Chapter 5

Various spectra and energies of non-commuting graphs of finite rings

It is observed that spectral aspects of non-commuting graphs of finite rings (Γ_R) are yet not studied unlike non-commuting graphs of finite groups (Γ_G). In this chapter, we compute spectrum, energy, Laplacian spectrum, Laplacian energy, Signless Laplacian spectrum and Signless Laplacian energy of non-commuting graphs of certain classes of finite rings. These calculations allow us to determine which of the finite rings under consideration yields integral/ L-integral/ Q-integral and hyperenergetic/ L-hyperenergetic/ Q-hyperenergetic non-commuting graphs. Throughout the chapter, $\frac{R}{Z(R)}$ denotes an additive quotient group and p, q denote distinct primes. This chapter is based on our paper [86] submitted for publication.

5.1 Various spectra and energies

In this section, we compute various spectra and energies of non-commuting graphs of some finite rings. We take into account n-centralizer finite rings for $n = 4, 5$ and $p + 2$

along with finite rings with central quotients isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. In addition, non-commutative rings of order p^2, p^3, p^4, p^5, p^2q and p^3q are taken into consideration, where p and q are prime numbers. With the following theorem which arrives as a consequence of the Results 1.1.1–1.1.2 we begin the section.

Theorem 5.1.1. *If* $\mathcal{H} = \overline{mK_n}$ *then* $\operatorname{Spec}(\mathcal{H}) = \{(-n)^{m-1}, (0)^{(n-1)m}, ((m-1)n)^1\}$,
$\text{L-spec}(\mathcal{H}) = \{(0)^1, ((m-1)n)^{m(n-1)}, (mn)^{m-1}\}$ *and* $E(\mathcal{H}) = LE(\mathcal{H}) = 2n(m-1)$.

Proof. If $\mathcal{H} = \overline{mK_n}$ then it is a strongly regular graph where $|v(\mathcal{H})| = mn$, $k = (m-1)n = \mu, \lambda = (m-2)n$. We have $\lambda - \mu = -n$, $k - \mu = 0$ and so $\sqrt{(\lambda-\mu)^2 + 4(k-\mu)} = n$. Therefore,

$$\frac{\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}{2} = -n, \frac{\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}{2} = 0$$

$$\text{and } \frac{2k + (|v(\mathcal{H})| - 1)(\lambda - \mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}} = 2m - mn - 1.$$

Hence, $\operatorname{Spec}(\mathcal{H}) = \{(-n)^{m-1}, (0)^{(n-1)m}, ((m-1)n)^1\}$ (by Result 1.1.1) and so $E(\mathcal{H}) = (m-1)|-n| + 0 + n|m-1| = 2n(m-1)$.

Since $\text{L-spec}(\mathcal{H}) = \{(0)^m, (n)^{m(n-1)}\}$ we have, by Result 1.1.2,

$$\text{L-spec}(\overline{\mathcal{H}}) = \{(0)^1, ((m-1)n)^{m(n-1)}, (mn)^{m-1}\}.$$

We have $\frac{2|e(\mathcal{H})|}{|v(\mathcal{H})|} = \frac{2}{mn}\left(\frac{mn(mn-1)}{2} - \frac{mn(n-1)}{2}\right) = (m-1)n$. If follows that

$$|0 - n(m-1)| = n(m-1), |n(m-1) - n(m-1)| = 0 \text{ and } |mn - (m-1)n| = n.$$

Hence, $LE(\mathcal{H}) = n(m-1) + 0 + n(m-1) = 2n(m-1)$. □

Theorem 5.1.2. *If* $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ *and* $|Z(R)| = \eta$ *then*

$$\operatorname{Spec}(\Gamma_R) = \{(-\eta(p-1))^p, (0)^{\eta(p^2-1)-p-1}, (\eta p(p-1))^1\},$$
$$\text{Q-spec}(\Gamma_R) = \{((p^2-p)\eta)^{(p^2-1)\eta-p-1}, ((p-1)^2\eta)^p, ((2p^2-2p)\eta)^1\},$$
$$\text{L-spec}(\Gamma_R) = \{(0)^1, (\eta p(p-1))^{\eta(p^2-1)-p-1}, (\eta(p^2-1))^p\}$$

and $$E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p(p-1)\eta.$$

Proof. By Result 1.4.30, we have $\overline{\Gamma_R} = (p+1)K_{(p-1)\eta}$. This implies $|v(\Gamma_R)| = (p^2-1)\eta$ and $\Gamma_R = K_{p+1.(p-1)\eta}$. Therefore, Theorem 5.1.1 leads to the expression of $\text{Spec}(\Gamma_R)$, $\text{L-spec}(\Gamma_R)$, $E(\Gamma_R)$ and $LE(\Gamma_R)$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) =&(x-(p^2-1)\eta+(p-1)\eta)^{(p+1)((p-1)\eta-1)}(x-(p^2-1)\eta+2(p-1)\eta)^{p+1} \\ &\times\left(1-\frac{(p^2-1)\eta}{x-(p^2-1)\eta+2(p-1)\eta}\right) \\ =&\,(x-p(p-1)\eta)^{(p^2-1)\eta-p-1}(x-(p-1)^2\eta)^p(x-2p(p-1)\eta). \end{aligned}$$

Thus, $\text{Q-spec}(\Gamma_R) = \{((p^2-p)\eta)^{(p^2-1)\eta-p-1}, ((p-1)^2\eta)^p, ((2p^2-2p)\eta)^1\}$.

Number of edges of $\overline{\Gamma_R}$ is $\frac{(p-1)(p^2-1)\eta^2-(p^2-1)\eta}{2}$. Therefore, $|e(\Gamma_R)| = \frac{1}{2}((p^2-1)^2\eta^2 -(p^2-1)\eta-(p-1)(p^2-1)\eta^2+(p^2-1)\eta) = \frac{p(p-1)(p^2-1)\eta^2}{2}$. Now,

$$\begin{aligned} \left|p(p-1)\eta-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p(p-1)\eta-(p^2-p)\eta\right| = 0, \\ \left|(p-1)^2\eta-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= |(1-p)\eta| = (p-1)\eta, \\ \left|2p(p-1)\eta-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|(p^2-p)\eta\right| = (p^2-p)\eta. \end{aligned}$$

Thus, $LE^+(\Gamma_R) = ((p^2-1)\eta-p-1)\times 0 + p\times(p-1)\eta+(p^2-p)\eta = 2p(p-1)\eta$. □

The following results give various spectra and energies of Γ_R for some n-centralizer rings.

Theorem 5.1.3. *Let R be a finite n-centralizer ring and $|Z(R)| = \eta$.*

(a) *If $n=4$ then* $\text{Spec}(\Gamma_R) = \{(-\eta)^2, (0)^{3\eta-3}, (2\eta)^1\}$,
$\text{Q-spec}(\Gamma_R) = \{(2\eta)^{3\eta-3}, (\eta)^2, (4\eta)^1\}$, $\text{L-spec}(\Gamma_R) = \{(0)^1, (2\eta)^{3\eta-3}, (3\eta)^2\}$
and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 4\eta$.

(b) *If $n=5$ then* $\text{Spec}(\Gamma_R) = \{(-2\eta)^3, (0)^{8\eta-4}, (6\eta)^1\}$,
$\text{Q-spec}(\Gamma_R) = \{(6\eta)^{8\eta-4}, (4\eta)^3, (12\eta)^1\}$, $\text{L-spec}(\Gamma_R) = \{(0)^1, (6\eta)^{8\eta-4}, (8\eta)^3\}$
and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 12\eta$.

(c) *If $n = p+2$ and $|R| = p^k$ for $k\in\mathbb{N}$ then*

Spec$(\Gamma_R) = \{(-(p-1)\eta)^p, (0)^{(p^2-1)\eta-p-1}, ((p^2-p)\eta)^1\}$,
Q-spec$(\Gamma_R) = \{((p^2-p)\eta)^{(p^2-1)\eta-p-1}, ((p-1)^2\eta)^p, ((2p^2-2p)\eta)^1\}$,
L-spec$(\Gamma_R) = \{(0)^1, (\eta(p^2-p))^{\eta(p^2-1)-p-1}, (\eta(p^2-1))^p\}$
and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p(p-1)\eta$.

Proof. If $n = 4$ then $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (conf. Result 1.3.2) and if $n = 5$ then $\frac{R}{Z(R)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ (conf. Result 1.3.3). Further, if $|R| = p^k$ and $n = p+2$ then $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ (conf. Result 1.3.4). Therefore, Theorem 5.1.2 leads to the conclusion. □

Theorem 5.1.4. *Let* Pr(R) *be the commuting probability of* R *and* $|Z(R)| = \eta$.

(a) *If* Pr$(R) = \frac{5}{8}$ *then* Spec$(\Gamma_R) = \{(-\eta)^2, (0)^{3\eta-3}, (2\eta)^1\}$,
Q-spec$(\Gamma_R) = \{(2\eta)^{3\eta-3}, (\eta)^2, (4\eta)^1\}$, L-spec$(\Gamma_R) = \{(0)^1, (2\eta)^{3\eta-3}, (3\eta)^2\}$
and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 4\eta$.

(b) *If* Pr$(R) = \frac{p^2+p-1}{p^3}$, *where* p *is the smallest prime divisor of* $|R|$, *then*
Spec$(\Gamma_R) = \{(-\eta(p-1))^p, (0)^{\eta(p^2-1)-p-1}, (\eta p(p-1))^1\}$,
Q-spec$(\Gamma_R) = \{((p^2-p)\eta)^{(p^2-1)\eta-p-1}, ((p-1)^2\eta)^p, ((2p^2-2p)\eta)^1\}$,
L-spec$(\Gamma_R) = \{(0)^1, (\eta p(p-1))^{\eta(p^2-1)-p-1}, (\eta(p^2-1))^p\}$
and $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p(p-1)\eta$.

Proof. We have $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_p \times \mathbb{Z}_p$ if Pr$(R) = \frac{5}{8}$ and $\frac{p^2+p-1}{p^3}$ (see Results 1.3.7 and 1.3.8, in the second case p is assumed to be the smallest prime divisor of $|R|$). Therefore, Theorem 5.1.2 leads to the conclusion. □

Theorem 5.1.5. *Let* R *be a non-commutative ring.*

(a) *If* $|R| = p^2$ *then* Spec$(\Gamma_R) = \{(-(p-1))^p, (0)^{p^2-p-2}, (p(p-1))^1\}$, Q-spec$(\Gamma_R) = \{(p^2-p)^{p^2-p-2}, ((p-1)^2)^p, (2p^2-2p)^1\}$, L-spec$(\Gamma_R) = \{(0)^1, (p(p-1))^{p^2-p-2}, (p^2-1)^p\}$ *and* $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p(p-1)$.

(b) *If* $|R| = p^3$ *with unity then* Spec$(\Gamma_R) = \{(p-p^2)^p, (0)^{p^3-2p-1}, (p^3-p^2)^1\}$, Q-spec$(\Gamma_R) = \{(p^3-p^2)^{p^3-2p-1}, (p(p-1)^2)^p, (2p^3-2p^2)^1\}$, L-spec$(\Gamma_R) = \{(0)^1, (p^3-p^2)^{p^3-2p-1}, (p^3-p)^p\}$ *and* $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p^2(p-1)$.

