Chapter 6

A Three-Step Two-Grid DG Method for the Kelvin-Voigt Model

In this chapter, we propose and study a two-grid algorithm based on DG approximation for the equations of motion arising in Kelvin-Voigt viscoelastic fluid flow model. Similar to the previous chapter, the first step consists of discretizing the nonlinear system utilizing DG method in the space direction and solving the system on a coarse grid $\mathcal{E}_H$ with grid size H. Then, by employing the coarse grid solution, Newton's iteration type linearization is carried out and we find the approximate solutions for the resulting system on a fine grid $\mathcal{E}_h$ with size h in the second step. However, unlike the previous chapter, a third step is introduced, which is a correction step for the solutions of the second step. A modified final solution is produced in this third step, by solving a different linear problem on the fine grid.
Optimal $\mathbf{L}^2$ and energy-norm error estimates for velocity and L^2-norm error estimates for pressure are derived for an appropriate choice of coarse and fine grid parameters. We further discretize the two-grid DG model in time, using the backward Euler method and derive the fully discrete error estimates. Finally, numerical results are presented to confirm the efficiency of the proposed scheme.

6.1 Introduction

We recall here the DG weak formulation of the Kelvin-Voigt viscoelastic fluid flow (3.1)-(3.3) on the discontinuous spaces $\mathbf{X}$ and M from Chapter 3: Find $(\mathbf{u}(t), p(t)) \in \mathbf{X} \times M,\ t > 0$, such that

$$(\mathbf{u}_t(t), \mathbf{v}) + \kappa\, a(\mathbf{u}_t(t),\, \mathbf{v}) + \nu\, a(\mathbf{u}(t),\, \mathbf{v}) + c^{\mathbf{u}(t)}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v})$$

$$+b(\mathbf{v},\, p(t)) = (\mathbf{f}(t),\, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{X}, \tag{6.1}$$

$$b(\mathbf{u}(t),\, q) = 0 \quad \forall\, q \in M, \tag{6.2}$$

$$(\mathbf{u}(0),\, \mathbf{v}) = (\mathbf{u}_0,\, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{X}. \tag{6.3}$$

For the Kelvin-Voigt model with forcing term $\mathbf{f} = \mathbf{0}$, the two-grid technique has been applied by Bajpai *et al.* [11, 16] with classical finite element approximation for spatial discretization. A second order accurate backward difference scheme for time discretization has been employed in [16]. Also, optimal velocity error estimates in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}^1)$-norms with $h = \mathcal{O}(h^{2-\theta})$ and $h = \mathcal{O}(h^{5-2\theta})$, respectively, and pressure error estimates in $L^\infty(L^2)$-norm when $h = \mathcal{O}(h^{5-2\theta})$, have been derived, where $\theta > 0$ is arbitrarily small. The semi-discrete scheme of [16] has been discretized in time by the Crank-Nicolson scheme in [11], and optimal fully discrete error estimates are established.

However, to the best of our knowledge, the two-grid techniques in conjunction with the DG methods for the Kelvin-voigt model have never been applied. Taking a leaf out of Chapter 5, we aim to obtain optimal estimates in energy norm for velocity and in L^2-norm for pressure approximations, for similar relation between H and h. Nevertheless, to attain optimal $\mathbf{L}^2$-norm estimates of velocity, further correction is required as has been carried out in the CG case [16]. Thus, following the algorithm employed in [16], we have implemented a similar type of algorithm, but in DG set up, for the Kelvin-Voigt model in this chapter.

The two-grid DG algorithm proposed in this chapter involves the following three steps:

- **Step 1:** Solve the nonlinear system over a coarse mesh $\mathcal{E}_H$ to obtain an approximate solution, say $\mathbf{u}_H$.
- **Step 2:** Linearize the nonlinear system with one Newton iteration around the coarse grid solution $\mathbf{u}_H$ and solve it over a fine mesh $\mathcal{E}_h$ to obtain the solution, say $\mathbf{u}_h$.
- **Step 3:** Correct the solution $\mathbf{u}_h$ obtained in **Step 2** with the help of $\mathbf{u}_h$ and $\mathbf{u}_H$ over the fine mesh to provide a modified final solution $\mathbf{u}_h^*$.

Applying the above algorithm, we now introduce the DG two-grid semi-discrete scheme for (6.1)-(6.3), which is described as follows:

Step 1 (Nonlinear system on $\mathcal{E}_H$): Find $(\mathbf{u}_H, p_H) \in \mathbf{X}_H \times M_H$ such that for all

$(\boldsymbol{\phi}_H, q_H) \in \mathbf{X}_H \times M_h$ for $\mathbf{u}_H(0) = \mathbf{P}_H\mathbf{u}_0$ and $t > 0$

$$\left.\begin{aligned}(\mathbf{u}_{Ht}(t), \boldsymbol{\phi}_H) + \kappa\, a\,(\mathbf{u}_{Ht}(t), \boldsymbol{\phi}_H) + \nu\, a\,(\mathbf{u}_H(t), \boldsymbol{\phi}_H) + c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t), \mathbf{u}_H(t), \boldsymbol{\phi}_H)&\\ +b\,(\boldsymbol{\phi}_H, p_H(t)) = (\mathbf{f}(t), \boldsymbol{\phi}_H),&\\ b(\mathbf{u}_H(t), q_H) = 0.&\end{aligned}\right\} \tag{6.4}$$

Step 2 (Update on $\mathcal{E}_h$ with one Newton iteration): Find $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times M_h$ such that for all $(\boldsymbol{\phi}_h, q_h) \in \mathbf{X}_h \times M_h$ for $\mathbf{u}_h(0) = \mathbf{P}_h\mathbf{u}_0$ and $t > 0$

$$\left.\begin{aligned}(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) + \kappa\, a\,(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{u}_h(t), \boldsymbol{\phi}_h) + c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t), \mathbf{u}_H(t), \boldsymbol{\phi}_h)&\\ +c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t), \mathbf{u}_h(t), \boldsymbol{\phi}_h) + b\,(\boldsymbol{\phi}_h, p_h(t)) = (\mathbf{f}(t), \boldsymbol{\phi}_h)&\\ + c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t), \mathbf{u}_H(t), \boldsymbol{\phi}_h),&\\ b(\mathbf{u}_h(t), q_h) = 0.&\end{aligned}\right\} \tag{6.5}$$

Step 3 (Correct on $\mathcal{E}_h$): Find $(\mathbf{u}_h^*, p_h^*) \in \mathbf{X}_h \times M_h$ such that for all $(\boldsymbol{\phi}_h, q_h) \in \mathbf{X}_h \times M_h$ for $\mathbf{u}_h^*(0) = \mathbf{P}_h\mathbf{u}_0$ and $t > 0$

$$\left.\begin{aligned}(\mathbf{u}_{ht}^*(t), \boldsymbol{\phi}_h) + \kappa\, a\,(\mathbf{u}_{ht}^*(t), \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{u}_h^*(t), \boldsymbol{\phi}_h) + c^{\mathbf{u}_h^*(t)}(\mathbf{u}_h^*(t), \mathbf{u}_H(t), \boldsymbol{\phi}_h)&\\ +c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t), \mathbf{u}_h^*(t), \boldsymbol{\phi}_h) + b\,(\boldsymbol{\phi}_h, p_h^*(t)) = (\mathbf{f}(t), \boldsymbol{\phi}_h) + c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t), \mathbf{u}_h(t), \boldsymbol{\phi}_h)&\\ +c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t), \mathbf{u}_H(t) - \mathbf{u}_h(t), \boldsymbol{\phi}_h),&\\ b(\mathbf{u}_h^*(t), q_h) = 0.&\end{aligned}\right\} \tag{6.6}$$

Let us recall the subspace $\mathbf{V}_\lambda$ of $\mathbf{X}_\lambda$:

$$\mathbf{V}_\lambda \;=\; \{\mathbf{v}_\lambda \in \mathbf{X}_\lambda :\; b\,(\mathbf{v}_\lambda,\, q_\lambda) = 0,\; \forall q_\lambda \in M_\lambda\},$$

where $\lambda = H, h$.

An equivalent DG two-grid semi-discrete algorithm corresponding to the scheme (6.4)–(6.6) on the space $\mathbf{V}_\lambda$ is the following:

Step 1 (Nonlinear system on $\mathcal{E}_H$): Find $\mathbf{u}_H \in \mathbf{V}_H$ such that for all $\boldsymbol{\phi}_H \in \mathbf{V}_H$ for $\mathbf{u}_H(0) = \mathbf{P}_H\mathbf{u}_0$ and $t > 0$

$$\begin{aligned}(\mathbf{u}_{Ht}(t), \boldsymbol{\phi}_H) + \kappa\, a\,(\mathbf{u}_{Ht}(t), \boldsymbol{\phi}_H) + \nu\, a\,(\mathbf{u}_H(t), \boldsymbol{\phi}_H) + c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t), \mathbf{u}_H(t), \boldsymbol{\phi}_H)&\\ = (\mathbf{f}(t), \boldsymbol{\phi}_H).&\end{aligned} \tag{6.7}$$

Step 2 (Update on $\mathcal{E}_h$ with one Newton iteration): Find $\mathbf{u}_h \in \mathbf{V}_h$ such that for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ for $\mathbf{u}_h(0) = \mathbf{P}_h\mathbf{u}_0$ and $t > 0$

$$(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) + \kappa\, a\,(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{u}_h(t), \boldsymbol{\phi}_h) + c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t), \mathbf{u}_H(t), \boldsymbol{\phi}_h)$$

$$+c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t),\mathbf{u}_h(t),\boldsymbol{\phi}_h) = (\mathbf{f}(t),\boldsymbol{\phi}_h) + c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t),\mathbf{u}_H(t),\boldsymbol{\phi}_h). \qquad (6.8)$$

Step 3 (Correct on $\mathcal{E}_h$): Find $\mathbf{u}_h^* \in \mathbf{V}_h$ such that for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ for $\mathbf{u}_h^*(0) = \mathbf{P}_h\mathbf{u}_0$ and $t > 0$

$$\begin{aligned}(\mathbf{u}_{ht}^*(t),\boldsymbol{\phi}_h) + \kappa\, a\,(\mathbf{u}_{ht}^*(t),\boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{u}_h^*(t),\boldsymbol{\phi}_h) + c^{\mathbf{u}_h^*(t)}(\mathbf{u}_h^*(t),\mathbf{u}_H(t),\boldsymbol{\phi}_h)\\ +c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t),\mathbf{u}_h^*(t),\boldsymbol{\phi}_h) = (\mathbf{f}(t),\boldsymbol{\phi}_h) + c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t),\mathbf{u}_h(t),\boldsymbol{\phi}_h)\\ +c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t),\mathbf{u}_H(t)-\mathbf{u}_h(t),\boldsymbol{\phi}_h).\end{aligned} \qquad (6.9)$$

In this case, the estimates presented in Lemmas 2.7 (see Chapter 2) and 5.4 (see Chapter 5) of the trilinear form $c(\cdot,\cdot,\cdot)$ and upwinding term $l(\cdot,\cdot,\cdot)$ are not sufficient for optimal scaling between h and H, and demand some new improved estimates of $c(\cdot,\cdot,\cdot)$ and $l(\cdot,\cdot,\cdot)$. For this reason, we have introduced some new interpolated Sobolev and trace inequalities (see Lemma 6.4), which result in improved estimates of $c(\cdot,\cdot,\cdot)$ and $l(\cdot,\cdot,\cdot)$, and hence in improved scaling between h and H. We have used here a fully discrete approximation considering a first-order backward Euler method in the temporal direction.
The major findings of this chapter are summarised below:

- *A priori* bounds of the semi-discrete two-grid DG solutions along with optimal error estimates for the semi-discrete two-grid DG velocity approximation in $\mathbf{L}^2$-norm when $h = \mathcal{O}\big(H^{\min\left(r+1-\theta,\frac{3r+2-2\theta}{r+1}\right)}\big)$, in energy norm for $h = \mathcal{O}(H^{\frac{3r+2-2\theta}{r}})$ and pressure approximation in L^2-norm when $h = \mathcal{O}(H^{\frac{3r+2-2\theta}{r}})$ are derived for $t > 0$, where $\theta > 0$ is arbitrarily small. These convergence estimates have been established by utilizing newly derived interpolated Sobolev and trace inequalities, and modified Sobolev-Stokes's projection $\mathbf{S}_h^{so}$ (for $\mathbf{S}_h^{so}$ see (3.28) of Chapter 3).

- Under the smallness assumption on the given data and for $t > 0$, uniform in time velocity and pressure error estimates in $\mathbf{L}^2$, energy and L^2-norms, respectively, are established.

- *A priori* bounds of the fully discrete two-grid DG solutions and optimal order convergence rates for the fully-discrete backward Euler velocity and pressure approximations are achieved. Numerical experiments are carried out to show the performance of the scheme.

The rest of the chapter contains the following sections: We discuss about *a priori* estimates of semi-discrete solutions in Section 6.2. Some auxiliary inequalities and estimates are derived in Section 6.3. In Section 6.4, the semi-discrete error analysis is carried out. Fully discrete scheme is presented in Section 6.5. We have employed the backward Euler method, and optimal error estimates for the velocity and pressure are derived. We carry out numerical experiments in Section 6.6, and the results are analyzed. Finally, Section 6.7 concludes this chapter by summarizing the results briefly. Throughout this chapter, we will use C, $K(>0)$ as generic constants that depend on the given data, ν, κ, α, K_1, K_2, C_2 but do not depend on h and Δt. Note that, K and C may grow algebraically with ν^{-1}. Further, the notations $K(t)$ and K_T will be used when they grow exponentially in time.

6.2 *A priori* and Regularity Bounds

In this section, we present *a priori* and regularity bounds to the discrete velocity for all three steps, which will be used in our later analysis.
In Lemma 6.1, we recall **Step 1** *a priori* estimates from Lemma 3.1 of Chapter 3, which play crucial role in the derivation of **Step 2** and **Step 3** error estimates.

Lemma 6.1. *Suppose* $0<\alpha<\frac{\nu K_1}{2(C_2+\kappa K_2)}$. *Then, for the semi-discrete DG velocity* $\boldsymbol{u}_H(t)$, $t>0$ *of* **step 1**, *the following holds true:*

$$\|\boldsymbol{u}_H(t)\|^2+\kappa\|\boldsymbol{u}_H(t)\|_{\varepsilon}^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{u}_H(s)\|_{\varepsilon}^2\,ds\leq C, \tag{6.10}$$

$$\limsup_{t\to\infty}\|\boldsymbol{u}_H(t)\|_{\varepsilon}\leq\frac{C_2}{K_1\nu}\|\mathbf{f}\|_{L^\infty(L^2(\Omega))}, \tag{6.11}$$

where C *is a positive constant.*

Next in Lemma 6.2, we derive *a priori* estimates of **Step 2** solution $\mathbf{u}_h$.

Lemma 6.2. *Let* $0<\alpha<\frac{\nu K_1}{2(C_2+\kappa K_2)}$. *Then, for the semi-discrete DG velocity* $\boldsymbol{u}_h(t)$, $t>0$ *of* **step 2**, *there exists a constant* $K>0$, *such that, the following holds true*

$$\|\boldsymbol{u}_h(t)\|^2+\kappa\|\boldsymbol{u}_h(t)\|_{\varepsilon}^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{u}_h(s)\|_{\varepsilon}^2\,ds\leq K(t), \tag{6.12}$$

$$\|\boldsymbol{u}_{ht}(t)\|^2+\kappa\|\boldsymbol{u}_{ht}(t)\|_{\varepsilon}^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\boldsymbol{u}_{hs}(s)\|^2+\kappa\|\boldsymbol{u}_{hs}(s)\|_{\varepsilon}^2)\,ds\leq K(t), \tag{6.13}$$

$$\|\boldsymbol{u}_{htt}(t)\|^2+\kappa\|\boldsymbol{u}_{htt}(t)\|_{\varepsilon}^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\boldsymbol{u}_{hss}(s)\|^2+\kappa\|\boldsymbol{u}_{hss}(s)\|_{\varepsilon}^2)\,ds\leq K(t), \tag{6.14}$$

where $K(t)$ *grows exponentially in time.*

Proof. Choose $\boldsymbol{\phi}_h = \mathbf{u}_h$ in (6.8), and apply estimate (1.14), coercivity property from Lemma 1.6, positivity property (1.19), the Cauchy-Schwarz inequality and Young's inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_h\|^2 + \frac{\kappa}{2}\frac{d}{dt}\big(a\left(\mathbf{u}_h, \mathbf{u}_h\right)\big) + \nu K_1 \|\mathbf{u}_h\|_\varepsilon^2 \leq \frac{\nu K_1}{4}\|\mathbf{u}_h\|_\varepsilon^2 + C\|\mathbf{f}\|^2 + |c^{\mathbf{u}_h}(\mathbf{u}_h,\, \mathbf{u}_H,\, \mathbf{u}_h)| + |c^{\mathbf{u}_H}(\mathbf{u}_H,\, \mathbf{u}_H,\, \mathbf{u}_h)|. \tag{6.15}$$

The estimates (2.55), (2.57) and Young's inequality lead to the following bound:

$$\begin{aligned} |c^{\mathbf{u}_h}(\mathbf{u}_h,\, \mathbf{u}_H,\, \mathbf{u}_h)| + |c^{\mathbf{u}_H}(\mathbf{u}_H,\, \mathbf{u}_H,\, \mathbf{u}_h)| &\leq C\big(\|\mathbf{u}_h\|^{1/2}\|\mathbf{u}_h\|_\varepsilon^{3/2}\|\mathbf{u}_H\|_\varepsilon + \|\mathbf{u}_H\|_\varepsilon^2\|\mathbf{u}_h\|_\varepsilon\big) \\ &\leq \frac{\nu K_1}{4}\|\mathbf{u}_h\|_\varepsilon^2 + C\big(\|\mathbf{u}_h\|^2\|\mathbf{u}_H\|_\varepsilon^4 + \|\mathbf{u}_H\|_\varepsilon^4\big). \end{aligned}$$

Multiplying (6.15) by $e^{2\alpha t}$, integrating from 0 to t, and applying (1.14), Lemmas 1.6 and 1.7, and the above inequality, we obtain

$$\begin{aligned} e^{2\alpha t}\|\mathbf{u}_h(t)\|^2 + K_1\kappa e^{2\alpha t}\|\mathbf{u}_h(t)\|_\varepsilon^2 + (\nu K_1 - 2\alpha C_2 - 2\alpha\kappa K_2)\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\, ds \\ \leq \|\mathbf{u}_h(0)\|^2 + K_2\kappa\|\mathbf{u}_h(0)\|_\varepsilon^2 + C\int_0^t e^{2\alpha s}\|\mathbf{f}(s)\|^2\, ds \\ +C\int_0^t e^{2\alpha s}\big(\|\mathbf{u}_h(s)\|^2\|\mathbf{u}_H(s)\|_\varepsilon^4 + \|\mathbf{u}_H(s)\|_\varepsilon^4\big)\, ds. \end{aligned}$$

Now, multiplying the above inequality by $e^{-2\alpha t}$ and using (6.10), the fact

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\, ds = \frac{1}{2\alpha}(1 - e^{-2\alpha t}),$$

Gronwall's inequality and choosing $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$, we arrive at the estimate (6.12). Substitute $\boldsymbol{\phi}_h = \mathbf{u}_{ht}$ in (6.8), and use (1.14), Lemmas 1.6 and 1.7, (2.55), the Cauchy-Schwarz inequality and Young's inequality to find

$$\|\mathbf{u}_{ht}\|^2 + K_1\kappa\|\mathbf{u}_{ht}\|_\varepsilon^2 \leq C(\|\mathbf{u}_h\|_\varepsilon^2 + \|\mathbf{u}_h\|_\varepsilon^2\|\mathbf{u}_H\|_\varepsilon^2 + \|\mathbf{f}\|^2 + \|\mathbf{u}_H\|_\varepsilon^4). \tag{6.16}$$

An application of (6.10) and (6.12) leads to

$$\|\mathbf{u}_{ht}\|^2 + K_1\kappa\|\mathbf{u}_{ht}\|_\varepsilon^2 \leq C. \tag{6.17}$$

Multiplying (6.16) by $e^{2\alpha t}$, integrating from 0 to t, and applying (6.10) and (6.12), we find (6.13).

