Chapter 4

Some restricted overpartition functions

4.1 Introduction

In this chapter, we prove arithmetic properties of some restricted overpartition functions in which the parts are from certain residue classes of 8. We consider the following restricted overpartition functions:

$$\overline{p}_{3,5}(n) = \text{number of overpartitions of } n \text{ into parts congruent to } \pm 3 \text{ modulo } 8,$$

$$\overline{p}_{3,4,5}(n) = \text{number of overpartitions of } n \text{ into parts congruent to } \pm 3 \text{ or } 4 \text{ modulo } 8,$$

$$\overline{p}_{1,7}(n) = \text{number of overpartitions of } n \text{ into parts congruent to } \pm 1 \text{ modulo } 8,$$

$$\overline{p}_{1,4,7}(n) = \text{number of overpartitions of } n \text{ into parts congruent to } \pm 1 \text{ or } 4 \text{ modulo } 8.$$

By convention, we also set $\overline{p}_{3,5}(0) = \overline{p}_{3,4,5}(0) = \overline{p}_{1,7}(0) = \overline{p}_{1,4,7}(0) = 1$.

Example 4.1.1. $n = 9$

Then $\overline{p}_{3,5}(9) = 2$, $\overline{p}_{3,4,5}(9) = 6$, $\overline{p}_{1,7}(9) = 8$, $\overline{p}_{1,4,7}(9) = 16$, and the relevant partitions are given by

$$3+3+3 = \overline{3}+3+3,$$

$$5+4 = \overline{5}+4 = 5+\overline{4} = \overline{5}+\overline{4} = 3+3+3 = \overline{3}+3+3,$$

$$9 = \overline{9} = 7+1+1 = \overline{7}+1+1 = 7+\overline{1}+1 = \overline{7}+\overline{1}+1$$
$$= 1+1+1+1+1+1+1+1+1 = \overline{1}+1+1+1+1+1+1+1+1,$$
$$9 = \overline{9} = 7+1+1 = \overline{7}+1+1 = 7+\overline{1}+1 = \overline{7}+\overline{1}+1 = 4+4+1$$
$$= \overline{4}+4+1 = 4+4+\overline{1} = \overline{4}+4+\overline{1} = 4+1+1+1+1+1$$
$$= \overline{4}+1+1+1+1+1 = 4+\overline{1}+1+1+1+1 = \overline{4}+\overline{1}+1+1+1+1$$
$$= 1+1+1+1+1+1+1+1+1 = \overline{1}+1+1+1+1+1+1+1+1.$$

Now, let $\text{pod}(n)$ denote the number of partitions of n into non-repeating odd parts, that is, odd parts must be distinct (and even parts are unrestricted). This function has appeared in several earlier works. For example, see the papers by Andrews [4, 5, 6], Berkovich and Garvan [18], Alladi [2, 3]. The function has also been studied by several authors from the arithmetical point of view. For example, see the papers by Hirschhorn and Sellers [42], Radu and Sellers [54], Lovejoy and Osburn [48], Cui, Gu and Ma [30], Wang [63], Fang, Xue and Yao [31], and some other papers cited thereon.

The functions $\overline{p}_{3,4,5}(n)$ and $\overline{p}_{1,4,7}(n)$ are connected to $\text{pod}(n)$ by the following relations.

Theorem 4.1.2. *For any positive integer n, we have*

$$\overline{p}_{3,4,5}(n) = \sum_{k=-\infty}^{\infty} (-1)^{k(k+1)/2} \text{pod}\left(n-(2k^2-k)\right), \qquad (4.1.1)$$
$$\overline{p}_{1,4,7}(n) = \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} \text{pod}\left(n-(2k^2-k)\right). \qquad (4.1.2)$$

Example 4.1.3. $n = 9$

We have

$$\overline{p}_{3,4,5}(9) = \text{pod}(9) - \text{pod}(8) + \text{pod}(6) - \text{pod}(3)$$
$$= 13 - 10 + 5 - 2 = 6,$$
$$\overline{p}_{1,4,7}(9) = \text{pod}(9) + \text{pod}(8) - \text{pod}(6) - \text{pod}(3)$$

$$= 13 + 10 - 10 - 5 - 2 = 16.$$

which are in agreement with Example 4.1.1.

In Section 4.2, we present a proof of Theorem 4.1.2.

Clearly, the generating functions of $\overline{p}_{3,4}, \overline{p}_{1,7}, \overline{p}_{3,4,5}$, and $\overline{p}_{1,4,7}(n)$ are given by

$$\sum_{n=0}^{\infty} \overline{p}_{3,5}(n)q^n = \frac{(-q^3;q^8)_\infty(-q^5;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}, \quad \sum_{n=0}^{\infty} \overline{p}_{1,7}(n)q^n = \frac{(-q;q^8)_\infty(-q^7;q^8)_\infty}{(q;q^8)_\infty(q^7;q^8)_\infty},$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,4,5}(n)q^n = \frac{(-q^3;q^8)_\infty(-q^4;q^8)_\infty(-q^5;q^8)_\infty}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}, \tag{4.1.3}$$

$$\sum_{n=0}^{\infty} \overline{p}_{1,4,7}(n)q^n = \frac{(-q;q^8)_\infty(-q^4;q^8)_\infty(-q^7;q^8)_\infty}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}. \tag{4.1.4}$$

Since $(-q^\ell;q^8)_\infty \equiv (q^\ell;q^8)_\infty \pmod 2$, it is clear that for all $n > 0$,

$$\overline{p}_{3,5}(n) \equiv \overline{p}_{3,4,5}(n) \equiv \overline{p}_{1,7}(n) \equiv \overline{p}_{1,4,7}(n) \equiv 0 \pmod 2.$$

We present the following congruences.

Theorem 4.1.4. *For any non-negative integer n, we have*

$$\overline{p}_{3,5}(8n+7) \equiv 0 \pmod{2^2}, \tag{4.1.5}$$

$$\overline{p}_{3,5}(16n+14) \equiv 0 \pmod{2^2}, \tag{4.1.6}$$

$$\overline{p}_{3,5}(32n+28) \equiv 0 \pmod{2^2}, \tag{4.1.7}$$

$$\overline{p}_{3,5}(16n+1) \equiv 0 \pmod{2^3}, \tag{4.1.8}$$

$$\overline{p}_{3,5}(32n+14) \equiv 0 \pmod{2^3}, \tag{4.1.9}$$

$$\overline{p}_{3,5}(16n+7) \equiv 0 \pmod{2^4}, \tag{4.1.10}$$

$$\overline{p}_{3,5}(32n+2) \equiv 0 \pmod{2^4}, \tag{4.1.11}$$

$$\overline{p}_{3,4,5}(8n+7) \equiv 0 \pmod{2^2}, \tag{4.1.12}$$

$$\overline{p}_{3,4,5}(16n+15) \equiv 0 \pmod{2^3}, \tag{4.1.13}$$

$$\overline{p}_{3,4,5}(16n+1) \equiv 0 \pmod{2^4}, \tag{4.1.14}$$

$$\overline{p}_{3,4,5}(32n+r) \equiv 0 \pmod{2^4}, \quad \textit{where } r \in \{2, 14\}, \tag{4.1.15}$$

