \setminus Z(H, G)$ and $x \nleftrightarrow g$ then $x \leftrightarrow g^2$.*

Proof. Since $g \neq 1$ and $x \nleftrightarrow g$ it follows that $[x, g] = g^{-1}$. We have

$$[x, g^2] = [x, g][x, g]^g = g^{-2} \neq g, g^{-1}. \tag{4.2.1}$$

If $g^2 \in Z(H,G)$ then, by equation (4.2.1), we have $g^{-2} = [x, g^2] = 1$; a contradiction. Therefore, $g^2 \in H \setminus Z(H,G)$. Hence, $x \leftrightarrow g^2$. □

Theorem 4.2.3. *Let $g \in H \setminus Z(H,G)$ and $o(g) \neq 3$. Then* $\operatorname{diam}(\Delta^g_{H,G}) \leq 3$.

Proof. If $g^2 = 1$ then, by Theorem 4.2.1, we have $\operatorname{diam}(\Delta^g_{H,G}) = 2$. Therefore, we assume that $g^2 \neq 1$. Let x, y be any two vertices of $\Delta^g_{H,G}$ such that $x \not\leftrightarrow y$. Therefore, $[x,y] = g$ or g^{-1}. If $x \leftrightarrow g$ and $y \leftrightarrow g$ then $x \leftrightarrow g \leftrightarrow y$ and so $d(x,y) = 2$. If $x \not\leftrightarrow g$ and $y \not\leftrightarrow g$ then, by Lemma 4.2.2, we have $x \leftrightarrow g^2 \leftrightarrow y$ and so $d(x,y) = 2$. Therefore, we are not going to consider these two situations in the following cases.

Case 1: $x, y \in H$

Suppose that one of x, y is adjacent to g and the other is not. Without any loss we assume that $x \not\leftrightarrow g$ and $y \leftrightarrow g$. Then $[x,g] = g^{-1}$ and $[y,g] \neq g, g^{-1}$. By Lemma 4.2.2, we have $x \leftrightarrow g^2$.

Consider the element $yg \in H$. If $yg \in Z(H,G)$ then $[y,g^2] = 1 \neq g, g^{-1}$. Therefore, $x \leftrightarrow g^2 \leftrightarrow y$ and so $d(x,y) = 2$.

If $yg \notin Z(H,G)$ then we have $[x,yg] = [x,g][x,y]^g = g^{-1}[x,y]^g \neq g, g^{-1}$. In addition, $[y,yg] = [y,g] \neq g, g^{-1}$. Hence, $x \leftrightarrow yg \leftrightarrow y$ and so $d(x,y) = 2$.

Case 2: One of x, y belongs to H and the other does not.

Without any loss, assume that $x \in H$ and $y \notin H$. If $x \not\leftrightarrow g$ and $y \leftrightarrow g$ then, by Lemma 4.2.2, we have $x \leftrightarrow g^2$. In addition, $[g,g^2] = 1 \neq g, g^{-1}$ and so $g^2 \leftrightarrow g$. Therefore, $x \leftrightarrow g^2 \leftrightarrow g \leftrightarrow y$ and hence $d(x,y) \leq 3$. If $x \leftrightarrow g$ and $y \not\leftrightarrow g$ then $[x,g] \neq g, g^{-1}$ and $[y,g] = g^{-1}$. By Lemma 4.2.2, we have $y \leftrightarrow g^2$. Consider the element $xg \in H$. If $xg \in Z(H,G)$ then $[x,g^2] = 1 \neq g, g^{-1}$. Therefore, $x \leftrightarrow g^2$ and so $y \leftrightarrow g^2 \leftrightarrow x$. Thus $d(x,y) = 2$.

If $xg \notin Z(H,G)$ then we have $[y,xg] = [y,g][y,x]^g = g^{-1}[y,x]^g \neq g, g^{-1}$. In addition, $[x,xg] = [x,g] \neq g, g^{-1}$. Hence, $y \leftrightarrow xg \leftrightarrow x$ and so $d(x,y) = 2$.

Case 3: $x, y \notin H$.

Suppose that one of x, y is adjacent to g and the other is not. Without any loss, we assume that $x \not\leftrightarrow g$ and $y \leftrightarrow g$. Then, by Lemma 4.2.2, we have $x \leftrightarrow g^2$. In addition, $[g,g^2] = 1 \neq g, g^{-1}$ and so $g^2 \leftrightarrow g$. Therefore, $x \leftrightarrow g^2 \leftrightarrow g \leftrightarrow y$ and hence $d(x,y) \leq 3$.

