Chapter 5

Arithmetic identities for 10-cores and self-conjugate 10-cores

5.1 Introduction

From (1.11.1) and (1.11.2), the generating functions for $c_{10}(n)$ and $sc_{10}(n)$ are given by

$$\sum_{n=0}^{\infty} c_{10}(n)q^n = \frac{f_{10}^{10}}{f_1}, \tag{5.1.1}$$

and

$$\sum_{n=0}^{\infty} sc_{10}(n)q^n = (-q;q^2)_\infty f_{20}^5. \tag{5.1.2}$$

In this chapter, we find several arithmetic identities for $c_{10}(n)$ and $sc_{10}(n)$ by finding some exact generating function representations. A few such representations involving $c_{10}(n)$ are presented in the following theorem.

Theorem 5.1.1. *Set*

$$A := \frac{f_5^{15}}{f_{10}^6},\ B := \frac{f_1 f_5^{10}}{f_2 f_{10}},\ C := f_2^2 f_5^7,\ D := \frac{f_2^4 f_{10}^6}{f_5},\ E := \frac{f_1^2 f_{10}^{10}}{f_5^3},$$
$$F := \frac{f_2 f_{10}^{13}}{f_1 f_5^4},\ G := \frac{f_2^2 f_5^{16}}{f_1 f_{10}^8},\ H := \frac{f_2 f_5^{11}}{f_{10}^3},\ I := f_1 f_5^6 f_{10}^2,\ J := \frac{f_1^2 f_5 f_{10}^7}{f_2},$$
$$K := \frac{f_{10}^{10}}{f_1},\ \textit{and}\ L := \frac{f_{10}^{15}}{f_2 f_5^5}.$$

We have

$$\sum_{n=0}^{\infty} c_{10}(5n+4)q^n = 5\frac{f_2^{10}f_5^5}{f_1^6} = 5\left(C + 6qD + 19q^2E + 100q^3F\right), \tag{5.1.3}$$

$$\sum_{n=0}^{\infty} c_{10}(5^2n+24)q^n = 5^2\left(16H + 144qI + 735q^2J + 3107q^3K + 72q^4L\right), \tag{5.1.4}$$

$$\sum_{n=0}^{\infty} c_{10}(5^3n+74)q^n = 5^3\Bigg(115A + 115qB + 3053qC + 17301q^2D + 59962q^3E + 312500q^4F\Bigg), \tag{5.1.5}$$

$$\sum_{n=0}^{\infty} c_{10}(5^4n+74)q^n = 5^4\Bigg(23G + 49552qH + 452928q^2I + 2294752q^3J + 9715353q^4K + 201088q^5L\Bigg), \tag{5.1.6}$$

$$\sum_{n=0}^{\infty} c_{10}(5^5n+1949)q^n = 5^5\Bigg(358459A + 355584qB + 9553273qC + 54077814q^2D + 187362811q^3E + 976562500q^4F\Bigg), \tag{5.1.7}$$

$$\sum_{n=0}^{\infty} c_{10}(5^6n+1949)q^n = 5^5\Bigg(358459G + 774266656qH + 7077040464q^2I + 35855384771q^3J + 151802153879q^4K + 3142946984q^5L\Bigg), \tag{5.1.8}$$

$$\sum_{n=0}^{\infty} c_{10}(5^7n+48824)q^n = 5^6\Bigg(5600887222A + 5556079847qB + 149269685209qC + 844965400737q^2D + 2927544257138q^3E + 15258789062500q^4F\Bigg), \tag{5.1.9}$$

$$\sum_{n=0}^{\infty} c_{10}(5^8n+48824)q^n = 5^6\Bigg(5600887222G + 12097917004448qH + 110578752736512q^2I + 560240404818368q^3J$$

$$+ 2371908645544457q^4K + 49108581884672q^5L\Bigg).$$

(5.1.10)

$$\sum_{n=0}^{\infty} c_{10}(5^9n + 1220699)q^n = 5^7(87513867192926A + 86813756290176qB + 2332338806145097qC + 13202584349933046q^2D + 45742879050457579q^3E + 238418579101562500q^4F). \quad (5.1.11)$$

$$\sum_{n=0}^{\infty} c_{10}(5^{10}n + 1220699)q^n = 5^7(87513867192926G + 189029953126163184qH + 1727793011729516496q^2I + 8753756325060050419q^3J + 37061072587744123431q^4K + 76732158750006877 6q^5L). \quad (5.1.12)$$

$$\sum_{n=0}^{\infty} c_{10}(5^{11}n + 30517574)q^n = 5^8(1367404174814312733A + 1356464941415196983qB + 36442793847894877801qC + 206290380470101790193q^2D + 714732485160934381682q^3E + 3725290298461914062500q^4F). \quad (5.1.13)$$

$$\sum_{n=0}^{\infty} c_{10}(5^{12}n + 30517574)q^n = 5^8(1367404174814312733G + 2953593017597470996272qH + 26996765808282456207168q^2I + 136777442579021219701152q^3J + 579079259183482723825273q^4K + 1198939980476539376140 8q^5L). \quad (5.1.14)$$

The following congruences are immediate from (5.1.3)–(5.1.14).