We now differentiate (6.8) with respect to t to obtain

$$\begin{aligned} (\mathbf{u}_{htt}, \boldsymbol{\phi}_h) + \kappa\, a\left(\mathbf{u}_{htt}, \boldsymbol{\phi}_h\right) + \nu\, a\left(\mathbf{u}_{ht}, \boldsymbol{\phi}_h\right) + c^{\mathbf{u}_h}(\mathbf{u}_{ht}, \mathbf{u}_H, \boldsymbol{\phi}_h) \\ +c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_{Ht}, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_{Ht}, \mathbf{u}_h, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{u}_{ht}, \boldsymbol{\phi}_h) \end{aligned}$$

$$= (\mathbf{f}_t, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_{Ht}, \mathbf{u}_H, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{u}_{Ht}, \boldsymbol{\phi}_h). \tag{6.18}$$

Replace $\boldsymbol{\phi}_h$ by $\mathbf{u}_{htt}$ in (6.18), and apply (1.14), Lemmas 1.6 and 1.7, (2.55), the Cauchy-Schwarz inequality and Young's inequality to arrive at

$$\begin{aligned} \|\mathbf{u}_{htt}\|^2 + K_1\kappa\|\mathbf{u}_{htt}\|_\varepsilon^2 \leq C(&\|\mathbf{f}_t\|^2 + \|\mathbf{u}_{ht}\|_\varepsilon^2 + \|\mathbf{u}_{ht}\|_\varepsilon^2\|\mathbf{u}_H\|_\varepsilon^2 + \|\mathbf{u}_h\|_\varepsilon^2\|\mathbf{u}_{Ht}\|_\varepsilon^2 \\ &+ \|\mathbf{u}_{Ht}\|_\varepsilon^2\|\mathbf{u}_H\|_\varepsilon^2). \end{aligned} \tag{6.19}$$

Now, using (6.10), (6.11) and (6.13), one can obtain

$$\|\mathbf{u}_{htt}\|^2 + K_1\kappa\|\mathbf{u}_{htt}\|_\varepsilon^2 \leq C. \tag{6.20}$$

Finally, multiply (6.19) by $e^{2\alpha t}$, integrate from 0 to t, and employ (6.10), (6.11) and (6.13) to arrive at (6.14) and this conclude the rest of the proof. □

In the following lemma, we state *a priori* estimates of **Step 3** semi-discrete solution $\mathbf{u}_h^*$. These estimates can be obtained from (6.9) and as in Lemma 6.2. Hence the proof is skipped.

Lemma 6.3. *Let* $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. *Then, for the semi-discrete DG velocity* $\boldsymbol{u}_h^*(t)$, $t > 0$ *of* **step 3**, *the following holds true*

$$\|\boldsymbol{u}_h^*(t)\|^2 + \kappa\|\boldsymbol{u}_h^*(t)\|_\varepsilon^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{u}_h^*(s)\|_\varepsilon^2\, ds \leq K(t), \tag{6.21}$$

$$\|\boldsymbol{u}_{ht}^*(t)\|^2 + \kappa\|\boldsymbol{u}_{ht}^*(t)\|_\varepsilon^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\boldsymbol{u}_{hs}^*(s)\|^2 + \kappa\|\boldsymbol{u}_{hs}^*(s)\|_\varepsilon^2)\, ds \leq K(t), \tag{6.22}$$

$$\|\boldsymbol{u}_{htt}^*(t)\|^2 + \kappa\|\boldsymbol{u}_{htt}^*(t)\|_\varepsilon^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\boldsymbol{u}_{hss}^*(s)\|^2 + \kappa\|\boldsymbol{u}_{hss}^*(s)\|_\varepsilon^2)\, ds \leq K(t). \tag{6.23}$$

Now from the coercivity result in Lemma 1.6, the positivity (1.19), the inf-sup condition in Lemma 1.8, Lemmas 6.2 and 6.3, and following [98, Lemma 3.4], the existence and uniqueness of the discrete solutions of (6.5) (or (6.8)) and (6.6) (or (6.9)) in **Step 2** and **Step 3**, respectively, will follow easily.

6.3 Some Useful Estimates

We begin this section by deriving some new interpolated Sobolev and trace inequalities. Then, we focus on estimating the terms $c(\cdot,\cdot,\cdot)$ and $l(\cdot,\cdot,\cdot)$.
In the following lemma, we establish interpolated Sobolev and trace inequalities, which will be useful for our future analysis.

Lemma 6.4. *For $p = \frac{2}{1-\theta}$, $0 < \theta < 1$ and $\mathbf{v} \in \boldsymbol{H}^1(\mathcal{E}_h)$, we have*

$$\|\mathbf{v}\|_{L^p(\Omega)} \leq C\|\mathbf{v}\|^{1-\theta}\|\mathbf{v}\|_{\varepsilon}^{\theta} + Ch^{-\theta}\|\mathbf{v}\|, \tag{6.24}$$

$$\|\mathbf{v}\|_{L^q(e)} \leq Ch^{-\frac{1}{q}}\|\mathbf{v}\|_{L^q(E)} + C\|\mathbf{v}\|_{L^{2(q-1)}(E)}^{\frac{q-1}{q}}\|\nabla\mathbf{v}\|_{L^2(E)}^{\frac{1}{q}}, \quad \forall q > 2, \tag{6.25}$$

$$\|\mathbf{v}\|_{L^p(e)} \leq Ch^{-\frac{1}{p}}\|\mathbf{v}\|_{L^p(E)} + C\|\mathbf{v}\|_{L^2(E)}^{\frac{1-\theta}{2}}\|\nabla\mathbf{v}\|_{L^2(E)}^{\frac{1+\theta}{2}}, \tag{6.26}$$

where

$$\boldsymbol{H}^1(\mathcal{E}_h) = \{\mathbf{v} \in \boldsymbol{L}^2(\Omega) : \quad \mathbf{v}|_E \in \boldsymbol{H}^1(E), \quad \forall E \in \mathcal{E}_h\}.$$

Proof. From [106, eq. (2.10)], we know for $\mathbf{v} \in W^{1,2}(E)$ satisfying

$$\int_E \mathbf{v}(x)\, dx = 0, \tag{6.27}$$

the following inequality holds

$$\|\mathbf{v}\|_{L^p(E)} \leq C\|\mathbf{v}\|_{L^2(E)}^{1-\theta}\|\nabla\mathbf{v}\|_{L^2(E)}^{\theta}. \tag{6.28}$$

For an arbitrary function $\mathbf{v} \in W^{1,2}(E)$ that does not satisfy (6.27), we define a function

$$\mathbf{w}(x) = \mathbf{v}(x) - \mathcal{W}, \tag{6.29}$$

where $\mathcal{W} = \frac{1}{|E|}\int_E \mathbf{v}(x)\, dx$.
Since $\mathbf{w}$ satisfies (6.27), we have

$$\|\mathbf{w}\|_{L^p(E)} \leq C\|\mathbf{w}\|_{L^2(E)}^{1-\theta}\|\nabla\mathbf{w}\|_{L^2(E)}^{\theta}. \tag{6.30}$$

The triangle inequality yields

$$\|\mathbf{v}\|_{L^p(E)} - |\mathcal{W}||E|^{1/p} \leq \|\mathbf{w}\|_{L^p(E)},$$
$$\|\mathbf{w}\|_{L^2(E)} \leq \|\mathbf{v}\|_{L^2(E)} + |\mathcal{W}||E|^{1/2}.$$

Thus, the above two inequalities and (6.30) lead to

$$\|\mathbf{v}\|_{L^p(E)} \leq |\mathcal{W}||E|^{1/p} + C(\|\mathbf{v}\|_{L^2(E)} + |\mathcal{W}||E|^{1/2})^{1-\theta}\|\nabla\mathbf{v}\|_{L^2(E)}^{\theta}.$$

We now apply Hölder's inequality to bound $|\mathcal{W}|$ as

$$|\mathcal{W}| \leq |E|^{-1/s}\|\mathbf{v}\|_{L^s(E)}, \quad \forall s \geq 2. \tag{6.31}$$

An application of (6.31) with $s = 2$ and note that $|E|^{-\theta/2} \leq Ch_E^{-\theta}$ to find

$$\|\mathbf{v}\|_{L^p(E)} \leq C\|\mathbf{v}\|_{L^2(E)}^{1-\theta}\|\nabla\mathbf{v}\|_{L^2(E)}^{\theta} + Ch_E^{-\theta}\|\mathbf{v}\|_{L^2(E)}.$$

Taking p-th power on both side of the above inequality, summing over $E \in \mathcal{E}_h$ and then $1/p$-th power of the resulting inequality to find

$$\|\mathbf{v}\|_{L^p(\Omega)} \leq C \sum_{E\in\mathcal{E}_h} \|\mathbf{v}\|^{1-\theta}_{L^2(E)}\|\nabla\mathbf{v}\|^{\theta}_{L^2(E)} + Ch^{-\theta} \sum_{E\in\mathcal{E}_h} \|\mathbf{v}\|_{L^2(E)}.$$

A use of Jensen's inequality leads to (6.24). The second estimate (6.25) is derived in [96, eq. (7.3)]. To derive the third estimate, we again consider the construction (6.29). Hence, one can write

$$\|\mathbf{w}\|_{L^{2(p-1)}(E)} \leq C\|\mathbf{w}\|^{1-\gamma}_{L^2(E)}\|\nabla\mathbf{w}\|^{\gamma}_{L^2(E)}, \tag{6.32}$$

where $2(p-1) = \dfrac{2}{1-\gamma}$ and which implies $\gamma = \dfrac{2\theta}{1+\theta}$. Consider (6.25) with q and $\mathbf{v}$ replaced by p and $\mathbf{w}$, respectively and substitute (6.32) in the resulting inequality, we arrive at

$$\|\mathbf{w}\|_{L^p(e)} \leq Ch^{-\frac{1}{p}}\|\mathbf{w}\|_{L^p(E)} + C\|\mathbf{w}\|^{\frac{1-\theta}{2}}_{L^2(E)}\|\nabla\mathbf{w}\|^{\frac{1+\theta}{2}}_{L^2(E)}. \tag{6.33}$$

To derive the above estimate for $\mathbf{v}$, let us utilize (6.29) and the obvious relation

$$\|\mathbf{v}\|_{L^p(E)} - |\mathcal{W}||E|^{1/p} \leq \|\mathbf{w}\|_{L^p(E)} \leq \|\mathbf{v}\|_{L^p(E)} + |\mathcal{W}||E|^{1/p}, \tag{6.34}$$

$$\|\mathbf{v}\|_{L^p(e)} - |\mathcal{W}||e|^{1/p} \leq \|\mathbf{w}\|_{L^p(e)} \leq \|\mathbf{v}\|_{L^p(e)} + |\mathcal{W}||e|^{1/p}. \tag{6.35}$$

The relations (6.33) and (6.35) yield

$$\|\mathbf{v}\|_{L^p(e)} \leq |\mathcal{W}||e|^{1/p} + Ch^{-\frac{1}{p}}\|\mathbf{w}\|_{L^p(E)} + C\|\mathbf{w}\|^{\frac{1-\theta}{2}}_{L^2(E)}\|\nabla\mathbf{w}\|^{\frac{1+\theta}{2}}_{L^2(E)}.$$

Apply the right hand side part of (6.34) in the above inequality

$$\begin{aligned}\|\mathbf{v}\|_{L^p(e)} \leq& |\mathcal{W}||e|^{1/p} + Ch^{-\frac{1}{p}}(\|\mathbf{v}\|_{L^p(E)} + |\mathcal{W}||E|^{1/p}) \\ &+ C(\|\mathbf{v}\|_{L^2(E)} + |\mathcal{W}||E|^{1/2})^{\frac{1-\theta}{2}}\|\nabla\mathbf{v}\|^{\frac{1+\theta}{2}}_{L^2(E)}.\end{aligned}$$

Finally, we employ (6.31) with $s = 2$ and $s = p$, and observe that $|e| \leq Ch$ to establish (6.26). □

The next lemma derives some estimates of $c(\cdot,\cdot,\cdot)$ which will be useful for our future error analysis.

Lemma 6.5. *There exists a positive constant C, which is independent of h, such that for all $\boldsymbol{\Theta}, \mathbf{v}, \boldsymbol{w} \in \boldsymbol{X}$ and $\boldsymbol{\phi}_h \in \boldsymbol{X}_h$, the following estimates hold true:*

$$|c^{\boldsymbol{\Theta}}(\mathbf{v},\, \boldsymbol{w},\, \boldsymbol{\phi}_h)| \leq C\big(\|\mathbf{v}\|^{1-\theta}\|\mathbf{v}\|^{\theta}_{\varepsilon} + h^{-\theta}\|\mathbf{v}\| + h^{\frac{1-\theta}{2}}\|\mathbf{v}\|^{\frac{1-\theta}{2}}\|\mathbf{v}\|^{\frac{1+\theta}{2}}_{\varepsilon}\big)\|\boldsymbol{w}\|_{\varepsilon}\|\boldsymbol{\phi}_h\|_{\varepsilon}, \tag{6.36}$$

$$|c^{\mathbf{v}}(\mathbf{v},\, \boldsymbol{w},\, \boldsymbol{\phi}_h)| \leq C\|\mathbf{v}\|_{\varepsilon}\big(\|\boldsymbol{w}\|^{1-\theta}\|\boldsymbol{w}\|^{\theta}_{\varepsilon} + h^{-\theta}\|\boldsymbol{w}\| + h^{\frac{1-\theta}{2}}\|\boldsymbol{w}\|^{\frac{1-\theta}{2}}\|\boldsymbol{w}\|^{\frac{1+\theta}{2}}_{\varepsilon}\big)\|\boldsymbol{\phi}_h\|_{\varepsilon}. \tag{6.37}$$

Proof. With the help of Green's formula, we can write

$$\int_E (\nabla\cdot\mathbf{v})\mathbf{w}\cdot\boldsymbol{\phi}_h = \int_E \nabla\cdot(\mathbf{w}\otimes\mathbf{v})\cdot\boldsymbol{\phi}_h - \int_E(\mathbf{v}\cdot\nabla\mathbf{w})\cdot\boldsymbol{\phi}_h$$
$$= -\int_E(\mathbf{v}\cdot\nabla\boldsymbol{\phi}_h)\cdot\mathbf{w} + \int_{\partial E}(\mathbf{v}^{int}\cdot\mathbf{n}_E)\mathbf{w}^{int}\cdot\boldsymbol{\phi}_h^{int} - \int_E(\mathbf{v}\cdot\nabla\mathbf{w})\cdot\boldsymbol{\phi}_h.$$

The above allows us to arrive at the following reformulation:

$$\begin{aligned} c^{\boldsymbol{\Theta}}(\mathbf{v},\,\mathbf{w},\,\boldsymbol{\phi}_h) =& \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E(\mathbf{v}\cdot\nabla\mathbf{w})\cdot\boldsymbol{\phi}_h - \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E(\mathbf{v}\cdot\nabla\boldsymbol{\phi}_h)\cdot\mathbf{w} \\ & - \sum_{E\in\mathcal{E}_h}\int_{\partial E_-}|\{\mathbf{v}\}\cdot\mathbf{n}_E|(\mathbf{w}^{int}-\mathbf{w}^{ext})\cdot\boldsymbol{\phi}_h^{int} - \frac{1}{2}\sum_{e\in\Gamma_h}\int_e[\mathbf{v}]\cdot\mathbf{n}_e\{\mathbf{w}\cdot\boldsymbol{\phi}_h\} \\ & + \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_{\partial E}(\mathbf{v}^{int}\cdot\mathbf{n}_E)\mathbf{w}^{int}\cdot\boldsymbol{\phi}_h^{int} \\ =& N_1+N_2+N_3+N_4+N_5. \end{aligned} \tag{6.38}$$

The terms N_1 and N_2 are bounded using (6.24), (1.14) and Hölder's inequality as follows:

$$\begin{aligned} |N_1| &\le C\sum_{E\in\mathcal{E}_h}\|\mathbf{v}\|_{L^p(E)}\|\nabla\mathbf{w}\|_{L^2(E)}\|\boldsymbol{\phi}_h\|_{L^q(E)} \\ &\le C\|\mathbf{v}\|^{1-\theta}\|\mathbf{v}\|_\varepsilon^\theta\|\mathbf{w}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon + Ch^{-\theta}\|\mathbf{v}\|\|\mathbf{w}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon, \end{aligned} \tag{6.39}$$

$$\begin{aligned} |N_2| &\le C\sum_{E\in\mathcal{E}_h}\|\mathbf{v}\|_{L^p(E)}\|\nabla\boldsymbol{\phi}_h\|_{L^2(E)}\|\mathbf{w}\|_{L^q(E)} \\ &\le C\|\mathbf{v}\|^{1-\theta}\|\mathbf{v}\|_\varepsilon^\theta\|\mathbf{w}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon + Ch^{-\theta}\|\mathbf{v}\|\|\mathbf{w}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon, \end{aligned} \tag{6.40}$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and $p=\frac{2}{1-\theta}$. Applying Hölder's inequality, we can bound N_3 as

$$|N_3| \le \sum_{e\in\Gamma_h}\|\{\mathbf{v}\}\cdot\mathbf{n}_e\|_{L^p(e)}\|[\mathbf{w}]\|_{L^2(e)}\|\boldsymbol{\phi}_h\|_{L^q(e)}.$$