$$\overline{p}_{1,7}(8n+7) \equiv 0 \pmod{2^2}, \tag{4.1.16}$$

$$\overline{p}_{1,7}(16n+14) \equiv 0 \pmod{2^2}, \tag{4.1.17}$$

$$\overline{p}_{1,7}(32n+28) \equiv 0 \pmod{2^2}, \tag{4.1.18}$$

$$\overline{p}_{1,7}(16n+9) \equiv 0 \pmod{2^3}, \tag{4.1.19}$$

$$\overline{p}_{1,7}(32n+30) \equiv 0 \pmod{2^3}, \tag{4.1.20}$$

$$\overline{p}_{1,7}(16n+15) \equiv 0 \pmod{2^4}, \tag{4.1.21}$$

$$\overline{p}_{1,7}(32n+18) \equiv 0 \pmod{2^4}, \tag{4.1.22}$$

$$\overline{p}_{1,4,7}(8n+7) \equiv 0 \pmod{2^2}, \tag{4.1.23}$$

$$\overline{p}_{1,4,7}(16n+7) \equiv 0 \pmod{2^3}, \tag{4.1.24}$$

$$\overline{p}_{1,4,7}(16n+9) \equiv 0 \pmod{2^4}, \tag{4.1.25}$$

$$\overline{p}_{1,4,7}(32n+r) \equiv 0 \pmod{2^4}, \quad \textit{where } r \in \{18, 30\}. \tag{4.1.26}$$

The power of 2 in the modulus of each of the above congruences is sharp. Furthermore, we have checked numerically that there is no congruence modulo 2^5 for arguments of the form $An+B$ where $A = 4, 8, 16, 32$, or 64 and $B = 0, 1, \ldots, A-1$.

We state some internal congruences modulo 2^2 and 2^3 in the following theorem.

Theorem 4.1.5. *For all nonnegative integers n and j,*

$$\overline{p}_{3,5}(2^j n) \equiv \overline{p}_{3,5}(n) \pmod{2^2}, \tag{4.1.27}$$

$$\overline{p}_{1,7}(2^j n) \equiv \overline{p}_{1,7}(n) \pmod{2^2}. \tag{4.1.28}$$

For n being an integer not of the form $(4k+2)^2$, where k is any nonnegative integer, we have

$$\overline{p}_{3,5}(2n) \equiv \overline{p}_{3,5}(n) \pmod{2^3}, \tag{4.1.29}$$

$$\overline{p}_{1,7}(2n) \equiv \overline{p}_{1,7}(n) \pmod{2^3}. \tag{4.1.30}$$

For n not being the square of an odd integer and j being any nonnegative integer, we have

$$\overline{p}_{3,5}(2^j n) \equiv \overline{p}_{3,5}(n) \pmod{2^3}, \tag{4.1.31}$$

$$\overline{p}_{1,7}(2^j n) \equiv \overline{p}_{1,7}(n) \pmod{2^3}. \tag{4.1.32}$$

We prove Theorem 4.1.5 in Section 4.3. We derive some infinite families of congruences modulo 2^2 and 2^3 from the previous two theorems.

Corollary 4.1.6. *For all nonnegative integers n and j,*

$$\overline{p}_{3,5}\left(2^j(8n+7)\right) \equiv 0 \pmod{2^2}, \tag{4.1.33}$$

$$\overline{p}_{3,5}\left(2^j(32n+14)\right) \equiv 0 \pmod{2^3}, \tag{4.1.34}$$

$$\overline{p}_{1,7}\left(2^j(8n+7)\right) \equiv 0 \pmod{2^2}, \tag{4.1.35}$$

$$\overline{p}_{1,7}\left(2^j(32n+30)\right) \equiv 0 \pmod{2^3}. \tag{4.1.36}$$

For $2n$ not being a triangular number and j being any nonnegative integer,

$$\overline{p}_{3,5}\left(2^j(16n+1)\right) \equiv 0 \pmod{2^3}. \tag{4.1.37}$$

For $2n+1$ not being a triangular number and j being any nonnegative integer,

$$\overline{p}_{1,7}\left(2^j(16n+9)\right) \equiv 0 \pmod{2^3}. \tag{4.1.38}$$

Remark 4.1.7. *The congruences of* (4.1.6) *and* (4.1.7) *are special cases of* (4.1.33)*. Similarly* (4.1.17) *and* (4.1.18) *are special cases of* (4.1.35)*.*

Proof. The congruences of (4.1.5) and (4.1.27) immediately imply (4.1.33).

Similarly, (4.1.16) and (4.1.28) imply (4.1.35).

Since $32n+14$ is not the square of an odd number for any n, (4.1.9) and (4.1.29) imply (4.1.34).

Similarly, $32n+30$ is not the square of an odd number for any n. Hence (4.1.20) and (4.1.30) imply (4.1.36).

Now, for k being any nonnegative integer,

$$\begin{aligned} 16n+1 &\neq (2k+1)^2 \\ \implies 2n &\neq \frac{k(k+1)}{2}. \end{aligned}$$

Hence, (4.1.8) and (4.1.29) imply (4.1.37).

Similarly, for k being any nonnegative integer,

$$\begin{aligned} 16n+9 &\neq (2k+1)^2 \\ \implies 2n+1 &\neq \frac{k(k+1)}{2}. \end{aligned}$$

Hence, (4.1.19) and (4.1.30) imply (4.1.38). □

In the following theorem, we present the explicit generating functions implying congruences in (4.1.5)–(4.1.26).

Theorem 4.1.8. *We have*

$$\sum_{n=0}^{\infty} \overline{p}_{3,5}(8n+7)q^n = 4q\,\frac{\psi(-q^2)\psi(-q^4)f(-q^2,-q^{14})}{\varphi^2(-q)\varphi(-q^2)}, \tag{4.1.39}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \overline{p}_{3,5}(16n+14)q^n = 4\,\frac{\varphi^4(q)\psi(-q^2)}{\varphi^9(-q^2)}\Big(&2f(-q,q^3)\psi^2(q^2)\varphi(-q^8) \\ &+ qf(-q^2,-q^{14})\varphi^2(q)\psi(-q^4)\Big), \end{aligned} \tag{4.1.40}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \overline{p}_{3,5}(32n+28)q^n = 4\frac{\varphi^8(q)}{\varphi^{17}(-q^2)}\Bigg(&2f(-q,q^3)\varphi(-q^4)\psi(q^2) \\ &\times \Big(\varphi(q)\psi(-q^4)\left(2\varphi^4(q)-\varphi^4(-q)\right) \\ &+ \varphi(-q^8)\psi(q^4)\left(4\varphi^4(q)-\varphi^4(-q)\right)\Big) \\ &+ qf(-q^2,-q^{14})\varphi^2(q)\psi(-q^2)\Big(\psi(-q^4)\left(4\varphi^4(q)-3\varphi^4(-q)\right) \\ &+ 16\varphi(q)\varphi(q^2)\varphi(-q^8)\psi^2(q^2)\Big)\Bigg), \end{aligned} \tag{4.1.41}$$