Corollary 5.1.2. *For any nonnegative integer n, we have*

$$c_{10}(5n+4) \equiv 0 \pmod{5},$$

$$c_{10}(5^2n+24) \equiv 0 \pmod{5^2},$$

$$c_{10}(5^3n+74) \equiv 0 \pmod{5^3},$$

$$c_{10}(5^4n+74) \equiv 0 \pmod{5^4},$$

$$c_{10}(5^5n+1949) \equiv 0 \pmod{5^5},$$

$$c_{10}(5^7n+48824) \equiv 0 \pmod{5^6},$$

$$c_{10}(5^9n+1220699) \equiv 0 \pmod{5^7},$$

$$c_{10}(5^{11}n+30517574) \equiv 0 \pmod{5^8}.$$

We note that, there is a fairly regular pattern occurring in the congruences above, coming from the generating functions (5.1.5)–(5.1.14). This leads us to the following conjecture.

Conjecture 5.1.1. *For each integer $j \geq 5$,*

$$c_{10}\left(5^{2j-5}n + \frac{5^{2j-4}-33}{8}\right) \equiv 0 \pmod{5^j}. \tag{5.1.15}$$

We also note that, the above family of congruences for $c_{10}(n)$ modulo arbitrary powers of 5 is not related to the well-known congruences for $c_5(n)$, the 5-core partition function. See [34, Corollary 1(1)].

We find that (5.1.3)–(5.1.10) can also be employed to find linear recurrence relations for $c_{10}(n)$. We state the results in the following theorem.

Theorem 5.1.3. *For any nonnegative integer n, we have*

$$23\ c_{10}(5^7n+80074) - 5 \times 358459\ c_{10}(5^5n+3199) - 5^4 \times 572159\ c_{10}(5^3n+124)$$
$$- 5^{11} \times 341\ c_{10}(5n+1) = 0, \tag{5.1.16}$$

$$23\ c_{10}(5^7n+95699) - 5\times 358459\ c_{10}(5^5n+3824) - 5^4\times 572159\ c_{10}(5^3n+149)$$

$$-\,5^{11}\times 341\ c_{10}(5n+2) = 0, \tag{5.1.17}$$

$$170333\ c_{10}(5^9n+2783199) - 5\times 2658646014\ c_{10}(5^7n+111324)$$

$$-\,5^6\times 69610828\ c_{10}(5^5n+4449) - 5^7\times 8879785986\ c_{10}(5^3n+49)$$

$$+\,5^{10}\times 17544442967\ c_{10}(5n+3) = 0, \tag{5.1.18}$$

$$170333\ c_{10}(5^9n+1611324) - 5\times 2658646014\ c_{10}(5^7n+64449)$$

$$-\,5^6\times 69610828\ c_{10}(5^5n+2574) - 5^7\times 8879785986\ c_{10}(5^3n+99)$$

$$+\,5^{10}\times 17544442967\ c_{10}(5n) = 0. \tag{5.1.19}$$

In the next theorem we present some exact generating function representations related to $sc_{10}(n)$.

Theorem 5.1.4. *We have*

$$\sum_{n=0}^{\infty} sc_{10}(5n+4)q^n = \frac{f_2^2 f_4^3 f_5 f_{20}}{f_1^2}, \tag{5.1.20}$$

$$\sum_{n=0}^{\infty} sc_{10}(5^2n+24)q^n = 5f_1^2 f_5 f_{10}^2 + 25q\frac{f_2 f_{10}^5}{f_1} + \frac{f_1^2 f_4^3 f_5 f_{20}}{f_2 f_{10}} + 5q\frac{f_4^3 f_{10}^2 f_{20}}{f_1}, \tag{5.1.21}$$

$$\sum_{n=0}^{\infty} sc_{10}(5^3n+74)q^n = 5\left(5f_1 f_2^2 f_5^2 + 31q\frac{f_5^4 f_{10}^2}{f_1} + 32q^2\frac{f_{10}^7}{f_2 f_5} + q^4\frac{f_{10}^2 f_{20}^5}{f_4 f_5}\right), \tag{5.1.22}$$

$$\sum_{n=0}^{\infty} sc_{10}(5^4n+74)q^n = 5^2\left(\frac{f_2^4 f_5^3}{f_{10}^2} + 31qf_1^2 f_5 f_{10}^2 + 156q^2\frac{f_2 f_{10}^5}{f_1} + q^4\frac{f_2^2 f_{20}^5}{f_1 f_4}\right), \tag{5.1.23}$$

$$\sum_{n=0}^{\infty} sc_{10}(5^5n+1949)q^n = 5^4\left(5f_1 f_2^2 f_5^2 + 31q\frac{f_5^4 f_{10}^2}{f_1} + 32q^2\frac{f_{10}^7}{f_2 f_5}\right) + 5^2q\frac{f_2^2 f_4^3 f_5 f_{20}}{f_1^2}. \tag{5.1.24}$$

The following congruences readily follow from the above theorem.

