Chapter 6

Arithmetic identities for some analogues of 5-core partition function

6.1 Introduction

In the notation of (1.2.4), the generating function (1.11.1) of $c_t(n)$ may be recast as

$$\sum_{n=0}^{\infty} c_t(n)q^n = \frac{f^t(-q^t)}{f(-q)}. \tag{6.1.1}$$

Recently, Gireesh, Ray and Shivashankar [33, Eq. (1.2)] considered an analogue $\overline{a}_t(n)$ of $c_t(n)$ with $f(-q)$ is replaced by $\varphi(-q)$ in (6.1.1), namely,

$$\sum_{n=0}^{\infty} \overline{a}_t(n)q^n = \frac{\varphi^t(-q^t)}{\varphi(-q)}.$$

They obtained some arithmetic identities and multiplicative formulas for $\overline{a}_3(n)$, $\overline{a}_4(n)$ and $\overline{a}_8(n)$ by using Ramanujan's theta functions (It is to be noted that Theorem 1.1 in their paper [33] holds true only for $\alpha = 1$. The induction process in the proof of the theorem is not quite correct). Employing the theory of modular forms they also studied the arithmetic density of $\overline{a}_t(n)$ and found the following Ramanujan type congruence for $\overline{a}_5(n)$ [33, Theorem 1.10]: For all $n \geq 0$,

$$\overline{a}_5(20n+6) \equiv 0 \pmod 5. \tag{6.1.2}$$

Note that

$$\sum_{n=0}^{\infty}\overline{a}_5(n)q^n = \frac{\varphi^5(-q^5)}{\varphi(-q)}$$
$$= 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 14q^5 + 20q^6 + 24q^7 + \cdots . \qquad (6.1.3)$$

In this chapter, we revisit the function $\overline{a}_5(n)$ in conjunction with $c_5(n)$ as well as another function $\overline{b}_5(n)$ defined by

$$\sum_{n=0}^{\infty}\overline{b}_5(n)q^n = \frac{\psi^5(-q^5)}{\psi(-q)} = 1+q+q^2+2q^3+3q^4-q^5+2q^7-2q^9+6q^{10}+\cdots , \qquad (6.1.4)$$

where $\psi(-q)$ is defined in (1.2.3).

Note that, the sequence $(c_5(n))$ is A053723 in [58]. We have recently added the sequences $(\overline{a}_5(n))$ and $(\overline{b}_5(n))$ as A368490 and A368495, respectively, in [58].

We state our results in the following theorems and corollaries. In the sequel, we assume that $c_5(n) = \overline{a}_5(n) = \overline{b}_5(n) = 0$ for $n < 0$.

A recurrence relation for $\overline{a}_5(n)$ and some relations between $\overline{a}_5(n)$ and $c_5(n)$ are stated in the following theorem.

Theorem 6.1.1. *For any nonnegative integer n,*

$$\overline{a}_5(5n+2) = 4c_5(5n+1), \qquad (6.1.5)$$

$$\overline{a}_5(5n+3) = 4c_5(5n+2), \qquad (6.1.6)$$

$$\overline{a}_5(10n+1) = 2c_5(10n), \qquad (6.1.7)$$

$$\overline{a}_5(10n+9) = 2c_5(10n+8), \qquad (6.1.8)$$

$$\overline{a}_5(20n+6) = 10c_5(10n+2), \qquad (6.1.9)$$

$$\overline{a}_5(20n+14) = 10c_5(10n+6). \qquad (6.1.10)$$

Furthermore, for any integer $k \geq 2$,

$$\overline{a}_5(5^k n) = \left(\frac{5^k-1}{4}\right)\overline{a}_5(5n) - \left(\frac{5^k-5}{4}\right)\overline{a}_5(n). \qquad (6.1.11)$$

The following corollary is immediate from the above theorem.