$$\sum_{n=0}^{\infty}\overline{p}_{3,5}(16n+1)q^n = 8q\ \frac{\varphi^3(-q^4)\psi(q^4)f(-q,-q^7)}{\varphi^5(-q)}, \tag{4.1.42}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,5}(32n+14)q^n = 8\ &\frac{\psi(-q)}{\varphi^9(-q)}\Big(f(-q^3,-q^5)\psi^2(q)\varphi(-q^4)\left(2\varphi^4(q)-\varphi^4(-q)\right)\\ &+2qf(-q,-q^7)\psi^2(q^2)\Big(\psi(-q^2)\left(4\varphi^4(q)-\varphi^4(-q)\right)\\ &-4\varphi^2(q)\psi^2(q)\varphi(-q^4)\Big)\Big),\end{aligned} \tag{4.1.43}$$

$$\sum_{n=0}^{\infty}\overline{p}_{3,5}(16n+7)q^n = 16q\ \frac{\psi(-q)\psi(-q^2)\psi^2(q^2)f(-q,-q^7)}{\varphi^5(-q)}, \tag{4.1.44}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,5}(32n+2)q^n = 16q\ &\frac{\psi(q)\psi(-q)\psi(-q^2)}{\varphi^9(-q)}\Big(2f(-q,-q^7)\big(2\varphi^4(q)\varphi(q^2)\\ &+\varphi(-q)\varphi^4(-q^4)\big)-f(-q^3,-q^5)\varphi^4(-q)\psi(q^4)\Big),\end{aligned} \tag{4.1.45}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{p}_{3,4,5}(8n+7)q^n = 4\ &\frac{\psi(-q^2)\psi(-q^4)}{\varphi^3(-q)\varphi(-q^2)}\Big(\varphi(q^4)f(-q,q^3)\\ &+q\varphi(q^2)f(-q^2,-q^{14})\Big),\end{aligned} \tag{4.1.46}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{p}_{3,4,5}(16n+15)q^n = 8\ &\frac{\psi(-q)\psi(-q^2)\varphi(q^2)f(-q,-q^7)}{\varphi^7(-q)}\\ &\times\Big(2\varphi^3(q)+\varphi^2(-q)\varphi(q^2)\Big),\end{aligned} \tag{4.1.47}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(16n+1)q^n = 16q\ &\frac{\varphi(-q^2)\varphi(-q^4)\psi(q^4)}{\varphi^7(-q)}\Big(\varphi^3(q)f(-q,-q^7)\\ &+\varphi^2(-q)\psi(q^4)f(-q^3,-q^5)\Big),\end{aligned} \tag{4.1.48}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{p}_{3,4,5}(32n+2)q^n = 16q\ &\frac{\varphi(-q^2)\varphi(-q^4)\psi(q^4)}{\varphi^{13}(-q)}\\ &\times\Bigg(f(-q^3,-q^5)\psi(q^4)\varphi^2(-q)\Big(32\varphi^6(q)+\varphi^6(-q)\Big)\\ &+f(-q,-q^7)\Big(32\varphi^9(q)-24\varphi^5(q)\varphi^4(-q)-7\varphi^3(q)\varphi^6(-q)\\ &+2\varphi(q)\varphi^8(-q)+4\varphi^4(q)\varphi^4(-q)\varphi(q^2)\\ &-6\varphi^2(q)\varphi^6(-q)\varphi(q^2)-\varphi^8(-q)\varphi(q^2)\Bigg),\end{aligned} \tag{4.1.49}$$

$$\sum_{n=1}^{\infty}\overline{p}_{3,4,5}(32n+14)q^n = 16\frac{\psi(-q)\psi(-q^2)f(-q^3,-q^5)}{\varphi^{13}(-q)}\Bigg(32\varphi^{10}(q)-32\varphi^9(q)\varphi(q^2)$$
$$+24\varphi^8(q)\varphi^2(-q)-16\varphi^7(q)\varphi^2(-q)\varphi(q^2)$$
$$-22\varphi^6(q)\varphi^4(-q)+20\varphi^5(q)\varphi^4(-q)\varphi(q^2)$$
$$-13\varphi^4(q)\varphi^6(-q)+7\varphi^3(q)\varphi^6(-q)\varphi(q^2)+2\varphi^2(q)\varphi^8(-q)$$
$$-\varphi(q)\varphi^8(-q)\varphi(q^2)-2q\varphi^8(-q)\psi^2(q^4),\Bigg), \tag{4.1.50}$$

$$\sum_{n=0}^{\infty}\overline{p}_{1,7}(8n+7)q^n = 4\ \frac{\varphi^2(q)\psi(-q^2)\psi(-q^4)f(-q^6,-q^{10})}{\varphi^5(-q)}, \tag{4.1.51}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,7}(16n+14)q^n = 4\ \frac{\phi^4(q)\psi(-q^2)}{\varphi^9(-q^2)}\Bigg(2\psi^2(q^2)\varphi(-q^8)f(q,-q^3)$$
$$+\varphi^2(q)\psi(-q^4)f(-q^6,-q^{10})\Bigg), \tag{4.1.52}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,7}(32n+28)q^n = 4\ \frac{\varphi^8(q)}{\varphi^{17}(-q^2)}\Bigg(2f(q,-q^3)\varphi(-q^4)\psi(q^2)$$
$$\times\Big(\varphi(-q^8)\psi(q^4)\left(4\varphi^4(q)-\varphi^4(-q)\right)$$
$$+\varphi(q)\psi(-q^4)\left(2\varphi^4(q)-\varphi^4(-q)\right)\Big)$$
$$+f(-q^6,-q^{10})\Big(16\varphi(-q^4)\varphi(-q^8)\psi^6(q)$$
$$+\varphi^2(q)\psi(-q^2)\psi(-q^4)\left(4\varphi^4(q)-3\varphi^4(-q)\right)\Big)\Bigg), \tag{4.1.53}$$

$$\sum_{n=0}^{\infty}\overline{p}_{1,7}(16n+9)q^n = 8\ \frac{\varphi^3(-q^4)\psi(q^4)f(-q^3,-q^5)}{\varphi^5(-q)}, \tag{4.1.54}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,7}(32n+30)q^n = 8\ \frac{\psi(-q)f(-q,-q^7)}{\varphi^9(-q)}\Bigg(\psi^2(q)\varphi(-q^4)\left(2\varphi^4(q)-\varphi^4(-q)\right)$$
$$+\left(\varphi(q)+\varphi(q^2)\right)\Big(\varphi(q^2)\psi(-q^2)\left(4\varphi^4(q)-\varphi^4(-q)\right)$$
$$+4\varphi^3(q)\varphi(q^2)\psi(q^2)\varphi(-q^4)\Big)\Bigg), \tag{4.1.55}$$