Thus $d(x,y) \leq 3$ for all $x, y \in G \setminus Z(H,G)$. Hence the result follows. □

The rest part of this chapter is devoted to the study of connectivity of $\Delta^g_{H,D_{2n}}$, where $D_{2n} = \langle a, b : a^n = b^2 = 1, bab^{-1} = a^{-1}\rangle$ is the dihedral group of order $2n$. It is well-known that $Z(D_{2n}) = \{1\}$, the commutator subgroup $D'_{2n} = \langle a\rangle$ if n is odd and $Z(D_{2n}) = \{1, a^{\frac{n}{2}}\}$ and $D'_{2n} = \langle a^2\rangle$ if n is even. By Result 1.4.26, it follows that $\Delta^g_{D_{2n}}$ is disconnected if $n = 3, 4, 6$. Therefore, we consider $n \geq 8$ and $n \geq 5$ accordingly, as n is even or odd in the following results.

Theorem 4.2.4. *Consider the graph $\Delta^g_{H,D_{2n}}$, where $n\,(\geq 8)$ is even.*

(a) *If $H = \langle a\rangle$ then $\Delta^g_{H,D_{2n}}$ is connected and* $\operatorname{diam}(\Delta^g_{H,D_{2n}}) = 2$.

(b) *Let $H = \langle a^{\frac{n}{2}}, a^r b\rangle$ for $0 \leq r < \frac{n}{2}$. Then $\Delta^g_{H,D_{2n}}$ is connected with diameter 2 if $g = 1$ and $\Delta^g_{H,D_{2n}}$ is not connected if $g \neq 1$.*

(c) *If $H = \langle a^r b\rangle$ for $1 \leq r \leq n$ then $\Delta^g_{H,D_{2n}}$ is not connected.*

Proof. Since n is even we have $g = a^{2i}$ for $1 \leq i \leq \frac{n}{2}$.

(a) **Case 1:** $g = 1$

Since H is abelian, the induced subgraph of $\Delta^g_{H,D_{2n}}$ on $H\setminus Z(H, D_{2n})$ is empty. Thus, we need to see the adjacency of these vertices with those in $D_{2n}\setminus H$. Suppose that $[a^r b, a^j] = 1$ and $[b, a^j] = 1$ for every integers r, j such that $1 \leq r, j \leq n-1$. Then $a^{2j} = a^0$ or a^n and so $j = 0$ or $j = \frac{n}{2}$. Therefore, every vertex in $H \setminus Z(H, D_{2n})$ is adjacent to all the vertices in $D_{2n} \setminus H$. Thus $\Delta^g_{H,D_{2n}}$ is connected and $\operatorname{diam}(\Delta^g_{H,D_{2n}}) = 2$.

Case 2: $g \neq 1$

Since H is abelian, the induced subgraph of $\Delta^g_{H,D_{2n}}$ on $H \setminus Z(H, D_{2n})$ is a complete graph. Therefore, it is sufficient to prove that no vertex in $D_{2n} \setminus H$ is isolated. If $g \neq g^{-1}$ then $g \neq a^{\frac{n}{2}}$. Suppose that $[a^r b, a^j] = g$ and $[b, a^j] = g$ for every integers r, j such that $1 \leq r, j \leq n-1$. Then $a^{2j} = a^{2i}$ and so $j = i$ or $j = \frac{n}{2} + i$. If $[a^r b, a^j] = g^{-1}$ and $[b, a^j] = g^{-1}$ for every integers r, j such that $1 \leq r, j \leq n-1$ then $a^{2j} = a^{n-2i}$ and so $j = n - i$ or $j = \frac{n}{2} - i$. Therefore, there exists an integer j such that $1 \leq j \leq n-1$ and $j \neq i, \frac{n}{2} + i, n - i$ and $\frac{n}{2} - i$ for which a^j is adjacent to all the vertices in $D_{2n}\setminus H$. If $g = g^{-1}$ then $g = a^{\frac{n}{2}}$. Suppose that $[a^r b, a^j] = g$ and $[b, a^j] = g$ for every integers r, j such that $1 \leq r, j \leq n-1$ then $a^{2j} = a^{\frac{n}{2}}$ and so $j = \frac{n}{4}$ or $j = \frac{3n}{4}$. Therefore, there exists an integer

j such that $1 \leq j \leq n-1$ and $j \neq \frac{n}{4}$ and $\frac{3n}{4}$ for which a^j is adjacent to all the vertices in $D_{2n} \setminus H$. Thus $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$.