Corollary 6.1.2. *For any nonnegative integer n and $k \geq 2$,*

$$\overline{a}_5(20n+6) \equiv 0 \pmod{10}, \tag{6.1.12}$$

$$\overline{a}_5(20n+14) \equiv 0 \pmod{10}, \tag{6.1.13}$$

and

$$4\overline{a}_5(5^k n) \equiv 5\overline{a}_5(n) - \overline{a}_5(5n) \pmod{5^k}.$$

Note that (6.1.12) implies (6.1.2). However, even stronger results implying (6.1.12) and (6.1.13) are stated in Corollary 6.1.5.

Now we state some recurrence relations for $\overline{b}_5(n)$.

Theorem 6.1.3. *For any nonnegative integer n and $k \geq 2$, we have*

$$\overline{b}_5(4n+3) = 2\overline{b}_5(2n) \tag{6.1.14}$$

and

$$\overline{b}_5\left(5^k(n+3)-3\right) = \left(\frac{5^k-1}{4}\right)\overline{b}_5(5n+12) - \left(\frac{5^k-5}{4}\right)\overline{b}_5(n). \tag{6.1.15}$$

Next we state some identities connecting $\overline{b}_5(n)$ with $\overline{a}_5(n)$ and $c_5(n)$.

Theorem 6.1.4. *For any nonnegative integer n, we have*

$$\overline{b}_5(4n+1) = c_5(n) - 2\overline{b}_5(2n-1), \tag{6.1.16}$$

$$\overline{b}_5(10n) = \frac{1}{2}c_5(10n+2), \tag{6.1.17}$$

$$\overline{b}_5(10n+1) = c_5(5n+1), \tag{6.1.18}$$

$$\overline{b}_5(10n+2) = \frac{1}{4}\overline{a}_5(2n+1) + \frac{1}{2}c_5(2n), \tag{6.1.19}$$

$$\overline{b}_5(10n+3) = c_5(5n+2), \tag{6.1.20}$$

$$\overline{b}_5(10n+4) = \frac{1}{2}c_5(10n+6), \tag{6.1.21}$$

$$\overline{b}_5(10n+6) = 0, \tag{6.1.22}$$

$$\overline{b}_5(10n+8) = 0, \tag{6.1.23}$$

$$\overline{b}_5(20n+5) = -c_5(5n+1), \tag{6.1.24}$$

$$\overline{b}_5(20n+7) = \frac{1}{2}\overline{a}_5(2n+1) + c_5(2n), \tag{6.1.25}$$

$$\overline{b}_5(20n+9) = -c_5(5n+2), \tag{6.1.26}$$

$$\overline{b}_5(20n+15) = 0, \tag{6.1.27}$$

$$\overline{b}_5(20n+19) = 0. \tag{6.1.28}$$

From (6.1.9), (6.1.10), (6.1.17) and (6.1.21) we arrive at the following corollary, implying the congruence of (6.1.2) by Gireesh et al.[33, Theorem 1.10].

Corollary 6.1.5. *For n being any non-negative integer,*

$$\overline{a}_5(20n+6) = 20\overline{b}_5(10n), \tag{6.1.29}$$

$$\overline{a}_5(20n+14) = 20\overline{b}_5(10n+4). \tag{6.1.30}$$

In the following corollary some infinite families of congruences are stated.

Corollary 6.1.6. *For any nonnegative integer n and $k \geq 2$,*

$$4\overline{b}_5\left(5^k(n+3)-3\right) \equiv 5\overline{b}_5(n) - \overline{b}_5(5n+12) \pmod{5^k},$$
$$\overline{b}_5(5^k(20n+18)-3) \equiv 0 \pmod{\frac{5^k-1}{4}},$$

and

$$\overline{b}_5(5^k(20n+22)-3) \equiv 0 \pmod{\frac{5^k-1}{4}}.$$

Proof. The first congruence readily follows from (6.1.15). Again, from (6.1.27), (6.1.28) and (6.1.15) it follows that, for any positive odd integer n and $k \geq 2$,

$$\overline{b}_5(5^k(20n+18)-3) = \left(\frac{5^k-1}{4}\right)\overline{b}_5(100n+87),$$
$$\overline{b}_5(5^k(20n+22)-3) = \left(\frac{5^k-1}{4}\right)\overline{b}_5(100n+107),$$

which implies the last two congruences in the corollary. □

The results of this chapter have been submitted for publication [14].