Proof. Note that $|Z(R)| = 1$ if $|R| = p^2$ and $|Z(R)| = p$ if $|R| = p^3$ with unity respectively. In both the cases $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, Theorem 5.1.2 leads to the conclusion. □

Now we compute various spectra and energies of Γ_R for higher order finite non-commutative ring.

Theorem 5.1.6. *Let $|R| = p^4$ with unity.*

(a) *Suppose that $Z(R)$ has p elements. Then*

$\text{Spec}(\Gamma_R) = \{(-p(p-1))^{p^2+p}, (0)^{p^4-p^2-2p-1}, (p^4-p^2)^1\}$,

$\text{Q-spec}(\Gamma_R) = \{(p^4-p^2)^{p^4-p^2-2p-1}, (p^4-2p^2+p)^{p^2+p}, (2p^4-2p^2)^1\}$,

$\text{L-spec}(\Gamma_R) = \{(0)^1, (p^4-p^2)^{p^4-p^2-2p-1}, (p^4-p)^{p^2+p}\}$ *and*

$E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p^2(p^2-1)$ *or*

$\text{Spec}(\Gamma_R) = \left\{(0)^{p^4-p-l_1-l_2}, (p-p^2)^{l_1-1}, (p-p^3)^{l_2-1}, (x_1)^1, (x_2)^1\right\}$, *where x_1, x_2 are the roots of the polynomial* $x^2 - x(p^4-p^3-p^2+p) - (p^5-p^4-p^3+p^2)(l_1+l_2-1)$,

$\text{Q-spec}(\Gamma_R) = \{(p^4-p^2)^{l_1(p^2-p-1)}, (p^4-p^3)^{l_2(p^3-p-1)}, (p^4-2p^2+p)^{l_1-1}, (p^4-2p^3+p)^{l_2-1}, (x_1)^1, (x_2)^1\}$, *where x_1, x_2 are the roots of the polynomial* $x^2 - x(3p^4-2p^3-2p^2+p) + p^8-2p^7-2p^6+6p^5-2p^4-2p^3+p^2+l_1(p^2-p)(p^4-2p^3+p)+l_2(p^3-p)(p^4-2p^2+p)$,

$\text{L-spec}(\Gamma_R) = \{(0)^1, (p^4-p^3)^{l_2(p^3-p-1)}, (p^4-p^2)^{l_1(p^2-p-1)}, (p^4-p)^{l_1+l_2-1}\}$ *and* $LE(\Gamma_R) = \frac{2}{p^2+p+1}[(p^4-p^2)(l_1+pl_2)+(p^6-p^5-p^4+p^2)l_1l_2]$, *where* $l_1 + l_2(p+1) = p^2+p+1$.

(b) *Suppose that $Z(R)$ has p^2 elements. Then*

$\text{Spec}(\Gamma_R) = \{(-p^3+p^2)^p, (0)^{p^4-p^2-p-1}, (p^4-p^3)^1\}$, $\text{Q-spec}(\Gamma_R) = \{(p^4-p^3)^{p^4-p^2-p-1}, (p^4-2p^3+p^2)^p, (2p^4-2p^3)^1\}$, $\text{L-spec}(\Gamma_R) = \{(0)^1, (p^4-p^3)^{p^4-p^2-p-1}, (p^4-p^2)^p\}$ *and* $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p(p^3-p^2)$.

Proof. (a) Result 1.4.31(a)(i) gives $\overline{\Gamma_R} = (1+p+p^2)K_{(p^2-p)}$ or $l_1K_{(p^2-p)} \cup l_2K_{(p^3-p)}$, when $l_1 + l_2(p+1) = p^2+p+1$.

If $\overline{\Gamma_R} = (p^2+p+1)K_{(p^2-p)}$ then by Theorem 5.1.1 we have $\text{Spec}(\Gamma_R) = \{(-p^2+p)^{p^2+p}, (0)^{p^4-p^2-2p-1}, (p^4-p^2)^1\}$, $\text{L-spec}(\Gamma_R) = \{(0)^1, (p^4-p^2)^{p^4-p^2-2p-1}, (p^4-p)^{p^2+p}\}$

and $E(\Gamma_R) = LE(\Gamma_R) = 2p^2(p^2-1)$. Here, $|v(\Gamma_R)| = p^4 - p$ and $\Gamma_R = K_{p^2+p+1.p^2-p}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) =&(x-(p^4-p)+p^2-p)^{(p^2+p+1)(p^2-p-1)}(x-(p^4-p)+2(p^2-p))^{p^2+p+1} \\ &\times\left(1-\frac{p^4-p}{x-(p^4-p)+2(p^2-p)}\right) \\ =&(x-(p^4-p^2))^{p^4-p^2-2p-1}(x-(p^4-2p^2+p))^{p^2+p}(x-(2p^4-2p^2)). \end{aligned}$$

Therefore, Q-spec$(\Gamma_R) = \{(p^4-p^2)^{p^4-p^2-2p-1}, (p^4-2p^2+p)^{p^2+p}, (2p^4-2p^2)^1\}$.

Number of edges of Γ_R^c is $\frac{p^6-p^5-p^4-p^3+p^2+p}{2}$. Therefore,

$$|e(\Gamma_R)| = \frac{p^8-2p^5-p^4+p^2+p}{2} - \frac{p^6-p^5-p^4-p^3+p^2+p}{2} = \frac{p^3(p^3-1)(p^2-1)}{2}.$$

Now,

$$\begin{aligned} \left|p^4-p^2-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p^4-p^2-p^4+p^2\right| = 0, \\ \left|p^4-2p^2+p-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|-p^2+p\right| = p^2-p, \\ \left|2p^4-2p^2-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p^4-p^2\right| = p^4-p^2. \end{aligned}$$

Thus, $LE^+(\Gamma_R) = (p^4-p^2-2p-1)\times 0 + (p^2+p)\times(p^2-p) + p^4-p^2 = 2p^2(p^2-1)$.

If $\overline{\Gamma_R} = l_1K_{(p^2-p)} \cup l_2K_{(p^3-p)}$ then $\Gamma_R = K_{l_1.p^2-p, l_2.p^3-p}$. This implies $|v(\Gamma_R)| = p^4-p$.

Using Result 1.1.4(a), we have

$$\begin{aligned} P_{\Gamma_R}(x) =& x^{(p^4-p)-(l_1+l_2)}\prod_{i=1}^{2}(x+p_i)^{a_i-1}\left(\prod_{i=1}^{2}(x+p_i)-\sum_{j=1}^{2}a_jp_j\prod_{i=1,i\neq j}^{2}(x+p_i)\right) \\ =& x^{p^4-p-l_1-l_2}(x+p^2-p)^{l_1-1}(x+p^3-p)^{l_2-1} \\ &\times((x+p^2-p)(x+p^3-p)-l_1(p^2-p)(x+p^3-p)-l_2(p^3-p)(x+p^2-p)) \\ =& x^{p^4-p-l_1-l_2}(x-(p-p^2))^{l_1-1}(x-(p-p^3))^{l_2-1} \\ &\times(x^2-x(p^4-p^3-p^2+p)-(p^5-p^4-p^3+p^2)(l_1+l_2-1)). \end{aligned}$$

Thus, Spec$(\Gamma_R) = \left\{(0)^{p^4-p-l_1-l_2}, (p-p^2)^{l_1-1}, (p-p^3)^{l_2-1}, (x_1)^1, (x_2)^1\right\}$, where x_1, x_2 are the roots of the polynomial $x^2-x(p^4-p^3-p^2+p)-(p^5-p^4-p^3+p^2)(l_1+l_2-1)$.

Using Result 1.1.4(b), we have

$$\begin{aligned}
Q_{\Gamma_R}(x) =&\prod_{i=1}^{2}(x-(p^4-p)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(p^4-p)+2p_i)^{a_i}\\
&\times\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(p^4-p)+2p_i}\right)\\
=&(x-(p^4-p)+p^2-p)^{l_1(p^2-p-1)}(x-(p^4-p)+p^3-p)^{l_2(p^3-p-1)}\\
&\times(x-(p^4-p)+2p^2-2p)^{l_1}(x-(p^4-p)+2p^3-2p)^{l_2}\\
&\times\left(1-\frac{l_1(p^2-p)}{x-(p^4-p)+2p^2-2p}-\frac{l_2(p^3-p)}{x-(p^4-p)+2p^3-2p}\right)\\
=&(x-(p^4-p^2))^{l_1(p^2-p-1)}(x-(p^4-p^3))^{l_2(p^3-p-1)}(x-(p^4-2p^2+p))^{l_1}\\
&\times(x-(p^4-2p^3+p))^{l_2}\left(1-\frac{l_1(p^2-p)}{x-(p^4-2p^2+p)}-\frac{l_2(p^3-p)}{x-(p^4-2p^3+p)}\right)\\
=&(x-(p^4-p^2))^{l_1(p^2-p-1)}(x-(p^4-p^3))^{l_2(p^3-p-1)}(x-(p^4-2p^2+p))^{l_1-1}\\
&\times(x-(p^4-2p^3+p))^{l_2-1}(x^2-x(3p^4-2p^3-2p^2+p)+p^8-2p^7-2p^6+6p^5\\
&-2p^4-2p^3+p^2+l_1(p^2-p)(p^4-2p^3+p)+l_2(p^3-p)(p^4-2p^2+p)).
\end{aligned}$$

Thus, Q-spec$(\Gamma_R)=\Big\{(p^4-p^2)^{l_1(p^2-p-1)},(p^4-p^3)^{l_2(p^3-p-1)},(p^4-2p^2+p)^{l_1-1},$ $(p^4-2p^3+p)^{l_2-1},(x_1)^1,(x_2)^1\Big\}$, where x_1,x_2 are the roots of the polynomial $x^2-x(3p^4-2p^3-2p^2+p)+p^8-2p^7-2p^6+6p^5-2p^4-2p^3+p^2+l_1(p^2-p)(p^4-2p^3+p)+l_2(p^3-p)(p^4-2p^2+p)$.

Using Result 1.4.31(a)(i), we have L-spec$(\overline{\Gamma_R})=\{(0)^{l_1+l_2},\ (p^2-p)^{l_1(p^2-p-1)},(p^3-p)^{l_2(p^3-p-1)}\}$. Therefore, Result 1.1.2 yields L-spec$(\Gamma_R)=\{(0)^1,(p^4-p^3)^{l_2(p^3-p-1)},(p^4-p^2)^{l_1(p^2-p-1)},(p^4-p)^{l_1+l_2-1}\}$.

