Chapter 2

Number of representations of n as a sum of generalized polygonal numbers

2.1 Introduction

Recall the definition of the *partial Bell polynomials* from Section 1.3 and that of the *polygonal numbers* from Section 1.4. Jha [44, 45] has obtained two identities that connect certain sums over the divisors of n to the number of representations of n as sums of squares and sums of triangular numbers, respectively. Our objective is to show that these results can be generalized to the number of representations of n as a sum of any specific generalized polygonal number. We also obtain some corollaries, including Jha's results (2.3.1) and (2.3.6). The contents of this chapter appeared in [11].

The next section contains some lemmas and the main theorem. In the final section, we present some corollaries, including Jha's results (2.3.1) and (2.3.6).

2.2 Lemmas and Theorem 2.2.3

In the following, we state and prove two lemmas that lead us to the main theorem, i.e., Theorem 2.2.3.

Lemma 2.2.1. *Let n be a positive integer. Then we have the following result.*

$$\sum_{d|n}\frac{1}{d}\left((-1)^d\delta_1\left(\frac{n}{d},s-2\right)+\delta_2\left(\frac{n}{d},s-2\right)\right)$$
$$=\frac{1}{n!}\sum_{k=1}^{n}(-1)^k(k-1)!B_{n,k}(G_s'(0),G_s''(0),\ldots,G_s^{n-k+1}(0)),$$

where we define $\delta_1(m,v)$ and $\delta_2(m,v)$ for $v\geq 2$ as follows:

$$\delta_1(m,v)=\begin{cases}2, & \text{if } m\equiv 1(\text{mod } 2),\ v=2,\\ 1, & \text{if } m\equiv 1 \text{ or } (v-1)(\text{mod } v),\ v\geq 3,\\ 0, & \text{otherwise},\end{cases}$$

and

$$\delta_2(m,v)=\begin{cases}1, & \text{if } m\equiv 0(\text{mod } v),\ v\geq 2,\\ 0, & \text{otherwise}.\end{cases}$$

Proof. By Jacobi triple product identity [19, p. 35, Entry 19], we have

$$\begin{aligned}G_s(q)&=f(q,q^{s-3})\\ &=(-q;q^{s-2})_\infty(-q^{s-3};q^{s-2})_\infty(q^{s-2};q^{s-2})_\infty\\ &=\prod_{j=0}^{\infty}\left((1+q^{(s-2)j+1})(1+q^{(s-2)j+s-3})(1-q^{(s-2)(j+1)})\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\log G_s(q) &= \sum_{j=0}^{\infty}\left(\log(1+q^{(s-2)j+1})+\log(1+q^{(s-2)j+s-3})+\log(1-q^{(s-2)(j+1)})\right)\\
&= -\sum_{j=0}^{\infty}\sum_{\ell=1}^{\infty}\left(\frac{(-1)^\ell}{\ell}q^{((s-2)j+1)\ell}+\frac{(-1)^\ell}{\ell}q^{((s-2)j+s-3)\ell}+\frac{1}{\ell}q^{(s-2)(j+1)\ell}\right)\\
&= -\sum_{\substack{j\geq 1\\ j\equiv 1 \pmod{s-2}}}\sum_{\ell\geq 1}\frac{(-1)^\ell}{\ell}q^{j\ell}-\sum_{\substack{j\geq 1\\ j\equiv -1 \pmod{s-2}}}\sum_{l\geq 1}\frac{(-1)^\ell}{\ell}q^{j\ell}\\
&\quad -\sum_{\substack{j\geq 1\\ j\equiv 0 \pmod{s-2}}}\sum_{\ell\geq 1}\frac{1}{\ell}q^{j\ell}\\
&= -\sum_{n\geq 1}q^n\left(\sum_{d|n}\frac{1}{d}\left((-1)^d\delta_1\left(\frac{n}{d},s-2\right)+\delta_2\left(\frac{n}{d},s-2\right)\right)\right),\end{aligned} \tag{2.2.1}$$

where the given definitions of $\delta_1(m,v)$ and $\delta_2(m,v)$ follow naturally.

