Chapter 1

Introduction

In this thesis, we prove identities that connect certain sums over the divisors of n to the number of representations of n as a sum of s-gonal numbers, for any integer $s \geq 3$. We also find several results involving the partition function, its restrictions and generalizations. We find a result that connects the so-called n-color partition function to some well known arithmetic functions and some counting theorems related to the n-color partitions. We prove congruences involving some restricted overpartition functions. We prove exact generating functions involving the 10-core and self-conjugate 10-core partition functions, thereby deriving some congruences and recurrence relations. Finally, we also deal with some analogues of the t-core partition function.

The thesis comprises of six chapters, including this introductory chapter. In the following sections, we present the background material, objective of the thesis and an overview of the subsequent chapters.

1.1 q-products

For any complex number a and q, with $|q| < 1$, we define

$$(a;q)_0 := 1,$$
$$(a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k), \quad n \geq 1,$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n.$$

Subsequently, for any positive integer j, we use $f_j := (q^j;q^j)_\infty$.

1.2 Ramanujan's theta functions

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1.$$

Jacobi's famous triple product identity [19, p. 35, Entry 19] is given below.

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.2.1}$$

We refer to Berndt's book [19] for various properties satisfied by $f(a,b)$.

Consider the following special cases of $f(a,b)$:

$$\left.\begin{aligned} \varphi(q) &:= f(q,q) = \frac{f_2^5}{f_1^2 f_4^2}, \\ \varphi(-q) &= f(-q,-q) = \frac{f_1^2}{f_2}, \end{aligned}\right\} \tag{1.2.2}$$

$$\left.\begin{aligned} \psi(q) &:= f(q,q^3) = \frac{f_2^2}{f_1}, \\ \psi(-q) &= f(-q,-q^3) = \frac{f_1 f_4}{f_2}, \end{aligned}\right\} \tag{1.2.3}$$

and

$$\left.\begin{aligned} f(q) &:= f(q,-q^2) = \frac{f_2^3}{f_1 f_4}, \\ f(-q) &= f(-q,-q^2) = f_1. \end{aligned}\right\} \tag{1.2.4}$$

The q-product representations in the above arise from (1.2.1) and manipulation of the q-products.

1.3 Partial Bell polynomials

The partial Bell polynomials are the polynomials $B_{n,k} \equiv B_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ in an infinite number of variables defined by the formal double series expansion:

$$\sum_{n,k\geq 0} B_{n,k}\frac{t^n}{n!}u^k = \exp\left(u\sum_{m\geq 1} x_m\frac{t^m}{m!}\right).$$

For more equivalent definitions, exact expressions and further results involving the Bell polynomials, we refer to [28, Chap. 3.3].

1.4 Polygonal numbers

For an integer $s \geq 3$, the generalized n^{th} s-gonal number is defined by

$$F_s(n) := \frac{(s-2)n^2 - (s-4)n}{2}, \quad n \in \mathbb{Z}.$$

Henceforth, we call these numbers as s-gonal numbers.

The generating function $G_s(q)$ of $F_s(n)$ is given by

$$G_s(q) := \sum_{n=-\infty}^{\infty} q^{F_s(n)} = f(q, q^{s-3}).$$

Also note the exceptional case that $G_3(q)$ generates each triangular number twice while $G_6(q)$ generates only once.

1.5 Partitions and the partition generating function

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a nonnegative integer n is a finite sequence of non-increasing positive integer *parts* $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

The partition function $p(n)$ is defined as the number of partitions of n. For example, $p(5)$=7, since there are seven partitions of 5, namely,

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), \text{ and } (1,1,1,1,1).$$

By convention, $p(0) = 1$. The generating function for $p(n)$, due to Euler, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

1.6 Ramanujan's partition congruences

Ramanujan[55] found three congruence properties for $p(n)$ modulo 5, 7 and 11. For any nonnegative integer n, they are given by

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.6.1}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{1.6.2}$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

He also found the exact generating functions of $p(5n+4)$ and $p(7n+5)$ as given below:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\,\frac{f_5^5}{f_1^6}, \tag{1.6.3}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\,\frac{f_7^3}{f_1^4} + 49q\frac{f_7^7}{f_1^8}, \tag{1.6.4}$$

which immediately imply (1.6.1) and (1.6.2), respectively. It can also be shown from the above generating functions that

$$p(25n+24) \equiv 0 \pmod{25} \tag{1.6.5}$$

and

$$p(49n+47) \equiv 0 \pmod{49}.$$

In 1939, Zuckerman[72] found the generating functions of $p(25n+24)$, $p(49n+47)$ and $p(13n+6)$ analogous to (1.6.3) and (1.6.4). In particular, he showed that

$$\begin{aligned}\sum_{n\geq 0} p(25n+24)q^n =& 63\times 5^2\frac{f_5^6}{f_1^7} + 52\times 5^5 q\frac{f_5^{12}}{f_1^{13}} + 63\times 5^7 q^2\frac{f_5^{18}}{f_1^{19}} + 6\times 5^{10}q^3\frac{f_5^{24}}{f_1^{25}}\\ &+ 5^{12}q^4\frac{f_5^{30}}{f_1^{31}},\end{aligned} \tag{1.6.6}$$

which readily shows (1.6.5).