(b) **Case 1:** $g = 1$

We have $[a^{\frac{n}{2}+r}b, a^r b] = 1$ for every integer r such that $1 \leq r \leq n$. Therefore, the induced subgraph of $\Delta^g_{H,D_{2n}}$ on $H \setminus Z(H, D_{2n})$ is empty. Thus, we need to see the adjacency of these vertices with those in $D_{2n} \setminus H$. Suppose $[a^r b, a^i] = 1$ and $[a^{\frac{n}{2}+r}b, a^i] = 1$ for every integer i such that $1 \leq i \leq n-1$. Then $a^{2i} = a^n$ and so $i = \frac{n}{2}$. Therefore, for every integer i such that $1 \leq i \leq n-1$ and $i \neq \frac{n}{2}$, a^i is adjacent to both $a^r b$ and $a^{\frac{n}{2}+r}b$. In addition, we have $[a^s b, a^r b] = a^{2(s-r)}$ and $[a^{\frac{n}{2}+r}b, a^s b] = a^{2(\frac{n}{2}+r-s)}$ for every integer s such that $1 \leq s \leq n$. Suppose $[a^s b, a^r b] = 1$ and $[a^{\frac{n}{2}+r}b, a^s b] = 1$. Then $s = r$ or $s = \frac{n}{2} + r$. Therefore, for every integer s such that $1 \leq s \leq n$ and $s \neq r, \frac{n}{2} + r$, $a^s b$ is adjacent to both $a^r b$ and $a^{\frac{n}{2}+r}b$. Thus $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$.

Case 2: $g \neq 1$

If $H = \langle a^{\frac{n}{2}}, a^r b\rangle = \{1, a^{\frac{n}{2}}, a^r b, a^{\frac{n}{2}+r}b\}$ for $0 \leq r < \frac{n}{2}$ then $H \setminus Z(H, D_{2n}) = \{a^r b, a^{\frac{n}{2}+r}b\}$. We have $[a^r b, a^i] = a^{2i} = [a^{\frac{n}{2}+r}b, a^i]$ for every integer i such that $1 \leq i \leq \frac{n}{2} - 1$. That is, $[a^r b, a^i] = g$ and $[a^{\frac{n}{2}+r}b, a^i] = g$ for every integer i such that $1 \leq i \leq \frac{n}{2} - 1$. Thus a^i is an isolated vertex in $D_{2n} \setminus H$. Hence, $\Delta^g_{H,D_{2n}}$ is not connected.

(c) **Case 1:** $g = 1$

We have $[a^{\frac{n}{2}+r}b, a^r b] = 1$ for every integer r such that $1 \leq r \leq n$. Thus $a^{\frac{n}{2}+r}b$ is an isolated vertex in $D_{2n} \setminus H$. Hence, $\Delta^g_{H,D_{2n}}$ is not connected.

Case 2: $g \neq 1$

If $H = \langle a^r b\rangle = \{1, a^r b\}$ for $1 \leq r \leq n$ then $H \setminus Z(H, D_{2n}) = \{a^r b\}$. We have $[a^r b, a^i] = a^{2i} = g$ for every integer i such that $1 \leq i \leq \frac{n}{2} - 1$. Thus a^i is an isolated vertex in $D_{2n} \setminus H$. Hence, $\Delta^g_{H,D_{2n}}$ is not connected. □

Theorem 4.2.5. *Consider the graph $\Delta^g_{H,D_{2n}}$, where $n\,(\geq 8)$ and $\frac{n}{2}$ are even.*

(a) *If $H = \langle a^2\rangle$ then $\Delta^g_{H,D_{2n}}$ is connected with diameter 2 if and only if $g \notin \langle a^4\rangle$.*

(b) *If $H = \langle a^2, b\rangle$ or $\langle a^2, ab\rangle$ then $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) \leq 3$.*

Proof. Since n is even, we have $g = a^{2i}$ for $1 \leq i \leq \frac{n}{2}$.

(a) **Case 1:** $g = 1$

We know that the vertices in H commutes with all the odd powers of a. That is, any vertex in $\Delta^g_{H,D_{2n}}$ of the form a^i, where i is an odd integer and $1 \leq i \leq n-1$, is not adjacent with any vertex. Hence, $\Delta^g_{H,D_{2n}}$ is not connected.

