Chapter 3

The n-color partition function and some counting theorems

3.1 Introduction

The *n-color partitions* are defined in Section 1.9. Merca and Schmidt [52, 53] found some decompositions for the partition function $p(m)$ in terms of the classical elementary functions, namely, the Möbius function and Euler's totient. In this chapter, we find some connections between n-color partition function (equivalently, the plane partition function) PL(m) and elementary arithmetic functions and their divisor sums. The contents of this chapter appeared in [10].

We define an associated function $T_k^r(n)$ in two separate scenarios:

1. For $r \leq k$, $T_k^r(n) = \frac{1}{k} \times$ (the number of k's in the n-color partitions of n with the smallest part at least r).

2. For $r > k$, $T_k^r(n) =$ the number of the n-color partitions of $n - k$ with the smallest part at least r except the possibility of the part $k < r$ being present in only one color.

We consider the following three examples. First we consider $T_3^2(5)$. We note that the number of 3's in the n-color partitions of 5 with the smallest part at least 2 is 6, which is evident from the relevant partitions $3_1 + 2_1, 3_2 + 2_2, 3_3 + 2_1, 3_1 + 2_2, 3_2 + 2_1$ and $3_3 + 2_2$. Therefore, $T_3^2(5) = \frac{1}{3} \times 6 = 2$.

Next, we consider $T_2^3(5)$. Here $n-k=5-2=3$. The n-color partitions of 3 with the smallest part at least 3 except the possibility of the part 2 being present in only one color are $3_1, 3_2$ and 3_3. Hence, $T_2^3(5)=3$.

Finally, we consider $T_2^3(7)$. In this case, $n-k=7-2=5$ and the n-color partitions of 5 with the smallest part at least 3 except the possibility of the part 2 being present in only one color are $5_1, 5_2, 5_3, 5_4, 5_5, 3_1+2, 3_2+2,\ 3_3+2$. Thus, $T_2^3(7)=8$.

The generating function of $T_k^r(m)$ is given in the following lemma.

Lemma 3.1.1. *We have*

$$\sum_{m=k}^{\infty} T_k^r(m)q^m = \frac{q^k}{1-q^k}\prod_{m=r}^{\infty}\frac{1}{(1-q^m)^m}.$$

We have the following main theorem that connects $\mathrm{PL}(m)$, $T_k^r(m)$, and elementary arithmetic functions.

Theorem 3.1.2. *Let $A(m)$ be an arithmetic function for $m \geqslant 1$ and $B(m)$ be its divisor sum, that is,*

$$B(m) = \sum_{d|m} A(d).$$

Also, define the functions $\ell_r(m)$ for $m \geqslant 0$ and $r \geqslant 1$ recursively as

$$\ell_1(m) = 1,$$

and

$$\ell_r(m) = \sum_{k=0}^{\lfloor m/r \rfloor} \binom{r+k-1}{k} \ell_{r-1}(m-kr), \text{ for } r \geqslant 2. \tag{3.1.1}$$

Additionally, we set

$$\ell_0(m) = \begin{cases} 1, & \text{if } m=0, \\ 0, & \text{for } m \geqslant 1. \end{cases}$$

Then for $m \geqslant 1$ and $1 \leqslant r \leqslant m$, we have

$$\sum_{k=1}^{m} \mathrm{PL}(m-k)B(k) = \sum_{k=1}^{m}\sum_{j=0}^{m-k} A(k)T_k^r(m-j)\ell_{r-1}(j). \tag{3.1.2}$$

Recently, Andrews and Deutsch [8] used a counting technique of Erdös to derive certain theorems that involve counting parts of the integer partition. The following theorem is one such result.

Theorem 3.1.3. *Given $k \geq 1$, Let $S_k(n)$ be the number of appearances of k in the partitions of n. Also, in each partition of n, we count the number of times a part divisible by k appears uniquely; then sum these numbers over all the partitions of n. Let this number be $S_{|k}(n)$. Then,*

$$S_{|k}(n) = S_{2k}(n+k).$$

In this chapter, we generalize the above theorem to the case of counting the number of times a part congruent to $s \pmod{k}$ appears uniquely for some s satisfying $0 \leqslant s < k$, then summing these numbers over all the integer partitions of n. Furthermore, we apply the same techniques to give counting theorems for n-color partitions involving the counting function $T_k^1(n)$, which is a special case of $T_k^r(n)$ defined earlier.