In [55], Ramanujan also offered a more general conjecture for congruences of $p(n)$ modulo arbitrary powers of 5, 7 and 11. In particular, if $\alpha \geq 1$ and if δ_α is the reciprocal modulo 5^α of 24, then

$$p(5^\alpha n + \delta_\alpha) \equiv 0 \pmod{5^\alpha}.$$

In his unpublished manuscript [56, pp. 133–177] (See also [22]), Ramanujan gave a proof of the above. Hirschhorn and Hunt [38] gave an elementary proof of the above by finding the generating function of $p(5^\alpha n + \delta_\alpha)$.

1.7 The Rogers-Ramanujan continued fraction

For $|q| < 1$, the famous Rogers-Ramanujan continued fraction $\mathcal{R}(q)$ is defined by

$$\mathcal{R}(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots .$$

In [57], Rogers proved that this continued fraction has the q-product representation

$$\mathcal{R}(q) = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

We refer to Andrews and Berndt's book [7] for a detailed treatment of $\mathcal{R}(q)$.

1.8 m-dissection

If $P(q)$ denotes a power series in q, then the m-dissection of $P(q)$ is given by

$$P(q) = \sum_{k=0}^{m-1} q^k P_k(q^m),$$

where P_k's are power series in q^k.

We note the following 5-dissections of theta functions from [37, p. 85, Eq. (8.1.1) and p. 89, Eq. (8.4.4)] and [19, p. 49, Corollary]:

$$f_1 = \frac{1}{R(q^5)} - q - q^2 R(q^5), \tag{1.8.1}$$

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\Big(R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2R(q^5)^{-2} + 3q^3R(q^5)^{-1} + 5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\Big), \tag{1.8.2}$$

$$\varphi(q) = \varphi(q^{25}) + 2qf(q^{15}, q^{35}) + 2q^4f(q^5, q^{45}), \tag{1.8.3}$$

$$\psi(q) = f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}), \tag{1.8.4}$$

where $R(q) = q^{1/5}/\mathcal{R}(q) = \dfrac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}$.

1.9 The n-color partitions

An n-color partition (also called a partition with "n copies of n") of a positive integer m is a partition in which a part of size n can appear in n different colors denoted by subscripts in $n_1, n_2, \ldots, n_n$ and the parts satisfy the order:

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < \cdots.$$

Let $\mathrm{PL}(m)$ denote the number of n-color partitions of m. For example, $\mathrm{PL}(4) = 13$ since there are 13 n-color partitions of 4, namely, $(4_1), (4_2), (4_3), (4_4), (3_1, 1_1), (3_2, 1_1), (3_3, 1_1), (2_1, 2_1), (2_2, 2_1), (2_2, 2_2), (2_1, 1_1, 1_1), (2_2, 1_1, 1_1)$, and $(1_1, 1_1, 1_1, 1_1)$. The generating function of $\mathrm{PL}(m)$ is given by

$$\sum_{m=0}^{\infty} \mathrm{PL}(m)q^m = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m}. \tag{1.9.1}$$

MacMahon [49, p. 1421] observed that the right side of (1.9.1) also generates the number of plane partitions of m (Also see [61, Corollary 7.20.3]), where a plane partition π of m is an array of non-negative integers,

$$\begin{matrix} m_{11} & m_{12} & m_{13} & \cdots \\ m_{21} & m_{22} & m_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

such that $\sum m_{ij} = m$ and the rows and columns are in decreasing order, that is, $m_{ij} \geq m_{(i+1)j}$, $m_{ij} \geq m_{i(j+1)}$, for all $i, j \geq 1$. For example, the plane partitions of 4 are

$$\begin{pmatrix} 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & & \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & \\ 1 & \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

For further reading on n-color partitions and plane partitions we refer to [1, 49, 59, 60].

1.10 Overpartitions

An overpartition of n is a non-increasing sequence of positive integers whose sum is n, in which the first occurrence of a number may be overlined. The overpartition function, denoted by $\overline{p}(n)$, is the number of overpartitions of n, with the convention that $\overline{p}(0) = 1$. For example, $\overline{p}(3) = 8$ because there are 8 overpartitions of 3, namely $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1$, and $\overline{1}+1+1$. Since the overlined parts of an overpartition form a partition into distinct parts and the non-overlined parts of an overpartition form an unrestricted partition, the generating function of $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

After the publication of the paper [29] by Corteel and Lovejoy in 2004, overpartitions have been studied quite extensively from various points of view. For example, many arithmetic properties of overpartitions as well as overpartitions into odd parts have been proved by a number of mathematicians. For more information and references, see [23, 24, 25, 26, 32, 39, 40, 41, 43, 46, 47, 48, 50, 51, 62, 65, 66, 67, 68, 69, 70, 71].