$$\sum_{n=0}^{\infty}\overline{p}_{1,7}(16n+15)q^n = 16\ \frac{\psi(-q)\psi(-q^2)\psi^2(q^2)f(-q^3,-q^5)}{\varphi^5(-q)}, \tag{4.1.56}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,7}(32n+18)q^n = 16\ \frac{\varphi(-q^2)\varphi(-q^4)\psi(q^4)}{\varphi^9(-q)}\Bigg(f(-q^3,-q^5)\Big(\varphi^5(q) + 4\varphi^4(q)\varphi(q^2) - \varphi^4(-q)\varphi(q^2) - 2\varphi^3(q)\varphi^2(-q)\Big) - qf(-q,-q^7)\varphi^4(-q)\psi(q^4)\Bigg), \tag{4.1.57}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,4,7}(8n+7)q^n = 4\ \frac{\psi(-q^2)\psi(-q^4)}{\varphi^3(-q)\varphi(-q^2)}\big(\varphi(q^2)f(-q^6,-q^{10}) + \varphi(q^4)f(q,-q^3)\big), \tag{4.1.58}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,4,7}(16n+7)q^n = 8\ \frac{\psi(-q)\psi(-q^2)\varphi(q^2)f(-q^3,-q^5)}{\varphi^7(-q)} \times \big(2\varphi^3(q) - \varphi^2(-q)\varphi(q^2)\big), \tag{4.1.59}$$

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(16n+9)q^n = 16\ \frac{\varphi(-q^2)\psi(q^2)\psi(-q^2)}{\varphi^7(-q)}\Big(f(-q^3,-q^5)\varphi^3(q) + qf(-q,-q^7)\varphi^2(-q)\psi(q^4)\Big), \tag{4.1.60}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,4,7}(32n+18)q^n = 16\frac{\varphi(-q^2)\varphi(-q^4)f(-q,-q^7)}{\varphi^{13}(-q)}\Big(16\varphi^{10}(q) + 16\varphi^9(q)\varphi(q^2) + 4\varphi^8(q)\varphi^2(-q) - 17\varphi^6(q)\varphi^4(-q) - 14\varphi^5(q)\varphi^4(-q)\varphi(q^2) - 3\varphi^4(q)\varphi^6(-q) - 2\varphi^3(q)\varphi^6(-q)\varphi(q^2) + 3\varphi^2(q)\varphi^8(-q) + 3\varphi(q)\varphi^6(-q)\varphi^3(q^2) - q\varphi^8(-q)\psi^2(q^4),\Big), \tag{4.1.61}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,4,7}(32n+30)q^n = 16\frac{\psi(-q)\psi(-q^2)}{\varphi^{13}(-q)}\Bigg(f(-q^3,-q^5)\psi(q^4)\varphi^8(-q)\Big(\varphi(q) + \varphi(q^2)\Big) + f(-q,-q^7)\Big(32\varphi^{10}(q) + 24\varphi^8(q)\varphi^2(-q) - 22\varphi^6(q)\varphi^4(-q) - 13\varphi^4(q)\varphi^6(-q) + \varphi^2(q)\varphi^8(-q) + 32\varphi^9(q)\varphi(q^2) + 16\varphi^7(q)\varphi^2(-q)\varphi(q^2) - 20\varphi^5(q)\varphi^4(-q)\varphi(q^2) - 7\varphi^3(q)\varphi^6(-q)\varphi(q^2)\Big)\Bigg). \tag{4.1.62}$$

In Section 4.4, we prove Theorem 4.1.8. However, for brevity, we present only the proofs of (4.1.48) and (4.1.60). The other generating functions can be proven in a similar fashion.

The contents of this chapter have been submitted for publication [12].

We now end this section with a lemma containing some useful identities from [19, p. 40, p. 45, and p. 49].

Lemma 4.1.9. *We have*

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2), \tag{4.1.63}$$

$$\psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2), \tag{4.1.64}$$

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{4.1.65}$$

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4), \tag{4.1.66}$$

$$\psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}). \tag{4.1.67}$$

Furthermore, if $ab = cd$*, then*

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc), \tag{4.1.68}$$

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{4.1.69}$$

4.2 Proof of Theorem 4.1.2

Proof of Theorem 4.1.2. Setting $a = -q$ and $b = q^3$ in (1.2.1), we have

$$f(-q, q^3) = \sum_{k=-\infty}^{\infty} (-1)^{k(k+1)/2} q^{2k^2-k} = (q; -q^4)_\infty (-q^3; -q^4)_\infty (-q^4; -q^4)_\infty. \tag{4.2.1}$$

Now,

$$(q; -q^4)_\infty = \prod_{j=0}^{\infty} \left(1 - q(-q^4)^j\right) = \prod_{j=0}^{\infty} \left(1 - q^{8j+1}\right)\left(1 + q^{8j+5}\right) = (q; q^8)_\infty (-q^5; q^8)_\infty,$$

$$(-q^3; -q^4)_\infty = \prod_{j=0}^{\infty} \left(1 + q^3(-q^4)^j\right) = \prod_{j=0}^{\infty} \left(1 + q^{8j+3}\right)\left(1 - q^{8j+7}\right) = (-q^3; q^8)_\infty (q^7; q^8)_\infty,$$

$$(-q^4;-q^4)_\infty = \prod_{j=0}^{\infty}\left(1+q^4(-q^4)^j\right) = \prod_{j=0}^{\infty}\left(1+q^{8j+4}\right)\left(1-q^{8j+8}\right) = (-q^4;q^8)_\infty(q^8;q^8)_\infty.$$

Therefore, from (4.2.1), we find that

$$\begin{aligned}\sum_{k=-\infty}^{\infty}(-1)^{k(k+1)/2}q^{2k^2-k} &= (q;q^8)_\infty(-q^3;q^8)_\infty(-q^4;q^8)_\infty(-q^5;q^8)_\infty(q^7;q^8)_\infty(q^8;q^8)_\infty \\ &= \frac{(q;q^2)_\infty(-q^3;q^8)_\infty(-q^4;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} \\ &= \frac{(q;q^2)_\infty(q^4;q^4)_\infty(-q^3;q^8)_\infty(-q^4;q^8)_\infty(-q^5;q^8)_\infty}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty} \\ &= \frac{(q^2;q^2)_\infty(-q^3;q^8)_\infty(-q^4;q^8)_\infty(-q^5;q^8)_\infty}{(-q;q^2)_\infty(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty},\end{aligned}$$

where we have used Euler's identity $(-q;q)_\infty = 1/(q;q^2)_\infty$ in the last equality. With the aid of (4.1.3), the above may be recast as

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{k=-\infty}^{\infty}(-1)^{k(k+1)/2}q^{2k^2-k}. \tag{4.2.2}$$

Now, the generating function of pod(n), the number of partitions of n into non-repeating odd parts, is given by

$$\sum_{n=0}^{\infty}\text{pod}(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{4.2.3}$$

Thus, (4.2.2) reduces to

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(n)q^n = \left(\sum_{m=0}^{\infty}\text{pod}(m)q^m\right)\left(\sum_{k=-\infty}^{\infty}(-1)^{k(k+1)/2}q^{2k^2-k}\right).$$

Equating the coefficients of q^n from both sides of the above, we arrive at (4.1.1).