Corollary 5.1.5. *For any nonnegative integer n, we have*

$$sc_{10}(5^3n+74) \equiv 0 \pmod 5$$

and

$$sc_{10}(5^4n+74) \equiv 0 \pmod{5^2}.$$

We also find the following linear recurrence relation for $sc_{10}(n)$.

Theorem 5.1.6. *For any nonnegative integer n, we have*

$$sc_{10}(5^6n+64449) - 5^3 sc_{10}(5^4n+2574) - 5^2 sc_{10}(5^2n+99) + 5^5 sc_{10}(n) = 0. \tag{5.1.25}$$

The material within this chapter has been submitted for publication [13].

We organize the rest of the chapter in the following way. We state and prove some preliminaries and useful lemmas in the next section. In Sections 5.3–5.6, we prove Theorems 5.1.1, 5.1.3, 5.1.4, and 5.1.6, respectively.

5.2 Preliminaries and useful lemmas

Recall the definitions of $\varphi(q)$ and $\psi(q)$ and their 5-dissections from (1.2.2), (1.2.3), (1.8.3) and (1.8.4) respectively. In the following, we state some additional lemmas which will be used in the subsequent sections.

Lemma 5.2.1. [19, page 258, Entry 9 and page 262, Entry 10] *We have*

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7) = 4q(-q;q^2)_\infty f_5 f_{20}. \tag{5.2.1}$$

We present some useful identities involving f_k's in the following lemma.

Lemma 5.2.2. *We have*

$$\frac{f_2^2}{f_1^4} = \frac{f_{10}^2}{f_5^4} + 4q\frac{f_2 f_{10}^5}{f_1^3 f_5^5}, \tag{5.2.2}$$

$$\frac{f_2^3}{f_1} = \frac{f_5^3}{f_{10}} + q\frac{f_1 f_{10}^4}{f_2 f_5^2}, \tag{5.2.3}$$
$$\frac{f_2^5}{f_1^5} = \frac{f_{10}}{f_5} + 5q\frac{f_2 f_{10}^4}{f_1^3 f_5^2}, \tag{5.2.4}$$
$$\frac{f_2^{10}}{f_1^4 f_4^4} = \frac{f_{10}^{10}}{f_5^4 f_{20}^4} + 4q\frac{f_2^2 f_5 f_{20}}{f_1 f_4}, \tag{5.2.5}$$
$$\frac{f_1 f_4}{f_2} = \frac{f_{10}^8}{f_2 f_5^3 f_{20}^3} - q\frac{f_2 f_5^2 f_{20}^2}{f_1 f_4 f_{10}^2}, \tag{5.2.6}$$
$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_1^2 f_{10}^6}{f_2 f_5^4 f_{20}^2} + 4q\frac{f_4 f_{10} f_{20}}{f_5^2}, \tag{5.2.7}$$
$$\frac{f_2 f_5^3}{f_{10}^2} = \frac{f_4^2 f_{10}^3}{f_1 f_{20}^2} - q\frac{f_2^6 f_5^2 f_{20}^2}{f_1^3 f_4^2 f_{10}^3}. \tag{5.2.8}$$

Proof. The identities (5.2.2)–(5.2.4) are simply Equations (2.6), (2.7), (2.29) in [15]. Replacing q by $-q$ in (5.2.2) and (5.2.3), we easily arrive at (5.2.5) and (5.2.6), respectively. By (1.2.2) and (1.2.3), the identity (5.2.7) is equivalent to the identity [19, page 278]

$$\varphi(q)\varphi(-q^5) - \varphi(-q)\varphi(q^5) = 4q f_4 f_{20},$$

whereas the identity (5.2.8) is equivalent to the identity [17, page 509]

$$\varphi(q^5)\psi(q^2) - q\varphi(q)\psi(q^{10}) = f_1 f_5.$$

□

We also recall the following useful identities.

Lemma 5.2.3. [15, Lemma 1.3] *If $R(q)$ is as defined in the previous lemma, then*

$$R(q)R^2(q^2) - \frac{q^2}{R(q)R^2(q^2)} = \frac{f_2 f_5^5}{f_1 f_{10}^5},$$
$$\frac{R^3(q^2)}{R(q)} + q^2\frac{R(q)}{R^3(q^2)} = \frac{f_2 f_5^5}{f_1 f_{10}^5} + 4q^2\frac{f_1 f_{10}^5}{f_2 f_5^5} - 2q.$$

Next, suppose that $[q^{5n+r}]\{F(q)\}$ for $r \in \{0,1,2,3,4\}$ denotes the terms after extraction of the terms involving q^{5n+r} of $F(q)$, dividing by q^r and then replacing q^5 by q. We have stated some useful extraction formulas in the following lemma.