The rest of the chapter is organized as follows. In Sections 3.2 and 3.3, we prove Lemma 3.1.1 and Theorem 3.1.2, respectively. In Section 3.4, we present some corollaries and a detailed work out example. In Section 3.5, we present an interesting identity involving $\mathrm{PL}(n)$ and Euler's totient ϕ that is analogous to a recent result of Merca and Schmidt [52]. In the final section, we present a generalization of Theorem 3.1.3 and some counting theorems for n-color partitions involving $T_k^1(n)$.

3.2 Proof of Lemma 3.1.1

If $G(q)$ denotes the generating function of the number of n-color partitions of m with the least part being at least r, then

$$G(q) = \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m}.$$

Marking the part $k \geq r$ with a counter u, let

$$G(q,u) = \frac{1}{(1-q^r)^r \cdots (1-q^{k-1})^{k-1}(1-uq^k)^k(1-q^{k+1})^{k+1}\cdots}.$$

Note that $G(q,1) = G(q)$. Each term of $G(q,u)$ is of the form $\ell \times u^j \times q^m$ where j is the number of part k present in the n-color partitions of m and ℓ is the number of such n-color partitions where j number of part k are present. If we take derivative with respect to u then the term becomes $\ell \times j \times u^{j-1} \times q^m$ and the terms without u vanish. Next, taking $u = 1$ helps to sum up the q^m terms for each m, and we get the required generating function.

Hence, taking symbolic derivative at $u = 1$, we obtain the generating function of $k \times T_k^r(m)$ for $r \leq k$ as

$$\begin{aligned} \left.\frac{dG(u)}{du}\right|_{u=1} &= \frac{1}{(1-q^r)^r} \cdots \frac{1}{(1-q^{k-1})^{k-1}} \frac{kq^k}{(1-q^k)^{k+1}} \frac{1}{(1-q^{k+1})^{k+1}} \cdots \\ &= \frac{kq^k}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m}. \end{aligned}$$

In case of $r > k$, we consider $h(m)$ to be the number of the n-color partitions of m with the least part being r except the possibility of the part $k < r$ being present in only one color. Then

$$\sum_{m=0}^{\infty} h(m)q^m = \frac{1}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m},$$

from which, we have

$$\sum_{m=0}^{\infty} h(m)q^{m+k} = \frac{q^k}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m},$$

which can be rewritten, after adjusting the index of the sum on the left side, as

$$\sum_{m=k}^{\infty} h(m-k)q^m = \frac{q^k}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m}.$$

Of course, the above gives the required generating function of $T_k^r(m)$ for $r > k$.

3.3 Proof of Theorem 3.1.2

Observe that

$$\left(\prod_{m=1}^{\infty}\frac{1}{(1-q^m)^m}\right)\left(\sum_{k=1}^{\infty}\frac{A(k)q^k}{1-q^k}\right) = \left(\sum_{m=0}^{\infty}\mathrm{PL}(m)q^m\right)\left(\sum_{k=1}^{\infty}B(k)q^k\right)$$
$$= \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\mathrm{PL}(m-k)B(k)\right)q^m. \qquad (3.3.1)$$

Again,

$$\begin{aligned}
&\left(\prod_{m=1}^{\infty}\frac{1}{(1-q^m)^m}\right)\left(\sum_{k=1}^{\infty}\frac{A(k)q^k}{1-q^k}\right)\\
&=\left(\prod_{m=1}^{r-1}\frac{1}{(1-q^m)^m}\right)\left(\prod_{m=r}^{\infty}\frac{1}{(1-q^m)^m}\sum_{k=1}^{\infty}\frac{A(k)q^k}{1-q^k}\right)\\
&=\left(\sum_{j=0}^{\infty}\ell_{r-1}(j)q^j\right)\left(\sum_{k=1}^{\infty}A(k)\sum_{m=k}^{\infty}T_k^r(m)q^m\right)\\
&=\left(\sum_{j=0}^{\infty}\ell_{r-1}(j)q^j\right)\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m}A(k)T_k^r(m)q^m\right)\\
&=\sum_{m=1}^{\infty}\left(\sum_{j=0}^{m-1}\sum_{k=1}^{m-j}A(k)T_k^r(m-j)\ell_{r-1}(j)\right)q^m\\
&=\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\sum_{j=0}^{m-k}A(k)T_k^r(m-j)\ell_{r-1}(j)\right)q^m. \qquad (3.3.2)
\end{aligned}$$

Comparing (3.3.1) and (3.3.2) we arrived at the desired result.