Again, setting $a=q$ and $b=-q^3$ in (1.2.1), and proceeding as above, we find that

$$\sum_{k=-\infty}^{\infty}(-1)^{k(k-1)/2}q^{2k^2-k} = \frac{(q^2;q^2)_\infty(-q;q^8)_\infty(-q^4;q^8)_\infty(-q^7;q^8)_\infty}{(-q;q^2)_\infty(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty},$$

which, by (4.1.4) and (4.2.3), can be rewritten as

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(n)q^n = \left(\sum_{m=0}^{\infty}\text{pod}(m)q^m\right)\left(\sum_{k=-\infty}^{\infty}(-1)^{k(k-1)/2}q^{2k^2-k}\right).$$

Equating the coefficients of q^n from both sides of the above, we arrive at (4.1.2) to finish the proof of Theorem 4.1.2.

4.3 Proof of Theorem 4.1.5

Proof of (4.1.27). The generating function (4.1.3) of $\overline{p}_{3,4,5}(n)$ may be rewritten as

$$\sum_{n=0}^{\infty}\overline{p}_{3,5}(n)q^n = \frac{f(q^3,q^5)}{f(-q^3,-q^5)} = \frac{f(-q,-q^7)f(q^3,q^5)}{f(-q,-q^7)f(-q^3,-q^5)}$$
$$= \frac{\psi(q)f(-q,-q^7)f(q^3,q^5)}{(q^2;q^2)_\infty(q^8;q^8)_\infty^2}. \tag{4.3.1}$$

Now, setting $a=q, b=q^7, c=-q^3, d=-q^5$ in (4.1.68) and (4.1.69), and then using (1.2.3), we find that

$$f(q,q^7)f(-q^3,-q^5)+f(-q,-q^7)f(q^3,q^5) = 2\psi(-q^4)f(-q^6,-q^{10}), \tag{4.3.2}$$
$$f(q,q^7)f(-q^3,-q^5)-f(-q,-q^7)f(q^3,q^5) = 2q\psi(-q^4)f(-q^2,-q^{14}). \tag{4.3.3}$$

Subtracting (4.3.3) from (4.3.2), we have

$$f(-q,-q^7)f(q^3,q^5) = \psi(-q^4)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right). \tag{4.3.4}$$

Invoking (4.3.4) and (4.1.67) in (4.3.1), and then manipulating the q-products, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{3,5}(n)q^n = \frac{\psi(q)\psi(-q^4)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)}{(q^2;q^2)_\infty(q^8;q^8)_\infty^2} \tag{4.3.5}$$
$$= \frac{(q^4;q^4)_\infty(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty(q^8;q^8)_\infty^3}\left(f(q^6,q^{10})+qf(q^2,q^{14})\right)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right).$$

Extracting the even terms from both sides of the above,

$$\sum_{n=0}^{\infty}\overline{p}_{3,5}(2n)q^n = \frac{(q^2;q^2)_\infty(q^8;q^8)_\infty}{(q;q)_\infty(q^4;q^4)_\infty^3}\left(f(q^3,q^5)f(-q^3,-q^5)-qf(q,q^7)f(-q,-q^7)\right) \tag{4.3.6}$$

Now, setting $a=q^3, b=q^5, c=-q^3, d=-q^5$ in (4.1.68), we have

$$f(q^3,q^5)f(-q^3,-q^5) = f(-q^6,-q^{10})f(-q^8,-q^8) = f(-q^6,-q^{10})\varphi(-q^8). \tag{4.3.7}$$

Setting $a=q, b=q^7, c=-q, d=-q^7$ in (4.1.68), we have

$$f(q,q^7)f(-q,-q^7) = f(-q^2,-q^{14})f(-q^8,-q^8) = f(-q^2,-q^{14})\varphi(-q^8). \tag{4.3.8}$$

Employing (1.2.2), (4.1.3), (4.3.7) and (4.3.8) in (4.3.6), and then manipulating the q-products, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,5}(2n)q^n &= \frac{(q^2;q^2)_\infty(q^8;q^8)_\infty}{(q;q)_\infty(q^4;q^4)_\infty^3}\varphi(-q^8)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)\\ &= \frac{(q^2;q^2)_\infty(q^8;q^8)_\infty^3}{(q;q)_\infty(q^4;q^4)_\infty^3(q^{16};q^{16})_\infty}\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right) \qquad (4.3.9)\\ &= \frac{(q^8;q^8)_\infty^6}{(q^4;q^4)_\infty^4(q^{16};q^{16})_\infty^2}\sum_{n=0}^{\infty}\overline{p}_{3,5}(n)q^n\\ &\equiv \sum_{n=0}^{\infty}\overline{p}_{3,5}(n)q^n \pmod 4.\end{aligned}$$

Hence, for any nonnegative integer n,

$$\overline{p}_{3,5}(2n)\equiv\overline{p}_{3,5}(n) \pmod{2^2}.$$

The result of (4.1.27) immediately follows from the above equation.

Proof of (4.1.28). In a similar manner as that of the above, we can show that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{1,7}(2n)q^n &= \frac{(q^8;q^8)_\infty^6}{(q^4;q^4)_\infty^4(q^{16};q^{16})_\infty^2}\sum_{n=0}^{\infty}\overline{p}_{1,7}(n)q^n\\ &\equiv \sum_{n=0}^{\infty}\overline{p}_{1,7}(n)q^n \pmod 4. \qquad (4.3.10)\end{aligned}$$

Hence, for any nonnegative integer n,

$$\overline{p}_{1,7}(2n)\equiv\overline{p}_{1,7}(n) \pmod{2^2}.$$

The result of (4.1.28) immediately follows from the above equation.

Proof of (4.1.29). Subtracting (4.3.5) from (4.3.9), and employing (1.2.3) and (4.1.67), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{p}_{3,5}(2n)-\overline{p}_{3,5}(n))q^n\\ &= \frac{\psi(q)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)}{(q^2;q^2)_\infty}\left(\frac{(q^8;q^8)_\infty^3}{(q^4;q^4)_\infty^3(q^{16};q^{16})_\infty}-\frac{(q^4;q^4)_\infty(q^{16};q^{16})_\infty}{(q^8;q^8)_\infty^3}\right)\end{aligned}$$

$$= \frac{\psi(q)f(-q,q^3)}{(q^2;q^2)_\infty}\left(\frac{(q^8;q^8)_\infty^3}{(q^4;q^4)_\infty^3(q^{16};q^{16})_\infty} - \frac{(q^4;q^4)_\infty(q^{16};q^{16})_\infty}{(q^8;q^8)_\infty^3}\right).$$