Here $|v(\Gamma_R)|=p^4-p$, $|e(\Gamma_R)|=\frac{p^3(p-1)(p^2-1)(l_1+pl_2)}{2}$ and so $\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}=\frac{(p^4-p^2)(l_1+pl_2)}{p^2+p+1}$. Therefore

$$\begin{aligned}
\left|p^4-p^3-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|&=\frac{(p^3-p^2)l_1}{p^2+p+1},\\
\left|p^4-p^2-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|&=\frac{(p^4-p^2)l_2}{p^2+p+1},\\
\left|p^4-p-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|&=\frac{(p^2-p)(l_1+(p+1)^2l_2)}{p^2+p+1}
\end{aligned}$$

and so

$$LE(\Gamma_R) = \frac{(p^4-p^2)(l_1+pl_2)}{p^2+p+1} + \frac{(p^3-p^2)(p^3-p-1)l_1l_2}{p^2+p+1} + \frac{(p^4-p^2)(p^2-p-1)l_1l_2}{p^2+p+1} + \frac{(p^2-p)(l_1+(p+1)^2l_2)(l_1+l_2-1)}{p^2+p+1}.$$

Consequently, we obtain the necessary expression.

(b) The expressions for Spec(Γ_R), L-spec(Γ_R), $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Result 1.4.31(a)(ii) and Theorem 5.1.1. By Result 1.4.31(a)(ii), we have $\overline{\Gamma_R} = (p+1)K_{(p^3-p^2)}$. This implies $|v(\Gamma_R)| = p^4 - p^2$ and $\Gamma_R = K_{p+1.p^3-p^2}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) =& (x-(p^4-p^2)+p^3-p^2)^{(p+1)(p^3-p^2-1)}(x-(p^4-p^2)+2(p^3-p^2))^{p+1} \\ &\times \left(1-\frac{p^4-p^2}{x-(p^4-p^2)+2(p^3-p^2)}\right) \\ =& (x-(p^4-p^3))^{p^4-p^2-p-1}(x-(p^4-2p^3+p^2))^p(x-(2p^4-2p^3)). \end{aligned}$$

Therefore, Q-spec(Γ_R) $= \{(p^4-p^3)^{p^4-p^2-p-1}, (p^4-2p^3+p^2)^p, (2p^4-2p^3)^1\}$.

Number of edges of $\overline{\Gamma_R}$ is $\frac{(p+1)(p^3-p^2)(p^3-p^2-1)}{2}$. Thus, $|e(\Gamma_R)| = \frac{(p^4-p^2)(p^4-p^2-1)}{2} - \frac{(p+1)(p^3-p^2)(p^3-p^2-1)}{2} = \frac{p^5(p^2-1)(p-1)}{2}$. Now,

$$\begin{aligned} \left|p^4-p^3-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p^4-p^3-p^4+p^3\right| = 0, \\ \left|p^4-2p^3+p^2-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|-p^3+p^2\right| = p^3-p^2, \\ \left|2p^4-2p^3-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p^4-p^3\right| = p^4-p^3. \end{aligned}$$

Thus, $LE^+(\Gamma_R) = (p^4-p^2-p-1)\times 0 + p\times(p^3-p^2) + p^4 - p^3 = 2(p^4-p^3)$. □

Theorem 5.1.7. *Let $|R| = p^5$ with unity and $Z(R)$ is not a field.*

(a) *Suppose that $Z(R)$ has p^2 elements. Then*

Spec(Γ_R) $= \{(-p^3+p^2)^{p^2+p},\ (0)^{p^5-2p^2-p-1},\ (p^5-p^3)^1\}$,

Q-spec(Γ_R) $= \{(p^5-p^3)^{p^5-2p^2-p-1}, (p^5-2p^3+p^2)^{p^2+p}, (2p^5-2p^3)^1\}$,

L-spec(Γ_R) $= \{(0)^1, (p^5-p^3)^{p^5-2p^2-p-1},\ (p^5-p^2)^{p^2+p}\}$ *and*

$E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2(p^5 - p^3)$ *or*

$\text{Spec}(\Gamma_R) = \left\{(0)^{p^5-p^2-l_1-l_2}, (p^2-p^3)^{l_1-1}, (p^2-p^4)^{l_2-1}, (x_1)^1, (x_2)^1\right\}$, *where* x_1, x_2 *are the roots of the polynomial* $x^2 - x(p^5-p^4-p^3+p^2) - (p^7-p^6-p^5+p^4)(l_1+l_2-1)$,

$\text{Q-spec}(\Gamma_R) = \left\{(p^5-p^3)^{l_1(p^3-p^2-1)}, (p^5-p^4)^{l_2(p^4-p^2-1)}, (p^5-2p^3+p^2)^{l_1-1}, (p^5-2p^4 +p^2)^{l_2-1}, (x_1)^1, (x_2)^1\right\}$, *where* x_1, x_2 *are the roots of the polynomial* $x^2 - x(3p^5-2p^4-2p^3+p^2) + p^{10} - 2p^9 - 2p^8 + 6p^7 - 2p^6 - 2p^5 + p^4 + l_1(p^3-p^2)(p^5-2p^4+p^2) + l_2(p^4-p^2)(p^5-2p^3+p^2)$,

$\text{L-spec}(\Gamma_R) = \{(0)^1, (p^5-p^4)^{l_2(p^4-p^2-1)}, (p^5-p^3)^{l_1(p^3-p^2-1)}, (p^5-p^2)^{l_1+l_2-1}\}$

and $LE(\Gamma_R) = \frac{2}{p^2+p+1}[(p^5-p^3)(l_1+pl_2) + (p^8-p^7-p^6+p^5-p^4+p^3)l_1l_2]$, *where* $l_1 + l_2(p+1) = p^2+p+1$.

(b) *If* $|Z(R)| = p^3$ *then*

$\text{Spec}(\Gamma_R) = \{(-p^4+p^3)^p, (0)^{p^5-p^3-p-1}, (p^5-p^4)^1\}$,

$\text{Q-spec}(\Gamma_R) = \{(p^5-p^4)^{p^5-p^3-p-1}, (p^5-2p^4+p^3)^p, (2p^5-2p^4)^1\}$,

$\text{L-spec}(\Gamma_R) = \{(0)^1, (p^5-p^4)^{p^5-p^3-p-1}, (p^5-p^3)^p\}$ *and*

$E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2(p^5-p^4)$.

Proof. (a) By Result 1.4.31(b)(i), $\overline{\Gamma_R} = (p^2+p+1)K_{p^2(p-1)}$ or $l_1K_{p^2(p-1)} \cup l_2K_{p^2(p^2-1)}$, when $l_1 + l_2(p+1) = p^2+p+1$, if $|Z(R)| = p^2$.

If $\overline{\Gamma_R} = (p^2+p+1)K_{(p^3-p^2)}$ then Theorem 5.1.1 yields $\text{Spec}(\Gamma_R) = \{(-p^3+p^2)^{p^2+p}, (0)^{p^5-2p^2-p-1}, (p^5-p^3)^1\}$, $\text{L-spec}(\Gamma_R) = \{(0)^1, (p^5-p^3)^{p^5-2p^2-p-1}, (p^5-p^2)^{p^2+p}\}$ and $E(\Gamma_R) = LE(\Gamma_R) = 2(p^5-p^3)$. Here, $|v(\Gamma_R)| = p^5-p^2$ and $\Gamma_R = K_{p^2+p+1.p^3-p^2}$. Using Result 1.1.4(b), we have

$$\begin{aligned}
Q_{\Gamma_R}(x) =& (x-(p^5-p^2)+p^3-p^2)^{(p^2+p+1)(p^2-p-1)}(x-(p^5-p^2)+2(p^3-p^2))^{p^2+p+1} \\
&\times \left(1 - \frac{p^5-p^2}{x-(p^5-p^2)+2(p^3-p^2)}\right) \\
=& (x-(p^5-p^3))^{p^5-2p^2-p-1}(x-(p^5-2p^3+p^2))^{p^2+p}(x-(2p^5-2p^3)).
\end{aligned}$$

Therefore, $\text{Q-spec}(\Gamma_R) = \{(p^5-p^3)^{p^5-2p^2-p-1}, (p^5-2p^3+p^2)^{p^2+p}, (2p^5-2p^3)^1\}$.

Number of edges of $\overline{\Gamma_R}$ is $\frac{p^8-p^7-2p^5+p^4+p^2}{2}$. Therefore, $|e(\Gamma_R)| = \frac{p^{10}-2p^7-p^5+p^4+p^2}{2}$ $- \frac{p^8-p^7-2p^5+p^4+p^2}{2} = \frac{p^5(p^3-1)(p^2-1)}{2}$. Now,

$$\left|p^5 - p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^5 - p^3 - p^5 + p^3\right| = 0,$$
$$\left|p^5 - 2p^3 + p^2 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|-p^3 + p^2\right| = p^3 - p^2,$$
$$\left|2p^5 - 2p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^5 - p^3\right| = p^5 - p^3.$$

Thus, $LE^+(\Gamma_R) = (p^5 - 2p^2 - p - 1) \times 0 + (p^2 + p) \times (p^3 - p^2) + p^5 - p^3 = 2(p^5 - p^3)$.

If $\overline{\Gamma_R} = l_1 K_{(p^3-p^2)} \cup l_2 K_{(p^4-p^2)}$ then $\Gamma_R = K_{l_1.p^3-p^2, l_2.p^4-p^2}$. This implies $|v(\Gamma_R)| = p^5 - p^2$. Using Result 1.1.4(a), we have

$$\begin{aligned}
P_{\Gamma_R}(x) =& x^{(p^5-p^2)-(l_1+l_2)} \prod_{i=1}^{2} (x+p_i)^{a_i-1} \left(\prod_{i=1}^{2}(x+p_i) - \sum_{j=1}^{2} a_j p_j \prod_{i=1, i\neq j}^{2} (x+p_i) \right) \\
=& x^{p^5-p^2-l_1-l_2} (x + p^3 - p^2)^{l_1-1} (x + p^4 - p^2)^{l_2-1} \\
& \times ((x+p^3-p^2)(x+p^4-p^2) - l_1(p^3-p^2)(x+p^4-p^2) - l_2(p^4-p^2)(x+p^3-p^2)) \\
=& x^{p^5-p^2-l_1-l_2} (x - (p^2-p^3))^{l_1-1} (x-(p^2-p^4))^{l_2-1} \\
& \times (x^2 - x(p^5-p^4-p^3+p^2) - (p^7-p^6-p^5+p^4)(l_1+l_2-1)).
\end{aligned}$$

Thus, $\text{Spec}(\Gamma_R) = \left\{(0)^{p^5-p^2-l_1-l_2}, (p^2-p^3)^{l_1-1}, (p^2-p^4)^{l_2-1}, (x_1)^1, (x_2)^1\right\}$, where x_1, x_2 are the roots of the polynomial $x^2 - x(p^5-p^4-p^3+p^2) - (p^7-p^6-p^5+p^4)(l_1+l_2-1)$.