We arrange the rest of the chapter as follows. In Section 6.2, we provide some preliminary lemmas. Section 6.3 is devoted to proving the identities stated in Theorem 6.1.1. The proofs of Theorems 6.1.3 and Theorem 6.1.4 are given in Section 6.4 and Section 6.5 respectively. Finally, we conclude the paper with an observation on the sign of $\overline{b}_5(n)$.

6.2 Preliminary lemmas

In the following lemma, we state some known theta function identities.

Lemma 6.2.1. *We have*

$$\frac{\varphi^5(q^5)}{\varphi(q)}+4q\frac{f^5(q^5)}{f(q)}=\varphi(q)\varphi^3(q^5), \tag{6.2.1}$$

$$\frac{\psi^5(-q^5)}{\psi(-q)}-\frac{\psi^5(q^5)}{\psi(q)}=4q^3\frac{\psi^5(q^{10})}{\psi(q^2)}+2q\frac{f^5(-q^{20})}{f(-q^4)}, \tag{6.2.2}$$

$$\psi^2(q)-q\psi^2(q^5)=\frac{f(-q^5)\varphi(-q^5)}{\chi(-q)}=f(q,q^4)f(q^2,q^3), \tag{6.2.3}$$

$$\frac{f_5^5}{f_1}-4q^3\frac{f_{20}^5}{f_4}=\frac{f^5(q^5)}{f(q)}+2q\frac{f_{10}^5}{f_2}. \tag{6.2.4}$$

Proof. Identity (6.2.1) is taken from Entry 9.(ii) of [19, Chap. 19]. For the proofs of (6.2.2) and (6.2.3), we refer to Entry 15 and Entry 18 of [20, Chap. 36]. Identity (6.2.4) can be found in [16, Eq. (4.7)]. □

In the following lemma we recall two results involving the 5-core partition function.

Lemma 6.2.2. *For n being any non-negative integer,*

$$c_5(4n+1)=c_5(2n), \tag{6.2.5}$$

$$c_5(5n+4)=5c_5(n). \tag{6.2.6}$$

Proof. See [16, Eq. (4.8)] and [34, Eq. (5.1)]. □

6.3 Proof of Theorem 6.1.1

Proofs of (6.1.5) *and* (6.1.6). Replacing q by $-q$ in (6.2.1), we have

$$\frac{\varphi^5(-q^5)}{\varphi(-q)} = 4q\frac{f_5^5}{f_1} + \varphi(-q)\varphi^3(-q^5), \tag{6.3.1}$$

which, by (1.11.1) and (6.1.3), may be recast as

$$\sum_{n=0}^{\infty} \overline{a}_5(n)q^n = 4\sum_{n=0}^{\infty} c_5(n)q^{n+1} + \varphi(-q)\varphi^3(-q^5). \tag{6.3.2}$$

Replacing q by $-q$ in (1.8.3) and then using the resulting identity in the above, we have

$$\sum_{n=0}^{\infty} \overline{a}_5(n)q^n = 4\sum_{n=0}^{\infty} c_5(n)q^{n+1} + \varphi^3(-q^5)\left(\varphi(-q^{25}) - 2qf(-q^{15},-q^{35}) + 2q^4 f(-q^5,-q^{45})\right). \tag{6.3.3}$$

Equating the coefficients of q^{5n+2} and q^{5n+3} from both sides of the above, we arrive at (6.1.5) and (6.1.6), respectively.