Using (1.2.2), (1.2.3) and (4.1.65), the above equation becomes

$$\begin{aligned}\sum_{n=0}^{\infty}(\overline{p}_{3,5}(2n)-\overline{p}_{3,5}(n))q^n &= \frac{(q^{16};q^{16})_\infty\psi(q)f(-q,q^3)}{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}\left(\varphi(q^4)-\varphi(-q^4)\right)\\ &= \frac{(q^{16};q^{16})_\infty\psi(q)f(-q,q^3)}{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}\times 4q^4\psi(q^{32})\\ &= 4q^4\frac{(q^2;q^2)_\infty(q^{16};q^{16})_\infty(q^{64};q^{64})_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2(q^{32};q^{32})_\infty}f(-q,q^3).\end{aligned} \tag{4.3.11}$$

Since $(q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod 2$, the above equation leads to the following congruence.

$$\sum_{n=0}^{\infty}(\overline{p}_{3,5}(2n)-\overline{p}_{3,5}(n))q^n \equiv 4q^4\frac{(q^{32};q^{32})_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}f(-q,q^3) \pmod 8. \tag{4.3.12}$$

Now, (1.2.1), (1.2.3) and standard q-product manipulations imply,

$$\begin{aligned}f(-q,q^3) &= \prod_{n=1}^{\infty}\left(1-(-1)^nq^{4n+1}\right)\left(1+(-1)^nq^{4n+3}\right)\left(1-(-1)^nq^{4n+4}\right)\\ &\equiv \prod_{n=1}^{\infty}\left(1+q^{4n+1}\right)\left(1+q^{4n+3}\right)\left(1-q^{4n+4}\right) \pmod 2\\ &\equiv f(q,q^3)\\ &\equiv \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}\\ &\equiv (q;q)_\infty(q^2;q^2)_\infty \pmod 2.\end{aligned} \tag{4.3.13}$$

Utilizing (4.3.13) in (4.3.12) and employing Jacobi's identity[37, Eq. (1.7.1)], we have

$$\begin{aligned}\sum_{n=0}^{\infty}(\overline{p}_{3,5}(2n)-\overline{p}_{3,5}(n))q^n &\equiv 4q^4(q^{32};q^{32})_\infty^3 \pmod 8\\ &\equiv 4q^4\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{16k(k+1)}\\ &\equiv 4\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{(4k+2)^2} \pmod 8.\end{aligned}$$

The above congruence immediately implies (4.1.29), which in turn implies the infinite family of congruences in (4.1.31).

Proof of (4.1.30). In a similar manner as that of the above proof, we have

$$\sum_{n=0}^{\infty}(\overline{p}_{1,7}(2n)-\overline{p}_{1,7}(n))q^n \equiv 4\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{(4k+2)^2} \pmod 8. \tag{4.3.14}$$

Clearly, (4.1.30) follows from the above congruence, thereby implying the the infinite family of congruences in (4.1.32).

4.4 Proof of Theorem 4.1.8

Proof of (4.1.48). With the help of (1.2.1), (1.2.2) and (1.2.3), the generating function (4.1.3) of $\overline{p}_{3,4,5}(n)$ may be rewritten as

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(n)q^n &= \frac{\varphi(-q^8)f(q^3,q^5)}{\varphi(-q^4)f(-q^3,-q^5)} = \frac{\varphi(-q^8)f(-q,-q^7)f(q^3,q^5)}{\varphi(-q^4)f(-q,-q^7)f(-q^3,-q^5)}\\ &= \frac{\varphi(-q^8)\psi(q)f(-q,-q^7)f(q^3,q^5)}{\varphi(-q^4)(q^2;q^2)_\infty(q^8;q^8)_\infty^2}.\end{aligned} \tag{4.4.1}$$

Invoking (4.3.4) and (4.1.67) in (4.4.1), and then manipulating the q-products, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(n)q^n\\ &= \frac{\varphi(-q^8)\psi(-q^4)}{\varphi(-q^4)(q^2;q^2)_\infty(q^8;q^8)_\infty^2}\left(f(q^6,q^{10})+qf(q^2,q^{14})\right)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)\\ &= \frac{\left(f(q^6,q^{10})+qf(q^2,q^{14})\right)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)}{(q^2;q^2)_\infty(q^4;q^4)_\infty}.\end{aligned} \tag{4.4.2}$$

Extracting the odd terms from both sides of the above, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(2n+1)q^n = \frac{f(q,q^7)f(-q^3,-q^5)-f(-q,-q^7)f(q^3,q^5)}{(q;q)_\infty(q^2;q^2)_\infty}, \tag{4.4.3}$$

which, by (1.2.3), (4.1.67) and (4.3.3) gives

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(2n+1)q^n = 2q\frac{\psi(q)\psi(-q^4)f(-q^2,-q^{14})}{(q^2;q^2)_\infty^3}$$

$$= 2q\frac{\psi(-q^4)f(-q^2,-q^{14})}{(q^2;q^2)_\infty^3}\left(f(q^6,q^{10})+qf(q^2,q^{14})\right). \tag{4.4.4}$$

Extracting the even terms from both sides of the above, we find that

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(4n+1)q^n = 2q\frac{\psi(-q^2)f(q,q^7)f(-q,-q^7)}{(q;q)_\infty^3}. \tag{4.4.5}$$

From (4.3.8) and (4.4.5), we have

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(4n+1)q^n = 2q\frac{\psi(-q^2)\varphi(-q^8)f(-q^2,-q^{14})}{(q;q)_\infty^3}, \tag{4.4.6}$$

which, with the aid of (1.2.2) and (1.2.3), may be rewritten as

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(4n+1)q^n = 2q\frac{\varphi(-q^8)f(-q^2,-q^{14})\varphi(q)\psi(q)}{\varphi^3(-q^2)\varphi(-q^4)}. \tag{4.4.7}$$

Employing (4.1.67) and (4.1.65) in (4.4.7), extracting the even terms from both sides of the resulting identity, and then using (4.1.63), (4.3.4), (4.3.8), and (4.1.65), we find the following 2-dissection of $\overline{p}_{3,4,5}(8n+1)$:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(8n+1)q^n \\
&= 2q\frac{\varphi(q^2)\varphi(-q^4)f(q,q^7)f(-q,-q^7)}{\varphi(-q^2)\varphi^3(-q)} + 4q\frac{\psi(q^4)\varphi(-q^4)f(-q,-q^7)f(q^3,q^5)}{\varphi(-q^2)\varphi^3(-q)} \\
&= 2q\frac{\varphi(q^2)\varphi(-q^4)f(q,q^7)f(-q,-q^7)\varphi^3(q)}{\varphi^7(-q^2)} + 4q\frac{\psi(q^4)\varphi(-q^4)f(-q,-q^7)f(q^3,q^5)\varphi^3(q)}{\varphi^7(-q^2)} \\
&= 2q\frac{\varphi(q^2)\varphi(-q^4)\varphi(-q^8)f(-q^2,-q^{14})}{\varphi^7(-q^2)}\left(\varphi(q^4)+2q\psi(q^8)\right)^3 + 4q\frac{\psi(q^4)\psi(-q^4)\varphi(-q^4)}{\varphi^7(-q^2)} \\
&\quad\times\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)\left(\varphi(q^4)+2q\psi(q^8)\right)^3.
\end{aligned} \tag{4.4.8}$$