Now, let the Taylor series expansion of $G_s(q)$ be

$$G_s(q)=\sum_{n\geq 0}g_n\frac{q^n}{n!}.$$

Then, from [28, p. 140, (5a) and (5b)], we have the following result:

$$\log G_s(q)=\sum_{n\geq 1}L_n\frac{q^n}{n!}, \tag{2.2.2}$$

where

$$L_n=L_n(g_1,g_2,\ldots,g_n)=\sum_{k=1}^{n}(-1)^k(k-1)!B_{n,k}(g_1,g_2,\ldots,g_n).$$

Comparing (2.2.1) and (2.2.2), we arrive at the desired result. □

Lemma 2.2.2. *Let $t_{s,j}(n)$ denote the number of representations of n as a sum of j s-gonal numbers. Then, we have*

$$B_{n,k}(G_s'(0),G_s''(0),\cdots,G_s^{n-k+1}(0))=\frac{n!}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}t_{s,j}(n).$$

Proof. The proof is similar to the one given in [45, Lemma 2]. So we omit. □

Theorem 2.2.3. *For all positive integers n, s with $s \geq 4$, we have*

$$\sum_{d|n} \frac{1}{d}\left((-1)^d \delta_1\left(\frac{n}{d}, s-2\right) + \delta_2\left(\frac{n}{d}, s-2\right)\right) = \sum_{j=1}^{n} \frac{(-1)^j}{j}\binom{n}{j} t_{s,j}(n).$$

Proof. Our proof of the theorem is essentially similar to the one given in [45, Theorem 1]. From Lemma 2.2.1 and Lemma 2.2.2, we have

$$\begin{aligned}
&\sum_{d|n} \frac{1}{d}\left((-1)^d \delta_1(\frac{n}{d}, s-2) + \delta_2(\frac{n}{d}, s-2)\right) \\
&= \frac{1}{n!}\sum_{k=1}^{n}(-1)^k (k-1)! \frac{n!}{k!} \sum_{j=1}^{k} (-1)^{k-j}\binom{k}{j} t_{s,j}(n) \\
&= \sum_{k=1}^{n}\sum_{j=1}^{k} \frac{(-1)^j}{k}\binom{k}{j} t_{s,j}(n) \\
&= \sum_{j=1}^{n}(-1)^j t_{s,j}(n) \sum_{k=j}^{n} \frac{1}{k}\binom{k}{j} \\
&= \sum_{j=1}^{n}(-1)^j \frac{1}{j}\binom{n}{j} t_{s,j}(n),
\end{aligned}$$

where we have used the result

$$\sum_{k=j}^{n} \frac{1}{k}\binom{k}{j} = \frac{1}{j}\binom{n}{j}$$

that can be derived easily from the identity

$$\binom{k}{j-1} = \binom{k+1}{j} - \binom{k}{j}.$$

□

2.3 Corollaries

In this section, we present the results in [44, 45] and some other interesting results as corollaries to Main Theorem 2.2.3.

Corollary 2.3.1. (Jha [45, Theorem 1]) *For any positive integer n, we have*

$$\sum_{\substack{d|n \\ d \text{ odd}}} \frac{2(-1)^n}{d} = \sum_{j=1}^{n} \frac{(-1)^j}{j}\binom{n}{j} t_{4,j}(n). \tag{2.3.1}$$

Proof. Setting $s = 4$ in Theorem 2.2.3, we find that

$$\sum_{\substack{d|n \\ \frac{n}{d} \text{ odd}}} \frac{2(-1)^d}{d} + \sum_{\substack{d|n \\ \frac{n}{d} \text{ even}}} \frac{1}{d} = \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{4,j}(n). \tag{2.3.2}$$

To prove that (2.3.2) is equivalent to (2.3.1), it is enough to show that

$$\sum_{\substack{d|n \\ d \text{ odd}}} \frac{2(-1)^n}{d} = \sum_{\substack{d|n \\ \frac{n}{d} \text{ odd}}} \frac{2(-1)^d}{d} + \sum_{\substack{d|n \\ \frac{n}{d} \text{ even}}} \frac{1}{d}. \tag{2.3.3}$$

We complete it by considering the following three possible cases of n.

Case I, n is odd: In this case, it is easily seen that both sides of (2.3.3) become

$$-\sum_{\substack{d|n \\ d \text{ odd}}} \frac{2}{d}.$$

Case II, $n = 2^k$, where $k \geq 1$: In this case, the left-hand side of (2.3.3) is equal to 2.