Proofs of (6.1.7) *and* (6.1.8). Multiplying both sides of (6.2.4) by $\dfrac{f_5^5 f_2}{f_{10}^5 f_1}$, we have

$$\frac{\varphi^5(-q^5)}{\varphi(-q)} - 4q^3\frac{\psi^5(-q^5)}{\psi(-q)} = \frac{\varphi^5(-q^{10})}{\varphi(-q^2)} + 2q\frac{f_5^5}{f_1}.$$

which, by (1.11.1), (6.1.3) and (6.1.4), yields

$$\sum_{n=0}^{\infty} \overline{a}_5(n)q^n - 4\sum_{n=0}^{\infty} \overline{b}_5(n)q^{n+3} = \sum_{n=0}^{\infty} \overline{a}_5(n)q^{2n} + 2\sum_{n=0}^{\infty} c_5(n)q^{n+1}. \tag{6.3.4}$$

Comparing the coefficients of q^{2n+1} from both sides, we find that

$$\overline{a}_5(2n+1) - 4\overline{b}_5(2n-2) = 2c_5(2n). \tag{6.3.5}$$

Replacing n by $5n$ and $5n+4$, we obtain

$$\overline{a}_5(10n+1) = 4\overline{b}_5(10n-2) + 2c_5(10n)$$

and

$$\overline{a}_5(10n+9) = 4\overline{b}_5(10n+6) + 2c_5(10n+8),$$

respectively. Using (6.1.22) and (6.1.23) in the above, we arrive at (6.1.7) and (6.1.8).

Proofs of (6.1.9) *and* (6.1.10). Equating the coefficients of q^{2n} from both sides of (6.3.4), we have

$$\overline{a}_5(2n) - 4\overline{b}_5(2n-3) = \overline{a}_5(n) + 2c_5(2n-1),. \tag{6.3.6}$$

From (6.3.5) and (6.3.6), it follows that

$$\overline{a}_5(4n+2) - 4\overline{b}_5(4n-1) = \overline{a}_5(2n+1) + 2c_5(4n+1), \tag{6.3.7}$$

$$\overline{a}_5(4n) - 4\overline{b}_5(4n-3) = \overline{a}_5(2n) + 2c_5(4n-1), \tag{6.3.8}$$

$$\overline{a}_5(4n+1) - 4\overline{b}_5(4n-2) = 2c_5(4n), \tag{6.3.9}$$

$$\overline{a}_5(4n+3) - 4\overline{b}_5(4n) = 2c_5(4n+2). \tag{6.3.10}$$

Again, employing (1.11.1) and (6.1.4), it follows from (6.2.2) that

$$\sum_{n=0}^{\infty}\overline{b}_5(n)q^n - \sum_{n=0}^{\infty}\overline{b}_5(n)(-q)^n = 4\sum_{n=0}^{\infty}(-1)^n\overline{b}_5(n)q^{2n+3} + 2\sum_{n=0}^{\infty}c_5(n)q^{4n+1}. \tag{6.3.11}$$

Equating the coefficients of q^{4n+3} from both sides of the above, we have

$$\overline{b}_5(4n+3) = 2\overline{b}_5(2n). \tag{6.3.12}$$

It follows from (6.3.5) and (6.3.12) that

$$\overline{a}_5(2n+1) = 2c_5(2n) + 2\overline{b}_5(4n-1).$$

Using (6.2.5) and the above identity in (6.3.7), we obtain

$$\overline{a}_5(4n+2) = 3\overline{a}_5(2n+1) - 2c_5(2n), \tag{6.3.13}$$

which by replacement of n with $5n+1$ yields

$$\overline{a}_5(20n+6) = 3\overline{a}_5(10n+3) - 2c_5(10n+2). \tag{6.3.14}$$

Again, replacing n by $2n$ in (6.1.6), we have

$$\overline{a}_5(10n+3) = 4c_5(10n+2). \tag{6.3.15}$$

It follows from (6.3.14) and (6.3.15) that

$$\overline{a}_5(20n+6) = 10c_5(10n+2),$$

which is (6.1.9).

Next, replacing n by $5n+3$ in (6.3.13), we have

$$\overline{a}_5(20n+14) = 3\overline{a}_5(10n+7) - 2c_5\left(10n+6\right). \tag{6.3.16}$$

Again, replacing n by $2n+1$ in (6.1.5), we have

$$\overline{a}_5(10n+7) = 4c_5(10n+6). \tag{6.3.17}$$

It follows from (6.3.16) and (6.3.17) that

$$\overline{a}_5(20n+14) = 10c_5(10n+6),$$

which is (6.1.10).