An application of trace inequalities (6.25) and (6.26), one can find

$$\begin{aligned} &\|\{\mathbf{v}\}\cdot\mathbf{n}_e\|_{L^p(e)}\|[\mathbf{w}]\|_{L^2(e)}\|\boldsymbol{\phi}_h\|_{L^q(e)} \\ &\le C\sum_{i,j=1}^{2}\left(\|\mathbf{v}\|_{L^p(E_i)} + h_{E_i}^{\frac{1-\theta}{2}}\|\mathbf{v}\|_{L^2(E_i)}^{\frac{1-\theta}{2}}\|\nabla\mathbf{v}\|_{L^2(E_i)}^{\frac{1+\theta}{2}}\right)\frac{1}{|e|^{\frac{1}{2}}}\|[\mathbf{w}]\|_{L^2(e)} \\ &\qquad\times\left(\|\boldsymbol{\phi}_h\|_{L^q(E_j)} + h_{E_j}^{\frac{1}{q}}\|\boldsymbol{\phi}_h\|_{L^{2(q-1)}(E_j)}^{\frac{q-1}{q}}\|\nabla\boldsymbol{\phi}_h\|_{L^2(E_j)}^{\frac{1}{q}}\right). \end{aligned}$$

Using interpolation inequality (6.24), we obtain

$$\begin{aligned} |N_3| \le& C\|\mathbf{v}\|^{1-\theta}\|\mathbf{v}\|_\varepsilon^\theta\|\mathbf{w}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon + Ch^{-\theta}\|\mathbf{v}\|\|\mathbf{w}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon \\ & + Ch^{\frac{1-\theta}{2}}\|\mathbf{v}\|^{\frac{1-\theta}{2}}\|\mathbf{v}\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{w}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon. \end{aligned}$$

Next, switch the sum of N_5 from elements to the edges. Then we consider this sum's contribution to any interior edge e. Let E_r and E_s be the two elements adjacent to e, with exterior normal $\mathbf{n}_r$ and $\mathbf{n}_s$. This implies

$$\int_e (\mathbf{v}|_{E_r}\cdot\mathbf{n}_r)\mathbf{w}|_{E_r}\cdot\boldsymbol{\phi}_h|_{E_r} + \int_e (\mathbf{v}|_{E_s}\cdot\mathbf{n}_s)\mathbf{w}|_{E_s}\cdot\boldsymbol{\phi}_h|_{E_s} = \int_e [(\mathbf{v}\cdot\mathbf{n}_e)\mathbf{w}\cdot\boldsymbol{\phi}_h].$$

By applying the above equality, one can obtain

$$N_4 + N_5 = \frac{1}{2}\sum_{e\in\Gamma_h}\int_e \{\mathbf{v}\}\cdot\mathbf{n}_e[\mathbf{w}\cdot\boldsymbol{\phi}_h].$$

Noting that, $[\mathbf{w}\cdot\boldsymbol{\phi}_h] = \{\mathbf{w}\}\cdot[\boldsymbol{\phi}_h] + [\mathbf{w}]\cdot\{\boldsymbol{\phi}_h\}$, and using trace inequalities (6.25) and (6.26), and Hölder's inequality, we obtain

$$\begin{aligned}
N_4 + N_5 &= \frac{1}{2}\sum_{e\in\Gamma_h}\int_e \{\mathbf{v}\}\cdot\mathbf{n}_e\{\mathbf{w}\}\cdot[\boldsymbol{\phi}_h] + \frac{1}{2}\sum_{e\in\Gamma_h}\int_e \{\mathbf{v}\}\cdot\mathbf{n}_e[\mathbf{w}]\cdot\{\boldsymbol{\phi}_h\}\\
\le& C\sum_{e\in\Gamma_h}|e|^{\frac{1}{p}}\|\mathbf{v}\|_{L^p(e)}|e|^{\frac{1}{q}}\|\mathbf{w}\|_{L^q(e)}\frac{1}{|e|^{\frac{1}{2}}}\|[\boldsymbol{\phi}_h]\|_{L^2(e)}\\
&+ C\sum_{e\in\Gamma_h}|e|^{\frac{1}{p}}\|\mathbf{v}\|_{L^p(e)}\frac{1}{|e|^{\frac{1}{2}}}\|[\mathbf{w}]\|_{L^2(e)}|e|^{\frac{1}{q}}\|\boldsymbol{\phi}_h\|_{L^q(e)}\\
\le& C\sum_{e\in\Gamma_h}\sum_{i,j=1}^{2}\left(\|\mathbf{v}\|_{L^p(E_i)} + h_{E_i}^{\frac{1-\theta}{2}}\|\mathbf{v}\|_{L^2(E_i)}^{\frac{1-\theta}{2}}\|\nabla\mathbf{v}\|_{L^2(E_i)}^{\frac{1+\theta}{2}}\right)\\
&\times\left(\|\mathbf{w}\|_{L^q(E_j)} + h_{E_j}^{\frac{1}{q}}\|\mathbf{w}\|_{L^{2(q-1)}(E_j)}^{\frac{q-1}{q}}\|\nabla\mathbf{w}\|_{L^2(E_j)}^{\frac{1}{q}}\right)\frac{1}{|e|^{\frac{1}{2}}}\|[\boldsymbol{\phi}_h]\|_{L^2(e)}\\
&+ C\sum_{e\in\Gamma_h}\sum_{i,j=1}^{2}\left(\|\mathbf{v}\|_{L^p(E_i)} + h_{E_i}^{\frac{1-\theta}{2}}\|\mathbf{v}\|_{L^2(E_i)}^{\frac{1-\theta}{2}}\|\nabla\mathbf{v}\|_{L^2(E_i)}^{\frac{1+\theta}{2}}\right)\frac{1}{|e|^{\frac{1}{2}}}\|[\mathbf{w}]\|_{L^2(e)}\\
&\times\left(\|\boldsymbol{\phi}_h\|_{L^q(E_j)} + h_{E_j}^{\frac{1}{q}}\|\boldsymbol{\phi}_h\|_{L^{2(q-1)}(E_j)}^{\frac{q-1}{q}}\|\nabla\boldsymbol{\phi}_h\|_{L^2(E_j)}^{\frac{1}{q}}\right)\\
\le& C\|\mathbf{v}\|^{1-\theta}\|\mathbf{v}\|_\varepsilon^{\theta}\|\mathbf{w}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon + Ch^{-\theta}\|\mathbf{v}\|\|\mathbf{w}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon + Ch^{\frac{1-\theta}{2}}\|\mathbf{v}\|^{\frac{1-\theta}{2}}\|\mathbf{v}\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{w}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon.
\end{aligned}$$

Combining the bounds of N_1, $N_2, \cdots, N_5$ in (6.38), we finally obtain the bound (6.36). The last inequality (6.37) can be derived employing the form (1.18) and following the steps involved in deriving (6.36). Hence the proof is skipped. This completes the proof of this lemma. □

The next lemma is an auxiliary result for the upwinding term $l(\cdot,\cdot,\cdot)$.

Lemma 6.6. *There is a positive constant C independent of h such that for all $\boldsymbol{w}, \boldsymbol{u}, \mathbf{v} \in \boldsymbol{X}$ and $\boldsymbol{\phi}_h \in \boldsymbol{X}_h$, the following estimates hold true*

$$\begin{aligned}
|l^{\boldsymbol{w}}(\boldsymbol{u},\mathbf{v},\boldsymbol{\phi}_h) - l^{\boldsymbol{u}}(\boldsymbol{u},\mathbf{v},\boldsymbol{\phi}_h)| \le& C\|\boldsymbol{u}-\boldsymbol{w}\|^{1-\theta}\|\boldsymbol{u}-\boldsymbol{w}\|_\varepsilon^{\theta}\|\mathbf{v}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon\\
&+ Ch^{-\theta}\|\boldsymbol{u}-\boldsymbol{w}\|\|\mathbf{v}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon\\
&+ Ch^{\frac{1-\theta}{2}}\|\boldsymbol{u}-\boldsymbol{w}\|^{\frac{1-\theta}{2}}\|\boldsymbol{u}-\boldsymbol{w}\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{v}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon.
\end{aligned}$$

Proof. For any $\boldsymbol{\theta} \in \mathbf{X}$ and let $e \in \Gamma_h \setminus \partial\Omega$ be an edge adjacent to E_1 and E_2 with $\mathbf{n}_e = \mathbf{n}_{E_1}$. The contribution of e to the term $l^{\boldsymbol{\theta}}(\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}_h)$ reduces to

$$\int_e (\{\mathbf{u}\} \cdot \mathbf{n}_e)[\mathbf{v}] \cdot \boldsymbol{\phi}_h^{\boldsymbol{\theta}},$$

where $\boldsymbol{\phi}_h^{\boldsymbol{\theta}}|_e = \boldsymbol{\phi}_h|_{E_1}$ if $\{\boldsymbol{\theta}\} \cdot \mathbf{n}_e < 0$, $\boldsymbol{\phi}_h^{\boldsymbol{\theta}}|_e = \boldsymbol{\phi}_h|_{E_2}$ if $\{\boldsymbol{\theta}\} \cdot \mathbf{n}_e > 0$, and $\boldsymbol{\phi}_h^{\boldsymbol{\theta}}|_e = \mathbf{0}$ if $\{\boldsymbol{\theta}\} \cdot \mathbf{n}_e = 0$. In a similar way, if $e \in \partial\Omega \cap E$, then we have $\mathbf{n}_e = \mathbf{n}_{\partial\Omega}$. Then, the contribution corresponding to e is

$$\int_e (\mathbf{u} \cdot \mathbf{n}_e)\mathbf{v} \cdot \boldsymbol{\phi}_h^{\boldsymbol{\theta}},$$

where $\boldsymbol{\phi}_h^{\boldsymbol{\theta}}|_e = \boldsymbol{\phi}_h|_E$ if $\boldsymbol{\theta} \cdot \mathbf{n}_e < 0$ and $\boldsymbol{\phi}_h^{\boldsymbol{\theta}}|_e = \mathbf{0}$ otherwise. Set $B = l^{\mathbf{w}}(\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}_h) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}_h)$. Then, following the above notations, B can be rewritten as

$$B = \sum_{e \in \Gamma_h} \int_e (\{\mathbf{u}\} \cdot \mathbf{n}_e)[\mathbf{v}] \cdot (\boldsymbol{\phi}_h^{\mathbf{w}} - \boldsymbol{\phi}_h^{\mathbf{u}}).$$

The domain of integration can be partitioned as follows:

$$\Gamma_h = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3,$$

where

$$\begin{aligned}
\mathcal{G}_1 &= \{e : \{\mathbf{w}\} \cdot \mathbf{n}_e \neq 0 \text{ and } \{\mathbf{u}\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\}, \\
\mathcal{G}_2 &= \{e : \{\mathbf{w}\} \cdot \mathbf{n}_e = 0 \text{ and } \{\mathbf{u}\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\}, \\
\mathcal{G}_3 &= \Gamma_h \setminus (\mathcal{G}_1 \cup \mathcal{G}_2).
\end{aligned}$$

First we consider $\mathcal{G}_1$. For $e \in \mathcal{G}_1$, we decompose e into e_1 and e_2. e_1 is the part where $\{\mathbf{w}\} \cdot \mathbf{n}_e$ and $\{\mathbf{u}\} \cdot \mathbf{n}_e$ have the same sign and e_2 part is for the opposite signs of $\{\mathbf{w}\} \cdot \mathbf{n}_e$ and $\{\mathbf{u}\} \cdot \mathbf{n}_e$. On e_1, we then have $\boldsymbol{\phi}_h^{\mathbf{w}} - \boldsymbol{\phi}_h^{\mathbf{u}} = \mathbf{0}$. On e_2, $\boldsymbol{\phi}_h^{\mathbf{w}} - \boldsymbol{\phi}_h^{\mathbf{u}} = [\boldsymbol{\phi}_h]$, up to the sign. Using the fact of opposite signs, we can write

$$|\{\mathbf{u}\} \cdot \mathbf{n}_e| \leq |\{\mathbf{u} - \mathbf{w}\} \cdot \mathbf{n}_e|.$$

Applying Hölder's and Jensen's inequalities, and (1.14), (1.37), (6.24) and (6.26), we can deduce

$$\begin{aligned}
\left| \sum_{e \in \mathcal{G}_1} \int_e (\{\mathbf{u}\} \cdot \mathbf{n}_e)[\mathbf{v}] \cdot (\boldsymbol{\phi}_h^{\mathbf{w}} - \boldsymbol{\phi}_h^{\mathbf{u}}) \right| &\leq \sum_{e \in \mathcal{G}_1} \|\{\mathbf{u} - \mathbf{w}\}\|_{L^p(e)} \|[\mathbf{v}]\|_{L^2(e)} \|[\boldsymbol{\phi}_h]\|_{L^q(e)} \\
\leq C \sum_{e \in \mathcal{G}_1} \sum_{i,j=1}^{2} \Bigg(\|\mathbf{u} - \mathbf{w}\|_{L^{\frac{2}{1-\theta}}(E_i)} &+ h_{E_i}^{\frac{1-\theta}{2}} \|\mathbf{u} - \mathbf{w}\|_{L^2(E_i)}^{\frac{1-\theta}{2}} \|\nabla(\mathbf{u} - \mathbf{w})\|_{L^2(E_i)}^{\frac{1+\theta}{2}} \Bigg)
\end{aligned}$$

$$\times\left(\frac{\sigma_e}{|e|}\right)^{1/2}\|[\mathbf{v}]\|_{L^2(e)}\|\boldsymbol{\phi}_h\|_{L^q(E_j)}$$
$$\leq C\|\mathbf{u}-\mathbf{w}\|^{1-\theta}\|\mathbf{u}-\mathbf{w}\|_\varepsilon^\theta\|\mathbf{v}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon + Ch^{-\theta}\|\mathbf{u}-\mathbf{w}\|\|\mathbf{v}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon$$
$$+ Ch^{\frac{1-\theta}{2}}\|\mathbf{u}-\mathbf{w}\|^{\frac{1-\theta}{2}}\|\mathbf{u}-\mathbf{w}\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{v}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon.$$

Next, we consider $\mathcal{G}_2$. From the definition of $\mathcal{G}_2$, we have $|\{\mathbf{u}\}\cdot\mathbf{n}_e| \leq |\{\mathbf{u}-\mathbf{w}\}\cdot\mathbf{n}_e|$. Therefore, in a similar fashion as above we can show that

$$\Big|\sum_{e\in\mathcal{G}_2}\int_e(\{\mathbf{u}\}\cdot\mathbf{n}_e)[\mathbf{v}]\cdot\boldsymbol{\phi}_h^{\mathbf{u}}\Big| \leq C\|\mathbf{u}-\mathbf{w}\|^{1-\theta}\|\mathbf{u}-\mathbf{w}\|_\varepsilon^\theta\|\mathbf{v}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon + Ch^{-\theta}\|\mathbf{u}-\mathbf{w}\|\|\mathbf{v}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon$$
$$+ Ch^{\frac{1-\theta}{2}}\|\mathbf{u}-\mathbf{w}\|^{\frac{1-\theta}{2}}\|\mathbf{u}-\mathbf{w}\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{v}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon.$$

There is zero contribution of $\mathcal{G}_3$ to B. The combination of the above bounds completes the proof of this lemma. □

6.4 Semi-discrete Error Estimates

This section deals with the derivation of **Step 2** and **Step 3** semi-discrete error estimates of velocity and pressure.

6.4.1 Velocity Error Estimates

In this subsection, we derive the bounds of semi-discrete velocity error for two-grid algorithm. Let us define $\mathbf{e}_H = \mathbf{u}-\mathbf{u}_H$, $\mathbf{e}_h = \mathbf{u}-\mathbf{u}_h$ and $\mathbf{e}_h^* = \mathbf{u}-\mathbf{u}_h^*$. Below, we describe the error equations for **Step 2** and **Step 3**.