Now, the right-hand side of (2.3.3) becomes

$$\begin{aligned}\sum_{\substack{d|2^k \\ \frac{2^k}{d} \text{ odd}}} \frac{2(-1)^d}{d} + \sum_{\substack{d|2^k \\ \frac{2^k}{d} \text{ even}}} \frac{1}{d} &= \frac{1}{2^{k-1}} + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}}\right) \\ &= \frac{1}{2^{k-1}} + \frac{2^k - 1}{2^{k-1}} \\ &= 2.\end{aligned}$$

Thus, (2.3.3) holds good for this case.

Case III, $n = 2^k m$ where $k \geq 1$ and m is odd and greater than 1: The left-hand side of (2.3.3) becomes

$$\sum_{\substack{d|n \\ d \text{ odd}}} \frac{2}{d}. \tag{2.3.4}$$

Again, the right-hand side of (2.3.3) is

$$\begin{aligned}
&\sum_{\substack{d|2^km\\ \frac{2^km}{d}\text{ odd}}} \frac{2(-1)^d}{d} + \sum_{\substack{d|2^km\\ \frac{2^km}{d}\text{ even}}} \frac{1}{d}\\
&= \sum_{\substack{d|2^km\\ d\text{ odd}}} \frac{2}{2^kd} + \Bigg(\sum_{\substack{d|2^km\\ d\text{ odd}}} \frac{1}{d} + \sum_{\substack{d|2^km\\ d\text{ odd}}} \frac{1}{2d} + \sum_{\substack{d|2^km\\ d\text{ odd}}} \frac{1}{2^2d} + \cdots + \sum_{\substack{d|2^km\\ d\text{ odd}}} \frac{1}{2^{k-1}d}\Bigg)\\
&= \left(\frac{1}{2^{k-1}} + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}}\right) \sum_{\substack{d|n\\ d\text{ odd}}} \frac{1}{d}\\
&= \sum_{\substack{d|n\\ d\text{ odd}}} \frac{2}{d}.
\end{aligned} \tag{2.3.5}$$

From (2.3.4) and (2.3.5), we conclude that (2.3.3) holds good for this case as well. □

Corollary 2.3.2. (Jha [44, Theorem 1]) *For any positive integer n, we have*

$$\sum_{d|n} \frac{1+2(-1)^d}{d} = \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{6,j}(n). \tag{2.3.6}$$

Proof. Setting $s = 6$ in Theorem 2.2.3, we obtain

$$\sum_{\substack{d|n\\ \frac{n}{d}\text{ odd}}} \frac{(-1)^d}{d} + \sum_{\substack{d|n\\ \frac{n}{d}\equiv 0 \pmod 4}} \frac{1}{d} = \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{6,j}(n). \tag{2.3.7}$$

To prove the equivalence of (2.3.7) and (2.3.6), it is enough to show that

$$\sum_{\substack{d|n\\ \frac{n}{d}\text{ odd}}} \frac{(-1)^d}{d} + \sum_{\substack{d|n\\ \frac{n}{d}\equiv 0 \pmod 4}} \frac{1}{d} = \sum_{d|n} \frac{1+2(-1)^d}{d}. \tag{2.3.8}$$

We show it by considering three possible cases of n.

Case I, n is odd: In this case, we notice that both sides of (2.3.8) become

$$-\sum_{\substack{d|n\\ d\text{ odd}}} \frac{1}{d}.$$

Case II, $n = 2m$ with m odd: In this case, the left-hand side of (2.3.8) is

$$\sum_{\substack{d|2m \\ \frac{2m}{d} \text{ odd}}} \frac{(-1)^d}{d} + \sum_{\substack{d|2m \\ \frac{2m}{d} \equiv 0 \pmod 4}} \frac{1}{d} = \sum_{\substack{2d|2m \\ d \text{ odd}}} \frac{(-1)^{2d}}{2d}$$
$$= \frac{1}{2} \sum_{\substack{d|2m \\ d \text{ odd}}} \frac{1}{d}.$$

The right-hand side of (2.3.8) is

$$\sum_{\substack{d|2m \\ d \text{ odd}}} \frac{1+2(-1)^d}{d} + \sum_{\substack{d|2m \\ d \text{ even}}} \frac{1+2(-1)^d}{d} = -\sum_{\substack{d|2m \\ d \text{ odd}}} \frac{1}{d} + \frac{3}{2} \sum_{\substack{d|2m \\ d \text{ odd}}} \frac{1}{d}$$
$$= \frac{1}{2} \sum_{\substack{d|2m \\ d \text{ odd}}} \frac{1}{d}.$$

Thus, (2.3.8) holds good in this case.