Now we work out the definition of $\ell_r(m)$. As in the proof, for $r \geqslant 1$,

$$\sum_{m=0}^{\infty}\ell_r(m)q^m = \prod_{m=1}^{r}\frac{1}{(1-q^m)^m},$$

from which, for $r \geqslant 2$, we see that

$$\begin{aligned}
\sum_{m=0}^{\infty}\ell_r(m)q^m &= \left(\sum_{m=0}^{\infty}\ell_{r-1}(m)q^m\right)\left(\frac{1}{(1-q^r)^r}\right)\\
&=\left(\sum_{m=0}^{\infty}\ell_{r-1}(m)q^m\right)\left[\sum_{k=0}^{\infty}\binom{r+k-1}{k}q^{kr}\right]\\
&=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\lfloor m/r\rfloor}\binom{r+k-1}{k}\ell_{r-1}(m-kr)\right)q^m. \qquad (3.3.3)
\end{aligned}$$

Furthermore,

$$\sum_{m=0}^{\infty} \ell_1(m) q^m = \frac{1}{1-q} = \sum_{m=0}^{\infty} q^m. \tag{3.3.4}$$

From (3.3.3) and (3.3.4), we arrive at the definition of $\ell_r(m)$ for $r \geqslant 1$.

It remains to show that our definition for $\ell_0(m)$ is consistent with the result. That is, we need to prove that

$$\sum_{k=1}^{m} \mathrm{PL}(m-k) B(k) = \sum_{k=1}^{m} A(k) T_k^1(m). \tag{3.3.5}$$

We have

$$\begin{aligned}
&\left(\prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m}\right)\left(\sum_{k=1}^{\infty} \frac{A(k) q^k}{1-q^k}\right) \\
&= \sum_{k=1}^{\infty} A(k) \sum_{m=k}^{\infty} T_k^1(m) q^m \\
&= \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} A(k) T_k^1(m)\right) q^n. \qquad (3.3.6)
\end{aligned}$$

Comparing (3.3.1) and (3.3.6), we arrive at (3.3.5).

3.4 Corollaries and an example

In this section, we substitute $A(m)$ and $B(m)$ in Theorem 3.1.2 with some well known pairs of arithmetic functions to arrive at some interesting corollaries.

Corollary 3.4.1. *We have*

$$\sum_{k=1}^{m} \mathrm{PL}(m-k) = T_1^1(m).$$

Proof. Taking $A(m) = \left\lfloor \frac{1}{m} \right\rfloor$, $B(m) = \sum_{d|m} A(d) = 1$ and $r = 1$ in (3.1.2) we easily arrive at the corollary. □

Corollary 3.4.2. *For μ being the Möbius function, and $m \geq r \geq 1$, we have*

$$\mathrm{PL}(m-1) = \sum_{k=1}^{m} \sum_{j=0}^{m-k} \mu(k) T_k^r(m-j) \ell_{r-1}(j).$$

Proof. Take $A(m) = \mu(m)$. Hence

$$B(m) = \sum_{d|m} \mu(d) = \left\lfloor \frac{1}{m} \right\rfloor.$$

Putting these in (3.1.2) we arrive at the required result. $\square$

Corollary 3.4.3. *If $\tau(m)$ is the number of positive divisors of m for $m \geqslant 1$, and $m \geq r \geq 1$, then*

$$\sum_{k=1}^{m} \mathrm{PL}(m-k)\tau(k) = \sum_{k=1}^{m} \sum_{j=0}^{m-k} T_k^r(m-j)\ell_{r-1}(j).$$

Proof. This follows readily by substituting

$$A(m) = 1 \quad \text{and} \quad B(m) = \sum_{d|m} A(d) = \sum_{d|m} 1 = \tau(m)$$

in (3.1.2). $\square$

Corollary 3.4.4. *For $\lambda(m)$ being the Liouville function, and $m \geq r \geq 1$, we have*

$$\sum_{k=1}^{\lfloor \sqrt{m} \rfloor} \mathrm{PL}(m-k^2) = \sum_{k=1}^{m} \sum_{j=0}^{m-k} \lambda(k) T_k^r(m-j)\ell_{r-1}(j).$$

Proof. Let $A(m) = \lambda(m)$. It is well known that

$$B(m) = \sum_{d|m} A(d) = \sum_{d|m} \lambda(d) = \begin{cases} 1, & \text{if } m \text{ is a square;} \\ 0, & \text{otherwise.} \end{cases}.$$