Case 2: $g \neq 1$

Since H is abelian, the induced subgraph of $\Delta^g_{H,D_{2n}}$ on $H \setminus Z(H, D_{2n})$ is a complete graph. In addition, the vertices in H commutes with all the odd powers of a. That is, a vertex of the form a^i, where i is an odd integer, in $\Delta^g_{H,D_{2n}}$ is adjacent with all the vertices in H. We have $[a^r b, a^{2i}] = a^{4i}$ and $[b, a^{2i}] = a^{4i}$ for every integers r, i such that $1 \leq r \leq n-1$ and $1 \leq i \leq \frac{n}{2} - 1$. Thus, for $g \notin \langle a^4 \rangle$, every vertex of H is adjacent to the vertices of the form $a^r b$, where $1 \leq r \leq n$. Therefore, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$. Also, if $g = a^{4i}$ for some integer i where $1 \leq i \leq \frac{n}{4} - 1$ (i.e., $g \in \langle a^4 \rangle$) then the vertices $a^r b \in D_{2n} \setminus H$, where $1 \leq r \leq n$, will remain isolated. Hence, $\Delta^g_{H,D_{2n}}$ is disconnected in this case. This completes the proof of part (a).

(b) **Case 1:** $g = 1$

Suppose that $H = \langle a^2, b \rangle$. Then $a^{2i} \not\leftrightarrow a^j$ but $a^{2i} \leftrightarrow a^r b$ for all i, j, r such that $1 \leq i \leq \frac{n}{2} - 1$, $i \neq \frac{n}{4}$; $1 \leq j \leq n-1$ is an odd number and $1 \leq r \leq n$ because $[a^{2i}, a^j] = 1$ and $[a^{2i}, a^r b] = a^{4i}$. We need to find a path to a^j, where $1 \leq j \leq n-1$ is an odd number. We have $[a^j, b] = a^{2j} \neq 1$ and $a^j \in G \setminus H$ for all j such that $1 \leq j \leq n-1$ is an odd number. Therefore, $a^{2i} \leftrightarrow b \leftrightarrow a^j$. Hence, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) \leq 3$.

If $H = \langle a^2, ab \rangle$ then $a^{2i} \not\leftrightarrow a^j$ but $a^{2i} \leftrightarrow a^r b$ for all i, j, r such that $1 \leq i \leq \frac{n}{2} - 1$, $i \neq \frac{n}{4}$; $1 \leq j \leq n-1$ is an odd number and $1 \leq r \leq n$ because $[a^{2i}, a^j] = 1$ and $[a^{2i}, a^r b] = a^{4i}$. We need to find a path to a^j, where $1 \leq j \leq n-1$ is an odd number. We have $[a^j, ab] = a^{2j} \neq 1$ and $a^j \in G \setminus H$ for all j such that $1 \leq j \leq n-1$ is an odd number. Therefore, $a^{2i} \leftrightarrow ab \leftrightarrow a^j$. Hence, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) \leq 3$.

Case 2: $g \neq 1$

We have $\langle a^2 \rangle \subset H$. Therefore, if $g \notin \langle a^4 \rangle$ then every vertex in $\langle a^2 \rangle$ is adjacent to all other vertices in both the cases (as discussed in part (a)). Hence, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$. Suppose that $g = a^{4i}$ for some integer i, where $1 \leq i \leq \frac{n}{4} - 1$.

Suppose that $H = \langle a^2, b \rangle$. Then $a^{2i} \leftrightarrow a^j$ but $a^{2i} \not\leftrightarrow a^r b$ for all j, r such that $1 \leq j \leq n-1$ is an odd number and $1 \leq r \leq n$ because $[a^{2i}, a^j] = 1$ and $[a^{2i}, a^r b] = a^{4i}$.