Proof of (6.1.11). With the aid of (6.1.3), we recast (6.3.1) as

$$\sum_{n=0}^{\infty}\overline{a}_5(n)q^n = 4q\frac{f_5^5}{f_1} + \varphi(-q)\varphi^3(-q^5), \tag{6.3.18}$$

By using the 5-dissections of $\varphi(-q)$ from (1.8.3) and that of $1/f_1$ from (1.8.2) in the above identity and extracting the terms involving q^{5n} from both sides, we find that

$$\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n = 20q\frac{f_5^5}{f_1} + \varphi^3(-q)\varphi(-q^5). \tag{6.3.19}$$

Subtracting (6.3.18) from (6.3.19),

$$\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n - \sum_{n=0}^{\infty}\overline{a}_5(n)q^n = 16q\frac{f_5^5}{f_1} + \varphi(-q)\varphi(-q^5)\left(\varphi^2(-q) - \varphi^2(-q^5)\right). \tag{6.3.20}$$

Again, using (1.8.2) and (1.8.3) in the above and then extracting the q^{5n} terms, we obtain

$$\sum_{n=0}^{\infty}\overline{a}_5(25n)q^n-\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n$$
$$=80q\frac{f_5^5}{f_1}+\varphi(-q)\left(\varphi^3(-q^5)-24q\varphi(-q^5)f(-q^3,-q^7)f(-q,-q^9)\right)$$
$$-\varphi(-q)\varphi(-q^5).$$

Replacing q by $-q$ in (5.2.1) and then employing in the above identity, we find that

$$\sum_{n=0}^{\infty}\overline{a}_5(25n)q^n-\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n=80q\frac{f_5^5}{f_1}+5\varphi(-q)\varphi(-q^5)\left(\varphi^2(-q)-\varphi^2(-q^5)\right),$$

which, by (6.3.20), yields

$$\sum_{n=0}^{\infty}\overline{a}_5(25n)q^n-\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n=5\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n-5\sum_{n=0}^{\infty}\overline{a}_5(n)q^n.$$

Equating the coefficients of q^n from both sides, we find that, for any nonnegative integer n,

$$\overline{a}_5(25n)=6\overline{a}_5(5n)-5\overline{a}_5(n). \qquad (6.3.21)$$

Now (6.1.11) follows by mathematical induction on $k\geq 2$.

6.4 Proof of Theorem 6.1.3

Note that (6.1.14) is identical to (6.3.12). Therefore, we proceed to prove only (6.1.15).

Replacing q by $-q$ in (6.2.3), we have

$$q\psi^2(-q^5)=\frac{f(q^5)\varphi(q^5)}{\chi(q)}-\psi^2(-q).$$

Multiplying both sides of the above identity by $\dfrac{\psi^3(-q^5)}{\psi(-q)}$, we find that

$$q\frac{\psi^5(-q^5)}{\psi(-q)}=\frac{f_{10}^5}{f_2}-\psi(-q)\psi^3(-q^5), \qquad (6.4.1)$$

which, by (6.1.4), can be recast as

$$\sum_{n=0}^{\infty}\overline{b}_5(n)q^{n+1}=\frac{f_{10}^5}{f_2}-\psi(-q)\psi^3(-q^5). \tag{6.4.2}$$

Employing the 5-dissection of $\psi(-q)$ from (1.8.4) and that of $1/f_2$ from (1.8.2) in (6.4.1), and then extracting the terms involving q^{5n+3} from both sides of the resulting identity, we obtain

$$\sum_{n=0}^{\infty}\overline{b}_5(5n+2)q^{n}=5q\frac{f_{10}^5}{f_2}+\psi^3(-q)\psi(-q^5). \tag{6.4.3}$$

Multiplying (6.4.2) by q and subtracting from (6.4.3),

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{b}_5(5n+2)q^{n}-\sum_{n=0}^{\infty}\overline{b}_5(n)q^{n+2}\\&=4q\frac{f_{10}^5}{f_2}+\psi(-q)\psi(-q^5)\left(\psi^2(-q)+q\psi^2(-q^5)\right).\end{aligned} \tag{6.4.4}$$