Lemma 5.2.4. *Let $A, B, C, \ldots, K$ be as defined in Theorem 5.1.1. We have*

$$\left[q^{5n}\right]\{A\} = \frac{f_1^{15}}{f_2^6} = G - 16qH + 96q^2I - 256q^3J + 256q^4K - 1024q^5L, \tag{5.2.9}$$

$$\left[q^{5n}\right]\{qB\} = q\frac{f_1^{12}f_{10}^3}{f_2^5f_5} = qH - 12q^2I + 48q^3J - 64q^4K + 256q^5L, \tag{5.2.10}$$

$$\left[q^{5n}\right]\{qC\} = -qf_1^7f_{10}^2 = -qH + 7q^2I - 8q^3J - 16q^4K + 64q^5L, \tag{5.2.11}$$

$$\left[q^{5n}\right]\left\{q^2D\right\} = -5q^2\frac{f_2^6f_{10}^4}{f_1} = -5q^2I - 10q^3J - 5q^4K + 20q^5L, \tag{5.2.12}$$

$$\left[q^{5n}\right]\left\{q^3E\right\} = -q\frac{f_2^{10}f_5^2}{f_1^3} = -qH - 3q^2I - 3q^3J - q^4K + 4q^5L, \tag{5.2.13}$$

$$\left[q^{5n}\right]\left\{q^4F\right\} = q\frac{f_2^{15}f_5^3}{f_1^8f_{10}} = qH + 8q^2I + 38q^3J + 156q^4K + q^5L, \tag{5.2.14}$$

$$\left[q^{5n+3}\right]\{G\} = \frac{f_1^{16}f_{10}^2}{f_2^8f_5} = A - 624qB + 608qC - 512q^2D + 768q^3E, \tag{5.2.15}$$

$$\left[q^{5n+3}\right]\{qH\} = -\frac{f_1^{11}f_{10}}{f_2^3} = -A - qB + 12qC - 48q^2D + 112q^3E, \tag{5.2.16}$$

$$\left[q^{5n+3}\right]\left\{q^2I\right\} = -f_1^6f_2^2f_5 = -A - qB + 7qC - 8q^2D - 8q^3E, \tag{5.2.17}$$

$$\left[q^{5n+3}\right]\left\{q^3J\right\} = \frac{f_1f_2^7f_5^2}{f_{10}} = A + qB - 2qC - 7q^2D + 3q^3E, \tag{5.2.18}$$

$$\left[q^{5n+3}\right]\left\{q^4K\right\} = 5q\frac{f_2^{10}f_5^5}{f_1^6} = 5qC + 30q^2D + 95q^3E + 500q^4F, \tag{5.2.19}$$

$$\left[q^{5n+3}\right]\left\{q^5L\right\} = 5q^2\frac{f_2^{15}f_{10}^5}{f_1^5f_2^6} = 5q^2D + 25q^3E + 125q^4F. \tag{5.2.20}$$

Proof. We present the proof of only (5.2.10). The remaining identities can be accomplished in a similar fashion.

Employing (1.8.1) and (1.8.2), and then Lemma 5.2.3, we have

$$\begin{aligned}\left[q^{5n}\right]\{qB\} &= \left[q^{5n}\right]\left\{q\frac{f_1f_5^{10}}{f_2f_{10}}\right\}\\ &= \left[q^{5n}\right]\left\{q\frac{f_5^{10}f_{25}f_{50}^5}{f_{10}^7}\left(R(q^5) - q - \frac{q^2}{R(q^5)}\right)\left(R^4(q^{10}) + q^2R^3(q^{10})\right.\right.\\ &\quad \left.\left. + 2q^4R^2(q^{10}) + 3q^6R(q^{10}) + 5q^8 - \frac{3q^{10}}{R(q^{10})} + \frac{2q^{12}}{R^2(q^{10})} - \frac{q^{14}}{R^3(q^{10})}\right.\right.\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{q^{16}}{R^4(q^{10})}\Bigg)\Bigg\}\\
&= q\frac{f_1^{10}f_5f_{10}^5}{f_2^7}\left(2\left(R(q)R^2(q^2)-\frac{q^2}{R(q)R^2(q^2)}\right)\right.\\
&\quad \left.-\left(\frac{R^3(q^2)}{R(q)}+q^2\frac{R(q)}{R^3(q^2)}\right)-5q\right)\\
&= q\frac{f_1^{10}f_5f_{10}^5}{f_2^7}\left(\frac{f_2f_5^5}{f_1f_{10}^5}-3q-4q^2\frac{f_1f_{10}^5}{f_2f_5^5}\right)\\
&= q\frac{f_1^{10}f_5f_{10}^5}{f_2^7}\left(\left(\frac{f_2f_5^5}{f_1f_{10}^5}+q\right)-4q\left(1+q\frac{f_1f_{10}^5}{f_2f_5^5}\right)\right),
\end{aligned}
$$

which, by (5.2.3) and (5.2.2), reduces to

$$
\left[q^{5n}\right]\{qB\} = q\frac{f_1^{10}f_5f_{10}^5}{f_2^7}\left(\frac{f_2^4f_5^2}{f_1^2f_{10}^4}-4q\frac{f_2^3f_{10}}{f_1f_5^3}\right) = q\frac{f_1^{12}f_{10}^3}{f_2^5f_5}, \tag{5.2.21}
$$

which is the first equality of (5.2.10).