We need to find a path between a^{2i} and a^rb for all i, r such that $1 \leq i \leq \frac{n}{2} - 1$ and $1 \leq r \leq n$. We have $[a^j, b] = a^{2j} \neq a^{4i}$ and $a^j \in G \setminus H$ for all j such that $1 \leq j \leq n-1$ is an odd number. Therefore, $a^{2i} \leftrightarrow a^j \leftrightarrow b$. Consider the vertices of the form a^rb where $1 \leq r \leq n-1$. We have $[a^rb, b] = a^{2r}$. Suppose $[a^rb, b] = g$ then it gives $a^{2r} = a^{4i}$ which implies $r = 2i$ or $r = \frac{n}{2} + 2i$. Therefore, $b \leftrightarrow a^rb$ if and only if $r \neq 2i$ and $r \neq \frac{n}{2} + 2i$. Thus, we have $a^{2i} \leftrightarrow a^j \leftrightarrow b \leftrightarrow a^rb$, where $1 \leq r \leq n-1$ and $r \neq 2i$ and $r \neq \frac{n}{2} + 2i$. Again we know that $a^{\frac{n}{2}+2i}b, a^{2i}b \in H$ and $[a^{\frac{n}{2}+2i}b, a^{2i}b] = 1$, so $a^{\frac{n}{2}+2i}b \leftrightarrow a^{2i}b$. If we are able to find a path between a^j and any one of $a^{\frac{n}{2}+2i}b$ and $a^{2i}b$ then we are done. Now $[a^{2i}b, a^j] \neq a^{4i}$ and $[a^{\frac{n}{2}+2i}b, a^j] \neq a^{4i}$ for any odd number j such that $1 \leq j \leq n-1$ so we have $a^{\frac{n}{2}+2i}b \leftrightarrow a^j \leftrightarrow a^{2i}b$. Thus $a^{2i} \leftrightarrow a^j \leftrightarrow a^{2i}b$, $a^{2i} \leftrightarrow a^j \leftrightarrow a^{\frac{n}{2}+2i}b$, $a^rb \leftrightarrow b \leftrightarrow a^j \leftrightarrow a^{2i}b$ and $a^rb \leftrightarrow b \leftrightarrow a^j \leftrightarrow a^{\frac{n}{2}+2i}b$, where $1 \leq r \leq n-1$ and $r \neq 2i$ and $r \neq \frac{n}{2} + 2i$. Hence, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) \leq 3$.

If $H = \langle a^2, ab \rangle$ then $a^{2i} \leftrightarrow a^j$ but $a^{2i} \not\leftrightarrow a^rb$ for all j, r such that $1 \leq j \leq n-1$ is an odd number and $1 \leq r \leq n$ because $[a^{2i}, a^j] = 1$ and $[a^{2i}, a^rb] = a^{4i}$. We need to find a path between a^{2i} and a^rb for all i, r such that $1 \leq i \leq \frac{n}{2} - 1$ and $1 \leq r \leq n$. We have $[a^j, ab] = a^{2j} \neq a^{4i}$ and $a^j \in G \setminus H$ for all j such that $1 \leq j \leq n-1$ is an odd number. Thus, we have $a^{2i} \leftrightarrow a^j \leftrightarrow ab$. Consider the vertices of the form a^rb, where $2 \leq r \leq n$. We have $[a^rb, ab] = a^{2(r-1)}$. Suppose $[a^rb, ab] = g$ then it gives $a^{2(r-1)} = a^{4i}$ which implies $r = 2i+1$ or $r = \frac{n}{2} + 2i + 1$. Therefore, $ab \leftrightarrow a^rb$ if and only if $r \neq 2i+1$ and $r \neq \frac{n}{2} + 2i + 1$. Thus, we have $a^{2i} \leftrightarrow a^j \leftrightarrow ab \leftrightarrow a^rb$, where $2 \leq r \leq n$ and $r \neq 2i+1$ and $r \neq \frac{n}{2} + 2i + 1$. Again we know that $a^{\frac{n}{2}+2i+1}b, a^{2i+1}b \in H$ and $[a^{\frac{n}{2}+2i+1}b, a^{2i+1}b] = 1$, so $a^{\frac{n}{2}+2i+1}b \leftrightarrow a^{2i+1}b$. If we are able to find a path between a^j and any one of $a^{\frac{n}{2}+2i+1}b$ and $a^{2i+1}b$ then we are done. Now $[a^{2i+1}b, a^j] \neq a^{4i}$ and $[a^{\frac{n}{2}+2i+1}b, a^j] \neq a^{4i}$ for any odd number j such that $1 \leq j \leq n-1$ so we have $a^{\frac{n}{2}+2i+1}b \leftrightarrow a^j \leftrightarrow a^{2i+1}b$. Thus $a^{2i} \leftrightarrow a^j \leftrightarrow a^{2i+1}b$, $a^{2i} \leftrightarrow a^j \leftrightarrow a^{\frac{n}{2}+2i+1}b$, $a^rb \leftrightarrow ab \leftrightarrow a^j \leftrightarrow a^{2i+1}b$ and $a^rb \leftrightarrow ab \leftrightarrow a^j \leftrightarrow a^{\frac{n}{2}+2i+1}b$, where $2 \leq r \leq n$ and $r \neq 2i+1$ and $r \neq \frac{n}{2} + 2i + 1$. Hence, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) \leq 3$. □