Using Result 1.1.4(b), we have

$$\begin{aligned}
Q_{\Gamma_R}(x) =& \prod_{i=1}^{2} (x - (p^5-p^2) + p_i)^{a_i(p_i-1)} \prod_{i=1}^{2} (x - (p^5-p^2) + 2p_i)^{a_i} \\
& \times \left(1 - \sum_{i=1}^{2} \frac{a_i p_i}{x - (p^5-p^2) + 2p_i}\right) \\
=& (x - (p^5-p^2) + p^3 - p^2)^{l_1(p^3-p^2-1)} (x - (p^5-p^2) + p^4 - p^2)^{l_2(p^4-p^2-1)} \\
& \times (x - (p^5-p^2) + 2p^3 - 2p^2)^{l_1} (x - (p^5-p^2) + 2p^4 - 2p^2)^{l_2} \\
& \times \left(1 - \frac{l_1(p^3-p^2)}{x-(p^5-p^2)+2p^3-2p^2} - \frac{l_2(p^4-p^2)}{x-(p^5-p^2)+2p^4-2p^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (x-(p^5-p^3))^{l_1(p^3-p^2-1)}(x-(p^5-p^4))^{l_2(p^4-p^2-1)}(x-(p^5-2p^3+p^2))^{l_1} \\
&\quad \times (x-(p^5-2p^4+p^2))^{l_2}\left(1-\frac{l_1(p^3-p^2)}{x-(p^5-2p^3+p^2)}-\frac{l_2(p^4-p^2)}{x-(p^5-2p^4+p^2)}\right) \\
&= (x-(p^5-p^3))^{l_1(p^3-p^2-1)}(x-(p^5-p^4))^{l_2(p^4-p^2-1)}(x-(p^5-2p^3+p^2))^{l_1-1} \\
&\quad \times (x-(p^5-2p^4+p^2))^{l_2-1}(x^2-x(3p^5-2p^4-2p^3+p^2)+p^{10}-2p^9-2p^8+6p^7 \\
&\quad -2p^6-2p^5+p^4+l_1(p^3-p^2)(p^5-2p^4+p^2)+l_2(p^4-p^2)(p^5-2p^3+p^2)).
\end{aligned}
$$

Thus, Q-spec$(\Gamma_R)=\Big\{(p^5-p^3)^{l_1(p^3-p^2-1)},(p^5-p^4)^{l_2(p^4-p^2-1)},(p^5-2p^3+p^2)^{l_1-1},$ $(p^5-2p^4+p^2)^{l_2-1},(x_1)^1,(x_2)^1\Big\}$, where x_1,x_2 are the roots of the polynomial $x^2-x(3p^5-2p^4-2p^3+p^2)+p^{10}-2p^9-2p^8+6p^7-2p^6-2p^5+p^4+l_1(p^3-p^2)(p^5-2p^4+p^2)+l_2(p^4-p^2)(p^5-2p^3+p^2)$.

Using Result 1.4.31(b)(i), we have L-spec$(\overline{\Gamma_R})=\{(0)^{l_1+l_2},(p^3-p^2)^{l_1(p^3-p^2-1)},(p^4-p^2)^{l_2(p^4-p^2-1)}\}$. Thus, Result 1.1.2 yields L-spec$(\Gamma_R)=\{(0)^1,(p^5-p^4)^{l_2(p^4-p^2-1)},(p^5-p^3)^{l_1(p^3-p^2-1)},(p^5-p^2)^{l_1+l_2-1}\}$.

Here $|v(\Gamma_R)|=p^5-p^2$, $|e(\Gamma_R)|=\frac{(p^6-p^5)(p^2-1)(l_1+pl_2)}{2}$ and so $\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}=\frac{(p^5-p^3)(l_1+pl_2)}{p^2+p+1}$. Therefore

$$
\begin{aligned}
\left|p^5-p^4-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \frac{(p^4-p^3)l_1}{p^2+p+1}, \\
\left|p^5-p^3-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \frac{(p^5-p^3)l_2}{p^2+p+1}, \\
\left|p^5-p^2-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \frac{(p^3-p^2)(l_1+(p+1)^2l_2)}{p^2+p+1}
\end{aligned}
$$

and so

$$
\begin{aligned}
LE(\Gamma_R) = {} & \frac{(p^5-p^3)(l_1+pl_2)}{p^2+p+1}+\frac{(p^4-p^3)(p^4-p^2-1)l_1l_2}{p^2+p+1}+ \\
& \frac{(p^5-p^3)(p^3-p^2-1)l_1l_2}{p^2+p+1}+\frac{(p^3-p^2)(l_1+(p+1)^2l_2)(l_1+l_2-1)}{p^2+p+1}.
\end{aligned}
$$

Consequently, we obtain the necessary expression.

(b) The expressions for Spec(Γ_R), L-spec(Γ_R), $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Result 1.4.31(b)(ii) and Theorem 5.1.1.

By Result 1.4.31(b)(ii) we have $\overline{\Gamma_R} = (p+1)K_{(p^4-p^3)}$. This implies $|v(\Gamma_R)| = p^5 - p^3$ and $\Gamma_R = K_{p+1.p^4-p^3}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) =& (x - (p^5 - p^3) + p^4 - p^3)^{(p+1)(p^4-p^3-1)}(x - (p^5 - p^3) + 2(p^4 - p^3))^{p+1} \\ &\times \left(1 - \frac{p^5 - p^3}{x - (p^5 - p^3) + 2(p^4 - p^3)}\right) \\ =& (x - (p^5 - p^4))^{p^5-p^3-p-1}(x - (p^5 - 2p^4 + p^3))^p(x - (2p^5 - 2p^4)). \end{aligned}$$

Therefore, Q-spec$(\Gamma_R) = \{(p^5 - p^4)^{p^5-p^3-p-1}, (p^5 - 2p^4 + p^3)^p, (2p^5 - 2p^4)^1\}$.

Number of edges of $\overline{\Gamma_R}$ is $\frac{p^9-p^8-p^7+p^6-p^5+p^3}{2}$. Therefore, $|e(\Gamma_R)| = \frac{p^{10}-2p^8+p^6-p^5+p^3}{2} - \frac{p^9-p^8-p^7+p^6-p^5+p^3}{2} = \frac{p^7(p^2-1)(p-1)}{2}$. Now,

$$\begin{aligned} \left|p^5 - p^4 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p^5 - p^4 - p^5 + p^4\right| = 0, \\ \left|p^5 - 2p^4 + p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|-p^4 + p^3\right| = p^4 - p^3, \\ \left|2p^5 - 2p^4 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p^5 - p^4\right| = p^5 - p^4. \end{aligned}$$

Thus, $LE^+(\Gamma_R) = (p^5 - p^3 - p - 1) \times 0 + p \times (p^4 - p^3) + p^5 - p^4 = 2(p^5 - p^4)$. □

Theorem 5.1.8. *Let* $|R| = p^2q$ *and* $Z(R) = \{0\}$.

(a) *Suppose that "*$t \in \{p, q, p^2, pq\}$ *and* $(t-1)$ *divides* $(p^2q - 1)$*". Then*

Spec$(\Gamma_R) = \left\{(-t+1)^{\frac{p^2q-1}{t-1}-1}, (0)^{\frac{(t-2)(p^2q-1)}{t-1}}, (p^2q - t)^1\right\}$,

Q-spec$(\Gamma_R) = \{(p^2q - t)^{\frac{p^2q-1}{t-1}(t-2)}, (p^2q - 2t + 1)^{\frac{p^2q-t}{t-1}}, (2p^2q - 2t)^1\}$,

L-spec$(\Gamma_R) = \left\{(0)^1, (p^2q - t)^{\frac{(t-2)(p^2q-1)}{t-1}}, (p^2q - 1)^{\frac{p^2q-1}{t-1}-1}\right\}$ *and*

$E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2(p^2q - t)$.

(b) *If* $(p-1)l_1 + (q-1)l_2 + (p^2-1)l_3 + (pq-1)l_4 = p^2q - 1$ *then*

Spec$(\Gamma_R) = \Big\{(0)^{p^2q-1-l_1-l_2-l_3-l_4}, (1-p)^{l_1-1}, (1-q)^{l_2-1}, (1-p^2)^{l_3-1}, (1-pq)^{l_4-1}, (x_1)^1, (x_2)^1, (x_3)^1, (x_4)^1\Big\}$, *where* x_1, x_2, x_3, x_4 *are the roots of the polynomial* $(x+e)(x+f)(x+g)(x+h) - l_1e(x+f)(x+g)(x+h) - l_2f(x+e)(x+g)(x+h) - l_3g(x+e)(x+f)(x+h) - l_4h(x+e)(x+f)(x+g)$, *where* $e = p-1$, $f = q-1$, $g = p^2-1$ *and* $h = pq-1$, Q-spec$(\Gamma_R) = \Big\{(p^2q-p)^{l_1(p-2)}, (p^2q-q)^{l_2(q-2)}, (p^2q-p^2)^{l_3(p^2-2)}, (p^2q-pq)^{l_4(pq-2)}, (p^2q-2p+1)^{l_1-1}, (p^2q-2q+1)^{l_2-1}, (p^2q-2p^2+1)^{l_3-1}, (p^2q-2pq+1)^{l_4-1}, (x_1)^1, (x_2)^1, (x_3)^1, (x_4)^1\Big\}$, *where* x_1, x_2, x_3, x_4 *are the roots of the polynomial* $(x-a)(x-b)(x-c)(x-d) - l_1(p-1)(x-b)(x-c)(x-d) - l_2(q-1)(x-a)(x-c)(x-d) - l_3(p^2-1)(x-a)(x-b)(x-d) - l_4(pq-1)(x-a)(x-b)(x-c)$, *where* $a = p^2q-2p+1$, $b = p^2q-2q+1$, $c = p^2q-2p^2+1$ *and* $d = p^2q-2pq+1$ *and*

L-spec$(\Gamma_R) = \{(0)^1, (p^2q-pq)^{l_4(pq-2)}, (p^2q-p^2)^{l_3(p^2-2)}, (p^2q-q)^{l_2(q-2)}, (p^2q-p)^{l_1(p-2)}, (p^2q-1)^{l_1+l_2+l_3+l_4-1}\}$.

Proof. (a) The expressions for Spec(Γ_R), L-spec(Γ_R), $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Result 1.4.30(b)(i) and Theorem 5.1.1. By Result 1.4.30(b)(i), we have $\overline{\Gamma_R} = \frac{p^2q-1}{t-1}K_{t-1}$. This implies $\Gamma_R = K_{\frac{p^2q-1}{t-1}.t-1}$. Using Result 1.1.4(b), we have

$$\begin{aligned}
Q_{\Gamma_R}(x) =&(x-(p^2q-1)+t-1)^{\frac{p^2q-1}{t-1}(t-1-1)}(x-(p^2q-1)+2(t-1))^{\frac{p^2q-1}{t-1}} \\
&\times\left(1-\frac{p^2q-1}{x-(p^2q-1)+2(t-1)}\right) \\
=&(x-(p^2q-t))^{\frac{p^2q-1}{t-1}(t-2)}(x-(p^2q-2t+1))^{\frac{p^2q-t}{t-1}}(x-(2p^2q-2t)).
\end{aligned}$$

Therefore, Q-spec$(\Gamma_R) = \{(p^2q-t)^{\frac{p^2q-1}{t-1}(t-2)}, (p^2q-2t+1)^{\frac{p^2q-t}{t-1}}, (2p^2q-2t)^1\}$.