1.11 t-Core Partitions

The *Ferrers–Young diagram* of the partition λ of n is formed by arranging n nodes in k rows so that the ith row has λ_i nodes. The conjugate of a partition λ, denoted λ', is the partition whose Ferrers–Young diagram is the reflection along the main diagonal of the diagram of λ. A partition λ is *self-conjugate* if $\lambda = \lambda'$. The nodes are labeled by row and column coordinates as one would label the entries of a matrix. Let λ'_j denote the number of nodes in column j. The *hook number* $H(i,j)$ of the (i,j) node is defined as the number of nodes directly below and to the right of the node including the node itself. That is, $H(i,j) = \lambda_i + \lambda'_j - j - i + 1$. A partition is called a *t-core* if none of its hook numbers is divisible by t.

Example. The Ferrers–Young diagram of the partition $\lambda = (4,3,1,1)$ of 9 is

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \\ \bullet & & & \end{array}$$

The nodes $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,2)$, $(2,3)$, $(3,1)$ and $(4,1)$ have hook numbers 7, 4, 3, 1, 5, 2, 1, 2 and 1, respectively. Therefore, λ is a t-core for $t = 6$ and $t \geq 8$.

Granville and Ono [35] proved that for $t \geq 4$, every natural number n has a t-core, and thereby settling a conjecture of Brauer regarding the existence of defect zero characters for finite simple groups. Again, Baldwin, Depweg, Ford, Kunin and Sze [9] proved that if t is an integer with $t = 8$ or $t \geq 10$, then every integer $n > 2$ has a self-conjugate t-core. For a recent survey on t-core partitions we refer the readers to [27].

If $c_t(n)$ and $sc_t(n)$ denote the number of t-cores and self-conjugate t-cores, then the generating functions for $c_t(n)$ and $sc_t(n)$ are given by [34, Equations (2.1), (7.1a) and (7.1b)]

$$\sum_{n=0}^{\infty} c_t(n) q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}, \tag{1.11.1}$$

$$\sum_{n=0}^{\infty} sc_{2t}(n)q^n = (-q;q^2)_\infty (q^{4t};q^{4t})_\infty^t, \tag{1.11.2}$$

$$\sum_{n=0}^{\infty} sc_{2t+1}(n)q^n = \frac{(-q;q^2)_\infty (q^{4t+2};q^{4t+2})_\infty^t}{(-q^{2t+1};q^{4t+2})_\infty}. \tag{1.11.3}$$

1.12 Objective of the thesis

The objective of our thesis is to find arithmetical results on the number of representations of a positive integer as a sum of generalized polygonal numbers, n-color partitions, restricted overpartitions, 10-core and self-conjugate 10-core partitions and some analogues of the 5-core partitions.

1.13 Overview of the chapters

In this section, we present a brief account of the subsequent chapters.

Jha [44, 45] recently discovered connections between sums over divisors of n and the representations of n as sums of squares and triangular numbers. In Chapter 2, we establish a generalized result for s-gonal numbers, where $s \geq 3$, encompassing Jha's findings as corollaries.

Merca and Schmidt [52, 53] explored partition function decompositions using the Möbius function and Euler's totient. In Chapter 3, we define $T_k^r(m)$ for n-color partitions, linking it to the n-color partition function and key arithmetic functions like Möbius and Liouville. Employing Erdös' method, we establish counting theorems for n-color partitions, akin to results by Andrews and Deutsch [8].

In Chapter 4, we demonstrate arithmetic properties for overpartition functions with parts from selected residue classes modulo 8.

Chapter 5 is devoted to find new linear recurrence relations satisfied by the 10-core and self-conjugate 10-core partition functions. In the process, we also find several exact generating function representations and congruences satisfied by those functions.

Gireesh, Ray, and Shivashankar [33] explored the analogue $\overline{a}_t(n)$ of the t-core partition function $c_t(n)$ and established novel identities. In Chapter 6, we reexamine $\overline{a}_5(n)$ alongside $c_5(n)$ and another analogous function $\overline{b}_5(n)$. New recurrence relations for $\overline{a}_5$ and $\overline{b}_5$, as well as connections among $c_5(n)$, $\overline{a}_5(n)$, and $\overline{b}_5(n)$, are uncovered. Additionally, we prove a strengthened version of one of their congruences for $\overline{a}_5(n)$.