Case III, $n = 2^k m$ with m odd and $k \geq 2$: In this case, the left-hand side of (2.3.8) is

$$\sum_{\substack{d|2^k m \\ \frac{2^k m}{d} \text{ odd}}} \frac{(-1)^d}{d} + \sum_{\substack{d|2^k m \\ \frac{2^k m}{d} \equiv 0 \pmod 4}} \frac{1}{d}$$
$$= \frac{1}{2^k} \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{d} + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^{k-2}}\right) \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{d} \tag{2.3.9}$$
$$= \left(2 - \frac{3}{2^k}\right) \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{d}.$$

Again, the right-hand side of (2.3.8) is

$$\begin{aligned}
&\sum_{d|2^k m} \frac{1+2(-1)^d}{d} \\
&= \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1+2(-1)^d}{d} + \sum_{\substack{d|2^k m \\ d=2d_1,\ d_1 \text{ odd}}} \frac{1+2(-1)^d}{d} + \sum_{\substack{d|2^k m \\ d=2^2 d_1,\ d_1 \text{ odd}}} \frac{1+2(-1)^d}{d} \\
&\quad + \sum_{\substack{d|2^k m \\ d=2^3 d_1,\ d_1 \text{ odd}}} \frac{1+2(-1)^d}{d} + \cdots + \sum_{\substack{d|2^k m \\ d=2^k d_1,\ d_1 \text{ odd}}} \frac{1+2(-1)^d}{d} \qquad (2.3.10)\\
&= -\sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{d} + \frac{3}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\frac{1}{2^{k-1}}\right) \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{d} \\
&= \left(2-\frac{3}{2^k}\right) \sum_{\substack{d|2^k m \\ d \text{ odd}}} \frac{1}{d}.
\end{aligned}$$

From (2.3.9) and (2.3.10), we arrive at (2.3.8) for this case. □

Corollary 2.3.3. *For any positive integer n, we have*

$$\sum_{\substack{d|n \\ \frac{n}{d}\equiv 1 \text{ or } 2 \pmod 3}} \frac{(-1)^d}{d} + \sum_{\substack{d|n \\ \frac{n}{d}\equiv 0 \pmod 3}} \frac{1}{d} = \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{5,j}(n).$$

Proof. The result follows by setting $s = 5$ in Theorem 2.2.3. □

Corollary 2.3.4. *Let n be a positive integer and $\sigma(n)$ denote the sum of positive divisors of n. Let p be an odd prime such that $p \mid n$ and $p^2 \nmid n$. If $\frac{n}{p} \equiv 1$ or $p-1 \pmod p$, then*

$$\sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{p+2,j}(n) = \frac{\sigma(n)}{n} - \frac{2}{p}.$$

Otherwise,

$$\sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{p+2,j}(n) = \frac{\sigma(n)}{n} - \frac{1}{p}.$$

Proof. Let p be as stated in the corollary. Setting $s = p + 2$ in Theorem 2.2.3, it follows that, if $\frac{n}{p} \equiv 1$ or $p - 1 (\bmod p)$, then

$$\sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{p+2,j}(n) = -\frac{1}{p} + \sum_{\substack{d|n \\ d \neq p}} \frac{1}{d}.$$

Otherwise,

$$\sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} t_{p+2,j}(n) = \sum_{\substack{d|n \\ d \neq p}} \frac{1}{d}.$$

As

$$\sum_{\substack{d|n \\ d \neq p}} \frac{1}{d} = \sum_{d|n} \frac{1}{d} - \frac{1}{p} = \frac{1}{n} \sum_{d|n} \frac{n}{d} - \frac{1}{p} = \frac{1}{n} \sum_{d|n} d - \frac{1}{p} = \frac{\sigma(n)}{n} - \frac{1}{p},$$

we readily arrive at the desired results. □