Number of edges of $\overline{\Gamma_R}$ is $\frac{(p^2q-1)(t-2)}{2}$. Thus, $|e(\Gamma_R)| = \frac{(p^2q-1)(p^2q-2)}{2} - \frac{(p^2q-1)(t-2)}{2} = \frac{(p^2q-1)(p^2q-t)}{2}$. Now,

$$\begin{aligned}
\left|p^2q-t-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p^2q-t-p^2q+t\right| = 0, \\
\left|p^2q-2t+1-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= |-t+1| = t-1, \\
\left|2p^2q-2t-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p^2q-t\right| = p^2q-t.
\end{aligned}$$

Thus, $LE^+(\Gamma_R) = \left(\frac{p^2q-1}{t-1}(t-2)\right)\times 0 + \left(\frac{p^2q-t}{t-1}\right)\times(t-1) + p^2q-t = 2(p^2q-t)$.

(b) Result 1.4.30(b)(ii) gives $\overline{\Gamma_R} = l_1K_{p-1}\cup l_2K_{q-1}\cup l_3K_{p^2-1}\cup l_4K_{pq-1}$. This implies $\Gamma_R = K_{l_1.p-1, l_2.q-1, l_3.p^2-1, l_4.pq-1}$. Using Result 1.1.4(a), we have

$$P_{\Gamma_R}(x) = x^{(p^2q-1)-(l_1+l_2+l_3+l_4)}\prod_{i=1}^{4}(x+p_i)^{a_i-1}\left(\prod_{i=1}^{4}(x+p_i) - \sum_{j=1}^{4}a_jp_j\prod_{i=1,i\neq j}^{4}(x+p_i)\right)$$

$$
\begin{aligned}
&= x^{p^2q-1-l_1-l_2-l_3-l_4}(x+p-1)^{l_1-1}(x+q-1)^{l_2-1}(x+p^2-1)^{l_3-1}(x+pq-1)^{l_4-1}\\
&\quad \times ((x+p-1)(x+q-1)(x+p^2-1)(x+pq-1)\\
&\quad - l_1(p-1)(x+q-1)(x+p^2-1)(x+pq-1)\\
&\quad - l_2(q-1)(x+p-1)(x+p^2-1)(x+pq-1)\\
&\quad - l_3(p^2-1)(x+p-1)(x+q-1)(x+pq-1)\\
&\quad - l_4(pq-1)(x+p-1)(x+q-1)(x+p^2-1)).
\end{aligned}
$$

Thus, $\text{Spec}(\Gamma_R) = \Big\{(0)^{p^2q-1-l_1-l_2-l_3-l_4}, (1-p)^{l_1-1}, (1-q)^{l_2-1}, (1-p^2)^{l_3-1}, (1-pq)^{l_4-1}, (x_1)^1, (x_2)^1, (x_3)^1, (x_4)^1\Big\}$, where x_1, x_2, x_3, x_4 are the roots of the polynomial $(x+e)(x+f)(x+g)(x+h) - l_1e(x+f)(x+g)(x+h) - l_2f(x+e)(x+g)(x+h) - l_3g(x+e)(x+f)(x+h) - l_4h(x+e)(x+f)(x+g)$, where $e = p-1$, $f = q-1$, $g = p^2-1$ and $h = pq-1$.

Using Result 1.1.4(b), we have

$$
\begin{aligned}
Q_{\Gamma_R}(x) &= \prod_{i=1}^{4}(x-(p^2q-1)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{4}(x-(p^2q-1)+2p_i)^{a_i}\\
&\quad \times \left(1-\sum_{i=1}^{4}\frac{a_ip_i}{x-(p^2q-1)+2p_i}\right)\\
&=(x-(p^2q-1)+p-1)^{l_1(p-2)}(x-(p^2q-1)+q-1)^{l_2(q-2)}(x-(p^2q-1)+p^2-1)^{l_3(p^2-2)}\\
&\quad \times (x-(p^2q-1)+pq-1)^{l_4(pq-2)}(x-(p^2q-1)+2p-2)^{l_1}(x-(p^2q-1)+2q-2)^{l_2}\\
&\quad \times (x-(p^2q-1)+2p^2-2)^{l_3}(x-(p^2q-1)+2pq-2)^{l_4}\left(1-\frac{l_1(p-1)}{x-(p^2q-1)+2p-2}\right.\\
&\quad \left.-\frac{l_2(q-1)}{x-(p^2q-1)+2q-2}-\frac{l_3(p^2-1)}{x-(p^2q-1)+2p^2-2}-\frac{l_4(pq-1)}{x-(p^2q-1)+2pq-2}\right)\\
&=(x-(p^2q-p))^{l_1(p-2)}(x-(p^2q-q))^{l_2(q-2)}(x-(p^2q-p^2))^{l_3(p^2-2)}(x-(p^2q-pq))^{l_4(pq-2)}\\
&\quad \times (x-p^2q+2p-1)^{l_1}(x-p^2q+2q-1)^{l_2}(x-p^2q+2p^2-1)^{l_3}(x-p^2q+2pq-1)^{l_4}\\
&\quad \times\left(1-\frac{l_1(p-1)}{x-p^2q+2p-1}-\frac{l_2(q-1)}{x-p^2q+2q-1)}-\frac{l_3(p^2-1)}{x-p^2q+2p^2-1)}-\frac{l_4(pq-1)}{x-p^2q+2pq-1}\right)\\
&=(x-(p^2q-p))^{l_1(p-2)}(x-(p^2q-q))^{l_2(q-2)}(x-(p^2q-p^2))^{l_3(p^2-2)}(x-(p^2q-pq))^{l_4(pq-2)}\\
&\quad \times(x-p^2q+2p-1)^{l_1-1}(x-p^2q+2q-1)^{l_2-1}(x-p^2q+2p^2-1)^{l_3-1}(x-p^2q+2pq-1)^{l_4-1}
\end{aligned}
$$

$$\times((x-(p^2q-2p+1))(x-(p^2q-2q+1))(x-(p^2q-2p^2+1))(x-(p^2q-2pq+1))$$
$$-l_1(p-1)(x-(p^2q-2q+1))(x-(p^2q-2p^2+1))(x-(p^2q-2pq+1))$$
$$-l_2(q-1)(x-(p^2q-2p+1))(x-(p^2q-2p^2+1))(x-(p^2q-2pq+1))$$
$$-l_3(p^2-1)(x-(p^2q-2p+1))(x-(p^2q-2q+1))(x-(p^2q-2pq+1))$$
$$-l_4(pq-1)(x-(p^2q-2p+1))(x-(p^2q-2q+1))(x-(p^2q-2p^2+1))).$$

Thus, Q-spec$(\Gamma_R) = \Big\{(p^2q-p)^{l_1(p-2)}, (p^2q-q)^{l_2(q-2)}, (p^2q-p^2)^{l_3(p^2-2)}, (p^2q-pq)^{l_4(pq-2)}, (p^2q-2p+1)^{l_1-1}, (p^2q-2q+1)^{l_2-1}, (p^2q-2p^2+1)^{l_3-1}, (p^2q-2pq+1)^{l_4-1}, (x_1)^1, (x_2)^1, (x_3)^1, (x_4)^1\Big\}$, where x_1, x_2, x_3, x_4 are the roots of the polynomial $(x-a)(x-b)(x-c)(x-d) - l_1(p-1)(x-b)(x-c)(x-d) - l_2(q-1)(x-a)(x-c)(x-d) - l_3(p^2-1)(x-a)(x-b)(x-d) - l_4(pq-1)(x-a)(x-b)(x-c)$, where $a = p^2q-2p+1$, $b = p^2q-2q+1$, $c = p^2q-2p^2+1$ and $d = p^2q-2pq+1$.

Using Result 1.4.30(b)(ii), we have L-spec$(\overline{\Gamma_R}) = \{(0)^{l_1+l_2+l_3+l_4}, (p-1)^{(p-2)l_1}, (q-1)^{(q-2)l_2}, (p^2-1)^{(p^2-2)l_3}, (pq-1)^{(pq-2)l_4}\}$. Thus, Result 1.1.2 yields L-spec$(\Gamma_R) = \{(0)^1, (p^2q-pq)^{l_4(pq-2)}, (p^2q-p^2)^{l_3(p^2-2)}, (p^2q-q)^{l_2(q-2)}, (p^2q-p)^{l_1(p-2)}, (p^2q-1)^{l_1+l_2+l_3+l_4-1}\}$.

□

Theorem 5.1.9. *Let $|R| = p^3q$ with unity.*

(a) *If $Z(R)$ has pq elements then* Spec$(\Gamma_R) = \{(-p^2q+pq)^p, (0)^{(p+1)(p^2q-pq-1)}, (p^3q-p^2q)^1\}$, Q-spec$(\Gamma_R) = \{(p^3q-p^2q)^{(p+1)(p^2q-pq-1)}, (p^3q-2p^2q+pq)^p, (2p^3q-2p^2q)^1\}$, L-spec$(\Gamma_R) = \{(0)^1, (p^3q-p^2q)^{(p+1)(p^2q-pq-1)}, ((p+1)(p^2q-pq))^p\}$ *and* $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2(p^3q-p^2q)$.

(b) *Suppose that $Z(R)$ has p^2 elements.*

   (i) *If "$(p-1)$ divides $(pq-1)$" then*
   Spec$(\Gamma_R) = \left\{(-p^3+p^2)^{\frac{pq-1}{p-1}-1}, (0)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}, (p^3q-p^3)^1\right\}$,
   Q-spec$(\Gamma_R) = \left\{(p^3q-p^3)^{\frac{pq-1}{p-1}(p^3-p^2-1)}, (p^3q+p^2-2p^3)^{\frac{pq-p}{p-1}}, (2p^3q-2p^3)^1\right\}$
   L-spec$(\Gamma_R) = \left\{(0)^1, (p^3q-p^3)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}, (p^3q-p^2)^{\frac{pq-1}{p-1}-1}\right\}$ *and*
   $E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p^3(q-1)$.

(ii) *If* $(q-1) \mid (pq-1)$ *then*
$\text{Spec}(\Gamma_R) = \left\{(-p^2q+p^2)^{\frac{pq-1}{q-1}-1}, (0)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}, (p^3q-p^2)^1\right\}$,
$\text{Q-spec}(\Gamma_R) = \left\{(p^3q-p^2q)^{\frac{pq-1}{q-1}(p^2q-p^2-1)}, (p^3q+p^2-2p^2q)^{\frac{pq-q}{q-1}}, (2p^3q-2p^2q)^1\right\}$,
$\text{L-spec}(\Gamma_R) = \left\{(0)^1, (p^3q-p^2q)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}, (p^3q-p^2)^{\frac{pq-1}{q-1}-1}\right\}$ *and*
$E(\Gamma_R) = LE^+(\Gamma_R) = LE(\Gamma_R) = 2p^2(pq-q)$.