Now, to arrive at the second equality of (5.2.10), we employ (5.2.2) successively to (5.2.21). Accordingly, we have

$$
\begin{aligned}
\left[q^{5n}\right]\{qB\} &= q\frac{f_1^{12}f_{10}^3}{f_2^5f_5}\\
&= q\frac{f_1^8f_5^3f_{10}}{f_2^3}-4q^2\frac{f_1^9f_{10}^6}{f_2^4f_5^2}\\
&= q\frac{f_1^4f_5^7}{f_2f_{10}}-8q^2\frac{f_1^5f_5^2f_{10}^4}{f_2^2}+16q^3\frac{f_1^6f_{10}^9}{f_2^3f_5^3}\\
&= q\frac{f_2^2f_5^{11}}{f_{10}^3}-12q^2f_1f_5^6f_{10}^2+48q^3\frac{f_1^2f_5f_{10}^7}{f_2}-64q^4\frac{f_1^3f_{10}^{12}}{f_2^2f_5^4}\\
&= q\frac{f_2^2f_5^{11}}{f_{10}^3}-12q^2f_1f_5^6f_{10}^2+48q^3\frac{f_1^2f_5f_{10}^7}{f_2}-64q^4\frac{f_{10}^{10}}{f_1}+256q^5\frac{f_{10}^{15}}{f_2f_5^5},
\end{aligned}
$$

which is (5.2.10). □

5.3 Proof of Theorem 5.1.1

Proof of (5.1.3). Employing (1.8.1) in (5.1.1) and then extracting the terms involving q^{5n+4}, we find that

$$\sum_{n=0}^{\infty} c_{10}(5n+4)q^n = 5\frac{f_2^{10}f_5^5}{f_1^6},$$

which is the first equality of (5.1.3). To derive the second equality of (5.1.3) we apply (5.2.2) – (5.2.4) in the above identity. Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} c_{10}(5n+4)q^n &= 5\frac{f_2^{10}f_5^5}{f_1^6},\\
&= 5\left(\frac{f_2^5f_5^4f_{10}}{f_1} + 5q\frac{f_2^6f_5^3f_{10}^4}{f_1^4}\right),\\
&= 5\left(f_2^2f_5^7 + 6qf_1f_2f_5^2f_{10}^5 + 25q^2\frac{f_2^2f_5f_{10}^8}{f_1^2}\right)\\
&= 5\left(f_2^2f_5^7 + 6q\frac{f_2^4f_{10}^6}{f_5} + 19q^2\frac{f_1^2f_{10}^{10}}{f_5^3} + 100q^3\frac{f_2f_{10}^{13}}{f_1f_5^4}\right)\\
&= 5\left(C + 6qD + 19q^2E + 100q^3F\right),
\end{aligned}$$

which proves (5.1.3).

Proofs of (5.1.4)–(5.1.10). Multiplying (5.1.3) by q, extracting the terms q^{5n} on both sides and then employing (5.2.11)–(5.2.14), we find that

$$\begin{aligned}
\sum_{n=1}^{\infty} c_{10}(5^2n-1)q^n = 5\Big(& -qH + 7q^2I - 8q^3J - 16q^4K + 64q^5L + 6\big(-5q^2I - 10q^3J\\
& - 5q^4K + 20q^5L\big) + 19\big(-qH - 3q^2I - 3q^3J - q^4K + 4q^5L\big)\\
& + 100\big(qH + 8q^2I + 38q^3J + 156q^4K + q^5L\big)\Big)\\
= 5^2q\big(&16H + 144qI + 735q^2J + 3107q^3K + 72q^4L\big),
\end{aligned}$$

which is equivalent to (5.1.4).

To prove (5.1.5), we multiply both sides of (5.1.4) by q, extract the terms q^{5n+3}, and then employ the extraction formulas (5.2.16)–(5.2.20). Next, to prove (5.1.6), we extract the terms q^{5n} from both sides of (5.1.5) and then employ the extraction formulas (5.2.9)–(5.2.14). We proceed in a similar fashion to arrive at (5.1.7)–(5.1.10) to complete the proof of Theorem 5.1.1.

5.4 Proof of Theorem 5.1.3

We prove Theorem 5.1.3 by using the generating function representations given in Theorem 5.1.1. First, we have the following observations from Theorem 5.1.1.

Observation (i). The generating function representations (5.1.4), (5.1.6), (5.1.8), and (5.1.10) involve G, H, I, J, K, and L, where, by (5.1.1),

$$K = \sum_{n=0}^{\infty} c_{10}(n) q^n.$$

Observation (ii). Since, by (1.8.3), (1.8.4) and (1.8.1),

$$G = \frac{f_5^{16}\psi(q)}{f_{10}^8} = \frac{f_5^{16}}{f_{10}^8}\left(f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25})\right),$$
$$qH = q\frac{f_5^{11}f_{50}}{f_{10}^3}\left(R(q^{10}) - q^2 - \frac{q^4}{R(q^{10})}\right),$$
$$q^2 I = q^2 f_5^6 f_{10}^2 f_{25}\left(R(q^5) - q - \frac{q^2}{R(q^5)}\right),$$

and

$$q^3 J = q^3 f_5 f_{10}^7 \varphi(-q) = q^3 f_5 f_{10}^7\Big(\varphi(-q^{25}) - 2qf(-q^{15},-q^{35}) + 2q^4 f(-q^5,-q^{45})\Big),$$

we see that there are no terms of the forms q^{5n+2} and q^{5n+4} in the expansions of G as well as in qH and no terms of the forms q^{5n} and q^{5n+1} in the power series expansions of $q^2 I$ as well as in $q^3 J$.