Error equation for **Step 2**: From equations (6.1) and (6.8), and for each $\boldsymbol{\phi}_h \in \mathbf{V}_h$,

$$\begin{aligned}(\mathbf{e}_{ht},\boldsymbol{\phi}_h) + \kappa\, a(\mathbf{e}_{ht},\boldsymbol{\phi}_h) + \nu\, a(\mathbf{e}_h,\boldsymbol{\phi}_h) &= -c^{\mathbf{u}_h}(\mathbf{e}_h,\mathbf{u}_H,\boldsymbol{\phi}_h) - c^{\mathbf{u}_H}(\mathbf{u}_H,\mathbf{e}_h,\boldsymbol{\phi}_h)\\ &\quad -c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_H,\boldsymbol{\phi}_h) + \big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\phi}_h) - l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\phi}_h)\big)\\ &\quad +\big(l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\phi}_h) - l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\phi}_h)\big) - b(\boldsymbol{\phi}_h,p).\end{aligned} \tag{6.41}$$

Error equation for **Step 3**: From equations (6.1) and (6.9), and for each $\boldsymbol{\phi}_h \in \mathbf{V}_h$,

$$\begin{aligned}(\mathbf{e}_{ht}^*,\boldsymbol{\phi}_h) + \kappa\, a(\mathbf{e}_{ht}^*,\boldsymbol{\phi}_h) + \nu\, a(\mathbf{e}_h^*,\boldsymbol{\phi}_h) &= -c^{\mathbf{u}_h^*}(\mathbf{e}_h^*,\mathbf{u}_H,\boldsymbol{\phi}_h) - c^{\mathbf{u}_H}(\mathbf{u}_H,\mathbf{e}_h^*,\boldsymbol{\phi}_h)\\ &\quad -c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_h,\boldsymbol{\phi}_h) - c^{\mathbf{u}_h}(\mathbf{e}_h,\mathbf{e}_h,\boldsymbol{\phi}_h) - c^{\mathbf{u}_h}(\mathbf{e}_h,\mathbf{e}_H,\boldsymbol{\phi}_h)\\ &\quad +\big(l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\phi}_h) - l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\phi}_h) - l^{\mathbf{u}_h^*}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\phi}_h)\big)\\ &\quad +\big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_h,\boldsymbol{\phi}_h) - l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_h,\boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_h,\boldsymbol{\phi}_h) - l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_h,\boldsymbol{\phi}_h)\big) - b(\boldsymbol{\phi}_h,p).\end{aligned} \tag{6.42}$$

To derive optimal error estimates of $\mathbf{e}_h$ and $\mathbf{e}_h^*$ in $\mathbf{L}^2$ and energy-norms, we recall the following modified Sobolev-Stokes's projection $\mathbf{S}_h^{so}\mathbf{u} : [0,\infty) \to \mathbf{J}_h$ (see (3.28) of Chapter 3) satisfying

$$\kappa\, a(\mathbf{u}_t - \mathbf{S}_h^{so}\mathbf{u}_t, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u} - \mathbf{S}_h^{so}\mathbf{u}, \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, p) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h, \tag{6.43}$$

where $\mathbf{S}_h^{so}\mathbf{u}(0) = \mathbf{P}_h\mathbf{u}_0$. Let us decompose $\mathbf{e}_h$ and $\mathbf{e}_h^*$ with the help of $\mathbf{S}_h^{so}\mathbf{u}$ as

$$\mathbf{e}_h := (\mathbf{u} - \mathbf{S}_h^{so}\mathbf{u}) + (\mathbf{S}_h^{so}\mathbf{u} - \mathbf{u}_h) := \boldsymbol{\zeta} + \boldsymbol{\rho}, \tag{6.44}$$

$$\mathbf{e}_h^* := (\mathbf{u} - \mathbf{S}_h^{so}\mathbf{u}) + (\mathbf{S}_h^{so}\mathbf{u} - \mathbf{u}_h^*) := \boldsymbol{\zeta} + \boldsymbol{\Theta}, \tag{6.45}$$

where $\boldsymbol{\zeta} = \mathbf{u} - \mathbf{S}_h^{so}\mathbf{u}$, $\boldsymbol{\rho} = \mathbf{S}_h^{so}\mathbf{u} - \mathbf{u}_h$ and $\boldsymbol{\Theta} = \mathbf{S}_h^{so}\mathbf{u} - \mathbf{u}_h^*$.
A use of equations (6.41), (6.43) and (6.44), we find the equation in $\boldsymbol{\rho}$ as

$$\begin{aligned}(\boldsymbol{\rho}_t, \boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\rho}_t, \boldsymbol{\phi}_h) + \nu\, a(\boldsymbol{\rho}, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_H, \boldsymbol{\rho}, \boldsymbol{\phi}_h) &= -(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) - c^{\mathbf{u}_h}(\boldsymbol{\rho}, \mathbf{u}_H, \boldsymbol{\phi}_h) \\ - c^{\mathbf{u}_h}(\boldsymbol{\zeta}, \mathbf{u}_H, \boldsymbol{\phi}_h) - c^{\mathbf{u}_H}(\mathbf{u}_H, \boldsymbol{\zeta}, \boldsymbol{\phi}_h) &- c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\phi}_h) \\ + \big(l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h)\big) &+ \big(l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h) - l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h)\big).\end{aligned} \tag{6.46}$$

Furthermore, using (6.42), (6.43) and (6.45), we obtain the equation in $\boldsymbol{\Theta}$ as

$$\begin{aligned}&(\boldsymbol{\Theta}_t, \boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\Theta}_t, \boldsymbol{\phi}_h) + \nu\, a(\boldsymbol{\Theta}, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_H, \boldsymbol{\Theta}, \boldsymbol{\phi}_h) = -(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) - c^{\mathbf{u}_h^*}(\boldsymbol{\Theta}, \mathbf{u}_H, \boldsymbol{\phi}_h) \\ &- c^{\mathbf{u}_h^*}(\boldsymbol{\zeta}, \mathbf{u}_H, \boldsymbol{\phi}_h) - c^{\mathbf{u}_H}(\mathbf{u}_H, \boldsymbol{\zeta}, \boldsymbol{\phi}_h) - c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_h, \boldsymbol{\phi}_h) - c^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{e}_h, \boldsymbol{\phi}_h) \\ &- c^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{e}_H, \boldsymbol{\phi}_h) + \big(l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h) - l^{\mathbf{u}_h^*}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h)\big) \\ &+ \big(l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\phi}_h) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\phi}_h) - l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\phi}_h)\big).\end{aligned} \tag{6.47}$$

From (6.45), one can see that to derive optimal bounds for $\mathbf{e}_h^*$, we need to bound $\boldsymbol{\Theta}$ in an optimal way. The bounds of $\boldsymbol{\Theta}$ depend on the bounds of $\boldsymbol{\zeta}$, $\mathbf{e}_H$, $\mathbf{e}_h$ that are present on the right side of (6.47). The next lemma states the optimal estimates for $\boldsymbol{\zeta}$. The estimates of $\boldsymbol{\zeta}$ have been already established in Lemmas 3.2 and 3.3 of Chapter 3.

Lemma 6.7. *Suppose the assumption (**A2**) holds true and let $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. Then, for $t > 0$, $\boldsymbol{\zeta}$ satisfies the following estimates:*

$$\|\boldsymbol{\zeta}(t)\|^2 + h^2\|\boldsymbol{\zeta}(t)\|_\varepsilon^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\left(\boldsymbol{\zeta}(s)\|^2 + \|\boldsymbol{\zeta}_s(s)\|^2 + h^2\|\boldsymbol{\zeta}(s)\|_\varepsilon^2\right) ds \leq C h^{2r+2}.$$

And the following theorem provides estimates for the **Step 1** error $\mathbf{e}_H$. The estimates of $\mathbf{e}_H$ are already proved in Theorem 3.1 of Chapter 3.

Theorem 6.1. *Suppose the assumption (**A2**) holds true and let $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. In addition, let the semi-discrete initial velocity $\boldsymbol{u}_H(0) \in \boldsymbol{V}_H$ with $\boldsymbol{u}_H(0) = \boldsymbol{P}_H\boldsymbol{u}_0$. Then, there exists a constant $K > 0$, such that for $t > 0$,*

$$\|\boldsymbol{e}_H(t)\|^2 + H^2\|\boldsymbol{e}_H(t)\|_\varepsilon + H^2 e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{e}_H(s)\|_\varepsilon^2\, ds \leq K(t) H^{2r+2}.$$

Initially, we concentrate on finding estimates of $\mathbf{e}_h$. For that, we have to estimate $\boldsymbol{\rho}$. A combination of the estimate of $\boldsymbol{\rho}$ and $\boldsymbol{\zeta}$ will result in the estimates of $\mathbf{e}_h$.

Lemma 6.8. *Suppose the assumptions of Lemma 6.7 hold true. Then, for $t>0$, the following holds*

$$\|\boldsymbol{\rho}(t)\|^2+\kappa\|\boldsymbol{\rho}(t)\|_\varepsilon^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\rho}(s)\|_\varepsilon^2\,ds\le K(t)(h^{2r+2}+H^{4r+2-2\theta}).$$

Proof. Replace $\boldsymbol{\phi}_h$ by $\boldsymbol{\rho}$ in (6.46), and apply Lemma 1.6 and (1.19) to arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\rho}\|^2+\frac{\kappa}{2}\frac{d}{dt}\big(a(\boldsymbol{\rho},\boldsymbol{\rho})\big)+\nu K_1\|\boldsymbol{\rho}\|_\varepsilon^2 &\le -(\boldsymbol{\zeta}_t,\boldsymbol{\rho})-c^{\mathbf{u}_h}(\boldsymbol{\rho},\mathbf{u}_H,\boldsymbol{\rho})-c^{\mathbf{u}_h}(\boldsymbol{\zeta},\mathbf{u}_H,\boldsymbol{\rho})\\ &\quad -c^{\mathbf{u}_H}(\mathbf{u}_H,\boldsymbol{\zeta},\boldsymbol{\rho})-c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_H,\boldsymbol{\rho})+\big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\rho})-l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\rho})\big)\\ &\quad +\big(l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\rho})-l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\rho})\big).\end{aligned}\tag{6.48}$$

An application of estimate (2.57) and Young's inequality to $c^{\mathbf{u}_h}(\boldsymbol{\rho},\mathbf{u}_H,\boldsymbol{\rho})$ yields

$$|c^{\mathbf{u}_h}(\boldsymbol{\rho},\mathbf{u}_H,\boldsymbol{\rho})|\le\frac{\nu K_1}{64}\|\boldsymbol{\rho}\|_\varepsilon^2+C\|\boldsymbol{\rho}\|^2\|\mathbf{u}_H\|_\varepsilon^4.\tag{6.49}$$

Subtract and add $\mathbf{u}$ to the second argument of $c^{\mathbf{u}_h}(\boldsymbol{\zeta},\mathbf{u}_H,\boldsymbol{\rho})$, we find

$$c^{\mathbf{u}_h}(\boldsymbol{\zeta},\mathbf{u}_H,\boldsymbol{\rho})=-c^{\mathbf{u}_h}(\boldsymbol{\zeta},\mathbf{e}_H,\boldsymbol{\rho})+c^{\mathbf{u}_h}(\boldsymbol{\zeta},\mathbf{u},\boldsymbol{\rho}).\tag{6.50}$$

Now, estimate (6.36), Lemma 6.7, Theorem 6.1, the fact $h<H$ and $1-\theta>0$, and Young's inequality imply

$$\begin{aligned}|c^{\mathbf{u}_h}(\boldsymbol{\zeta},\mathbf{e}_H,\boldsymbol{\rho})|\le& C\|\boldsymbol{\zeta}\|^{1-\theta}\|\boldsymbol{\zeta}\|_\varepsilon^{\theta}\|\mathbf{e}_H\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon+Ch^{-\theta}\|\boldsymbol{\zeta}\|\|\mathbf{e}_H\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon\\ &+Ch^{\frac{1-\theta}{2}}\|\boldsymbol{\zeta}\|^{\frac{1-\theta}{2}}\|\boldsymbol{\zeta}\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{e}_H\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon\\ \le& Ch^{r+1-\theta}H^r\|\boldsymbol{\rho}\|_\varepsilon\le\frac{\nu K_1}{64}\|\boldsymbol{\rho}\|_\varepsilon^2+Ch^{2r}H^{2r+2-2\theta}.\end{aligned}\tag{6.51}$$

From (2.59), Young's inequality, Lemma 6.7 and assumption (**A2**), one can derive

$$|c^{\mathbf{u}_h}(\boldsymbol{\zeta},\mathbf{u},\boldsymbol{\rho})|\le C\|\mathbf{u}\|_2(\|\boldsymbol{\zeta}\|+h\|\boldsymbol{\zeta}\|_\varepsilon)\|\boldsymbol{\rho}\|_\varepsilon\le\frac{\nu K_1}{64}\|\boldsymbol{\rho}\|_\varepsilon^2+Ch^{2r+2}.$$

Combine the above two inequalities in (6.50) to arrive at

$$|c^{\mathbf{u}_h}(\boldsymbol{\zeta},\mathbf{u}_H,\boldsymbol{\rho})|\le\frac{\nu K_1}{32}\|\boldsymbol{\rho}\|_\varepsilon^2+Ch^{2r+2}+Ch^{2r}H^{2r+2-2\theta}.\tag{6.52}$$

Again, we rewrite the fourth term on the right hand side of (6.48) as follows

$$c^{\mathbf{u}_H}(\mathbf{u}_H,\boldsymbol{\zeta},\boldsymbol{\rho})=-c^{\mathbf{u}_H}(\mathbf{e}_H,\boldsymbol{\zeta},\boldsymbol{\rho})+c^{\mathbf{u}}(\mathbf{u},\boldsymbol{\zeta},\boldsymbol{\rho})-\big(l^{\mathbf{u}}(\mathbf{u},\boldsymbol{\zeta},\boldsymbol{\rho})-l^{\mathbf{u}_H}(\mathbf{u},\boldsymbol{\zeta},\boldsymbol{\rho})\big).\tag{6.53}$$

Apply (6.36), Lemma 6.7, Theorem 6.1 and Young's inequality, and observe that $h < H$ and $1-\theta > 0$ to obtain

$$\begin{aligned}
|c^{\mathbf{u}_H}(\mathbf{e}_H, \boldsymbol{\zeta}, \boldsymbol{\rho})| \leq & C\|\mathbf{e}_H\|^{1-\theta}\|\mathbf{e}_H\|_\varepsilon^\theta\|\boldsymbol{\zeta}\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon + CH^{-\theta}\|\mathbf{e}_H\|\|\boldsymbol{\zeta}\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon \\
& + Ch^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|_\varepsilon^{\frac{1+\theta}{2}}\|\boldsymbol{\zeta}\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon \\
\leq & Ch^r H^{r+1-\theta}\|\boldsymbol{\rho}\|_\varepsilon + Ch^{\frac{2r+1-\theta}{2}}H^{\frac{2r+1-\theta}{2}}\|\boldsymbol{\rho}\|_\varepsilon \\
\leq & \frac{\nu K_1}{64}\|\boldsymbol{\rho}\|_\varepsilon^2 + Ch^{2r}H^{2r+2-2\theta}.
\end{aligned} \tag{6.54}$$

Estimate (2.58), Young's inequality and assumption (**A2**) yield

$$|c^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\zeta}, \boldsymbol{\rho})| \leq C\|\mathbf{u}\|_2(\|\boldsymbol{\zeta}\| + h\|\boldsymbol{\zeta}\|_\varepsilon)\|\boldsymbol{\rho}\|_\varepsilon \leq \frac{\nu K_1}{64}\|\boldsymbol{\rho}\|_\varepsilon^2 + Ch^{2r+2}. \tag{6.55}$$

Using Lemmas 6.6 and 6.7, Theorem 6.1, and Young's inequality, one can obtain

$$\begin{aligned}
|l^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\zeta}, \boldsymbol{\rho}) - l^{\mathbf{u}_H}(\mathbf{u}, \boldsymbol{\zeta}, \boldsymbol{\rho})| \leq & C\|\mathbf{e}_H\|^{1-\theta}\|\mathbf{e}_H\|_\varepsilon^\theta\|\boldsymbol{\zeta}\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon + CH^{-\theta}\|\mathbf{e}_H\|\|\boldsymbol{\zeta}\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon \\
& + Ch^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|_\varepsilon^{\frac{1+\theta}{2}}\|\boldsymbol{\zeta}\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon \\
\leq & \frac{\nu K_1}{64}\|\boldsymbol{\rho}\|_\varepsilon^2 + Ch^{2r}H^{2r+2-2\theta}.
\end{aligned} \tag{6.56}$$

Substituting (6.54)-(6.56) in (6.53), we arrive at

$$|c^{\mathbf{u}_H}(\mathbf{u}_H, \boldsymbol{\zeta}, \boldsymbol{\rho})| \leq \frac{\nu K_1}{64}\|\boldsymbol{\rho}\|_\varepsilon^2 + Ch^{2r+2} + Ch^{2r}H^{2r+2-2\theta}. \tag{6.57}$$

A use of (6.36), Theorem 6.1 and Young's inequality, and observe that $h < H$ and $1-\theta > 0$ to find

$$\begin{aligned}
|c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\rho})| \leq & C\|\mathbf{e}_H\|^{1-\theta}\|\mathbf{e}_H\|_\varepsilon^{1+\theta}\|\boldsymbol{\rho}\|_\varepsilon + CH^{-\theta}\|\mathbf{e}_H\|\|\mathbf{e}_H\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon \\
& + Ch^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|_\varepsilon^{\frac{3+\theta}{2}}\|\boldsymbol{\rho}\|_\varepsilon \\
\leq & CH^{2r+1-\theta}\|\boldsymbol{\rho}\|_\varepsilon + Ch^{\frac{1-\theta}{2}}H^{\frac{4r+1-\theta}{2}}\|\boldsymbol{\rho}\|_\varepsilon \leq \frac{\nu K_1}{64}\|\boldsymbol{\rho}\|_\varepsilon^2 + CH^{4r+2-2\theta}.
\end{aligned} \tag{6.58}$$

Employing Lemma 6.6, Theorem 6.1 and Young's inequality, and similar to the above estimate, one can derive

$$\begin{aligned}
|l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\rho}) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\rho})| \leq & C\|\mathbf{e}_H\|^{1-\theta}\|\mathbf{e}_H\|_\varepsilon^{1+\theta}\|\boldsymbol{\rho}\|_\varepsilon + CH^{-\theta}\|\mathbf{e}_H\|\|\mathbf{e}_H\|_\varepsilon\|\boldsymbol{\rho}\|_\varepsilon \\
& + Ch^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|_\varepsilon^{\frac{3+\theta}{2}}\|\boldsymbol{\rho}\|_\varepsilon \\
\leq & \frac{\nu K_1}{64}\|\boldsymbol{\rho}\|_\varepsilon^2 + CH^{4r+2-2\theta}.
\end{aligned} \tag{6.59}$$

For the last term on the right hand side of (6.48), we will follow the proof of Lemma 6.6. Now, we can write

$$l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\rho}) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\rho}) = \sum_{e\in\Gamma_h}\int_e(\{\mathbf{u}\}\cdot\mathbf{n}_e)[\mathbf{e}_H]\cdot(\boldsymbol{\rho}^{\mathbf{u}_h} - \boldsymbol{\rho}^{\mathbf{u}}). \tag{6.60}$$

Here, we have to consider functions $\mathbf{u}_h$, $\mathbf{e}_H$ and $\boldsymbol{\rho}$ in place of $\mathbf{w}$, $\mathbf{v}$ and $\boldsymbol{\phi}_h$, respectively, of Lemma 6.6. Then, $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ will change accordingly. For non-zero contribution of the above integral on both $\mathcal{G}_1$ and $\mathcal{G}_2$, we can write

$$|\{\mathbf{u}\}\cdot\mathbf{n}_e| \le |\{\mathbf{u}-\mathbf{u}_h\}\cdot\mathbf{n}_e| \le |\{\boldsymbol{\zeta}\}\cdot\mathbf{n}_e| + |\{\boldsymbol{\rho}\}\cdot\mathbf{n}_e|. \tag{6.61}$$

With the help of the above relation, the fact that $[\mathbf{u}] = 0$ and Hölder's inequality, equation (6.60) becomes

$$\begin{aligned}|l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\rho}) - l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\rho})| \le &\sum_{e\in\mathcal{G}_1\cup\mathcal{G}_2} \|\{\boldsymbol{\zeta}\}\|_{L^p(e)}\|[\mathbf{e}_H]\|_{L^2(e)}\|\boldsymbol{\rho}^{\mathbf{u}_h}-\boldsymbol{\rho}^{\mathbf{u}}\|_{L^q(e)}\\ &+\sum_{e\in\mathcal{G}_1\cup\mathcal{G}_2} \|\{\boldsymbol{\rho}\}\|_{L^4(e)}\|[\mathbf{u}_H]\|_{L^2(e)}\|\boldsymbol{\rho}^{\mathbf{u}_h}-\boldsymbol{\rho}^{\mathbf{u}}\|_{L^4(e)}.\end{aligned}$$

Finally, similar to the bounds (6.49) and (6.51), we can estimate

$$|l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\rho}) - l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\rho})| \le \frac{\nu K_1}{32}\|\boldsymbol{\rho}\|_\varepsilon^2 + C\|\boldsymbol{\rho}\|^2\|\mathbf{u}_H\|_\varepsilon^4 + Ch^{2r}H^{2r+2-2\theta}. \tag{6.62}$$

We now substitute (6.49), (6.52), (6.57)-(6.59) and (6.62) in (6.48). Then multiply the resulting inequality by $e^{2\alpha t}$ and integrate with respect to time, and utilize (1.14), Lemmas 1.7, 1.6 and 6.7, the fact $\boldsymbol{\rho}(0) = \mathbf{0}$, the Cauchy-Schwarz and Young's inequalities to find

$$\begin{aligned}&e^{2\alpha t}\|\boldsymbol{\rho}(t)\|^2 + \kappa K_1 e^{2\alpha t}\|\boldsymbol{\rho}(t)\|_\varepsilon^2 + (\nu K_1 - 2\alpha C_2 - 2\alpha\kappa K_2)\int_0^t e^{2\alpha s}\|\boldsymbol{\rho}(s)\|_\varepsilon^2\\ &\le C\int_0^t e^{2\alpha s}\|\boldsymbol{\rho}(s)\|^2\|\mathbf{u}_H(s)\|_\varepsilon^4\,ds + C(h^{2r+2} + h^{2r}H^{2r+2-2\theta} + H^{4r+2-2\theta})\int_0^t e^{2\alpha s}\,ds.\end{aligned}$$

An application of Gronwall's lemma, (6.10), and with the choice $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$, and after a final multiplication by $e^{-2\alpha t}$ leads us to the desired estimate. □

Lemmas 6.7 and 6.8 will follow the following **Step 2** velocity error estimates:

$$\|\mathbf{e}_h\| \le K(t)(h^{r+1} + H^{2r+1-\theta}), \tag{6.63}$$

$$\|\mathbf{e}_h\|_\varepsilon \le K(t)(h^{r} + H^{2r+1-\theta}). \tag{6.64}$$

Remark 6.1. *Under the smallness condition on the data, that is,*

$$N = \sup_{\boldsymbol{\phi}_h,\mathbf{v}_h,\boldsymbol{w}_h\in \boldsymbol{V}_h} \frac{c^{\boldsymbol{\phi}_h}(\boldsymbol{w}_h,\mathbf{v}_h,\boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_\varepsilon^2\|\mathbf{v}_h\|_\varepsilon} \quad \text{and} \quad \frac{2NC_2}{K_1^2\nu^2}\|\mathbf{f}\| < 1. \tag{6.65}$$

the bounds (6.63) and (6.64) are uniform in time, that is,

$$\|\boldsymbol{e}_h\| \le C(h^{r+1} + H^{2r+1-\theta}),$$

$$\|\boldsymbol{e}_h\|_\varepsilon \le C(h^{r} + H^{2r+1-\theta}),$$

where the constant $C > 0$ is independent of time t.