(iii) *If* $pq-1 = (p-1)l_1 + (q-1)l_2$ *then* $\overline{\Gamma_R} = l_1K_{p^3-p^2} \cup l_2K_{p^2q-p^2}$ *and* $\text{Spec}(\Gamma_R) = \left\{(0)^{p^3q-p^2-l_1-l_2}, (p^2-p^3)^{l_1-1}, (p^2-p^2q)^{l_2-1}, (x_1)^1, (x_2)^1\right\}$, *where* x_1, x_2 *are the roots of the polynomial* $x^2 - x(p^3q-p^3-p^2q+p^2) - (p^5q-p^5-p^4q+p^4)(l_1+l_2-1)$, $\text{Q-spec}(\Gamma_R) = \{(p^3q-p^3)^{l_1(p^3-p^2-1)}, (p^3q-p^2q)^{l_2(p^2q-p^2-1)}, (p^3q-2p^3+p^2)^{l_1-1}, (p^3q-2p^2q+p^2)^{l_2-1}, (x_1)^1, (x_2)^1\}$, *where* x_1, x_2 *are the roots of the polynomial* $x^2 - x(3p^3q-2p^2q-2p^3+p^2) + p^6q^2 - 2p^6q - 2p^5q^2 + 6p^5q - 2p^4q - 2p^5 + p^4 + l_1(p^3-p^2)(p^3q-2p^2q+p^2) + l_2(p^2q-p^2)(p^3q-2p^3+p^2)$, $\text{L-spec}(\Gamma_R) = \{(0)^1, (p^3q-p^2q)^{l_2(p^2q-p^2-1)}, (p^3q-p^3)^{l_1(p^3-p^2-1)}, (p^3q-p^2)^{l_1+l_2-1}\}$ *and*

$$LE(\Gamma_R) = \begin{cases} \frac{2(p^3-p^2)[(q-1)(pl_1+ql_2)+(p^2q-p^2-1)(q-p)l_1l_2]}{pq-1}, & \text{if } p < q \\ \frac{2(p^2q-p^2)[(p-1)(pl_1+ql_2)+(p^3-p^2-1)(p-q)l_1l_2]}{pq-1}, & \text{if } p > q. \end{cases}$$

Proof. (a) The expressions for $\text{Spec}(\Gamma_R)$, $\text{L-spec}(\Gamma_R)$, $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Result 1.4.31(c) and Theorem 5.1.1. By Result 1.4.31(c), we have $\overline{\Gamma_R} = (p+1)K_{p^2q-pq}$. This implies $|v(\Gamma_R)| = p^3q - pq$ and $\Gamma_R = K_{p+1.p^2q-pq}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) =& (x-(p^3q-pq)+p^2q-pq)^{(p+1)(p^2q-pq-1)}(x-(p^3q-pq)+2(p^2q-pq))^{p+1} \\ &\times \left(1 - \frac{p^3q-pq}{x-(p^3q-pq)+2(p^2q-pq)}\right) \\ =& (x-(p^3q-p^2q))^{(p+1)(p^2q-pq-1)}(x-(p^3q-2p^2q+pq))^p(x-(2p^3q-2p^2q)). \end{aligned}$$

Therefore, $\text{Q-spec}(\Gamma_R) = \{(p^3q-p^2q)^{(p+1)(p^2q-pq-1)}, (p^3q-2p^2q+pq)^p, (2p^3q-2p^2q)^1\}$.

Number of edges of $\overline{\Gamma_R}$ is $\frac{(p+1)(p^2q-pq)(p^2q-pq-1)}{2}$. Therefore, $|e(\Gamma_R)| = \frac{(p^3q-pq)(p^3q-pq-1)}{2} - \frac{(p+1)(p^2q-pq)(p^2q-pq-1)}{2} = \frac{q^2(p-1)(p^5-p^3)}{2}$. Now

$$\left|p^3q - p^2q - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^3q - p^2q - p^3q + p^2q\right| = 0,$$

$$\left|p^3q - 2p^2q + pq - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|-p^2q + pq\right| = p^2q - pq,$$
$$\left|2p^3q - 2p^2q - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^3q - p^2q\right| = p^3q - p^2q.$$

Thus, $LE^+(\Gamma_R) = (p+1)(p^2q - pq - 1) \times 0 + p \times (p^2q - pq) + p^3q - p^2q = 2(p^3q - p^2q)$.

(b) (i) The expressions for Spec(Γ_R), L-spec(Γ_R), $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Result 1.4.31(d)(i) and Theorem 5.1.1. By Result 1.4.31(d)(i), we have $\overline{\Gamma_R} = \frac{pq-1}{p-1}K_{p^3-p^2}$. This implies $\Gamma_R = K_{\frac{pq-1}{p-1}.p^3-p^2}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) =& (x - (p^3q - p^2) + p^3 - p^2)^{\frac{pq-1}{p-1}(p^3-p^2-1)}(x - (p^3q - p^2) + 2(p^3 - p^2))^{\frac{pq-1}{p-1}} \\ &\times \left(1 - \frac{p^3q - p^2}{x - (p^3q - p^2) + 2(p^3 - p^2)}\right) \\ =& (x - (p^3q - p^3))^{\frac{pq-1}{p-1}(p^3-p^2-1)}(x - (p^3q - 2p^3 + p^2))^{\frac{pq-p}{p-1}}(x - (2p^3q - 2p^3)). \end{aligned}$$

Therefore, Q-spec(Γ_R) $= \{(p^3q - p^3)^{\frac{pq-1}{p-1}(p^3-p^2-1)}, (p^3q - 2p^3 + p^2)^{\frac{pq-p}{p-1}}, (2p^3q - 2p^3)^1\}$.

Number of edges of $\overline{\Gamma_R}$ is $\frac{(p^3q-p^2)(p^3-p^2-1)}{2}$. Therefore, $|e(\Gamma_R)| = \frac{(p^3q-p^2)(p^3q-p^2-1)}{2} - \frac{(p^3q-p^2)(p^3-p^2-1)}{2} = \frac{(p^3q-p^2)(p^3q-p^3)}{2}$. Now,

$$\left|p^3q - p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^3q - p^3 - p^3q + p^3\right| = 0,$$
$$\left|p^3q - 2p^3 + p^2 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|-p^3 + p^2\right| = p^3 - p^2,$$
$$\left|2p^3q - 2p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|p^3q - p^3\right| = p^3q - p^3.$$

Thus, $LE^+(\Gamma_R) = \left(\frac{pq-1}{p-1}(p^3 - p^2 - 1)\right) \times 0 + \left(\frac{pq-p}{p-1}\right) \times (p^3 - p^2) + p^3q - p^3 = 2(p^3q - p^3)$.
(b) (ii) The expressions for Spec(Γ_R), L-spec(Γ_R), $E(\Gamma_R)$ and $LE(\Gamma_R)$ follow from Result 1.4.31(d)(ii) and Theorem 5.1.1. By Result 1.4.31(d)(ii), we have $\overline{\Gamma_R} = \frac{pq-1}{q-1}K_{p^2q-p^2}$. This implies $\Gamma_R = K_{\frac{pq-1}{q-1}.p^2q-p^2}$. Using Result 1.1.4(b), we have

$$\begin{aligned} Q_{\Gamma_R}(x) =& (x - (p^3q - p^2) + p^2q - p^2)^{\frac{pq-1}{q-1}(p^2q-p^2-1)}(x - (p^3q - p^2) + 2(p^2q - p^2))^{\frac{pq-1}{q-1}} \\ &\times \left(1 - \frac{p^3q - p^2}{x - (p^3q - p^2) + 2(p^2q - p^2)}\right) \\ =& (x - (p^3q - p^2q))^{\frac{pq-1}{q-1}(p^2q-p^2-1)}(x - (p^3q - 2p^2q + p^2))^{\frac{pq-q}{q-1}}(x - (2p^3q - 2p^2q)). \end{aligned}$$

Therefore, Q-spec$(\Gamma_R) = \{(p^3q-p^2q)^{\frac{pq-1}{q-1}(p^2q-p^2-1)}, (p^3q-2p^2q+p^2)^{\frac{pq-q}{q-1}}, (2p^3q-2p^2q)^1\}$.

Number of edges of $\overline{\Gamma_R}$ is $\frac{p^2(p^2q-p^2-1)(pq-1)}{2}$. Therefore,

$$|e(\Gamma_R)| = \frac{(p^3q-p^2)(p^3q-p^2-1)}{2} - \frac{p^2(p^2q-p^2-1)(pq-1)}{2} = \frac{(p^3q-p^2)(p^3q-p^2q)}{2}.$$

Now,

$$\begin{aligned}
\left|p^3q-p^2q-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p^3q-p^2q-p^3q+p^2q\right| = 0,\\
\left|p^3q-2p^2q+p^2-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|-p^2q+p^2\right| = p^2q-p^2,\\
\left|2p^3q-2p^2q-\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| &= \left|p^3q-p^2q\right| = p^3q-p^2q.
\end{aligned}$$

Thus, $LE^+(\Gamma_R) = \left(\frac{pq-1}{q-1}(p^2q-p^2-1)\right)\times 0 + \left(\frac{pq-q}{q-1}\right)\times(p^2q-p^2)+p^3q-p^2q = 2(p^3q-p^2q)$.

(b) (iii) Result 1.4.31(d)(iii) gives $\overline{\Gamma_R} = l_1K_{p^3-p^2}\cup l_2K_{p^2q-p^2}$. This implies $\Gamma_R = K_{l_1.p^3-p^2, l_2.p^2q-p^2}$. Using Result 1.1.4(a), we have

$$\begin{aligned}
P_{\Gamma_R}(x) =& x^{(p^3q-p^2)-(l_1+l_2)}\prod_{i=1}^{2}(x+p_i)^{a_i-1}\left(\prod_{i=1}^{2}(x+p_i)-\sum_{j=1}^{2}a_jp_j\prod_{i=1,i\neq j}^{2}(x+p_i)\right)\\
&= x^{p^3q-p^2-l_1-l_2}(x+p^3-p^2)^{l_1-1}(x+p^2q-p^2)^{l_2-1}((x+p^3-p^2)(x+p^2q-p^2)\\
&\quad -l_1(p^3-p^2)(x+p^2q-p^2)-l_2(p^2q-p^2)(x+p^3-p^2))\\
&= x^{p^3q-p^2-l_1-l_2}(x-(p^2-p^3))^{l_1-1}(x-(p^2-p^2q))^{l_2-1}\\
&\quad \times(x^2-x(p^3q-p^3-p^2q+p^2)-(p^5q-p^5-p^4q+p^4)(l_1+l_2-1)).
\end{aligned}$$

Thus, Spec$(\Gamma_R) = \left\{(0)^{p^3q-p^2-l_1-l_2}, (p^2-p^3)^{l_1-1}, (p^2-p^2q)^{l_2-1}, (x_1)^1, (x_2)^1\right\}$, where x_1, x_2 are the roots of the polynomial $x^2-x(p^3q-p^3-p^2q+p^2)-(p^5q-p^5-p^4q+p^4)(l_1+l_2-1)$.