Observation (iii). By (1.8.2),

$$q^5 L = q^5 \frac{f_{10}^{15} f_{50}^5}{f_5^5 f_{10}^6}\Bigg(R^4(q^{10}) + q^2 R^3(q^{10}) + 2q^4 R^2(q^{10}) + 3q^6 R(q^{10}) + 5q^8 - \frac{3q^{10}}{R(q^{10})} + \frac{2q^{12}}{R^2(q^{10})} - \frac{q^{14}}{R^3(q^{10})} + \frac{q^{16}}{R^4(q^{10})}\Bigg).$$

Therefore, the above 5-dissection of $q^5 L$ contains terms of all forms q^{5n+r}, $r = 0, 1, \ldots, 4$.

It is now clear from the above observations that the terms G, qH and q^5L from (5.1.4), (5.1.6), (5.1.8), and (5.1.10) may be eliminated to obtain a couple of linear recurrence relations for $c_{10}(n)$. Similarly, q^2I, q^3J and q^5L may be eliminated to find two more linear recurrence relations. To that end, by eliminating q^5L from (5.1.4), (5.1.6), (5.1.8), and (5.1.10), we first find the following three identities:

$$\begin{aligned}
&9\sum_{n=0}^{\infty} c_{10}(5^4n+74)q^n - 25\times 25136q\sum_{n=0}^{\infty} c_{10}(5^2n+24)q^n \\
&= 625\Big(207G + 43792qH + 456768q^2I + 2177808q^3J + 9340625q^4K\Big), \qquad (5.4.1)
\end{aligned}$$

$$\begin{aligned}
&5\times 392868373\sum_{n=0}^{\infty} c_{10}(5^4n+74)q^n - 25136\sum_{n=0}^{\infty} c_{10}(5^6n+1949)q^n \\
&= 3125\Big(25747155G + 5446953680qH + 52597343040q^2I + 274533074640q^3J \\
&\quad + 1155986328125q^4K\Big), \qquad (5.4.2)
\end{aligned}$$

$$\begin{aligned}
&5\times 6138572735584\sum_{n=0}^{\infty} c_{10}(5^6n+1949)q^n - 392868373\sum_{n=0}^{\infty} c_{10}(5^8n+48824)q^n \\
&= 15625\Big(15193961075250G + 3214366876364000qH + 32964972373536000q^2I \\
&\quad + 151049280766716000q^3J + 672524871826171875q^4K\Big). \qquad (5.4.3)
\end{aligned}$$

Next, eliminating G and qH from (5.4.1)–(5.4.3), we obtain

$$\begin{aligned}
&23\sum_{n=0}^{\infty} c_{10}(5^6n+1949)q^n - 1792295\sum_{n=0}^{\infty} c_{10}(5^4n+74)q^n \\
&- 357599375\sum_{n=0}^{\infty} c_{10}(5^2n+24)q^{n+1} - 16650390625\sum_{n=0}^{\infty} c_{10}(n)q^{n+4} \\
&= 12057000000q^2I - 10443000000q^3J. \qquad (5.4.4)
\end{aligned}$$

Equating the coefficients of q^{5n+5} and q^{5n+6} on both sides of (5.4.4) and taking note of Observation (ii) above, we readily arrive at (5.1.16) and (5.1.17), respectively.

Again, eliminating q^2I and q^3J from (5.4.1)–(5.4.3), we find that

$$\begin{aligned}
&170333\sum_{n=0}^{\infty} c_{10}(5^8n+48824)q^n - 13293230070\sum_{n=0}^{\infty} c_{10}(5^6n+1949)q^n \\
&- 1087669187500\sum_{n=0}^{\infty} c_{10}(5^4n+74)q^n - 693733280156250\sum_{n=0}^{\infty} c_{10}(5^2n+24)q^{n+1}
\end{aligned}$$

$$+ 129633245849609375 \sum_{n=0}^{\infty} c_{10}(n) q^{n+4}$$
$$= -5062500000000G - 1071000000000000qH. \qquad (5.4.5)$$

Equating the coefficients of q^{5n+7} and q^{5n+4} on both sides of (5.4.5) and noting Observation (ii) above, we easily deduce (5.1.18) and (5.1.19), respectively. This completes the proof of Theorem 5.1.3.