Extracting the even terms from both sides of the above and then using (4.1.66), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(16n+1)q^n \\
&= 4q\frac{\varphi(q)\varphi(-q^2)\varphi(-q^4)\psi(q^4)f(-q,-q^7)}{\varphi^7(-q)}\left(3\varphi^2(q^2)+4q\psi^2(q^4)\right) \\
&\quad + 8q\frac{\psi(q^2)\psi(-q^2)\varphi(-q^2)\psi(q^4)f(-q^3,-q^5)}{\varphi^7(-q^2)}\left(3\varphi^2(q^2)+4q\psi^2(q^4)\right)
\end{aligned}$$

$$
\begin{aligned}
&- 4q\frac{\psi(q^2)\psi(-q^2)\varphi(q^2)\varphi(-q^2)f(-q,-q^7)}{\varphi^7(-q^2)}\left(\varphi^2(q^2)+12q\psi^2(q^4)\right)\\
&= 4q\frac{\varphi(q)\varphi(-q^2)\varphi(-q^4)\psi(q^4)f(-q,-q^7)}{\varphi^7(-q)}\left(\varphi^2(q)+2\varphi^2(q^2)\right)\\
&+ 8q\frac{\psi(q^2)\psi(-q^2)\varphi(-q^2)\psi(q^4)f(-q^3,-q^5)}{\varphi^7(-q^2)}\left(\varphi^2(q)+2\varphi^2(q^2)\right)\\
&- 4q\frac{\psi(q^2)\psi(-q^2)\varphi(q^2)\varphi(-q^2)f(-q,-q^7)}{\varphi^7(-q^2)}\left(3\varphi^2(q)-2\varphi^2(q^2)\right). \qquad (4.4.9)
\end{aligned}
$$

To simplify the above identity further, we now derive an expression for $f(-q^3,-q^5)$ in terms of $f(-q,-q^7)$. To that end, we set $a=q, b=q^3, c=d=-q^2$ in (4.1.68) and (4.1.69) to arrive at

$$
\begin{aligned}
\psi(q)\varphi(-q^2)+\psi(-q)\varphi(q^2) &= 2f^2(-q^3,-q^5),\\
\psi(q)\varphi(-q^2)-\psi(-q)\varphi(q^2) &= 2qf^2(-q,-q^7).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
f^2(-q^3,-q^5)+qf^2(-q,-q^7) &= \psi(q)\varphi(-q^2),\\
f^2(-q^3,-q^5)-qf^2(-q,-q^7) &= \psi(-q)\varphi(q^2).
\end{aligned}
$$

Dividing both sides of each of the above identities by $f(-q^3,-q^5)f(-q,-q^7)$, that is, by $(q;q^2)_\infty(q^8;q^8)^2_\infty$, and then using (1.2.2) and (1.2.3), we obtain

$$
\frac{f(-q^3,-q^5)}{f(-q,-q^7)}+q\frac{f(-q,-q^7)}{f(-q^3,-q^5)}=\frac{\psi(q)\varphi(-q^2)}{(q;q^2)_\infty(q^8;q^8)^2_\infty}=\frac{\varphi(q)}{\psi(q^4)} \qquad (4.4.10)
$$

and

$$
\frac{f(-q^3,-q^5)}{f(-q,-q^7)}-q\frac{f(-q,-q^7)}{f(-q^3,-q^5)}=\frac{\psi(-q)\varphi(q^2)}{(q;q^2)_\infty(q^8;q^8)^2_\infty}=\frac{\varphi(q^2)}{\psi(q^4)}. \qquad (4.4.11)
$$

Adding the above two identities, we find that

$$
2\psi(q^4)f(-q^3,-q^5)=\left(\phi(q)+\phi(q^2)\right)f(-q,-q^7). \qquad (4.4.12)
$$

Employing (4.4.12) in (4.4.9), we have

$$
\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(16n+1)q^n
$$

$$= 4q\frac{\varphi(q)\varphi(-q^2)\varphi(-q^4)\psi(q^4)f(-q,-q^7)}{\varphi^7(-q)}\left(\varphi^2(q)+2\varphi^2(q^2)\right)$$
$$+ 4q\frac{\psi(q^2)\psi(-q^2)\varphi(-q^2)f(-q,-q^7)}{\varphi^7(-q^2)}\left(\varphi(q)+\varphi(q^2)\right)\left(\varphi^2(q)+2\varphi^2(q^2)\right)$$
$$- 4q\frac{\psi(q^2)\psi(-q^2)\varphi(q^2)\varphi(-q^2)f(-q,-q^7)}{\varphi^7(-q^2)}\left(3\varphi^2(q)-2\varphi^2(q^2)\right).$$

With the aid of (4.1.64) and (4.1.66), the above further reduces to

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(16n+1)q^n$$
$$= 8q\frac{\varphi(-q^2)\varphi(-q^4)\psi(q^4)f(-q,-q^7)}{\varphi^7(-q)}\left(2\varphi^3(q)+\varphi^2(-q)\left(\varphi(q)+\varphi(q^2)\right)\right). \quad (4.4.13)$$

Employing the expression of $\varphi(q)+\varphi(q^2)$ from (4.4.12) in (4.4.13), we have

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(16n+1)q^n$$
$$= 16q\frac{\varphi(-q^2)\varphi(-q^4)\psi(q^4)}{\varphi^7(-q)}\left(\varphi^3(q)f(-q,-q^7)+\varphi^2(-q)\psi(q^4)f(-q^3,-q^5)\right),$$

which is (4.1.48).

Proof of (4.1.60). Applying (1.2.1), (1.2.2) and (1.2.3) in (4.1.4), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(n)q^n = \frac{\varphi(-q^8)f(q,q^7)}{\varphi(-q^4)f(-q,-q^7)} = \frac{\varphi(-q^8)f(q,q^7)f(-q^3,-q^5)}{\varphi(-q^4)(q;q^2)_\infty(q^8;q^8)^2_\infty}$$
$$= \frac{\varphi(-q^8)\psi(q)f(q,q^7)f(-q^3,-q^5)}{\varphi(-q^4)(q^2;q^2)_\infty(q^8;q^8)^2_\infty}. \quad (4.4.14)$$

Using (4.3.4), with q replaced by $-q$, and (4.1.67) in (4.4.14), we have

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(n)q^n = \frac{\varphi(-q^8)\psi(-q^4)}{\varphi(-q^4)(q^2;q^2)_\infty(q^8;q^8)^2_\infty}\left(f(q^6,q^{10})+qf(q^2,q^{14})\right)$$
$$\times\left(f(-q^6,-q^{10})+qf(-q^2,-q^{14})\right).$$