Using Result 1.1.4(b), we have

$$\begin{aligned}
Q_{\Gamma_R}(x) =& \prod_{i=1}^{2}(x-(p^3q-p^2)+p_i)^{a_i(p_i-1)}\prod_{i=1}^{2}(x-(p^3q-p^2)+2p_i)^{a_i}\\
&\times\left(1-\sum_{i=1}^{2}\frac{a_ip_i}{x-(p^3q-p^2)+2p_i}\right)
\end{aligned}$$

$$= (x - (p^3q - p^2) + p^3 - p^2)^{l_1(p^3-p^2-1)}(x - (p^3q - p^2) + p^2q - p^2)^{l_2(p^2q-p^2-1)}$$
$$\times (x - (p^3q - p^2) + 2p^3 - 2p^2)^{l_1}(x - (p^3q - p^2) + 2p^2q - 2p^2)^{l_2}$$
$$\times \left(1 - \frac{l_1(p^3 - p^2)}{x - (p^3q - p^2) + 2p^3 - 2p^2} - \frac{l_2(p^2q - p^2)}{x - (p^3q - p^2) + 2p^2q - 2p^2}\right)$$
$$= (x - (p^3q - p^3))^{l_1(p^3-p^2-1)}(x - (p^3q - p^2q))^{l_2(p^2q-p^2-1)}(x - (p^3q - 2p^3 + p^2))^{l_1}$$
$$\times (x - (p^3q - 2p^2q + p^2))^{l_2}\left(1 - \frac{l_1(p^3 - p^2)}{x - (p^3q - 2p^3 + p^2)} - \frac{l_2(p^2q - p^2)}{x - (p^3q - 2p^2q + p^2)}\right)$$
$$= (x - (p^3q - p^3))^{l_1(p^3-p^2-1)}(x - (p^3q - p^2q))^{l_2(p^2q-p^2-1)}(x - (p^3q - 2p^3 + p^2))^{l_1-1}$$
$$\times (x - (p^3q - 2p^2q + p^2))^{l_2-1}(x^2 - x(3p^3q - 2p^2q - 2p^3 + p^2) + p^6q^2 - 2p^6q$$
$$- 2p^5q^2 + 6p^5q - 2p^4q - 2p^5 + p^4 + l_1(p^3 - p^2)(p^3q - 2p^2q + p^2)$$
$$+ l_2(p^2q - p^2)(p^3q - 2p^3 + p^2)).$$

Thus, Q-spec$(\Gamma_R) = \Big\{(p^3q - p^3)^{l_1(p^3-p^2-1)}, (p^3q - p^2q)^{l_2(p^2q-p^2-1)}, (p^3q - 2p^3 + p^2)^{l_1-1}, (p^3q - 2p^2q + p^2)^{l_2-1}, (x_1)^1, (x_2)^1\Big\}$, where x_1, x_2 are the roots of the polynomial $x^2 - x(3p^3q - 2p^2q - 2p^3 + p^2) + p^6q^2 - 2p^6q - 2p^5q^2 + 6p^5q - 2p^4q - 2p^5 + p^4 + l_1(p^3 - p^2)(p^3q - 2p^2q + p^2) + l_2(p^2q - p^2)(p^3q - 2p^3 + p^2)$.

Using Result 1.4.31(d)(iii), we have L-spec$(\overline{\Gamma_R}) = \{(0)^{l_1+l_2}, (p^3 - p^2)^{l_1(p^3-p^2-1)}, (p^2q - p^2)^{l_2(p^2q-p^2-1)}\}$. Thus, Result 1.1.2 yields L-spec$(\Gamma_R) = \{(0)^1, (p^3q - p^2q)^{l_2(p^2q-p^2-1)}, (p^3q - p^3)^{l_1(p^3-p^2-1)}, (p^3q - p^2)^{l_1+l_2-1}\}$.

Here $|v(\Gamma_R)| = p^3q - p^2$, $|e(\Gamma_R)| = \frac{p^4(p-1)(q-1)(pl_1+ql_2)}{2}$ and so $\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|} = \frac{p^2(p-1)(q-1)(pl_1+ql_2)}{pq-1}$. Therefore

$$\left|p^3q - p^2q - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|\frac{p^2(p-1)(p-q)l_1}{pq-1}\right| = \begin{cases} \frac{(p^3-p^2)(q-p)l_1}{pq-1}, & \text{if } p < q \\ \frac{(p^3-p^2)(p-q)l_1}{pq-1}, & \text{if } p > q, \end{cases}$$

$$\left|p^3q - p^3 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \left|\frac{p^2(q-1)(q-p)l_2}{pq-1}\right| = \begin{cases} \frac{(p^2q-p^2)(q-p)l_2}{pq-1}, & \text{if } p < q \\ \frac{(p^2q-p^2)(p-q)l_2}{pq-1}, & \text{if } p > q, \end{cases}$$

and $\left|p^3q - p^2 - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right| = \frac{p^2((p-1)^2l_1 + (q-1)^2l_2)}{pq-1}.$

If $p < q$, then we have

$$LE(\Gamma_R) = \frac{p^2(p-1)(q-1)(pl_1+ql_2)}{pq-1} + \frac{(p^3-p^2)(q-p)(p^2q-p^2-1)l_1l_2}{pq-1} + \frac{(p^2q-p^2)(q-p)(p^3-p^2-1)l_1l_2}{pq-1} + \frac{p^2((p-1)^2l_1+(q-1)^2l_2)(l_1+l_2-1)}{pq-1}.$$

Consequently, we obtain the necessary expression.

If $p > q$, then we have

$$LE(\Gamma_R) = \frac{p^2(p-1)(q-1)(pl_1+ql_2)}{pq-1} + \frac{(p^3-p^2)(p-q)(p^2q-p^2-1)l_1l_2}{pq-1} + \frac{(p^2q-p^2)(p-q)(p^3-p^2-1)l_1l_2}{pq-1} + \frac{p^2((p-1)^2l_1+(q-1)^2l_2)(l_1+l_2-1)}{pq-1}.$$

Consequently, we obtain the necessary expression. □

With the following theorem we come to an end of this section.

Theorem 5.1.10. *If $S_1, S_2, \ldots, S_n$ are the non-identical centralizers of $s \in R \setminus Z(R)$, where R is a finite CC-ring and $|Z(R)| = \eta$, then* L-spec$(\Gamma_R) = \{(0)^1, (|R|-|S_n|)^{|S_n|-\eta-1}, \ldots, (|R|-|S_1|)^{|S_1|-\eta-1}, (|R|-\eta)^{n-1}\}$ *and* $LE(\Gamma_R) = \frac{2}{|R|-\eta}[|R|^2 - (\sum_{j=1}^{n}|S_j|^2) + (n-1)\eta^2]$. *In particular, if $|S| = |S_1| = |S_2| = \cdots = |S_n|$ then*

Spec$(\Gamma_R) = \{(-|S|+\eta)^{n-1}, (0)^{n(|S|-\eta-1)}, ((n-1)(|S|-\eta))^1\}$,

L-spec$(\Gamma_R) = \{(0)^1, ((n-1)(|S|-\eta))^{n(|S|-\eta-1)}, (n(|S|-\eta))^{n-1}\}$ *and*

$E(\Gamma_R) = LE(\Gamma_R) = 2(n-1)(|S|-\eta)$.

Proof. Result 1.4.32 gives $\overline{\Gamma_R} = \overset{n}{\underset{i=1}{\cup}} K_{|S_i|-\eta}$ and L-spec$(\overline{\Gamma_R}) = \{(0)^n, (|S_1|-\eta)^{|S_1|-\eta-1}, \ldots, (|S_n|-\eta)^{|S_n|-\eta-1}\}$. So, Result 1.1.2 yields L-spec$(\Gamma_R) = \{(0)^1, (|R|-|S_n|)^{|S_n|-\eta-1}, \ldots, (|R|-|S_1|)^{|S_1|-\eta-1}, (|R|-\eta)^{n-1}\}$.
Again, $|v(\Gamma_R)| = |R|-\eta$, $|R| = \sum_{j=1}^{n}|S_j| - (n-1)\eta$ and $\frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|} = \frac{|R|^2-(\sum_{j=1}^{n}|S_j|^2)+(n-1)\eta^2}{|R|-\eta}$.
Therefore, for $i = 1, 2, \ldots, n$, we have

$$\left| |R| - |S_i| - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|} \right| = \frac{\sum_{j=1}^{n}|S_j|^2 + \eta(|S_i| - \eta(n-1)) - (|S_i|+\eta)|R|}{|R|-\eta}$$

and
$$\left| |R| - \eta - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|} \right| = \frac{\sum_{j=1}^{n} |S_j|^2 - (n-2)\eta^2 - 2|R|\eta}{|R| - \eta}.$$

Thus
$$\begin{aligned} LE(\Gamma_R) = & \frac{|R|^2 - (\sum_{j=1}^{n} |S_j|^2) + (n-1)\eta^2}{|R| - \eta} \\ & + \sum_{i=1}^{n} (|S_i| - \eta - 1) \left(\frac{\sum_{j=1}^{n} |S_j|^2 + \eta(|S_i| - (n-1)\eta) - |R|(\eta + |S_i|)}{|R| - \eta} \right) \\ & + (n-1) \left(\frac{\sum_{j=1}^{n} |S_j|^2 - (n-2)\eta^2 - 2|R|\eta}{|R| - \eta} \right). \end{aligned}$$

Consequently, we obtain the necessary expression.

If $|S| = |S_1| = |S_2| = \cdots = |S_n|$ then $\overline{\Gamma_R} = nK_{|S|-\eta}$. Hence, Theorem 5.1.1 yields the result. □

5.2 Some consequences

Our findings, in the previous section, imply that Γ_R of the rings we analyze are L-integral. Now, we determine whether Γ_R of the rings considered in this chapter are L-hyperenergetic.

Theorem 5.2.1. *If $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then $LE(\Gamma_R) \leq LE(K_{|v(\Gamma_R)|})$.*

Proof. Since $|v(\Gamma_R)| = (p^2 - 1)\eta$, where $\eta = |Z(R)|$, we have
$$LE(K_{|v(\Gamma_R)|}) = 2((p^2 - 1)\eta - 1) \geq 2p(p-1)\eta.$$
Hence, Theorem 5.1.2 yields the result. □

Thus Γ_R is not L-hyperenergetic. As a consequence, Γ_R of all the rings taken into consideration in Theorem 5.1.3–5.1.5 are not L-hyperenergetic.