Again, using (1.8.2) and (1.8.4) in the above identity and extracting the terms involving q^{5n+4} from both sides, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{b}_5(25n+22)q^{n}-\sum_{n=0}^{\infty}\overline{b}_5(5n+2)q^{n}\\&=20q\frac{f_{10}^5}{f_2}+\psi(-q)\left(6\psi(-q^5)f(q^2,-q^3)f(-q,q^4)-q\psi^3(-q^5)\right)\\&\quad-\psi^3(-q)\psi(-q^5).\end{aligned} \tag{6.4.5}$$

Replacing q by $-q$ in (6.2.3) and employing in the above identity, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{b}_5(25n+22)q^{n}-\sum_{n=0}^{\infty}\overline{b}_5(5n+2)q^{n}\\&=20q\frac{f_{10}^5}{f_2}+5\psi(-q)\psi(-q^5)\left(\psi^2(-q)+q\psi^2(-q^5)\right).\end{aligned} \tag{6.4.6}$$

From (6.4.4) and (6.4.6) it follows that

$$\sum_{n=0}^{\infty}\overline{b}_5(25n+22)q^{n}-\sum_{n=0}^{\infty}\overline{b}_5(5n+2)q^{n}=5\sum_{n=0}^{\infty}\overline{b}_5(5n+2)q^{n}-5\sum_{n=0}^{\infty}\overline{b}_5(n)q^{n+2}.$$

Comparing the coefficients of q^n from both sides of the above equation, we find that, for any nonnegative integer n,

$$\overline{b}_5(25n+72)=6\overline{b}_5(5n+12)-5\overline{b}_5(n). \tag{6.4.7}$$

The general recurrence relation (6.1.15) now follows by mathematical induction on $k \geq 2$.

6.5 Proof of Theorem 6.1.4

Proofs of (6.1.16), (6.1.17) *and* (6.1.21). Equating the coefficients of q^{4n+1} from both sides of (6.3.11), have

$$\overline{b}_5(4n+1) = c_5(n) - 2\overline{b}_5(2n-1),$$

which is (6.1.16).

Replacing n by $n+1$ in (6.3.5) and rearranging the terms,

$$4\overline{b}_5(2n) = \overline{a}_5(2n+3) - 2c_5(2n+2). \tag{6.5.1}$$

Replacing n by $5n$ in the above identity and using (6.3.15), we have

$$\begin{aligned} 4\overline{b}_5(10n) &= \overline{a}_5(10n+3) - 2c_5(10n+2) \\ &= 4c_5(10n+2) - 2c_5(10n+2) \\ &= 2c_5(10n+2), \end{aligned}$$

which leads to (6.1.17).

Next, replacing n by $5n+2$ in (6.5.1) and employing (6.3.17), we obtain

$$\begin{aligned} 4\overline{b}_5(10n+4) &= \overline{a}_5(10n+7) - 2c_5(10n+6) \\ &= 4c_5(10n+6) - 2c_5(10n+6) \\ &= 2c_5(10n+6), \end{aligned}$$

implying (6.1.21).

Proofs of (6.1.18), (6.1.20), (6.1.22) *and* (6.1.23). Employing (1.8.4) in (6.4.2), we have

$$\sum_{n=0}^{\infty} \overline{b}_5(n) q^{n+1}$$

$$= \sum_{n=0}^{\infty} c_5(n)q^{2n} - \psi^3(-q^5)\left(f(q^{10}, -q^{15}) - qf(-q^5, q^{20}) - q^3\psi(-q^{25})\right). \qquad (6.5.2)$$

Comparing the coefficients of the terms involving q^{10n+2}, q^{10n+4}, q^{10n+7} and q^{10n+9} from both sides of the above identity, we arrive at the desired results of (6.1.18), (6.1.20), (6.1.22) and (6.1.23).

Proofs of (6.1.24), (6.1.26), (6.1.27) *and* (6.1.28). Replacing n by $5n+1$ in (6.1.16) and then applying (6.1.18),

$$\begin{aligned}\overline{b}_5(20n+5) &= c_5(5n+1) - 2\overline{b}_5(10n+1)\\ &= c_5(5n+1) - 2c_5(5n+1)\\ &= -c_5(5n+1),\end{aligned}$$

which proves (6.1.24).