Substituting the above in (3.1.2) we readily arrive at the corollary. $\square$

Corollary 3.4.5. *For $\alpha \geqslant 1$, let $\sigma_\alpha(m) = \sum_{d|m} d^\alpha$. Then, for $m \geq r \geq 1$,*

$$\sum_{k=1}^{m} \mathrm{PL}(m-k)\sigma_\alpha(k) = \sum_{k=1}^{m} \sum_{j=0}^{m-k} k^\alpha T_k^r(m-j)\ell_{r-1}(j). \tag{3.4.1}$$

Proof. Take $A(m) = m^\alpha$ so that

$$B(m) = \sum_{d|m} A(d) = \sum_{d|m} d^\alpha = \sigma_\alpha(m).$$

We substitute the above in (3.1.2) to arrive at the proffered identity. $\square$

Corollary 3.4.6. *For $m \geq r \geq 1$,*

$$\sum_{k=1}^{m} \mathrm{PL}(m-k)\log k = \sum_{\substack{1<k\leqslant m,\\ k=p^c, p\ prime, c\geqslant 1}} \sum_{j=0}^{m-k} T_k^r(m-j)\ell_{r-1}(j)\log p. \tag{3.4.2}$$

Proof. Take $A(m) = \Lambda(m)$, the Von Mangoldt function. Then

$$B(m) = \sum_{d|m} \Lambda(m) = \log m.$$

The result now follows by substituting these in (3.1.2). □

Example 3.4.7. We work out the case $m = 11$ and $r = 3$ in (3.4.2).

First of all, we generate the required values for $\ell_2(j)$. Using the definition (3.1.1), we have

$$\ell_2(m) = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{2+k-1}{k} \ell_1(m-2k) = \sum_{k=0}^{\lfloor m/2 \rfloor} (k+1)$$
$$= 1 + 2 + ... + (\lfloor m/2 \rfloor + 1).$$

Using the above, we have the values as shown in the following table.

j	0	1	2	3	4	5	6	7	8	9
$\ell_2(j)$	1	1	3	3	6	6	10	10	15	15

Setting $m = 11$, the left hand side of (3.4.2) becomes

$$\sum_{k=1}^{11} \mathrm{PL}(11-k)\log k.$$

The corresponding coefficients of the $\log k$ terms are given in the following table.

Table 3.1

$\log k$	$\log 2$	$\log 3$	$\log 5$	$\log 7$	$\log 11$
corresponding coefficients	497	190	49	13	1

Setting $m = 11$ and $r = 3$, the right hand side of (3.4.2) becomes

$$\sum_{\substack{1<k\leqslant 11,\\ k=p^c, p \text{ prime}, c\geqslant 1}} \sum_{j=0}^{11-k} T_k^3(11-j)\ell_2(j)\log p.$$

The coefficients of the $\log p$ terms are given in the following table, which matches with the values in Table 3.1 calculated for the left hand side of (3.4.2) for $m = 11$.

Table 3.2

$\log p$	corresponding coefficients
$\log 2$	$\sum_{j=0}^{9} T_2^3(11-j)\ell_2(j) + \sum_{j=0}^{7} T_4^3(11-j)\ell_2(j)$ $+ \sum_{j=0}^{3} T_8^3(11-j)\ell_2(j) = 497$
$\log 3$	$\sum_{j=0}^{8} T_3^3(11-j)\ell_2(j) + \sum_{j=0}^{2} T_9^3(11-j)\ell_2(j) = 190$
$\log 5$	$\sum_{j=0}^{6} T_5^3(11-j)\ell_2(j) = 49$
$\log 7$	$\sum_{j=0}^{4} T_7^3(11-j)\ell_2(j) = 13$
$\log 11$	1

As a demonstration, we now explicitly calculate the coefficient of $\log 3$, that is 190, from the combinatorial procedure.

To this end, for the partitions of 11 with the smallest part at least 3, the following table helps to calculate $T_3^3(11)$.

Integer partition	Number of corresponding n-color partitions	Total number of parts 3 present
(8,3)	24	24
(5,3,3)	30	60
(4,4,3)	30	30

Thus, $T_3^3(11) = \dfrac{1}{3}(24 + 60 + 30) = 38.$

Next, for the partitions of 10 with the smallest part at least 3, we have

Integer partition	Number of corresponding n-color partitions	Total number of parts 3 present
(7,3)	21	21
(4,3,3)	24	48

Therefore, $T_3^3(10) = \dfrac{1}{3}(21 + 48) = 23$.