Theorem 5.2.2. *Let $|R| = p^4$ with unity.*

(a) *Suppose that $|Z(R)|$ has p elements. Then $LE(\Gamma_R) < LE(K_{|v(\Gamma_R)|})$; and hence Γ_R is not L-hyperenergetic and if $l_1 + l_2(p+1) = p^2 + p + 1$ then $LE(\Gamma_R) > LE(K_{|v(\Gamma_R)|})$; and hence Γ_R is L-hyperenergetic.*

(b) *If* $|Z(R)| = p^2$, *then* $LE(\Gamma_R) < LE(K_{|v(\Gamma_R)|})$; *and hence* Γ_R *is not L-hyperenergetic.*

Proof. (a) Since $|v(\Gamma_R)| = p^4 - p$, $LE(K_{|v(\Gamma_R)|}) = 2(p^4 - p - 1)$. Therefore, Theorem 5.1.6(a) yields

$$LE(K_{|v(\Gamma_R)|}) - LE(\Gamma_R) = 2(p^2 - p - 1) > 0$$

and if $l_1 + l_2(p+1) = p^2 + p + 1$, we have

$$\begin{aligned} & LE(\Gamma_R) - LE(K_{|v(\Gamma_R)|}) \\ &= \frac{2(p^2+p+1) + 2\{(p^6 - p^5 - p^4 + p^2)l_1 - (p^4 - p^2)\}l_2}{p^2+p+1} > 0 \end{aligned}$$

as we observe that the term, $\{(p^6 - p^5 - p^4 + p^2)l_1 - (p^4 - p^2)\}l_2 > 0$, for all p and hence the result follows.

(b) We have $|v(\Gamma_R)| = p^4 - p^2$. Therefore $LE(K_{|v(\Gamma_R)|}) = 2(p^4 - p^2 - 1)$. From Theorem 5.1.6(b), $LE(\Gamma_R) = 2(p^4 - p^3) < 2(p^4 - p^2 - 1)$, as $p^3 - p^2 - 1 > 0$. Consequently, we get the required result. □

Theorem 5.2.3. *Let* R *has unity,* $|R| = p^5$ *and* $Z(R)$ *is not a field.*

(a) *Suppose that* $|Z(R)|$ *has* p^2 *elements. Then* $LE(\Gamma_R) < LE(K_{|v(\Gamma_R)|})$; *and hence* Γ_R *is not L-hyperenergetic and if* $l_1 + l_2(p+1) = p^2 + p + 1$ *then* $LE(\Gamma_R) > LE(K_{|v(\Gamma_R)|})$; *and hence* Γ_R *is L-hyperenergetic.*

(b) *If* $|Z(R)| = p^3$, *then* $LE(\Gamma_R) < LE(K_{|R|-|Z(R)|})$; *and hence* Γ_R *is not L-hyperenergetic.*

Proof. Since $Z(R)$ is not a field, $|Z(R)| = p^2$ or $|Z(R)| = p^3$.

Case 1: $|Z(R)| = p^2$

We have $|v(\Gamma_R)| = p^5 - p^2$. Therefore $LE(K_{|v(\Gamma_R)|}) = 2(p^5 - p^2 - 1)$. Theorem 5.1.7(a), yields

$$LE(K_{|v(\Gamma_R)|}) - LE(\Gamma_R) = 2(p^3 - p^2 - 1) > 0$$

and if $l_1 + l_2(p+1) = p^2 + p + 1$, we have

$$LE(\Gamma_R) - LE(K_{|v(\Gamma_R)|}) = \frac{2(-p^5 + 2p^2 + p + 1)}{p^2+p+1} +$$

$$\frac{2\{(p^8-p^7-p^6-p^4)l_1+(p^5l_1-p^5)+(p^3l_1+p^3)\}l_2}{p^2+p+1}$$
$$\geq \frac{p^4(p^4-p^3-p^2-p-1)+2p^3+2p^2+p+1}{p^2+p+1}$$
$$(\text{since } l_1, l_2 \geq 1)$$
$$> 0 \text{ (since } p^4-p^3-p^2-p-1>0 \text{ for all } p).$$

Hence the result follows.

Case 2: $|Z(R)| = p^3$

We have $|v(\Gamma_R)| = p^5 - p^3$. Therefore $LE(K_{|v(\Gamma_R)|}) = 2(p^5 - p^3 - 1)$. From Theorem 5.1.7(b), $LE(\Gamma_R) = 2(p^5 - p^4) < 2(p^5 - p^3 - 1)$, as $p^4 - p^3 - 1 > 0$. Consequently, we get the required result. □

Theorem 5.2.4. *If "$|R| = p^2q$ and $Z(R) = \{0\}$ and $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$" then Γ_R is not L-hyperenergetic.*

Proof. Since $|v(\Gamma_R)| = p^2q - 1$, $LE(K_{|v(\Gamma_R)|}) = 2(p^2q - 2)$. We know that $p^2q - t \leq p^2q - 2$ always, where $t \in \{p, q, p^2, pq\}$. Therefore, by Theorem 5.1.8(a) the result follows. □

Theorem 5.2.5. *Let $|R| = p^3q$ with unity.*

(a) *Then $LE(\Gamma_R) < LE(K_{|v(\Gamma_R)|})$ whenever $|Z(R)| = pq$; and hence Γ_R is not L-hyperenergetic.*

(b) *If $|Z(R)| = p^2$ and*

  (i) *$(p-1)$ divides $(pq-1)$ or $(q-1)$ divides $(pq-1)$, then Γ_R is not L-hyperenergetic.*

  (ii) *$pq-1 = (p-1)l_1+(q-1)l_2$, where $p \neq 2$ and $q \neq 3$, then Γ_R is L-hyperenergetic.*

Proof. Since $|Z(R)|$ is not a prime, $|Z(R)| = pq$ or $|Z(R)| = p^2$.

Case 1: $|Z(R)| = pq$

Since $|v(\Gamma_R)| = p^3q - pq$, $LE(K_{|v(\Gamma_R)|}) = 2(p^3q - pq - 1)$. From Theorem 5.1.9(a),

$$LE(\Gamma_R) = 2(p^3q - p^2q) < 2(p^3q - pq - 1), \text{ as } p^2q - pq - 1 > 0.$$

Hence the result follows.

Case 2: $|Z(R)| = p^2$

We have $|v(\Gamma_R)| = p^3q - p^2$. Therefore $LE(K_{|v(\Gamma_R)|}) = 2(p^3q - p^2 - 1)$.

Subcase 2.1: If $(p-1) \mid (pq-1)$ then from Theorem 5.1.9(b)(i), $LE(\Gamma_R) = 2(p^3q - p^3) < 2(p^3q - p^2 - 1)$, as $p^3 - p^2 - 1 > 0$.

Subcase 2.2: If $(q-1) \mid (pq-1)$ then from Theorem 5.1.9(b)(ii), $LE(\Gamma_R) = 2(p^3q - p^2q) < 2(p^3q - p^2 - 1)$, as $p^2q - p^2 - 1 > 0$.

Subcase 2.3: If $pq - 1 = (p-1)l_1 + (q-1)l_2$ and $p < q$ then from Theorem 5.1.9(b)(iii), we have

$$\begin{aligned}
LE(\Gamma_R) - LE(K_{|v(\Gamma_R)|}) &= \frac{2(pq-1)(1+p^2-p^2q)}{pq-1} + \\
&\quad \frac{2p^2(p-1)(q-p)l_1\{(p^2q-p^2-1)l_2+1\}}{pq-1} \\
&\geq \frac{(pq-1)(1+p^2-p^2q) + p^2(p-1)(q-p)(p^2q-p^2)}{pq-1} \\
&\qquad \text{(since } l_1, l_2 \geq 1) \\
&\geq \frac{p^2(q-1)[p^2(p-1)(q-p) - (pq-1)]}{pq-1} \\
&\geq \frac{p^3(q-1)[p(p-1)(q-p) - q]}{pq-1} \\
&\geq \frac{p^3(q-1)[p(q-p) - q]}{pq-1} \\
&> 0 \text{ (since } pq - p^2 - q > 0, \text{ for all } p \text{ and } q \text{ such that} \\
&\qquad p \neq 2 \text{ and } q \neq 3).
\end{aligned}$$

If $pq - 1 = (p-1)l_1 + (q-1)l_2$ and $p > q$ then from Theorem 5.1.9(b)(iii), we have

$$\begin{aligned}
LE(\Gamma_R) - LE(K_{|v(\Gamma_R)|}) &= \frac{2(pq-1)(1+p^2-p^3)}{pq-1} + \\
&\quad \frac{2p^2(q-1)(p-q)l_2\{(p^3-p^2-1)l_1+1\}}{pq-1} \\
&\geq \frac{(pq-1)(1+p^2-p^3) + p^2(q-1)(p-q)(p^3-p^2)}{pq-1} \\
&\qquad \text{(since } l_1, l_2 \geq 1) \\
&\geq \frac{p^2(p-1)[p^2(p-q)(q-1) - (pq-1)]}{pq-1}
\end{aligned}$$

$$\geq \frac{p^3(p-1)[p(p-q)(q-1)-q]}{pq-1}$$
$$\geq \frac{p^3(p-1)[p(p-q)-q]}{pq-1}$$
$$> 0 \text{ (since } p^2 - pq - q > 0, \text{ for all } p \text{ and } q).$$

This concludes the proof. □

The following theorem gives us some finite non-commutative rings R for which Γ_R is integral, Q-integral but not hyperenergetic as well as Q-hyperenergetic.

Theorem 5.2.6. *Γ_R is integral, Q-integral but not hyperenergetic and Q-hyperenergetic if*

(a) $\frac{R}{Z(R)}$ *is isomorphic to* $\mathbb{Z}_p \times \mathbb{Z}_p$.

(b) $|R| = p^4$ *with unity and*

  (i) $|Z(R)| = p$ *such that* $\Gamma_R = K_{p^2+p+1.p^2-p}$.

  (ii) $|Z(R)| = p^2$.

(c) $|R| = p^5$ *with unity and*

  (i) $|Z(R)| = p^2$ *such that* $\Gamma_R = K_{p^2+p+1.p^3-p^2}$.

  (ii) $|Z(R)| = p^3$.

(d) $|R| = p^2q$ *and* $Z(R) = \{0\}$ *such that* $t \in \{p, q, p^2, pq\}$ *and* $(t-1) \mid (p^2q-1)$.

(e) $|R| = p^3q$ *with unity and*

  (i) $|Z(R)| = pq$.

  (ii) $|Z(R)| = p^2$ *such that* $(p-1) \mid (pq-1)$ *or* $(q-1) \mid (pq-1)$.

Proof. For the aforementioned rings R, Spec(Γ_R) and Q-spec(Γ_R) contain only integers so Γ_R is integral as well as Q-integral. Also, $LE^+(\Gamma_R) = LE(\Gamma_R) = E(\Gamma_R)$. Therefore, in view of Theorems 5.2.1–5.2.5, Γ_R is not hyperenergetic and Q-hyperenergetic. □

Finally we wrap up this chapter noting that as a consequence of Theorem 5.2.6, Γ_R of all the rings taken into consideration in Theorem 5.1.3–5.1.5 are integral, Q-integral but not hyperenergetic as well as Q-hyperenergetic.

5.3 Conclusion

In this chapter, we have computed spectrum, energy, Laplacian spectrum, Laplacian energy, Signless Laplacian spectrum and Signless Laplacian energy of non-commuting graphs of finite non-commutative rings of order p^2, p^3, p^4, p^5, p^2q and p^3q, where both p and q are primes, along with certain other families of finite rings. We have observed that the non-commuting graphs of these rings are L-integral. Further, we have identified certain finite rings that yield integral, Q-integral and L-hyperenergetic non-commuting graphs.