Replacing n by $5n+2$ in (6.1.16) and using (6.1.20), we arrive at (6.1.26).

Similarly, replacing n by $5n+3$ in (6.1.14) and then employing (6.1.22), we have

$$\overline{b}_5(20n+15) = 2\overline{b}_5(10n+6) = 0,$$

which proves (6.1.27).

Finally, replacing n by $5n+4$ in (6.1.14) and utilizing (6.1.23), we obtain (6.1.28).

Proofs of (6.1.19) *and* (6.1.25). From (1.2.2) and (5.2.2), we see that

$$\begin{aligned}\varphi^3(-q)\varphi(-q^5) &= \frac{f_1^6 f_5^2}{f_2^3 f_{10}}\\ &= \frac{f_1^2 f_5^6}{f_2 f_{10}^3} - 4q\frac{f_1^3 f_5 f_{10}^2}{f_2^2}\\ &= \varphi(-q)\varphi^3(-q^5) - 4q\frac{f_5^5}{f_1} + 16q^2\frac{f_{10}^5}{f_2}.\end{aligned}$$

Utilizing (6.3.20), the above identity can be recast as

$$\sum_{n=0}^{\infty} \overline{a}_5(5n)q^n = \sum_{n=0}^{\infty} \overline{a}_5(n)q^n + 12q\frac{f_5^5}{f_1} + 16q^2\frac{f_{10}^5}{f_2}.$$

Extracting the terms with odd powers of q from both sides, we arrive at

$$\overline{a}_5(10n+5) = \overline{a}_5(2n+1) + 12c_5(2n). \qquad (6.5.3)$$

Now, replacing n by $10n+5$ in (6.3.6),

$$4\overline{b}_5(20n+7) = \overline{a}_5(20n+10) - \overline{a}_5(10n+5) - 2c_5(20n+9).$$

Employing (6.3.13) with n replaced by $5n+2$, and (6.2.5) with n replaced by $5n+2$, the above identity can be recast as

$$\begin{aligned} 4\overline{b}_5(20n+7) &= 3\overline{a}_5(10n+5) - 2c_5(10n+4) - \overline{a}_5(10n+5) - 2c_5(10n+4) \\ &= 2\overline{a}_5(10n+5) - 4c_5(10n+4) \\ &= 2\overline{a}_5(10n+5) - 20c_5(2n). \end{aligned}$$

Applying (6.5.3) in the above expression, we obtain

$$2\overline{b}_5(20n+7) = \overline{a}_5(2n+1) + 2c_5(2n)$$

which implies (6.1.25).

Finally, replacing n by $5n+1$ in (6.1.14) and then applying (6.1.25), we have

$$\begin{aligned} \overline{b}_5(10n+2) &= \frac{1}{2}\overline{b}_5(20n+7) \\ &= \frac{1}{4}\overline{a}_5(2n+1) + \frac{1}{2}c_5(2n), \end{aligned}$$

which is (6.1.19).

6.6 Concluding observation

We close this chapter with an observation on the sign of $\overline{b}_5(n)$.

For positive integers n, $\overline{b}_5(n)$ is 0 *for at least* 30%, *greater than* 0 *for at least* 52%, *and less than* 0 *for at least* 10%.

Identities (6.1.22), (6.1.23), (6.1.27) and (6.1.28) readily imply the observed frequency of zeroes. Similarly, (6.1.24) and (6.1.26) imply the frequency of negatives. From the identities of (6.1.5), (6.1.6), (6.1.7) and (6.1.8), we observe that the sequence $(\overline{a}_5(2n+1))$ is positive in at least 4 out of 5 cases. Together with (6.1.17)–(6.1.21) and (6.1.25), this implies that the frequency of positives is at least equal to

$$\frac{2 + 2 + 2 \times (4/5) + 2 + 2 + 1 \times (4/5)}{20},$$

that is, 52%.