For the partitions of 9 with the smallest part at least 3, the following table helps to calculate $T_3^3(9)$.

Integer partition	Number of corresponding n-color partitions	Total number of parts 3 present
(6,3)	18	18
(3,3,3)	10	30

Hence, $T_3^3(10) = \dfrac{1}{3}(18 + 30) = 16$.

For the partitions of 8 with the smallest part at least 3, we have the following table that helps to calculate $T_3^3(8)$.

Integer partition	Number of corresponding n-color partitions	Total number of parts 3 present
(5,3)	15	15

Thus, $T_3^3(10) = \dfrac{1}{3} \times 15 = 5$.

In a similar way, we calculate $T_3^3(7) = 4, T_3^3(6) = 4, T_3^3(3) = 1$ and $T_9^3(9) = 1$.

Now, using the table of $\ell_2(j)$ and the above values, we arrive at

$$\sum_{j=0}^{8} T_3^3(11-j)\ell_2(j) + \sum_{j=0}^{2} T_9^3(11-j)\ell_2(j) = 187 + 3 = 190,$$

which coincides with the coefficient 190 of $\log 3$ in Table 3.2.

3.5 A special identity involving Euler's totient ϕ

We recall from Hardy and Wright [36, Theorem 309] that,

$$\sum_{m=0}^{\infty} \frac{\phi(m)q^m}{1-q^m} = \frac{q}{(1-q)^2}. \tag{3.5.1}$$

Due to the existence of such a closed form, we can pursue a different approach for the case of ϕ function, as done in the paper by Merca and Schmidt [52].

Theorem 3.5.1. *For $m \geqslant 0$,*

$$\mathrm{PL}(m+2) - \mathrm{PL}(m) = \frac{1}{2}\sum_{k=3}^{m+5} \phi(k) T_k^3(m+5). \tag{3.5.2}$$

Proof. Notice that

$$\begin{aligned} &(1-q)(1-q^2)^2 \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \sum_{k=1}^{\infty} \frac{\phi(k)q^k}{1-q^k} \\ &= (q+q^2-3q^3-q^4+2q^5) \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} + \sum_{m=3}^{\infty}\sum_{k=3}^{m} \phi(k) T_k^3(m) q^m. \end{aligned} \tag{3.5.3}$$

Again, using the closed form (3.5.1), we have

$$\begin{aligned} &(1-q)(1-q^2)^2 \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \sum_{k=1}^{\infty} \frac{\phi(k)q^k}{1-q^k} \\ &= (q+q^2-q^3-q^4) \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \end{aligned} \tag{3.5.4}$$

From (3.5.3) and (3.5.4), we find that

$$\begin{aligned} \sum_{m=3}^{\infty} \left(\sum_{k=3}^{m} \phi(k) T_k^3(m) \right) q^m &= 2(q^3-q^5) \left(\sum_{m=0}^{\infty} \mathrm{PL}(m) q^m \right) \\ &= 2 \left(\sum_{m=3}^{\infty} \mathrm{PL}(m-3) - \sum_{m=5}^{\infty} \mathrm{PL}(m-5) \right) q^m. \end{aligned}$$

Equating the coefficients of q^{m+5}, for $m \geq 0$, from both sides of the above, we readily arrive at (3.5.2) to finish the proof. $\square$

Example 3.5.2. We verify Theorem 3.5.1 for the case $m = 6$.

The left side of (3.5.2) is PL(8) − PL(6) = 160 − 48 = 112.

On the other hand, the right side of (3.5.2) can be worked out as

$$\begin{aligned}
&\frac{1}{2}\sum_{k=3}^{11}\phi(k)T_k^3(11)\\
&=\frac{1}{2}\big(\phi(3)T_3^3(11)+\phi(4)T_4^3(11)+\phi(5)T_5^3(11)+\phi(6)T_6^3(11)+\phi(7)T_7^3(11)\\
&\quad+\phi(8)T_8^3(11)+\phi(11)T_{11}^3(11)\big)\\
&=\frac{1}{2}\left(2\times 38+2\times 22+4\times 12+2\times 5+6\times 4+4\times 3+10\times 1\right)\\
&=112.
\end{aligned}$$

Thus the result is verified for $n = 6$.