Proof. First, modify the bounds (6.49) and (6.62) by using the first relation of (6.65) as follows:

$$
\begin{aligned}
c^{\mathbf{u}_h}(\boldsymbol{\rho}, \mathbf{u}_H, \boldsymbol{\rho})| &\leq N\|\mathbf{u}_H\|_\varepsilon \|\boldsymbol{\rho}\|_\varepsilon^2, \\
|l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\rho}) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\rho})| &\leq N\|\mathbf{u}_H\|_\varepsilon \|\boldsymbol{\rho}\|_\varepsilon^2 + Ch^r H^{r+1-\theta}\|\boldsymbol{\rho}\|_\varepsilon.
\end{aligned}
$$

The other bounds in the proof of Lemma 6.8 will remain exactly the same. Thus, with the help of the above two bounds, we rewrite (6.48) to obtain

$$
\frac{d}{dt}\|\boldsymbol{\rho}\|^2 + \kappa \frac{d}{dt}\big(a(\boldsymbol{\rho}, \boldsymbol{\rho})\big) + 2(\nu K_1 - 2N\|\mathbf{u}_H\|_\varepsilon)\|\boldsymbol{\rho}\|_\varepsilon^2 \leq C(h^{r+1} + H^{2r+1-\theta})\|\boldsymbol{\rho}\|_\varepsilon.
$$

Following the steps of Remark 3.1, and applying (6.11) and (6.65), one can show that

$$
\limsup_{t\to\infty} \|\boldsymbol{\rho}(t)\|_\varepsilon \leq C(h^{r+1} + H^{2r+1-\theta}). \tag{6.66}
$$

From (1.14), we now obtain

$$
\limsup_{t\to\infty} \|\boldsymbol{\rho}(t)\| \leq C(h^{r+1} + H^{2r+1-\theta}). \tag{6.67}
$$

A combination of (6.66), (6.67) and Lemma 6.7 will lead to the uniform in time estimates. □

The following lemma establishes the bounds for $\boldsymbol{\Theta}$.

Lemma 6.9. *Let the assumptions of Lemma 6.7 be hold true. Then, there holds:*

$$
\|\boldsymbol{\Theta}\|^2 + \kappa\|\boldsymbol{\Theta}\|_\varepsilon^2 + +e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\Theta}(s)\|_\varepsilon^2\, ds \leq K(t)(h^{2r+2} + h^{2r}H^{2r+2-2\theta} + H^{6r+4-4\theta}).
$$

Proof. Let us choose $\boldsymbol{\phi}_h = \boldsymbol{\Theta}$ in (6.47), and use Lemma 1.6 and (1.19) to find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\Theta}\|^2 + \frac{\kappa}{2}\frac{d}{dt}\big(a(\boldsymbol{\Theta}, \boldsymbol{\Theta})\big) + \nu K_1\|\boldsymbol{\Theta}\|_\varepsilon^2 \leq -(\boldsymbol{\zeta}_t, \boldsymbol{\Theta}) - c^{\mathbf{u}_h^*}(\boldsymbol{\Theta}, \mathbf{u}_H, \boldsymbol{\Theta}) \\
&-c^{\mathbf{u}_h^*}(\boldsymbol{\zeta}, \mathbf{u}_H, \boldsymbol{\Theta}) - c^{\mathbf{u}_H}(\mathbf{u}_H, \boldsymbol{\zeta}, \boldsymbol{\Theta}) - c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_h, \boldsymbol{\Theta}) - c^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{e}_h, \boldsymbol{\Theta}) - c^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{e}_H, \boldsymbol{\Theta}) \\
&\qquad + \big(l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\Theta}) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\Theta})\big) + \big(l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\Theta}) - l^{\mathbf{u}_h^*}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\Theta})\big) \\
&\qquad + \big(l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\Theta}) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\Theta})\big) + \big(l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\Theta}) - l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\Theta})\big) \\
&\quad = Q_1 + Q_2 + \cdots + Q_{11}.
\end{aligned} \tag{6.68}
$$

Following the proof steps as in Lemma 6.8, we obtain

$$
|Q_1| \leq \frac{\nu K_1}{64}\|\boldsymbol{\Theta}\|_\varepsilon^2 + Ch^{2r+2}, \tag{6.69}
$$

$$
|Q_2| \leq \frac{\nu K_1}{64}\|\boldsymbol{\Theta}\|_\varepsilon^2 + C\|\boldsymbol{\Theta}\|^2\|\mathbf{u}_H\|_\varepsilon^4, \tag{6.70}
$$

$$|Q_3| \leq \frac{\nu K_1}{64}\|\mathbf{\Theta}\|_\varepsilon^2 + Ch^{2r+2} + Ch^{2r}H^{2r+2-2\theta}, \tag{6.71}$$

$$|Q_4| \leq \frac{\nu K_1}{64}\|\mathbf{\Theta}\|_\varepsilon^2 + Ch^{2r+2} + Ch^{2r}H^{2r+2-2\theta}. \tag{6.72}$$

Utilize (6.36), (6.64), Theorem 6.1 and Young's inequality to obtain

$$\begin{aligned}|Q_5| \leq & C\|\mathbf{e}_H\|^{1-\theta}\|\mathbf{e}_H\|_\varepsilon^{\theta}\|\mathbf{e}_h\|_\varepsilon\|\mathbf{\Theta}\|_\varepsilon + CH^{-\theta}\|\mathbf{e}_H\|\|\mathbf{e}_h\|_\varepsilon\|\mathbf{\Theta}\|_\varepsilon \\ & + Ch^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{e}_h\|_\varepsilon\|\mathbf{\Theta}\|_\varepsilon \\ \leq & CH^{r+1-\theta}(h^r + H^{2r+1-\theta})\|\mathbf{\Theta}\|_\varepsilon + Ch^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}H^{\frac{r(1+\theta)}{2}}(h^r + H^{2r+1-\theta})\|\mathbf{\Theta}\|_\varepsilon \\ \leq & \frac{\nu K_1}{64}\|\mathbf{\Theta}\|_\varepsilon^2 + C(h^{2r}H^{2r+2-2\theta} + H^{6r+4-4\theta}).\end{aligned} \tag{6.73}$$

With the help of (2.56), (6.64) and Young's inequality, and observing $r \geq 1$, we obtain

$$|Q_6| \leq C\|\mathbf{e}_h\|_\varepsilon\|\mathbf{e}_h\|_\varepsilon\|\mathbf{\Theta}\|_\varepsilon \leq \frac{\nu K_1}{64}\|\mathbf{\Theta}\|_\varepsilon^2 + C(h^{2r+2} + H^{8r+4-4\theta}). \tag{6.74}$$

To handle Q_7, let us rewrite it in the following manner

$$Q_7 = -\, c^{\mathbf{e}_h}(\mathbf{e}_h, \mathbf{e}_H, \mathbf{\Theta}) + \big(l^{\mathbf{e}_h}(\mathbf{e}_h, \mathbf{e}_H, \mathbf{\Theta}) - l^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{e}_H, \mathbf{\Theta})\big) = Q_{71} + Q_{72}.$$

Now, Q_{71} is bounded employing (6.37), (6.64), Theorem 6.1 and Young's inequality as follows:

$$\begin{aligned}|Q_{71}| \leq & C\|\mathbf{e}_h\|_\varepsilon\|\mathbf{e}_H\|^{1-\theta}\|\mathbf{e}_H\|_\varepsilon^{\theta}\|\mathbf{\Theta}\|_\varepsilon + CH^{-\theta}\|\mathbf{e}_h\|_\varepsilon\|\mathbf{e}_H\|\|\mathbf{\Theta}\|_\varepsilon \\ & + Ch^{\frac{1-\theta}{2}}\|\mathbf{e}_h\|_\varepsilon\|\mathbf{e}_H\|^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{\Theta}\|_\varepsilon \\ \leq & \frac{\nu K_1}{64}\|\mathbf{\Theta}\|_\varepsilon^2 + C(h^{2r}H^{2r+2-2\theta} + H^{6r+4-4\theta}).\end{aligned} \tag{6.75}$$

We now handle Q_{72} and Q_8 together. Furthermore, recall the proof of Lemma 6.6 to write Q_{72} and Q_8 as

$$Q_{72} = \sum_{e\in\Gamma_h}\int_e (\{\mathbf{e}_h\}\cdot\mathbf{n}_e)[\mathbf{e}_H]\cdot(\mathbf{\Theta}^{\mathbf{e}_h} - \mathbf{\Theta}^{\mathbf{u}_h}),$$

$$Q_8 = \sum_{e\in\Gamma_h}\int_e (\{\mathbf{u}\}\cdot\mathbf{n}_e)[\mathbf{e}_H]\cdot(\mathbf{\Theta}^{\mathbf{u}_h} - \mathbf{\Theta}^{\mathbf{u}}).$$

Let us partition Γ_h for Q_{72} as follows:

$$\Gamma_h = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3,$$

where

$$\mathcal{H}_1 = \{e : \{\mathbf{u}_h\}\cdot\mathbf{n}_e \neq 0 \text{ and } \{\mathbf{e}_h\}\cdot\mathbf{n}_e \neq 0 \text{ a.e on } e\},$$

$$\mathcal{H}_2 = \{e : \{\mathbf{u}_h\} \cdot \mathbf{n}_e = 0 \text{ and } \{\mathbf{e}_h\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\},$$
$$\mathcal{H}_3 = \Gamma_h \setminus (\mathcal{H}_1 \cup \mathcal{H}_2).$$

And consider the partition of Γ_h for Q_8.

$$\mathcal{J}_1 = \{e : \{\mathbf{u}_h\} \cdot \mathbf{n}_e \neq 0 \text{ and } \{\mathbf{u}\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\},$$
$$\mathcal{J}_2 = \{e : \{\mathbf{u}_h\} \cdot \mathbf{n}_e = 0 \text{ and } \{\mathbf{u}\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\},$$
$$\mathcal{J}_3 = \Gamma_h \setminus (\mathcal{J}_1 \cup \mathcal{J}_2).$$

The integrals of Q_{72} and Q_8 over $\mathcal{H}_3$ and $\mathcal{J}_3$, respectively are zero. Thus, the sum of Q_{72} and Q_8 become

$$Q_{72} + Q_8 = \sum_{e\in\mathcal{H}_1} \int_e (\{\mathbf{e}_h\} \cdot \mathbf{n}_e)[\mathbf{e}_H] \cdot (\boldsymbol{\Theta}^{\mathbf{e}_h} - \boldsymbol{\Theta}^{\mathbf{u}_h}) + \sum_{e\in\mathcal{J}_1} \int_e (\{\mathbf{u}\} \cdot \mathbf{n}_e)[\mathbf{e}_H] \cdot (\boldsymbol{\Theta}^{\mathbf{u}_h} - \boldsymbol{\Theta}^{\mathbf{u}})$$
$$+ \sum_{e\in\mathcal{H}_2} \int_e (\{\mathbf{u}\} \cdot \mathbf{n}_e)[\mathbf{e}_H] \cdot \boldsymbol{\Theta}^{\mathbf{e}_h} - \sum_{e\in\mathcal{J}_2} \int_e (\{\mathbf{u}\} \cdot \mathbf{n}_e)[\mathbf{e}_H] \cdot \boldsymbol{\Theta}^{\mathbf{u}}.$$

Notice that, $\mathcal{H}_2$ and $\mathcal{J}_2$ are equal. The reason behind this is that, on $\mathcal{H}_2$, $\{\mathbf{e}_h\} \cdot \mathbf{n}_e = \{\mathbf{u}\}\cdot\mathbf{n}_e$ a.e on e as $\{\mathbf{u}_h\}\cdot\mathbf{n}_e = 0$. Therefore, the last two integrals of the above equality cancel out each other. With an identical approach as the proof of Lemma 6.6, that is, we only have to consider opposite signs of $\{\mathbf{u}_h\} \cdot \mathbf{n}_e$ and $\{\mathbf{e}_h\} \cdot \mathbf{n}_e$ on $\mathcal{H}_1$ and $\{\mathbf{u}_h\} \cdot \mathbf{n}_e$ and $\{\mathbf{u}\} \cdot \mathbf{n}_e$ on $\mathcal{J}_1$, one can obtain

$$|Q_{72}| + |Q_8| \leq \sum_{e\in\mathcal{H}_1} \|\{\mathbf{e}_h\}\|_{L^q(e)}\|[\mathbf{e}_H]\|_{L^p(e)}\|[\boldsymbol{\Theta}]\|_{L^2(e)}$$
$$+ \sum_{e\in\mathcal{J}_1} \|\{\mathbf{e}_h\}\|_{L^q(e)}\|[\mathbf{e}_H]\|_{L^p(e)}\|[\boldsymbol{\Theta}]\|_{L^2(e)}.$$

Using (1.14), (6.24), (6.25), (6.26), (6.64), Theorem 6.1 and Young's inequality, we find

$$\begin{aligned}|Q_{72}| + |Q_8| \leq & C \sum_{e\in\mathcal{H}_1} \sum_{i,j=1}^{2} \left(\|\mathbf{e}_h\|_{L^q(E_i)} + h^{\frac{1}{q}} \|\mathbf{e}_h\|^{\frac{q-1}{q}}_{L^{2(q-1)}(E_i)} \|\nabla\mathbf{e}_h\|^{\frac{1}{q}}_{L^2(E_i)}\right) \\ & \times \left(\|\mathbf{e}_H\|_{L^{\frac{2}{1-\theta}}(E_j)} + h_{E_j}^{\frac{1-\theta}{2}} \|\mathbf{e}_H\|^{\frac{1-\theta}{2}}_{L^2(E_j)} \|\nabla\mathbf{e}_H\|^{\frac{1+\theta}{2}}_{L^2(E_j)}\right) \left(\frac{\sigma_e}{|e|}\right)^{1/2} \|[\boldsymbol{\Theta}]\|_{L^2(e)} \\ & + C \sum_{e\in\mathcal{J}_1} \sum_{i,j=1}^{2} \left(\|\mathbf{e}_h\|_{L^q(E_i)} + h^{\frac{1}{q}} \|\mathbf{e}_h\|^{\frac{q-1}{q}}_{L^{2(q-1)}(E_i)} \|\nabla\mathbf{e}_h\|^{\frac{1}{q}}_{L^2(E_i)}\right) \\ & \times \left(\|\mathbf{e}_H\|_{L^{\frac{2}{1-\theta}}(E_j)} + h_{E_j}^{\frac{1-\theta}{2}} \|\mathbf{e}_H\|^{\frac{1-\theta}{2}}_{L^2(E_j)} \|\nabla\mathbf{e}_H\|^{\frac{1+\theta}{2}}_{L^2(E_j)}\right) \left(\frac{\sigma_e}{|e|}\right)^{1/2} \|[\boldsymbol{\Theta}]\|_{L^2(e)} \\ \leq & C\|\mathbf{e}_h\|_\varepsilon \|\mathbf{e}_H\|^{1-\theta} \|\mathbf{e}_H\|^{\theta}_\varepsilon \|\boldsymbol{\Theta}\|_\varepsilon + CH^{-\theta}\|\mathbf{e}_h\|_\varepsilon\|\mathbf{e}_H\|\|\boldsymbol{\Theta}\|_\varepsilon \\ & + Ch^{\frac{1-\theta}{2}}\|\mathbf{e}_h\|_\varepsilon\|\mathbf{e}_H\|^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|_\varepsilon^{\frac{1+\theta}{2}}\|\boldsymbol{\Theta}\|_\varepsilon\end{aligned}$$

$$\leq \frac{\nu K_1}{64}\|\boldsymbol{\Theta}\|_{\varepsilon}^2 + C(h^{2r}H^{2r+2-2\theta} + H^{6r+4-4\theta}). \tag{6.76}$$

Q_9 can be estimated similar to (6.62). In this case, we have to consider

$$|\{\mathbf{u}\}\cdot\mathbf{n}_e| \leq |\{\mathbf{u}-\mathbf{u}_h^*\}\cdot\mathbf{n}_e| \leq |\{\boldsymbol{\zeta}\}\cdot\mathbf{n}_e| + |\{\boldsymbol{\Theta}\}\cdot\mathbf{n}_e|$$

in place of (6.61). Thus, one can show that

$$|Q_9| \leq \frac{\nu K_1}{32}\|\boldsymbol{\Theta}\|_{\varepsilon}^2 + C\|\boldsymbol{\Theta}\|^2\|\mathbf{u}_H\|_{\varepsilon}^4 + Ch^{2r}H^{2r+2-2\theta}. \tag{6.77}$$