5.5 Proof of Theorem 5.1.4

Proof of (5.1.20). From (5.1.2), (1.8.3) and (5.2.1), we have

$$\begin{aligned}4q\sum_{n=0}^{\infty} sc_{10}(n)q^n &= 4q(-q;q^2)_\infty f_{20}^5 \\ &= \frac{f_{20}^4}{f_5}\left(\varphi^2(q) - \varphi^2(q^5)\right) \\ &= \frac{f_{20}^4}{f_5}\Big(\left(\varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4 f(q^5,q^{45})\right)^2 - \varphi^2(q^5)\Big) \\ &= \frac{f_{20}^4}{f_5}\Big(4q\varphi(q^{25})f(q^{15},q^{35}) + 4q^2 f^2(q^{15},q^{35}) + 4q^4\varphi(q^{25})f(q^5,q^{45}) \\ &\quad + 4q^5 f(q^5,q^{45})f(q^{15},q^{35}) + 4q^8 f^2(q^5,q^{45})\Big).\end{aligned}$$

We extract and then use (5.2.1) to obtain

$$\sum_{n=0}^{\infty} sc_{10}(5n+4)q^n = \frac{f_4^4}{f_1} f(q,q^9)f(q^3,q^7) = \frac{f_4^4}{f_1}(-q;q^2)_\infty f_5 f_{20} = \frac{f_2^2 f_4^3 f_5 f_{20}}{f_1^2},$$

which is (5.1.20).

Proof of (5.1.21). It follows from (5.2.7) that

$$4q\frac{f_2^2 f_4^3 f_5 f_{20}}{f_1^2} = \frac{f_2^7 f_5^3}{f_1^4 f_{10}} - \frac{f_2 f_4^2 f_{10}^5}{f_5 f_{20}^2}.$$

Employing the above in (5.1.20) and then using (5.2.3) and (5.2.4), we find that

$$\begin{aligned}4\sum_{n=0}^{\infty} sc_{10}(5n+4)q^{n+1} &= f_1 f_2^2 f_5^2 + 5q\frac{f_2^3 f_5 f_{10}^3}{f_1^2} - \frac{f_2 f_4^2 f_{10}^5}{f_5 f_{20}^2} \\ &= f_1 f_2^2 f_5^2 + 5q\frac{f_5^4 f_{10}^2}{f_1} + 5q^2\frac{f_{10}^7}{f_2 f_5} - \frac{f_2 f_4^2 f_{10}^5}{f_5 f_{20}^2}.\end{aligned}$$

Now, employing (1.8.1) and (1.8.2) in the above identity, extracting the terms involving q^{5n} and proceeding as in the proof of Lemma 5.2.4, we find that

$$\begin{aligned}4\sum_{n=0}^{\infty} sc_{10}(25n+24)q^{n+1} &= \frac{f_2^4 f_5^3}{f_{10}^2} - \frac{f_2^3 f_4^2 f_{10}^3}{f_1 f_{20}^2} + 25q\frac{f_2^2 f_5^5}{f_1^2} + 25q^2\frac{f_2 f_{10}^5}{f_1}, \\ &= 25q\frac{f_2^5 f_5^2 f_{10}}{f_1^3} - q\frac{f_2^9 f_5^2 f_{20}^2}{f_1^3 f_4^2 f_{10}^3},\end{aligned}$$

where (5.2.3) and (5.2.8) have been applied in the last equality.

Next we divide both sides of the above by q and then apply (1.2.2), (5.2.1), (5.2.3) and (5.2.4). Accordingly, we have

$$\begin{aligned}4\sum_{n=0}^{\infty} sc_{10}(25n+24)q^{n} &= \frac{f_2^5 f_5^2 f_{10}}{f_1^3}\left(25 - \frac{\varphi^2(-q^2)}{\varphi^2(-q^{10})}\right) \\ &= \frac{f_2^5 f_5^2 f_{10}}{f_1^3}\left(24 + 4q^2\frac{f_2 f_{20}^5}{f_4 f_{10}^5}\right) \\ &= 4\frac{f_2^5 f_5^2 f_{10}}{f_1^3}\left(5 + \frac{f_4^3 f_{20}}{f_2 f_{10}^3}\right) \\ &= 4\left(f_1^2 f_5 f_{10}^2 + 5q\frac{f_2 f_{10}^5}{f_1}\right)\left(5 + \frac{f_4^3 f_{20}}{f_2 f_{10}^3}\right),\end{aligned}$$

which is equivalent to (5.1.21).

Proof of (5.1.22). From (5.2.3)–(5.2.5), we find that

$$\begin{aligned}4q\frac{f_{10}^2 f_4^3 f_{20}}{f_1} &= \frac{f_2^8 f_{10}^2}{f_1^4 f_5} - \frac{f_4^4 f_{10}^{12}}{f_2^2 f_5^5 f_{20}^4} \\ &= \frac{f_1 f_2^3 f_{10}^3}{f_5^2} + 5q\frac{f_2 f_{10}^5}{f_1} + 4q^2\frac{f_{10}^{10}}{f_5^5} - \frac{f_4 f_{10}^{15}}{f_2 f_5^5 f_{20}^5}.\end{aligned} \tag{5.5.1}$$

Multiplying both sides of (5.1.21) by 4 and then employing (5.5.1), we have

$$\begin{aligned}4\sum_{n=0}^{\infty} sc_{10}(25n+24)q^{n} = 20 f_1^2 f_5 f_{10}^2 &- 5\frac{f_4 f_{10}^{15}}{f_2 f_5^5 f_{20}^5} + 5\frac{f_1 f_2^3 f_{10}^3}{f_5^2} + 125q\frac{f_2 f_{10}^5}{f_1} \\ &+ 20q^2\frac{f_{10}^{10}}{f_5^5} + 4\frac{f_1^2 f_4^3 f_5 f_{20}}{f_2 f_{10}}.\end{aligned} \tag{5.5.2}$$