3.6 A generalization of Theorem 3.1.3 and some counting theorems for n-color partitions

We state a generalization of Theorem 3.1.3 as follows.

Theorem 3.6.1. *Given $k \geqslant 1$, in each partitions of n we count the number of times a part congruent to $s(\mathrm{mod}\ k)$ appears uniquely for some s satisfying $0 \leqslant s < k$, then sum these numbers over all the partitions of n. Let us call this $S_{s(k)}(n)$. Then*

$$S_{s(k)}(n) = S_{2k}(n+k-s) + S_{2k}(n-s) - S_{2k}(n-2s).$$

Proof. Approaching as in [8], assuming $n \geqslant 1$, $k \geqslant 1$, the generating function of $S_{s(k)}(n)$ is given by

$$\sum_{n\geqslant 1} S_{2k}(n)q^n = \sum_{j=1}^{\infty}\frac{q^{kj+s}}{\prod_{n\neq kj+s}(1-q^n)},$$

from which, we have

$$\sum_{n\geqslant 1} S_{2k}(n)q^n = \frac{1}{\prod_n (1-q^n)}\sum_{j=1}^{\infty} q^{kj+s}(1-q^{kj+s})$$

$$\begin{aligned}
&= \frac{1}{\prod_n (1-q^n)} \left(q^s \frac{q^k}{1-q^k} - q^{2s} \frac{q^{2k}}{1-q^{2k}} \right) \\
&= \frac{q^{2k}}{1-q^{2k}} \frac{1}{\prod_n (1-q^n)} \left(q^{-(k-s)} + q^s - q^{2s} \right) \\
&= \left(\sum_{n \geqslant 1} S_{2k}(n) q^n \right) \left(q^{-(k-s)} + q^s - q^{2s} \right).
\end{aligned}$$

Comparing the coefficients of q^n from both sides, we arrive at the desired result. □

The following theorem presents the case for n-color partitions in the spirit of Theorem 3.1.3. This shows how functions of the form $T_k^r(n)$ can be useful in such counting theorems.

Theorem 3.6.2. *In each n-color partitions of n, we count the number of times a part divisible by k appears uniquely, then sum these numbers over all the n-color partitions of n. Let us multiply this sum by $\frac{1}{k}$ and call it $T_{|k}(n)$. Then*

$$T_{|k}(n) = \sum_{j \geqslant 0} \Big(T_{2k}^1(n-(2j-1)k) + T_{2k}^1(n-2jk) + T_{2k}^1(n-(2j+1)k) \Big).$$

Proof. The generating function of $T_{|k}(n)$ is given by

$$k \sum_n T_{|k}(n) q^n = \sum_{j=1}^{\infty} \frac{kjq^{kj}}{(1-q^{kj})^{kj-1} \prod_{n \neq kj} (1-q^n)^n}.$$

Hence,

$$\begin{aligned}
\sum_n T_{|k}(n) q^n &= \frac{\sum_{j=1}^{\infty} jq^{kj}(1-q^{kj})}{\prod_n (1-q^n)^n} \\
&= \frac{1}{\prod_n (1-q^n)^n} \Big(\sum_{j=1}^{\infty} jq^{kj} - \sum_{j=1}^{\infty} jq^{2kj} \Big) \\
&= \frac{1}{\prod_n (1-q^n)^n} \left(\frac{q^k}{(1-q^k)^2} - \frac{q^{2k}}{(1-q^{2k})^2} \right) \\
&= \frac{1}{\prod_n (1-q^n)^n} \frac{q^{2k}}{(1-q^{2k})^2} \Big(q^{-k} + 1 + q^k \Big) \\
&= \Big(\sum_n T_{2k}^1(n) q^n \Big) \Big(\sum_{j=1}^{\infty} (q^{(2j-1)k} + q^{2jk} + q^{(2j+1)k}) \Big). \qquad (3.6.1)
\end{aligned}$$

Comparing the coefficients of q^n from both sides of (3.6.1), we obtain the desired result. □

In fact, we can also generalize this theorem to any part congruent to $s(\text{mod } k)$ as follows.