In addition, (1.14), (2.61), (6.64), Lemma 6.6, Theorem 6.1 and Young's inequality yield

$$\begin{aligned}|Q_{10}| \leq & C\|\mathbf{e}_H\|^{1-\theta}\|\mathbf{e}_H\|_{\varepsilon}^{\theta}\|\mathbf{e}_h\|_{\varepsilon}\|\boldsymbol{\Theta}\|_{\varepsilon} + CH^{-\theta}\|\mathbf{e}_H\|\|\mathbf{e}_h\|_{\varepsilon}\|\boldsymbol{\Theta}\|_{\varepsilon}\\ & + Ch^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|^{\frac{1-\theta}{2}}\|\mathbf{e}_H\|_{\varepsilon}^{\frac{1+\theta}{2}}\|\mathbf{e}_h\|_{\varepsilon}\|\boldsymbol{\Theta}\|_{\varepsilon}\\ \leq & \frac{\nu K_1}{64}\|\boldsymbol{\Theta}\|_{\varepsilon}^2 + C(h^{2r}H^{2r+2-2\theta} + H^{6r+4-4\theta})\end{aligned} \tag{6.78}$$

and

$$|Q_{11}| \leq C\|\mathbf{e}_h\|_{\varepsilon}\|\mathbf{e}_h\|_{\varepsilon}\|\boldsymbol{\Theta}\|_{\varepsilon} \leq \frac{\nu K_1}{64}\|\boldsymbol{\Theta}\|_{\varepsilon}^2 + C(h^{2r+2} + H^{8r+4-4\theta}). \tag{6.79}$$

Substitute the bounds (6.69)-(6.79) in (6.68). Multiply the resulting inequality by $e^{2\alpha t}$ and integrate with respect to time, and utilize Lemmas 1.7 and 1.6 and the fact $\boldsymbol{\Theta}(0) = \mathbf{0}$ to arrive at

$$\begin{aligned}& e^{2\alpha t}\|\boldsymbol{\Theta}(t)\|^2 + \kappa K_1\, e^{2\alpha t}\|\boldsymbol{\Theta}(t)\|_{\varepsilon}^2 + (\nu K_1 - 2\alpha C_2 - 2\alpha\kappa K_2)\int_0^t e^{2\alpha s}\|\boldsymbol{\Theta}(s)\|_{\varepsilon}^2\\ \leq & C\int_0^t e^{2\alpha s}\|\boldsymbol{\Theta}(s)\|^2\|\mathbf{u}_H(s)\|_{\varepsilon}^4\, ds\\ & + C(h^{2r+2} + h^{2r}H^{2r+2-2\theta} + H^{6r+4-4\theta} + H^{8r+4-4\theta})\int_0^t e^{2\alpha s}\, ds.\end{aligned}$$

Finally, using Gronwall's lemma, (6.10), and with the choice $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$, and multiplying by $e^{-2\alpha t}$, we arrive at the desired estimate. □

A combination of Lemmas 6.7 and 6.9 leads us to the following **Step 3** error estimates of the velocity.

Theorem 6.2. *Suppose the assumption (***A2***) holds true and let* $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. *In addition, let the semi-discrete initial velocity* $\boldsymbol{u}_h^*(0) \in \boldsymbol{V}_h$ *with* $\boldsymbol{u}_h^*(0) = \boldsymbol{P}_h\boldsymbol{u}_0$. *Then, there exists a constant* $K > 0$, *such that for* $t > 0$,

$$\|\boldsymbol{e}_h^*(t)\| \leq K(t)(h^{r+1} + h^rH^{r+1-\theta} + H^{3r+2-2\theta}),$$

and

$$\|\boldsymbol{e}_h^*(t)\|_\varepsilon \le K(t)(h^r + H^{3r+2-2\theta}).$$

Remark 6.2. *Under the condition (6.65) the estimates of Theorem 6.2 are uniform in time. This can be derived similar to Remark 6.1.*

6.4.2 Pressure Error Estimates

This subsection is devoted to the derivation of two-grid pressure error estimates. Before establishing the main result, we obtain the bounds for $\mathbf{e}_{ht}$ and $\mathbf{e}_{ht}^*$, which will play a significant role for achieving pressure error estimates.

Lemma 6.10. *Under the assumptions of Lemma 6.7, the error $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$ in* **Step 2** *for approximating the velocity satisfies*

$$\|\boldsymbol{e}_{ht}(t)\|^2 + \kappa\,\|\boldsymbol{e}_{ht}(t)\|_\varepsilon^2 \le K(t)(h^{2r} + H^{4r+2-2\theta}), \quad t > 0.$$

Proof. Consider (6.41) with $\boldsymbol{\phi}_h = \mathbf{P}_h\mathbf{e}_{ht} = \mathbf{e}_{ht} - (\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t)$, and employ Lemma 1.6 and definition of L^2-projection $\mathbf{P}_h$ to find

$$\begin{aligned}
\|\mathbf{P}_h\mathbf{e}_{ht}\|^2 + \kappa K_1\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon^2 &\le -\kappa\, a(\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \mathbf{P}_h\mathbf{e}_{ht}) - \nu\, a(\mathbf{P}_h\mathbf{e}_h, \mathbf{P}_h\mathbf{e}_{ht}) \\
&\quad - \nu\, a(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{P}_h\mathbf{e}_{ht}) - b(\mathbf{P}_h\mathbf{e}_{ht}, p) - c^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{u}_H, \mathbf{P}_h\mathbf{e}_{ht}) - c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{e}_h, \mathbf{P}_h\mathbf{e}_{ht}) \\
&\quad - c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_{ht}) + \big(l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_{ht}) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_{ht})\big) \\
&\quad + \big(l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_{ht}) - l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_{ht})\big).
\end{aligned} \tag{6.80}$$

A use of (2.56) with (6.64) to the fifth and sixth terms on the right hand side of (6.80) leads to

$$\begin{aligned}
|c^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{u}_H, \mathbf{P}_h\mathbf{e}_{ht})| + |c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{e}_h, \mathbf{P}_h\mathbf{e}_{ht})| &\le C\|\mathbf{e}_h\|_\varepsilon\|\mathbf{u}_H\|_\varepsilon\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon \\
&\le C(h^r + H^{2r+1-\theta})\|\mathbf{u}_H\|_\varepsilon\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon.
\end{aligned} \tag{6.81}$$

Using techniques which were used to derive (6.58) and Theorem 6.1, one can find

$$|c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_{ht})| \le CH^{2r+1-\theta}\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon. \tag{6.82}$$

Again, similar to (6.59) and using Theorem 6.1, we obtain

$$|l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_{ht}) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_{ht})| \le CH^{2r+1-\theta}\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon. \tag{6.83}$$

An application of (1.14), (6.64), (2.61) and Theorem 6.1 to the last term on the right hand side of (6.80) yields

$$\begin{aligned}|l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_{ht})-l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_{ht})| &\le C\|\mathbf{e}_h\|_\varepsilon\|\mathbf{e}_H\|_\varepsilon\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon\\ &\le C(h^r+H^{2r+1-\theta})H^r\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon. \end{aligned}\tag{6.84}$$

Using Lemmas 1.7 and 2.3, the first three terms on the right hand side of (6.80) are bounded as follows

$$\begin{aligned}&\kappa|a(\mathbf{u}_t-\mathbf{P}_h\mathbf{u}_t,\mathbf{P}_h\mathbf{e}_{ht})|+\nu|a(\mathbf{P}_h\mathbf{e}_h,\mathbf{P}_h\mathbf{e}_{ht})|+\nu|a(\mathbf{u}-\mathbf{P}_h\mathbf{u},\mathbf{P}_h\mathbf{e}_{ht})|\\ &\qquad\le Ch^r(\kappa|\mathbf{u}_t|_{r+1}+\nu|\mathbf{u}|_{r+1})\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon+C\nu\|\mathbf{P}_h\mathbf{e}_h\|_\varepsilon\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon.\end{aligned}\tag{6.85}$$

To bound the pressure term, we apply the definition of $\mathbf{V}_h$ and Lemma 2.4 to find

$$|b(\mathbf{P}_h\mathbf{e}_{ht},p)|=|b(\mathbf{P}_h\mathbf{e}_{ht},p-r_h(p))|\le Ch^r|p|_r\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon.\tag{6.86}$$

Collect the bounds (6.81)-(6.86) in (6.80), and use (6.10), Young's inequality and assumption (**A2**) to arrive at

$$\|\mathbf{P}_h\mathbf{e}_{ht}\|^2+\kappa K_1\|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon^2\le C\|\mathbf{P}_h\mathbf{e}_h\|_\varepsilon^2+C(h^{2r}+H^{4r+2-2\theta}).$$

An application of

$$\mathbf{e}_{ht}=\mathbf{P}_h\mathbf{e}_{ht}+(\mathbf{u}_t-\mathbf{P}_h\mathbf{u}_t),\quad \mathbf{P}_h\mathbf{e}_h=\mathbf{e}_h-(\mathbf{u}-\mathbf{P}_h\mathbf{u}),\tag{6.87}$$

triangle inequality, (6.64) and Lemma 2.2 leads to the completeness of the proof. □

Lemma 6.11. *Under the assumptions of Lemma 6.7, the error* $\boldsymbol{e}_h^*=\boldsymbol{u}-\boldsymbol{u}_h^*$ *in* **Step 3** *for approximating the velocity satisfies*

$$\|\boldsymbol{e}_{ht}^*(t)\|^2+\kappa\,\|\boldsymbol{e}_{ht}^*(t)\|_\varepsilon^2\le K(t)(h^{2r}+H^{6r+4-4\theta}),\quad t>0.$$

Proof. We choose $\boldsymbol{\phi}_h=\mathbf{P}_h\mathbf{e}_{ht}^*$ in (6.42) and use Lemma 1.6 and definition of L^2-projection $\mathbf{P}_h$ to find

$$\begin{aligned}\|\mathbf{P}_h\mathbf{e}_{ht}^*\|^2+\kappa K_1\|\mathbf{P}_h\mathbf{e}_{ht}^*\|_\varepsilon^2 &\le -\kappa\,a(\mathbf{u}_t-\mathbf{P}_h\mathbf{u}_t,\mathbf{P}_h\mathbf{e}_{ht}^*)-\nu\,a(\mathbf{P}_h\mathbf{e}_h^*,\mathbf{P}_h\mathbf{e}_{ht}^*)\\ -\nu\,a(\mathbf{u}-\mathbf{P}_h\mathbf{u},\mathbf{P}_h\mathbf{e}_{ht}^*)&-b(\mathbf{P}_h\mathbf{e}_{ht}^*,p)-c^{\mathbf{u}_h^*}(\mathbf{e}_h^*,\mathbf{u}_H,\mathbf{P}_h\mathbf{e}_{ht}^*)-c^{\mathbf{u}_H}(\mathbf{u}_H,\mathbf{e}_h^*,\mathbf{P}_h\mathbf{e}_{ht}^*)\\ &-c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_h,\mathbf{P}_h\mathbf{e}_{ht}^*)-c^{\mathbf{u}_h}(\mathbf{e}_h,\mathbf{e}_h,\mathbf{P}_h\mathbf{e}_{ht}^*)-c^{\mathbf{u}_h}(\mathbf{e}_h,\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_{ht}^*)\\ +\big(l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_{ht}^*)&-l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_{ht}^*)\big)+\big(l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_{ht}^*)-l^{\mathbf{u}_h^*}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_{ht}^*)\big)\\ +\big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_h,\mathbf{P}_h\mathbf{e}_{ht}^*)&-l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_h,\mathbf{P}_h\mathbf{e}_{ht}^*)\big)+\big(l^{\mathbf{u}}(\mathbf{u},\mathbf{e}_h,\mathbf{P}_h\mathbf{e}_{ht}^*)-l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_h,\mathbf{P}_h\mathbf{e}_{ht}^*)\big).\end{aligned}$$

The bounds for the terms on the right hand side of the above inequality are obtained in the same lines as the proof of Lemmas 6.9 and 6.10. Then proceed similar to the proof of Lemma 6.10 to complete the rest of the proof. □

The following theorem establishes a bound for **Step 2** semi-discrete pressure error.

Theorem 6.3. *Under the assumptions of Lemma 6.7, there exists a constant* $K > 0$*, such that, for all* $t > 0$*, the following error estimate holds true:*

$$\|(p - p_h)(t)\| \leq K(t)(h^r + H^{2r+1-\theta}).$$

Proof. For the **Step 2** pressure error estimate, we subtract (6.5) from (6.1) to arrive at

$$\begin{aligned} -b(\boldsymbol{\psi}_h, r_h(p) - p_h) &= (\mathbf{e}_{ht}, \boldsymbol{\psi}_h) + \kappa\, a(\mathbf{e}_{ht}, \boldsymbol{\psi}_h) + \nu\, a(\mathbf{e}_h, \boldsymbol{\psi}_h) + c^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{u}_H, \boldsymbol{\psi}_h) \\ &+ c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{e}_h, \boldsymbol{\psi}_h) + c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\psi}_h) - \big(l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\psi}_h) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\psi}_h)\big) \\ &- \big(l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\psi}_h) - l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\psi}_h)\big) + b(\boldsymbol{\psi}_h, p - r_h(p)), \quad \forall \boldsymbol{\psi}_h \in \mathbf{X}_h. \end{aligned} \tag{6.88}$$

Due to the discrete inf-sup condition presented in Lemma 1.8, there is $\boldsymbol{\psi}_h \in \mathbf{X}_h$ such that

$$b(\boldsymbol{\psi}_h, r_h(p) - p_h) = -\|r_h(p) - p_h\|^2, \tag{6.89}$$

$$\|\boldsymbol{\psi}_h\|_\varepsilon \leq \frac{1}{\beta^*}\|r_h(p) - p_h\|. \tag{6.90}$$

Apply (6.89) and (6.90) in (6.88) to obtain

$$\begin{aligned} \|r_h(p) - p_h\|^2 &= (\mathbf{e}_{ht}, \boldsymbol{\psi}_h) + \kappa\, a(\mathbf{P}_h\mathbf{e}_{ht}, \boldsymbol{\psi}_h) + \kappa\, a(\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \boldsymbol{\psi}_h) + \nu\, a(\mathbf{P}_h\mathbf{e}_h, \boldsymbol{\psi}_h) \\ &+ \nu\, a(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \boldsymbol{\psi}_h) + c^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{u}_H, \boldsymbol{\psi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{e}_h, \boldsymbol{\psi}_h) + c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\psi}_h) \\ &- \big(l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\psi}_h) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\psi}_h)\big) - \big(l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\psi}_h) - l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\psi}_h)\big) \\ &+ b(\boldsymbol{\psi}_h, p - r_h(p)). \end{aligned} \tag{6.91}$$

The terms on the right hand side of (6.91) can be bounded following the bounds (6.81)-(6.86) and the Cauchy-Schwarz inequality as follows

$$\begin{aligned} \|r_h(p) - p_h\|^2 \leq & C(\|\mathbf{e}_{ht}\| + \|\mathbf{P}_h\mathbf{e}_{ht}\|_\varepsilon + \|\mathbf{P}_h\mathbf{e}_h\|_\varepsilon + h^r + H^{2r+1-\theta} + h^r\|\mathbf{u}_H\|_\varepsilon \\ & + H^{2r+1-\theta}\|\mathbf{u}_H\|_\varepsilon + h^r|\mathbf{u}_t|_{r+1} + h^r|\mathbf{u}|_{r+1} + h^r|p|_r)\|\boldsymbol{\psi}_h\|_\varepsilon. \end{aligned}$$

Finally, a use of (6.90), the inequality $\|p - p_h\| \leq \|p - r_h(p)\| + \|r_h(p) - p_h\|$, approximation result (1.31), (6.10), (6.64), (6.87), assumption (**A2**), and Lemmas 2.2 and 6.10 leads us to the desired estimate. □

Next, we present an estimate for **Step 3** semi-discrete pressure error.