Extracting the odd terms and then using (4.3.2) and (4.1.67), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(2n+1)q^n$$
$$= \frac{\varphi(-q^4)\psi(-q^2)}{\varphi(-q^2)(q;q)_\infty(q^4;q^4)^2_\infty}\left(f(q,q^7)f(-q^3,-q^5)+f(-q,-q^7)f(q^3,q^5)\right),$$

$$= 2\frac{\psi(-q^2)\psi(-q^4)\varphi(-q^4)f(-q^6,-q^{10})}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^2\varphi(-q^2)}\left(f(q^6,q^{10})+qf(q^2,q^{14})\right). \tag{4.4.15}$$

Extracting the even terms, we find that

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(4n+1)q^n = 2\frac{\psi(-q)\psi(-q^2)\varphi(-q^2)f(q^3,q^5)f(-q^3,-q^5)}{(q;q)_\infty^2(q^2;q^2)_\infty^2\varphi(-q)}. \tag{4.4.16}$$

Employing (1.2.2), (1.2.3), (4.1.63) and (4.3.7) in (4.4.16), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(4n+1)q^n = 2\frac{\varphi(q)\psi(q)\psi(-q^2)\varphi(-q^8)f(-q^6,-q^{10})}{\varphi^2(-q^2)(q^2;q^2)_\infty^3},$$

from which, with the aid of (4.1.65) and (4.1.67), we have the following 2-dissection of the generating function of $\overline{p}_{1,4,7}(4n+1)$:

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(4n+1)q^n$$
$$= 2\frac{\psi(-q^2)\varphi(-q^8)f(-q^6,-q^{10})}{\varphi^2(-q^2)(q^2;q^2)_\infty^3}\left(\varphi(q^4)+2q\psi(q^8)\right)\left(f(q^6,q^{10})+qf(q^2,q^{14})\right). \tag{4.4.17}$$

Extracting the even terms and then using (4.3.7), (4.3.4), and (4.1.63), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(8n+1)q^n &= 2\frac{\psi(-q)\varphi(-q^4)f(-q^3,-q^5)}{\varphi^2(-q)(q;q)_\infty^3}\left(f(q^3,q^5)\varphi(q^2)+2qf(q,q^7)\psi(q^4)\right)\\ &= 2\frac{\varphi(-q^4)\varphi^3(q)}{\varphi^7(-q^2)}\Bigg(\varphi(q^2)\varphi(-q^8)f(-q^6,-q^{10})\\ &\qquad + 2q\psi(q^4)\psi(-q^4)\left(f(-q^6,-q^{10})+qf(-q^2,-q^{14})\right)\Bigg).\end{aligned}$$

Employing (4.1.65) in the above, we arrive at the following 2-dissection of the generating function of $\overline{p}_{1,4,7}(8n+1)$:

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(8n+1)q^n &= 2\frac{\varphi(-q^4)}{\varphi^7(-q^2)}\left(\varphi(q^4)+2q\psi(q^8)\right)^3\Bigg(\varphi(q^2)\varphi(-q^8)f(-q^6,-q^{10})\\ &\qquad + 2q^2\psi(q^4)\psi(-q^4)f(-q^2,-q^{14})+2q\psi(q^4)\psi(-q^4)f(-q^6,-q^{10})\Bigg)\\ &= 2\frac{\varphi(-q^4)}{\varphi^7(-q^2)}\left(A(q^2)+2qB(q^2)\right)\left(C(q^2)+2qD(q^2)\right), \end{aligned} \tag{4.4.18}$$

where

$$\begin{aligned}
A(q) &= \varphi^3(q^2) + 12q\varphi(q^2)\psi^2(q^4),\\
B(q) &= \psi(q^4)\left(3\varphi^2(q^2) + 4q\psi^2(q^4)\right),\\
C(q) &= \varphi(q)\varphi(-q^4)f(-q^3,-q^5) + 2q\psi(q^2)\psi(-q^2)f(-q,-q^7),\\
D(q) &= \psi(q^2)\psi(-q^2)f(-q^3,-q^5).
\end{aligned}$$

Extracting the odd terms from both sides of (4.4.18), we find that

$$\sum_{n=0}^{\infty} \overline{p}_{1,4,7}(16n+9)q^n = 4\frac{\varphi(-q^2)}{\varphi^7(-q)}\left(A(q)D(q) + B(q)C(q)\right). \tag{4.4.19}$$

Now, we simplify $B(q)C(q)$. Subtracting (4.4.11) from (4.4.10), we have

$$2q\psi(q^4)f(-q,-q^7) = \left(\varphi(q) - \varphi(q^2)\right)f(-q^3,-q^5). \tag{4.4.20}$$

Using (4.1.64) in the expression for $C(q)$ and then using (4.4.20), we have

$$\begin{aligned}
B(q)C(q) &= \left(\psi(q^4)\left(3\varphi^2(q^2) + 4q\psi^2(q^4)\right)\right)\\
&\quad\times\left(\varphi(q)\varphi(-q^4)f(-q^3,-q^5) + 2q\psi(q^4)\varphi(-q^4)f(-q,-q^7)\right)\\
&= \psi(q^4)\varphi(-q^4)f(-q^3,-q^5)\left(3\varphi^2(q^2) + 4q\psi^2(q^4)\right)\left(2\varphi(q) - \varphi(q^2)\right),
\end{aligned}$$

which, by an application of (4.1.64) again, gives

$$B(q)C(q) = \psi(q^2)\psi(-q^2)f(-q^3,-q^5)\left(3\varphi^2(q^2) + 4q\psi^2(q^4)\right)\left(2\varphi(q) - \varphi(q^2)\right). \tag{4.4.21}$$

On the other hand,

$$A(q)D(q) = \psi(q^2)\psi(-q^2)f(-q^3,-q^5)\left(\varphi^3(q^2) + 12q\varphi(q^2)\psi^2(q^4)\right). \tag{4.4.22}$$

Using (4.4.21) and (4.4.22) in (4.4.19), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{p}_{1,4,7}(16n+9)q^n = 8\frac{\varphi(-q^2)\psi(q^2)\psi(-q^2)f(-q^3,-q^5)}{\varphi^7(-q)}\Bigg(&\varphi^2(q^2)\left(\varphi(q) - \varphi(q^2)\right)\\
&+ 2\varphi(q)\varphi^2(q^2) + 4q\varphi(q)\psi^2(q^4) + 4q\varphi(q^2)\psi^2(q^4)\Bigg).
\end{aligned}$$

Employing (4.4.20) in the above, we have

$$\sum_{n=0}^{\infty} \overline{p}_{1,4,7}(16n+9)q^n = 16\frac{\varphi(-q^2)\psi(q^2)\psi(-q^2)f(-q^3,-q^5)}{\varphi^7(-q)}\Bigg(q\psi(q^4)\varphi^2(q^2)\frac{f(-q,-q^7)}{f(-q^3,-q^5)} + \varphi(q)\varphi^2(q^2) + 2q\varphi(q)\psi^2(q^4) + 2q\varphi(q^2)\psi^2(q^4)\Bigg).$$

Employing (4.1.66) and (4.4.20) again in the above, we arrive at (4.1.60).