Theorem 4.2.6. *Consider the graph $\Delta^g_{H,D_{2n}}$, where $n\,(\geq 8)$ is even and $\frac{n}{2}$ is odd.*

(a) *If $H = \langle a^2 \rangle$ then $\Delta^g_{H,D_{2n}}$ is not connected if $g = 1$ and $\Delta^g_{H,D_{2n}}$ is connected with $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$ if $g \neq 1$.*

(b) *If $H = \langle a^2, b \rangle$ or $\langle a^2, ab \rangle$ then $\Delta^g_{H,D_{2n}}$ is not connected if $g = 1$ and $\Delta^g_{H,D_{2n}}$ is* *connected with* $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$ *if* $g \neq 1$.

Proof. Since n is even, we have $g = a^{2i}$ for $1 \leq i \leq \frac{n}{2}$.

(a) **Case 1:** $g = 1$

We know that the vertices in H commute with all the odd powers of a. That is, any vertex of the form $a^i \in D_{2n} \setminus H$, where i is an odd integer, is not adjacent with any vertex in $\Delta^g_{H,D_{2n}}$. Hence, $\Delta^g_{H,D_{2n}}$ is not connected.

Case 2: $g \neq 1$

Since H is abelian, the induced subgraph of $\Delta^g_{H,D_{2n}}$ on $H \setminus Z(H, D_{2n})$ is a complete graph. In addition, the vertices in H commutes with all the odd powers of a. That is, a vertex of the form a^i, where i is an odd integer, in $\Delta^g_{H,D_{2n}}$ is adjacent with all the vertices in H. We claim that atleast one element of $H \setminus Z(H, D_{2n})$ is adjacent to all $a^r b$'s such that $1 \leq r \leq n$. Consider the following cases.

Subcase 1: $g^3 \neq 1$

If $[g, a^r b] = g$, i.e., $[a^{2i}, a^r b] = a^{2i}$ for all $1 \leq i \leq \frac{n}{2} - 1$ and $1 \leq r \leq n$ then we get $g = a^{2i} = 1$, a contradiction. If $[g, a^r b] = g^{-1}$, i.e., $[a^{2i}, a^r b] = a^{n-2i}$ for all $1 \leq i \leq \frac{n}{2} - 1$ and $1 \leq r \leq n$ then we get $g^3 = (a^{2i})^3 = a^{6i} = 1$, a contradiction. Therefore, g is adjacent to all other vertices of the form $a^r b$ such that $1 \leq r \leq n$.

Subcase 2: $g^3 = 1$

If $[g, a^r b] = g^{-1}$, i.e., $[a^{2i}, a^r b] = a^{2i}$ then $[ga^2, a^r b] = g^{-1}a^4$ for all $1 \leq i \leq \frac{n}{2} - 1$ and $1 \leq r \leq n$. Now, if $g^{-1}a^4 = g^{-1}$ then $a^4 = 1$, a contradiction since $a^n = 1$ for $n \geq 8$. If $g^{-1}a^4 = g$ then $a^{n-2i-4} = 1$ for all $1 \leq i \leq \frac{n}{2} - 1$, which is a contradiction since $1 \leq i \leq \frac{n}{2} - 1$. Therefore, ga^2 is adjacent to all other vertices of the form $a^r b$ such that $1 \leq r \leq n$.

Thus there exists a vertex in $H \setminus Z(H, D_{2n})$ which is adjacent to all other vertices in D_{2n}. Hence, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$.

(b) **Case 1:** $g = 1$

We know that the vertices in H commute with the vertex $a^{\frac{n}{2}}$. That is, the vertex $a^{\frac{n}{2}} \in D_{2n} \setminus H$ is not adjacent with any vertex in $\Delta^g_{H,D_{2n}}$. Hence, $\Delta^g_{H,D_{2n}}$ is not connected.

Case 2: $g \neq 1$

As shown in Case 2 of part (a), it can be seen that either g or ga^2 is adjacent to all other vertices. Hence, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$. □

Theorem 4.2.7. *Consider the graph $\Delta^g_{H,D_{2n}}$, where $n(\geq 5)$ is odd.*

(a) *If $H = \langle a \rangle$ then $\Delta^g_{H,D_{2n}}$ is connected and* $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$.

(b) *If $H = \langle a^r b \rangle$, where $1 \leq r \leq n$, then $\Delta^g_{H,D_{2n}}$ is connected with* $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$ *if $g = 1$ and $\Delta^g_{H,D_{2n}}$ is not connected if $g \neq 1$.*

Proof. Since n is odd, we have $g = a^i$ for $1 \leq i \leq n$.