Theorem 6.4. *Under the assumptions of Lemma 6.7 and for all $t > 0$, the following error estimate holds true:*

$$\|(p - p_h^*)(t)\| \leq K(t)(h^r + H^{3r+2-2\theta}).$$

Proof. From the equations (6.1) and (6.6), we obtain

$$\begin{aligned}
& - b(\boldsymbol{\phi}_h, r_h(p) - p_h^*) = (\mathbf{e}_{ht}^*, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{P}_h\mathbf{e}_{ht}^*, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{P}_h\mathbf{e}_h^*, \boldsymbol{\phi}_h) \\
& + \nu\, a(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \boldsymbol{\phi}_h) + c^{\mathbf{u}_h^*}(\mathbf{e}_h^*, \mathbf{u}_H, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{e}_h^*, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_h, \boldsymbol{\phi}_h) \\
& + c^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{e}_h, \boldsymbol{\phi}_h) + c^{\mathbf{u}_h}(\mathbf{e}_h, \mathbf{e}_H, \boldsymbol{\phi}_h) - \big(l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h)\big) \\
& - \big(l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h) - l^{\mathbf{u}_h^*}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h)\big) - \big(l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\phi}_h) - l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\phi}_h)\big) \\
& - \big(l^{\mathbf{u}}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\phi}_h) - l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_h, \boldsymbol{\phi}_h)\big) + b(\boldsymbol{\phi}_h, p - r_h(p)). \qquad (6.92)
\end{aligned}$$

We follow the arguments used in the derivations of (6.73)-(6.79), (6.81), (6.85)-(6.86) to bound the right hand side of (6.92). Then proceeding as in the proof of Theorem 6.3 with p_h replaced by p_h^* in (6.89) and (6.90), we obtain

$$\|r_h(p) - p_h^*\| \leq C(\|\mathbf{e}_{ht}^*\| + \|\mathbf{P}_h\mathbf{e}_{ht}^*\|_\varepsilon + \|\mathbf{P}_h\mathbf{e}_h^*\|_\varepsilon + h^r + H^{3r+2-2\theta} + H^{4r+2-2\theta}).$$

Use triangle inequality, (1.31), Lemmas 2.2 and 6.11, Theorem 6.2 and assumption (**A2**) to complete the proof of this theorem. □

Remark 6.3. *Under the condition (6.65) the estimate of Theorem 6.4 is uniform in time.*

6.5 Fully Discrete DG Two-Grid Method

For discretization in time variable, we employ the backward Euler scheme in this section. We describe below the backward Euler scheme for the semi-discrete DG Two-grid algorithm (6.4)-(6.6) as follows:

Step 1 (Nonlinear system on $\mathcal{E}_h$): Find $(\mathbf{U}_H^n, P_H^n)_{n\geq 1} \in \mathbf{X}_H \times M_H$ such that for all $(\boldsymbol{\phi}_H, q_H) \in \mathbf{X}_H \times M_H$ and for $\mathbf{U}_H^0 = \mathbf{P}_H\mathbf{u}_0$

$$\left.\begin{aligned}
(\partial_t\mathbf{U}_H^n, \boldsymbol{\phi}_H) + \kappa\, a\,(\partial_t\mathbf{U}_H^n, \boldsymbol{\phi}_H) + \nu\, a\,(\mathbf{U}_H^n, \boldsymbol{\phi}_H) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_H) & \\
+ b(\boldsymbol{\phi}_H, P_H^n) = (\mathbf{f}^n, \boldsymbol{\phi}_H), & \\
b(\mathbf{U}_H^n, q_H) = 0. &
\end{aligned}\right\} \qquad (6.93)$$

Step 2 (Update on $\mathcal{E}_h$ with one Newton iteration): Find $(\mathbf{U}_h^n, P_h^n)_{n\geq 1} \in \mathbf{X}_h \times M_h$ such that for all $(\boldsymbol{\phi}_h, q_h) \in \mathbf{X}_h \times M_h$ and for $\mathbf{U}_h^0 = \mathbf{P}_h \mathbf{u}_0$

$$\left.\begin{aligned} (\partial_t \mathbf{U}_h^n, \boldsymbol{\phi}_h) + \kappa\, a\,(\partial_t \mathbf{U}_h^n, \boldsymbol{\phi}_H) + \nu\, a\,(\mathbf{U}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) \\ + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, P_h^n) = (\mathbf{f}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h), \\ b(\mathbf{U}_h^n, q_h) = 0. \end{aligned}\right\} \tag{6.94}$$

Step 3 (Correct on $\mathcal{E}_h$): Find $(\mathbf{U}^n, P^n)_{n\geq 1} \in \mathbf{X}_h \times M_h$ such that for all $(\boldsymbol{\phi}_h, q_h) \in \mathbf{X}_h \times M_h$ and for $\mathbf{U}^0 = \mathbf{P}_h \mathbf{u}_0$

$$\left.\begin{aligned} (\partial_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa\, a\,(\partial_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{U}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^n}(\mathbf{U}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) \\ + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, P^n) = (\mathbf{f}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h) \\ + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{U}_H^n - \mathbf{U}_h^n, \boldsymbol{\phi}_h), \\ b(\mathbf{U}^n, q_h) = 0. \end{aligned}\right\} \tag{6.95}$$

The two-grid DG backward Euler scheme applied to (6.7)-(6.9) is described below in the form of the following algorithm:

Step 1 (Nonlinear system on $\mathcal{E}_h$): Find $\mathbf{U}_H^n \in \mathbf{V}_H$ such that for all $\boldsymbol{\phi}_H \in \mathbf{V}_H$ and for $\mathbf{U}_H^0 = \mathbf{P}_H \mathbf{u}_0$

$$(\partial_t \mathbf{U}_H^n, \boldsymbol{\phi}_H) + \kappa\, a\,(\partial_t \mathbf{U}_H^n, \boldsymbol{\phi}_H) + \nu\, a\,(\mathbf{U}_H^n, \boldsymbol{\phi}_H) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_H) = (\mathbf{f}^n, \boldsymbol{\phi}_H). \tag{6.96}$$

Step 2 (Update on $\mathcal{E}_h$ with one Newton iteration): Find $\mathbf{U}_h^n \in \mathbf{V}_h$ such that for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ and for $\mathbf{U}_h^0 = \mathbf{P}_h \mathbf{u}_0$

$$\begin{aligned} (\partial_t \mathbf{U}_h^n, \boldsymbol{\phi}_h) + \kappa\, a\,(\partial_t \mathbf{U}_h^n, \boldsymbol{\phi}_H) + \nu\, a\,(\mathbf{U}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h) \\ = (\mathbf{f}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h). \end{aligned} \tag{6.97}$$

Step 3 (Correct on $\mathcal{E}_h$): Find $\mathbf{U}^n \in \mathbf{V}_h$ such that for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ and for $\mathbf{U}^0 = \mathbf{P}_h \mathbf{u}_0$

$$\begin{aligned} (\partial_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \kappa\, a\,(\partial_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{U}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^n}(\mathbf{U}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}^n, \boldsymbol{\phi}_h) \\ = (\mathbf{f}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{U}_H^n - \mathbf{U}_h^n, \boldsymbol{\phi}_h). \end{aligned} \tag{6.98}$$

6.5.1 *A priori* Bounds

Below in Lemma 6.12, we state *a priori* bounds of the **Step 1** fully discrete solution $\mathbf{U}_H^n$. For a proof, one may refer to Lemma 3.7 of Chapter 3.

Lemma 6.12. *Let* $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. *Further, let* $\boldsymbol{U}_H^0 = \boldsymbol{P}_H \boldsymbol{u}_0$. *Then, there exists a constant* $C > 0$, *such that, the solution* $\{\boldsymbol{U}_H^n\}_{n\geq 1}$ *of (6.96) satisfies the following a priori bounds:*

$$\|\boldsymbol{U}_H^n\|^2 + \kappa \|\boldsymbol{U}_H^n\|_\varepsilon^2 + e^{-2\alpha t_M} \Delta t \sum_{n=1}^{M} e^{2\alpha t_n} \|\boldsymbol{U}_H^n\|_\varepsilon^2 \leq C, \quad n = 1, \cdots, M.$$

Now, we provide a proof of *a priori estimates* of the solution $\mathbf{U}_h^n$ of (6.97).

Lemma 6.13. *Choose* k_0 *small so that* $0 < \Delta t \leq k_0$ *and* $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. *Further, let* $\boldsymbol{U}_h^0 = \boldsymbol{P}_h \boldsymbol{u}_0$. *Then, there exists a constant* $K_T > 0$, *such that, the solution* $\{\boldsymbol{U}_h^n\}_{n\geq 1}$ *of (6.97) satisfies the following a priori bounds:*

$$\|\boldsymbol{U}_h^n\|^2 + \kappa \|\boldsymbol{U}_h^n\|_\varepsilon^2 + e^{-2\alpha t_M} \Delta t \sum_{n=1}^{M} e^{2\alpha t_n} \|\boldsymbol{U}_h^n\|_\varepsilon^2 \leq K_T, \quad n = 1, \cdots, M.$$

Proof. First of all, we choose $\boldsymbol{\phi}_h = \mathbf{U}_h^n$ in (6.97). Note that

$$(\partial_t \mathbf{U}_h^n, \mathbf{U}_h^n) = \frac{1}{2}\left(\frac{1}{\Delta t}\|\mathbf{U}_h^n\|^2 - \frac{1}{\Delta t}\|\mathbf{U}_h^{n-1}\|^2 + \Delta t \|\partial_t \mathbf{U}_h^n\|^2\right) \geq \frac{1}{2}\partial_t \|\mathbf{U}_h^n\|^2, \tag{6.99}$$

$$\begin{aligned} a(\partial_t \mathbf{U}_h^n, \mathbf{U}_h^n) &= \frac{1}{2}\left(\frac{1}{\Delta t} a(\mathbf{U}_h^n, \mathbf{U}_h^n) - \frac{1}{\Delta t} a(\mathbf{U}_h^{n-1}, \mathbf{U}_h^{n-1}) + \Delta t a(\partial_t \mathbf{U}_h^n, \partial_t \mathbf{U}_h^n)\right) \\ &\geq \frac{1}{2}\partial_t a(\mathbf{U}_h^n, \mathbf{U}_h^n), \end{aligned} \tag{6.100}$$

and from (1.19) and Lemma 1.6, we obtain

$$\begin{aligned} \partial_t \|\mathbf{U}_h^n\|^2 + \kappa\, \partial_t a(\mathbf{U}_h^n, \mathbf{U}_h^n) + 2\nu K_1 \|\mathbf{U}_h^n\|_\varepsilon^2 \leq{}& 2\|\mathbf{f}^n\|\|\mathbf{U}_h^n\| + 2|c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{U}_H^n, \mathbf{U}_h^n)| \\ &+ 2|c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{U}_h^n)|. \end{aligned} \tag{6.101}$$

A use of (2.55) and (2.57) yields

$$\begin{aligned} &2|c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{U}_H^n, \mathbf{U}_h^n)| + 2|c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{U}_h^n)| \\ &\quad \leq C\big(\|\mathbf{U}_h^n\|^{1/2}\|\mathbf{U}_H^n\|_\varepsilon \|\mathbf{U}_h^n\|_\varepsilon^{3/2} + \|\mathbf{U}_H^n\|_\varepsilon^2 \|\mathbf{U}_h^n\|_\varepsilon\big). \end{aligned} \tag{6.102}$$

Observe that

$$\begin{aligned} &\sum_{n=1}^{m} \Delta t e^{2\alpha t_n}\big(\partial_t \|\mathbf{U}_h^n\|^2 + \kappa\, \partial_t a(\mathbf{U}_h^n, \mathbf{U}_h^n)\big) \\ &= \sum_{n=1}^{m} e^{2\alpha t_n}\big(\|\mathbf{U}_h^n\|^2 - \|\mathbf{U}_h^{n-1}\|^2 + \kappa\, a(\mathbf{U}_h^n, \mathbf{U}_h^n) - \kappa\, a(\mathbf{U}_h^{n-1}, \mathbf{U}_h^{n-1})\big) \\ &= e^{2\alpha t_m}\big(\|\mathbf{U}_h^m\|^2 + \kappa\, a(\mathbf{U}_h^m, \mathbf{U}_h^m)\big) - \sum_{n=1}^{m-1} e^{2\alpha t_n}(e^{2\alpha \Delta t} - 1)\big(\|\mathbf{U}_h^n\|^2 + \kappa\, a(\mathbf{U}_h^n, \mathbf{U}_h^n)\big) \\ &\qquad - e^{2\alpha \Delta t}\big(\|\mathbf{U}_h^0\|^2 + \kappa\, a(\mathbf{U}_h^0, \mathbf{U}_h^0)\big). \end{aligned} \tag{6.103}$$

Multiply (6.101) by $\Delta t e^{2\alpha t_n}$, sum over $n = 1$ to m, and using (1.14), (6.102), Lemmas 1.6 and 1.7, the above equality and Young's inequality, we have

$$e^{2\alpha t_m}\|\mathbf{U}_h^m\|^2 + K_1\kappa\, e^{2\alpha t_m}\|\mathbf{U}_h^m\|_\varepsilon^2 + \left(\nu K_1 - \frac{(C_2+\kappa K_2)(e^{2\alpha\Delta t}-1)}{\Delta t}\right)\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\|\mathbf{U}_h^n\|_\varepsilon^2$$
$$\le e^{2\alpha\Delta t}\|\mathbf{U}_h^0\|^2 + K_2\kappa\, e^{2\alpha\Delta t}\|\mathbf{U}_h^0\|_\varepsilon^2 + C\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\big(\|\mathbf{U}_h^n\|^2\|\mathbf{U}_H^n\|_\varepsilon^4 + \|\mathbf{U}_H^n\|_\varepsilon^4 + \|\mathbf{f}\|^2\big). \tag{6.104}$$

Choose α in such a way that

$$1 + \frac{\nu K_1\Delta t}{C_2+\kappa\, K_2} \ge e^{2\alpha\Delta t},$$

which implies

$$\alpha \le \frac{\nu K_1}{2(C_2+\kappa\, K_2)}.$$

Using discrete Gronwall's inequality and Lemma 6.12, and multiplying the resulting inequality through out by $e^{-2\alpha t_m}$, we establish our desired estimate. $\square$

The next lemma present *a priori* bound for the solution $\mathbf{U}^n$ of (6.98).

Lemma 6.14. *Choose* k_0 *small so that* $0 < \Delta t \le k_0$ *and* $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. *Further, let* $\boldsymbol{U}^0 = \boldsymbol{P}_h\boldsymbol{u}_0$. *Then, the solution* $\{\boldsymbol{U}^n\}_{n\ge1}$ *of (6.98) satisfies the following bounds:*

$$\|\boldsymbol{U}^n\|^2 + \kappa\,\|\boldsymbol{U}^n\|_\varepsilon^2 + e^{-2\alpha t_M}\,\Delta t\sum_{n=1}^{M} e^{2\alpha t_n}\|\boldsymbol{U}^n\|_\varepsilon^2 \le K_T, \quad n = 1,\cdots, M.$$

The proof technique of the above estimates is quite similar to the proof of Lemma 6.13. Thus, we skip the proof.

6.5.2 Fully Discrete Error Estimates

Next, the error estimates of backward Euler method are discussed. Considering the semi-discrete scheme (6.7)-(6.9) at $t = t_n$ and subtracting from (6.96)-(6.98), we arrive at the error equation for all three steps:

Error equation for **Step 1**: Set $\mathbf{e}_H^n = \mathbf{U}_H^n - \mathbf{u}_H(t_n) = \mathbf{U}_H^n - \mathbf{u}_H^n$, for fixed $n\in\mathbb{N}$, $1 \le n \le M$. Then, for all $\boldsymbol{\phi}_H \in \mathbf{V}_H$,

$$(\partial_t\mathbf{e}_H^n, \boldsymbol{\phi}_H) + \kappa\, a(\partial_t\mathbf{e}_H^n, \boldsymbol{\phi}_H) + \nu\, a(\mathbf{e}_H^n, \boldsymbol{\phi}_H) = (\mathbf{u}_{Ht}^n - \partial_t\mathbf{u}_H^n, \boldsymbol{\phi}_H)$$
$$+\kappa\, a(\mathbf{u}_{Ht}^n - \partial_t\mathbf{u}_H^n, \boldsymbol{\phi}_H) - c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_H) + c^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \boldsymbol{\phi}_H). \tag{6.105}$$

Error equation for **Step 2**: Set $\mathbf{e}_h^n = \mathbf{U}_h^n - \mathbf{u}_h(t_n) = \mathbf{U}_h^n - \mathbf{u}_h^n$, for fixed $n\in\mathbb{N}$, $1 \le n \le M$. Then, for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$

$$(\partial_t\mathbf{e}_h^n, \boldsymbol{\phi}_h) + \kappa\, a(\partial_t\mathbf{e}_h^n, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{e}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{e}_h^n, \boldsymbol{\phi}_h) = (\mathbf{u}_{ht}^n - \partial_t\mathbf{u}_h^n, \boldsymbol{\phi}_h)$$

$$
\begin{aligned}
&+\kappa\, a(\mathbf{u}_{ht}^n-\partial_t\mathbf{u}_h^n,\,\boldsymbol{\phi}_h)-c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\,\mathbf{e}_H^n,\,\boldsymbol{\phi}_h)-c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\,\mathbf{u}_H^n,\,\boldsymbol{\phi}_h)-c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\,\mathbf{u}_h^n,\,\boldsymbol{\phi}_h)\\
&+c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\,\mathbf{U}_H^n,\,\boldsymbol{\phi}_h)+c^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\,\mathbf{e}_H^n,\,\boldsymbol{\phi}_h)-\big(l^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\,\mathbf{u}_H^n,\,\boldsymbol{\phi}_h)-l^{\mathbf{u}_h^n}(\mathbf{u}_h^n,\,\mathbf{u}_H^n,\,\boldsymbol{\phi}_h)\big)\\
&-\big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_h^n,\,\boldsymbol{\phi}_h)-l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_h^n,\,\boldsymbol{\phi}_h)\big)-\big(l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_H^n,\,\boldsymbol{\phi}_h)-l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_H^n,\,\boldsymbol{\phi}_h)\big).
\end{aligned}
\tag{6.106}
$$

Error equation for **Step 3**: Set $\mathbf{e}^n=\mathbf{U}^n-\mathbf{u}_h^*(t_n)=\mathbf{U}^n-\mathbf{u}_h^{*n}$, for fixed $n\in\mathbb{N}$, $1\le n\le M$. Then, for all $\boldsymbol{\phi}_h\in\mathbf{V}_h$

$$
\begin{aligned}
&(\partial_t\mathbf{e}^n,\boldsymbol{\phi}_h)+\kappa\, a(\partial_t\mathbf{e}^n,\boldsymbol{\phi}_h)+\nu\, a(\mathbf{e}^n,\,\boldsymbol{\phi}_h)+c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\,\mathbf{e}^n,\,\boldsymbol{\phi}_h)=(\mathbf{u}_{ht}^{*n}-\partial_t\mathbf{u}_h^{*n},\,\boldsymbol{\phi}_h)\\
&+\kappa\, a(\mathbf{u}_{ht}^{*n}-\partial_t\mathbf{u}_h^{*n},\,\boldsymbol{\phi}_h)-c^{\mathbf{U}^n}(\mathbf{e}^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)-c^{\mathbf{U}^n}(\mathbf{u}_h^{*n},\mathbf{e}_H^n,\boldsymbol{\phi}_h)-c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\mathbf{u}_h^{*n},\boldsymbol{\phi}_h)\\
&+c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\mathbf{e}_h^n,\boldsymbol{\phi}_h)+c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h)+c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\mathbf{e}_H^n,\boldsymbol{\phi}_h)-c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\mathbf{e}_h^n,\boldsymbol{\phi}_h)\\
&+c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\mathbf{u}_H^n-\mathbf{u}_h^n,\boldsymbol{\phi}_h)-\big(l^{\mathbf{U}^n}(\mathbf{u}_h^{*n},\mathbf{u}_H^n,\boldsymbol{\phi}_h)-l^{\mathbf{u}_h^{*n}}(\mathbf{u}_h^{*n},\mathbf{u}_H^n,\boldsymbol{\phi}_h)\big)\\
&-\big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\mathbf{u}_h^{*n},\boldsymbol{\phi}_h)-l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\mathbf{u}_h^{*n},\boldsymbol{\phi}_h)\big)+\big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h)-l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h)\big)\\
&+\big(l^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\mathbf{u}_H^n-\mathbf{u}_h^n,\boldsymbol{\phi}_h)-l^{\mathbf{u}_h^n}(\mathbf{u}_h^n,\mathbf{u}_H^n-\mathbf{u}_h^n,\boldsymbol{\phi}_h)\big).
\end{aligned}
\tag{6.107}
$$

The following lemma provides us bounds for the **Step 1** error $\mathbf{e}_H^n$.