Now we want to extract the terms involving q^{5n+2} from both sides of the above. The extraction for the first five terms on the right side of the equality may be accomplished easily by applying (1.8.1) and (1.8.2). For the last term, using (1.2.2), (1.8.1) and (1.8.3), we find that

$$\left[q^{5n+2}\right]\left\{4\frac{f_1^2 f_4^3 f_5 f_{20}}{f_2 f_{10}}\right\} = \left[q^{5n+2}\right]\left\{4\frac{f_4^3 f_5 f_{20}\varphi(-q)}{f_{10}}\right\}$$

$$= \left[q^{5n+2}\right] \left\{ 4\frac{f_5 f_{20} f_{100}^3}{f_{10}} \left(R(q^{20}) - q^4 - \frac{q^8}{R(q^{20})} \right)^3 \right.$$
$$\left. \times \left(\varphi(-q^{25}) - 2qf(-q^{15}, -q^{35}) + 2q^4 f(-q^5, -q^{45})\right) \right\}$$

$$= 20q^2 \frac{f_1 f_4 f_5^2 f_{20}^3}{f_2 f_{10}}. \tag{5.5.3}$$

Therefore, extraction of the terms involving q^{5n+2} from both sides of (5.5.2) yields

$$4\sum_{n=0}^{\infty} sc_{10}(125n+74)q^n = -20 f_1 f_2^2 f_5^2 + 105\frac{f_2^7 f_5^3}{f_1^4 f_{10}} + 20\frac{f_2^{10}}{f_1^5} - 5\frac{f_2^{11} f_{10}^3}{f_1^5 f_4^3 f_{20}}$$
$$+ 5q\frac{f_1^2 f_2 f_{10}^5}{f_5^3} + 20q^2 \frac{f_1 f_4 f_5^2 f_{20}^3}{f_2 f_{10}},$$

which, with the aid of (5.2.2)–(5.2.4), (5.2.6) and (5.2.7), reduces to

$$\sum_{n=0}^{\infty} sc_{10}(125n+74)q^n = 5\left(5 f_1 f_2^2 f_5^2 + 31q\frac{f_5^4 f_{10}^2}{f_1} + 32q^2 \frac{f_{10}^7}{f_2 f_5} + q^4 \frac{f_{10}^2 f_{20}^5}{f_4 f_5} \right),$$

which is (5.1.22).

Proofs of (5.1.23) *and* (5.1.24). We employ (1.8.1) and (1.8.2) in (5.1.22), extract the terms involving q^{5n}, and then simplify with the aid of the identities (5.2.2)–(5.2.4). Omitting details, we arrive at (5.1.23).

Next, with the aid of (5.1.2) we may recast (5.1.23) as

$$\sum_{n=0}^{\infty} sc_{10}(625n+74)q^n = 25\left(\frac{f_2^4 f_5^3}{f_{10}^2} + 31q f_1^2 f_5 f_{10}^2 + 156q^2 \frac{f_2 f_{10}^5}{f_1} \right) + 25\sum_{n=0}^{\infty} sc_{10}(n)q^{n+4}.$$

Employing (1.8.1) and (1.8.2) in the above, extracting the terms involving q^{5n+3}, and then using (5.1.20) and (5.2.2)–(5.2.4), we find that

$$\sum_{n=0}^{\infty} sc_{10}(3125n+1949)q^n = 625\left(5 f_1 f_2^2 f_5^2 + 31q\frac{f_5^4 f_{10}^2}{f_1} + 32q^2 \frac{f_{10}^7}{f_2 f_5} \right)$$
$$+ 25q\frac{f_2^2 f_4^3 f_5 f_{20}}{f_1^2},$$

which is (5.1.24).

5.6 Proof of Theorem 5.1.6

From (5.1.20), (5.1.22) and (5.1.24), we have

$$\sum_{n=0}^{\infty} sc_{10}(3125n+1949)q^n = 125\sum_{n=0}^{\infty} sc_{10}(125n+74)q^n + 25\sum_{n=0}^{\infty} sc_{10}(5n+4)q^{n+1} - 625q^4\frac{f_{10}^2 f_{20}^5}{f_4 f_5}.$$

Employing (1.8.1) in the above, extracting the terms involving q^{5n}, and then using (5.1.2), we obtain

$$\sum_{n=0}^{\infty} sc_{10}(15625n+1949)q^n = 125\sum_{n=0}^{\infty} sc_{10}(625n+74)q^n + 25\sum_{n=1}^{\infty} sc_{10}(25n-1)q^n - 3125\sum_{n=0}^{\infty} sc_{10}(n)q^{n+4}.$$

Equating the coefficients of q^{n+4} from both sides of the above, we see that for any nonnegative integer n,

$$sc_{10}(15625n+64449) = 125sc_{10}(625n+2574) + 25sc_{10}(25n+99) - 3125sc_{10}(n),$$

which is equivalent to (5.1.25).