(a) **Case 1:** $g = 1$

Since H is abelian, the induced subgraph of $\Delta^g_{H,D_{2n}}$ on $H \setminus Z(H, D_{2n})$ is empty. Therefore, we need to see the adjacency of these vertices with those in $D_{2n} \setminus H$. Suppose that $[a^r b, a^j] = 1$ and $[b, a^j] = 1$ for every integers r, j such that $1 \leq r, j \leq n-1$. Then $a^{2j} = a^n$ and so $j = \frac{n}{2}$, a contradiction. Therefore, for every integer j such that $1 \leq j \leq n-1$, a^j is adjacent to all the vertices in $D_{2n} \setminus H$. Thus $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$.

Case 2: $g \neq 1$

Since H is abelian, the induced subgraph of $\Delta^g_{H,D_{2n}}$ on $H \setminus Z(H, D_{2n})$ is a complete graph. Therefore, it is sufficient to prove that no vertex in $D_{2n} \setminus H$ is isolated. Since n is odd we have $g \neq g^{-1}$. If $[a^r b, a^j] = g$ and $[b, a^j] = g$ for every integers r, j such that $1 \leq r, j \leq n-1$ then $j = \frac{i}{2}$ or $j = \frac{n+i}{2}$. If $[a^r b, a^j] = g^{-1}$ and $[b, a^j] = g^{-1}$ for every integers r, j such that $1 \leq r, j \leq n-1$ then $j = \frac{n-i}{2}$ or $j = n - \frac{i}{2}$. Therefore, there exists an integer j such that $1 \leq j \leq n-1$ and $j \neq \frac{i}{2}, \frac{n+i}{2}, \frac{n-i}{2}$ and $n - \frac{i}{2}$ for which a^j is adjacent to all other vertices in $D_{2n} \setminus H$. Thus $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$.

(b) **Case 1:** $g = 1$

We have $[a^r b, a^j] \neq 1$ and $[b, a^j] \neq 1$ for every integers r, j such that $1 \leq r, j \leq n-1$. Thus, $a^r b$ is adjacent to a^j for every integer j such that $1 \leq j \leq n-1$. In addition, we have $[a^s b, a^r b] = a^{2(s-r)}$ for every integers r, s such that $1 \leq r, s \leq n$. Supposing that $[a^s b, a^r b] = 1$ then $s = r$ as $s = \frac{n}{2} + r$ is not possible. Therefore, for every integers r, s such that $1 \leq r, s \leq n$ and $s \neq r$, $a^s b$ is adjacent to $a^r b$. Thus $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$.

Case 2: $g \neq 1$

If i is even then $[a^{\frac{i}{2}}, a^r b] = a^i = g$ and so the vertex $a^{\frac{i}{2}}$ remains isolated. If i is odd then $n-i$ is even and we have $[a^{\frac{n-i}{2}}, a^r b] = a^{n-i} = g^{-1}$. Therefore, the vertex $a^{\frac{n-i}{2}}$ remains isolated. Hence, $\Delta^g_{H,D_{2n}}$ is not connected. $\square$

We conclude this chapter with the following theorem.

Theorem 4.2.8. *Consider the graph $\Delta^g_{H,D_{2n}}$, where $n(\geq 5)$ is odd.*

(a) *If $H = \langle a^d \rangle$, where $d|n$ and $o(a^d) = 3$, then $\Delta^g_{H,D_{2n}}$ is not connected.*

(b) *If $H = \langle a^d, b\rangle$, $\langle a^d, ab\rangle$ or $\langle a^d, a^2b\rangle$, where $d|n$ and $o(a^d) = 3$, then $\Delta^g_{H,D_{2n}}$ is connected with diameter 2 if $g \neq 1, a^d, a^{2d}$.*

(c) *If $H = \langle a^d, b\rangle$, where $d|n$ and $o(a^d) = 3$, then $\Delta^g_{H,D_{2n}}$ is connected and*

$$\text{diam}(\Delta^g_{H,D_{2n}}) = \begin{cases} 2, & \text{if } g = 1 \\ 3, & \text{if } g = a^d \text{ or } a^{2d}. \end{cases}$$

Proof. (a) Given $H = \{1, a^d, a^{2d}\}$. We have $[a^d, a^{2d}] = 1$, $[a^d, a^rb] = a^{2d}$ and $[a^{2d}, a^rb] = a^{4d} = a^d$ for all r such that $1 \leq r \leq n$. Therefore, $g = 1, a^d$ or a^{2d}. If $g = a^d$ or a^{2d} then $a^d \nleftrightarrow a^rb$ and $a^{2d} \nleftrightarrow a^rb$ for all r such that $1 \leq r \leq n$. Thus $\Delta^g_{H,D_{2n}}$ is disconnected. If $g = 1$ then the vertex $a \in D_{2n} \setminus H$ remains isolated because $[a^d, a] = 1 = [a^{2d}, a]$. Hence $\Delta^g_{H,D_{2n}}$ is not connected.