Theorem 3.6.3. *In each n-color partitions of n we count the number of times a part congruent to $s(\text{mod } k)$ appears uniquely for some s satisfying $0 \leqslant s < k$, then sum these numbers over all the n-color partitions of n. Let us call this $T_{s(k)}(n)$. Then,*

$$\begin{aligned} T_{s(k)}(n) =& (k+s)(T^1_{2k}(n+k-s) - T^1_{2k}(n-2s)) + sT^1_{2k}(n-s) \\ & + 2k \sum_{l \geqslant 1} T^1_{2k}(n+k-s-kl) - k \sum_{l \geqslant 1} T^1_{2k}(n-2s-2kl). \end{aligned}$$

Proof. The generating function of $T_{s(k)}(n)$ is given by

$$\sum_{n \geqslant 0} T_{s(k)}(n) q^n = \sum_{j \geqslant 1} \frac{(kj+s)q^{kj+s}}{(1-q^{kj+s})^{kj+s-1} \prod_{n \neq kj+s}(1-q^n)^n}.$$

Therefore,

$$\begin{aligned} &\sum_{n \geqslant 0} T_{s(k)}(n) q^n \\ &= \sum_{j \geqslant 1} \frac{(kj+s)q^{kj+s}}{(1-q^{kj+s})^{kj+s-1} \prod_{n \neq kj+s}(1-q^n)^n} \\ &= \sum_{j \geqslant 1} \frac{(kj+s)q^{kj+s}(1-q^{kj+s})}{\prod_{n \geqslant 1}(1-q^n)^n} \\ &= \frac{1}{\prod_{n \geqslant 1}(1-q^n)^n} \left(kq^s \sum_{j \geqslant 1} jq^{kj} + sq^s \sum_{j \geqslant 1} q^{kj} - kq^{2s} \sum_{j \geqslant 1} jq^{2kj} - sq^{2s} \sum_{j \geqslant 1} q^{2kj} \right) \\ &= \frac{1}{\prod_{n \geqslant 1}(1-q^n)^n} \left(kq^s \frac{q^k}{(1-q^k)^2} + sq^s \frac{q^k}{1-q^k} - kq^{2s} \frac{q^{2k}}{(1-q^{2k})^2} - sq^{2s} \frac{q^{2k}}{1-q^{2k}} \right) \\ &= \frac{q^{2k}}{1-q^{2k}} \frac{1}{\prod_{n \geqslant 1}(1-q^n)^n} \left(kq^{-(k-s)} \frac{1+q^k}{1-q^k} + sq^{-(k-s)}(1+q^k) - kq^{2s} \frac{1}{1-q^{2k}} - sq^{2s} \right) \\ &= \left(\sum_{n \geqslant 0} T^1_{2k}(n) q^n \right) \left(kq^{-(k-s)} \left(1 + 2 \sum_{l \geqslant 1} q^{kl} \right) + sq^{-(k-s)}(1+q^k) - kq^{2s} \left(l + \sum_{l \geqslant 1} q^{2kl} \right) \right) \\ &= \left(\sum_{n \geqslant 0} T^1_{2k}(n) q^n \right) \left(kq^{-(k-s)} + sq^{-(k-s)+sq^s-kq^{2s}-sq^{2s}+2kq^{-(k-s)}} \sum_{l \geqslant 1} q^{kl} - kq^{2s} \sum_{l \geqslant 1} q^{2kl} \right) \end{aligned}$$

$$= (k+s)\sum_n T^1_{2k}(n)q^{n-k+s} + s\sum_n T^1_{2k}(n)q^{n+s} - (k+s)\sum_n T^1_{2k}(n)q^{n+2s}$$
$$+ 2kq^{-(k-s)}\sum_n \left(\sum_{l\geqslant 1} T^1_{2k}(n-kl)\right) q^n - kq^{2s}\sum_n \left(\sum_{l\geqslant 1} T^1_{2k}(n-2kl)\right) q^n$$

$$= (k+s)\sum_n T^1_{2k}(n+k-s)q^n + s\sum_n T^1_{2k}(n-s)q^n - (k+s)\sum_n T^1_{2k}(n-2s)q^n$$
$$+ 2k\sum_n \left(\sum_{l\geqslant 1} T^1_{2k}(n+k-s-kl)\right) q^n - k\sum_n \left(\sum_{l\geqslant 1} T^1_{2k}(n-2s-2kl)\right) q^n.$$

Comparing the coefficient of q^n from both sides, we arrive at the desired result. □

It is to be noted that Theorem 3.6.2 can be obtained from Theorem 3.6.3 by putting $s = 0$, taking $T_{|k}(n) = \frac{1}{k}T_{0(k)}(n)$, and then rearranging the terms.