(b) If $g \neq 1, a^d, a^{2d}$ then a^d is adjacent to all other vertices, as discussed in part (a). Hence, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$.

(c) **Case 1:** $g = 1$

Since n is odd, we have $2i \neq n$ for all integers i such that $1 \leq i \leq n-1$. Therefore, if $g = 1$ then b is adjacent to all other vertices because $[a^i, b] = a^{2i}$ and $[a^rb, b] = a^{2r}$ for all integers i, r such that $1 \leq i, r \leq n-1$. Hence, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 2$.

Case 2: $g = a^d$ or a^{2d}

Since $[a^d, a^{2d}] = 1$ we have $a^d \leftrightarrow a^{2d}$. In addition, all the vertices of the form a^i commute among themselves, where $1 \leq i \leq n-1$. Therefore, $a^d \leftrightarrow a^i \leftrightarrow a^{2d}$ for all $1 \leq i \leq n-1$ such that $i \neq d, 2d$. Again, $[a^i, a^rb] = a^{2i} = [a^i, b]$ for all $1 \leq i, r \leq n-1$. If $[a^i, a^rb] = a^d$ or a^{2d} for all $1 \leq r \leq n$, then $i = 2d$ or d respectively. Therefore, $a^d \leftrightarrow a^i \leftrightarrow b$, $a^d \leftrightarrow a^i \leftrightarrow a^db$, $a^d \leftrightarrow a^i \leftrightarrow a^{2d}b$, $a^{2d} \leftrightarrow a^i \leftrightarrow b$, $a^{2d} \leftrightarrow a^i \leftrightarrow a^db$ and $a^{2d} \leftrightarrow a^i \leftrightarrow a^{2d}b$ for all $1 \leq i \leq n-1$ such that $i \neq d, 2d$. If $[a^rb, b] = a^d$ or a^{2d} for all $1 \leq r \leq n-1$, then $a^{2r} = a^d$ or a^{2d}; which gives $r = 2d$ or d respectively. Therefore, $a^d \leftrightarrow a^i \leftrightarrow b \leftrightarrow a^rb$, $a^{2d} \leftrightarrow a^i \leftrightarrow b \leftrightarrow a^rb$, $a^db \leftrightarrow a^i \leftrightarrow b \leftrightarrow a^rb$ and $a^{2d}b \leftrightarrow a^i \leftrightarrow b \leftrightarrow a^rb$ for all $1 \leq i, r \leq n-1$ such that $i, r \neq d, 2d$. Hence, $\Delta^g_{H,D_{2n}}$ is connected and $\text{diam}(\Delta^g_{H,D_{2n}}) = 3$. □

4.3 Conclusion

In this chapter, we have extended the notion of induced g-noncommuting graph of a finite group G by considering the graph $\Delta^g_{H,G}$, where H is a subgroup of G. In our study we have generalized Result 1.4.25, Result 1.4.27 and Result 1.4.28 (see Theorem 4.1.1, Theorem 4.2.1 and Theorem 4.2.3) among other results on $\Delta^g_{H,G}$. In Result 1.4.24, it has been shown that Δ^g_G is not a tree. Following this, we have considered the question whether $\Delta^g_{H,G}$ is a tree and we have shown that $\Delta^g_{H,G}$ is not a tree in general. In [71], Nasiri et al. have shown that $\mathrm{diam}(\Delta^g_G) \leq 4$ if Δ^g_G is connected. Furthermore, they have conjectured that $\mathrm{diam}(\Delta^g_G) \leq 2$ if Δ^g_G is connected. However, we have shown that this is not true in case of the graph $\Delta^g_{H,G}$, where H is a proper subgroup of G. In particular, we have identified a subgroup H of D_{2n} in Theorem 4.2.8 such that $\mathrm{diam}(\Delta^g_{H,D_{2n}}) = 3$ while discussing connectivity and diameter of $\Delta^g_{H,D_{2n}}$.