Lemma 6.15. *Suppose the assumptions of Lemma 6.12 hold true. Then, the following estimates hold true:*

$$
\|\boldsymbol{e}_H^n\|^2+\kappa\,\|\boldsymbol{e}_H^n\|_\varepsilon^2+e^{-2\alpha t_M}\Delta t\sum_{n=1}^{M}e^{2\alpha t_n}\|\boldsymbol{e}_H^n\|_\varepsilon^2\le K_T\Delta t^2. \tag{6.108}
$$

Proof. The estimate of this lemma is similar to Lemma 3.8 of Chapter 3. The only difference in the estimates of the nonlinear terms. For that, we set $\boldsymbol{\phi}_H=\mathbf{e}_H^n$ in (6.105) and rewrite the nonlinear terms to find

$$
\begin{aligned}
&(\partial_t\mathbf{e}_H^n,\mathbf{e}_H^n)+\kappa\, a(\partial_t\mathbf{e}_H^n,\mathbf{e}_H^n)+\nu a(\mathbf{e}_H^n,\mathbf{e}_H^n)+c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\mathbf{e}_H^n,\mathbf{e}_H^n)=(\mathbf{u}_{Ht}^n-\partial_t\mathbf{u}_H^n,\,\mathbf{e}_H^n)\\
&+\kappa\, a(\mathbf{u}_{Ht}^n-\partial_t\mathbf{u}_H^n,\,\mathbf{e}_H^n)+c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\mathbf{u}^n-\mathbf{u}_H^n,\mathbf{e}_H^n)-c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\mathbf{u}^n,\mathbf{e}_H^n)\\
&\qquad+l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\mathbf{u}_H^n,\mathbf{e}_H^n)-l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\mathbf{u}_H^n,\mathbf{e}_H^n).
\end{aligned}
\tag{6.109}
$$

With a similar technique as in the proof of Lemma 3.8, we can obtain

$$
|c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\mathbf{u}^n-\mathbf{u}_H^n,\mathbf{e}_H^n)|+|c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\mathbf{u}^n,\mathbf{e}_H^n)|\le C\|\mathbf{e}_H^n\|\|\mathbf{e}_H^n\|_\varepsilon.
$$

Since $\mathbf{u}^n$ is continuous, apply (2.60), Theorem 6.1, inequalities (1.35), (1.38) and (1.14) to arrive at

$$
\begin{aligned}
&|l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\mathbf{u}_H^n,\mathbf{e}_H^n)-l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\mathbf{u}_H^n,\mathbf{e}_H^n)|\\
&\quad=|l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\mathbf{u}^n-\mathbf{u}_H^n,\mathbf{e}_H^n)-l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\mathbf{u}^n-\mathbf{u}_H^n,\mathbf{e}_H^n)|
\end{aligned}
$$

$$\leq C\|\mathbf{e}_H^n\|_{L^4(\Omega)}\|\mathbf{u}-\mathbf{u}_H^n\|_\varepsilon\|\mathbf{e}_H^n\|_\varepsilon \leq C\|\mathbf{e}_H^n\|_\varepsilon\|\mathbf{u}-\mathbf{u}_H^n\|_\varepsilon\|\mathbf{e}_H^n\|_\varepsilon \leq C\|\mathbf{e}_H^n\|\|\mathbf{e}_H^n\|_\varepsilon.$$

Now, proceed similar to Lemma 3.8 will follow the desired estimate.

□

The next lemma establishes the bounds for the **Step 2** error $\mathbf{e}_h^n$.

Lemma 6.16. *Suppose the assumptions of Lemmas 6.8 and 6.13 hold true. Then, the following estimates hold true:*

$$\|\boldsymbol{e}_h^n\|^2+\kappa\,\|\boldsymbol{e}_h^n\|_\varepsilon^2+e^{-2\alpha t_M}\Delta t\sum_{n=1}^{M}e^{2\alpha t_n}\|\boldsymbol{e}_h^n\|_\varepsilon^2 \leq K_T\Delta t^2.$$

Proof. We put $\boldsymbol{\phi}_h=\mathbf{e}_h^n$ in error equation (6.106). A use of (1.19), Lemma 1.6, and (6.99) and (6.100) with $\mathbf{e}_h^n$ in place of $\mathbf{U}_h^n$ yields

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\mathbf{e}_h^n\|^2+\frac{\kappa}{2}\partial_t a(\mathbf{e}_h^n,\mathbf{e}_h^n)+\nu K_1\|\mathbf{e}_h^n\|_\varepsilon^2 \leq (\mathbf{u}_{ht}^n-\partial_t\mathbf{u}_h^n,\,\mathbf{e}_h^n)+\kappa\,a(\mathbf{u}_{ht}^n-\partial_t\mathbf{u}_h^n,\,\mathbf{e}_h^n)\\
&-c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\,\mathbf{u}_H^n,\,\mathbf{e}_h^n)-c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\,\mathbf{e}_H^n,\,\mathbf{e}_h^n)-c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\,\mathbf{u}_h^n,\,\mathbf{e}_h^n)+c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\,\mathbf{U}_H^n,\,\mathbf{e}_h^n)\\
&+c^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\,\mathbf{e}_H^n,\,\mathbf{e}_h^n)-\big(l^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\,\mathbf{u}_H^n,\,\mathbf{e}_h^n)-l^{\mathbf{u}_h^n}(\mathbf{u}_h^n,\,\mathbf{u}_H^n,\,\mathbf{e}_h^n)\big)\\
&-\big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_h^n,\,\mathbf{e}_h^n)-l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_h^n,\,\mathbf{e}_h^n)\big)-\big(l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_H^n,\,\mathbf{e}_h^n)-l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_H^n,\,\mathbf{e}_h^n)\big)\\
&=H_1+H_2+H_3+\cdots+H_{10}.
\end{aligned}\tag{6.110}$$

From (1.14), Lemma 1.7, (2.138), the Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned}
|H_1| &\leq C\,\Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hss}(s)\|^2\,ds\right)^{1/2}\|\mathbf{e}_h^n\|_\varepsilon\\
&\leq \frac{\nu K_1}{64}\|\mathbf{e}_h^n\|_\varepsilon^2+C\Delta t\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hss}(s)\|^2\,ds,
\end{aligned}\tag{6.111}$$

$$\begin{aligned}
|H_2| &\leq C\,\Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hss}(s)\|_\varepsilon^2\,ds\right)^{1/2}\|\mathbf{e}_h^n\|_\varepsilon\\
&\leq \frac{\nu K_1}{64}\|\mathbf{e}_h^n\|_\varepsilon^2+C\Delta t\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hss}(s)\|_\varepsilon^2\,ds.
\end{aligned}\tag{6.112}$$

An application of (2.57) and Young's inequality leads to

$$|H_3|\leq C\,\|\mathbf{e}_h^n\|^{1/2}\|\mathbf{u}_H^n\|_\varepsilon\|\mathbf{e}_h^n\|_\varepsilon^{3/2}\leq\frac{\nu K_1}{64}\|\mathbf{e}_h^n\|_\varepsilon^2+C\,\|\mathbf{u}_H^n\|_\varepsilon^4\|\mathbf{e}_h^n\|^2.\tag{6.113}$$

A use of (2.55) and Young's inequality leads to the following bound:

$$\begin{aligned}
&|H_4|+|H_5|+|H_6|+|H_7|\\
&\leq C\big(\|\mathbf{U}_h^n\|_\varepsilon\|\mathbf{e}_H^n\|_\varepsilon+\|\mathbf{e}_H^n\|_\varepsilon\|\mathbf{u}_h^n\|_\varepsilon+\|\mathbf{e}_H^n\|_\varepsilon\|\mathbf{U}_H^n\|_\varepsilon+\|\mathbf{u}_H^n\|_\varepsilon\|\mathbf{e}_H^n\|_\varepsilon\big)\|\mathbf{e}_h^n\|_\varepsilon
\end{aligned}$$

$$\leq \frac{\nu K_1}{64}\|\mathbf{e}_h^n\|_\varepsilon^2 + C\big(\|\mathbf{U}_h^n\|_\varepsilon^2\|\mathbf{e}_H^n\|_\varepsilon^2 + \|\mathbf{e}_H^n\|_\varepsilon^2\|\mathbf{u}_h^n\|_\varepsilon^2 + \|\mathbf{e}_H^n\|_\varepsilon^2\|\mathbf{U}_H^n\|_\varepsilon^2 + \|\mathbf{u}_H^n\|_\varepsilon^2\|\mathbf{e}_H^n\|_\varepsilon^2\big). \tag{6.114}$$

Use a result in (2.60), Lemma 2.6 and estimate (1.14), we find that

$$|H_8| \leq C\|\mathbf{e}_h^n\|_{L^4(\Omega)}\|\mathbf{u}_H^n\|_\varepsilon\|\mathbf{e}_h^n\|_\varepsilon \leq \frac{K_1\nu}{64}\|\mathbf{e}_h^n\|_\varepsilon^2 + C\,\|\mathbf{e}_h^n\|^2\|\mathbf{u}_H^n\|_\varepsilon^4 \tag{6.115}$$

$$|H_9| \leq C\|\mathbf{e}_H^n\|_{L^4(\Omega)}\|\mathbf{u}_h^n\|_\varepsilon\|\mathbf{e}_h^n\|_\varepsilon \leq \frac{K_1\nu}{64}\|\mathbf{e}_h^n\|_\varepsilon^2 + C\,\|\mathbf{e}_H^n\|_\varepsilon^2\|\mathbf{u}_h^n\|_\varepsilon^2 \tag{6.116}$$

$$|H_{10}| \leq C\|\mathbf{e}_H^n\|_{L^4(\Omega)}\|\mathbf{u}_H^n\|_\varepsilon\|\mathbf{e}_h^n\|_\varepsilon \leq \frac{K_1\nu}{64}\|\mathbf{e}_h^n\|_\varepsilon^2 + C\,\|\mathbf{e}_H^n\|_\varepsilon^2\|\mathbf{u}_H^n\|_\varepsilon^2. \tag{6.117}$$

Substitute (6.111)-(6.117) in (6.110), multiply the resulting inequality by $\Delta t e^{2\alpha t_n}$, sum from $n = 1$ to $m\,(\leq M)$, and employ (6.103) with $\mathbf{e}_h^n$ in place of $\mathbf{U}_h^n$, and Lemmas 1.6 and 1.7 to obtain

$$\begin{aligned}
&e^{2\alpha t_m}\big(\|\mathbf{e}_h^m\|^2 + \kappa K_1\|\mathbf{e}_h^m\|_\varepsilon^2\big) + \nu K_1\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\|\mathbf{e}_h^n\|_\varepsilon^2 \\
&\leq (e^{2\alpha\Delta t} - 1)\sum_{n=1}^{m-1} e^{2\alpha t_n}\big(\|\mathbf{e}_h^n\|^2 + K_2\kappa\|\mathbf{e}_h^n\|_\varepsilon^2\big) + C\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\|\mathbf{e}_h^n\|^2\|\mathbf{u}_H^n\|_\varepsilon^4 \\
&\quad + C\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\big(\|\mathbf{U}_h^n\|_\varepsilon^2\|\mathbf{e}_H^n\|_\varepsilon^2 + \|\mathbf{e}_H^n\|_\varepsilon^2\|\mathbf{u}_h^n\|_\varepsilon^2 + \|\mathbf{e}_H^n\|_\varepsilon^2\|\mathbf{U}_H^n\|_\varepsilon^2 + \|\mathbf{u}_H^n\|_\varepsilon^2\|\mathbf{e}_H^n\|_\varepsilon^2\big) \\
&\qquad + C\Delta t^2\sum_{n=1}^{m} e^{2\alpha t_n}\int_{t_{n-1}}^{t_n}\big(\|\mathbf{u}_{hss}(s)\|^2 + \|\mathbf{u}_{hss}(s)\|_\varepsilon^2\big)\,ds.
\end{aligned}$$

We now apply (6.10), (6.12), (6.14), (6.108), Lemmas 6.12, 6.13, the fact $e^{2\alpha\Delta t} - 1 \leq C(\alpha)\Delta t$ and discrete Gronwall's lemma to arrive at the desired result. □

For **Step 3** velocity error bound, we have the following lemma.

Lemma 6.17. *Suppose the assumptions of Theorem 6.2 and Lemma 6.14 hold true. Then, the following estimates hold true:*

$$\|\boldsymbol{e}^n\|^2 + \kappa\,\|\boldsymbol{e}^n\|_\varepsilon^2 + e^{-2\alpha t_M}\Delta t\sum_{n=1}^{M} e^{2\alpha t_n}\|\boldsymbol{e}^n\|_\varepsilon^2 \leq K_T\Delta t^2.$$

Proof. Substitute $\boldsymbol{\phi}_h = \mathbf{e}^n$ in (6.107). Applying (1.19) and Lemma 1.6, we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\mathbf{e}_h^n\|^2 + \frac{\kappa}{2}\partial_t a(\mathbf{e}_h^n, \mathbf{e}_h^n) + \nu K_1\|\mathbf{e}_h^n\|_\varepsilon^2 \leq (\mathbf{u}_{ht}^{*n} - \partial_t\mathbf{u}_h^{*n}, \mathbf{e}^n) + \kappa\, a(\mathbf{u}_{ht}^{*n} - \partial_t\mathbf{u}_h^{*n}, \mathbf{e}^n) \\
&- c^{\mathbf{U}^n}(\mathbf{e}^n, \mathbf{U}_H^n, \mathbf{e}^n) - c^{\mathbf{U}^n}(\mathbf{u}_h^{*n}, \mathbf{e}_H^n, \mathbf{e}^n) - c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}_h^{*n}, \mathbf{e}^n) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{e}_h^n, \mathbf{e}^n) \\
&+ c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}_h^n, \mathbf{e}^n) + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{e}_H^n, \mathbf{e}^n) - c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{e}_h^n, \mathbf{e}^n) + c^{\mathbf{U}_h^n}(\mathbf{e}_h^n, \mathbf{u}_H^n - \mathbf{u}_h^n, \mathbf{e}^n) \\
&- \big(l^{\mathbf{U}^n}(\mathbf{u}_h^{*n}, \mathbf{u}_H^n, \mathbf{e}^n) - l^{\mathbf{u}_h^{*n}}(\mathbf{u}_h^{*n}, \mathbf{u}_H^n, \mathbf{e}^n)\big) - \big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \mathbf{e}^n) - l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \mathbf{e}^n)\big) \\
&+ \big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^n, \mathbf{e}^n) - l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^n, \mathbf{e}^n)\big) + \big(l^{\mathbf{U}_h^n}(\mathbf{u}_h^n, \mathbf{u}_H^n - \mathbf{u}_h^n, \mathbf{e}^n) - l^{\mathbf{u}_h^n}(\mathbf{u}_h^n, \mathbf{u}_H^n - \mathbf{u}_h^n, \mathbf{e}^n)\big).
\end{aligned}$$

Following similar set of arguments as in Lemma 6.16, we can bound the right hand side terms of the above inequality and arrive at

$$\frac{1}{2}\partial_t\|\mathbf{e}_h^n\|^2 + \frac{\kappa}{2}\partial_t a(\mathbf{e}_h^n, \mathbf{e}_h^n) + \nu K_1\|\mathbf{e}_h^n\|_\varepsilon^2 \le C\Bigg(\Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}^*(s)\|^2\, ds\right)^{1/2}$$
$$+ \Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}^*(s)\|_\varepsilon^2\, ds\right)^{1/2} + \|\mathbf{u}_h^{*n}\|_\varepsilon\|\mathbf{e}_H^n\|_\varepsilon + \|\mathbf{U}_H^n\|_\varepsilon\|\mathbf{e}_h^n\|_\varepsilon + \|\mathbf{e}_H^n\|_\varepsilon\|\mathbf{u}_h^n\|_\varepsilon$$
$$+ \|\mathbf{U}_h^n\|_\varepsilon\|\mathbf{e}_H^n\|_\varepsilon + \|\mathbf{U}_h^n\|_\varepsilon\|\mathbf{e}_h^n\|_\varepsilon + \|\mathbf{e}_h^n\|_\varepsilon\|\mathbf{u}_H^n - \mathbf{u}_h^n\|_\varepsilon\Bigg)\|\mathbf{e}_h^n\|_\varepsilon$$
$$+ C\,\|\mathbf{e}^n\|^{1/2}(\|\mathbf{U}_H^n\|_\varepsilon + \|\mathbf{u}_H^n\|_\varepsilon)\|\mathbf{e}^n\|_\varepsilon^{3/2}.$$

Using (6.10), (6.12), (6.21), (6.23), (6.108), Lemmas 6.12, 6.13 and 6.16, Young's inequality, and proceed similarly as Lemma 6.16, we conclude the proof. □

Below, we present one of our main results from this chapter, which is **Step 3** fully discrete velocity error estimates, and an immediate consequence of Theorem 6.2 and Lemma 6.17.

Theorem 6.5. *Suppose the assumptions of Theorem 6.2 and Lemma 6.17 are satisfied. Then, the following estimates hold true:*

$$\|\boldsymbol{u}^n - \boldsymbol{U}^n\| \le K_T(h^{r+1} + h^r H^{r+1-\theta} + H^{3r+2-2\theta} + \Delta t),$$
$$\|\boldsymbol{u}^n - \boldsymbol{U}^n\|_\varepsilon \le K_T(h^r + H^{3r+2-2\theta} + \Delta t).$$

The next lemma presents the error bound related to **Step 3** pressure approximation.

Lemma 6.18. *Under the hypotheses of Lemma 6.17, the following estimates hold true:*

$$\|P^n - p_h^{*n}\| \le\; K_T\,\Delta t,\; 1 \le n \le M.$$

Proof. To find the pressure error estimates for **Step 3**, we subtract (6.6) with $t = t_n$ from (6.95) and arrive at

$$b(\boldsymbol{\phi}_h, P^n - p_h^{*n}) = -(\partial_t\mathbf{e}^n, \boldsymbol{\phi}_h) - \kappa\, a(\partial_t\mathbf{e}^n, \boldsymbol{\phi}_h) - \nu\, a(\mathbf{e}^n,\, \boldsymbol{\phi}_h) + (\mathbf{u}_{ht}^{*n} - \partial_t\mathbf{u}_h^{*n},\, \boldsymbol{\phi}_h)$$
$$+ \kappa\, a(\mathbf{u}_{ht}^{*n} - \partial_t\mathbf{u}_h^{*n},\, \boldsymbol{\phi}_h) - c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{e}^n,\, \boldsymbol{\phi}_h) - c^{\mathbf{U}^n}(\mathbf{e}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}^n}(\mathbf{u}_h^{*n}, \mathbf{e}_H^n, \boldsymbol{\phi}_h)$$
$$- c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{e}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{e}_H^n, \boldsymbol{\phi}_h)$$
$$- c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{e}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{e}_h^n, \mathbf{u}_H^n - \mathbf{u}_h^n, \boldsymbol{\phi}_h) - \big(l^{\mathbf{U}^n}(\mathbf{u}_h^{*n}, \mathbf{u}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{u}_h^{*n}}(\mathbf{u}_h^{*n}, \mathbf{u}_H^n, \boldsymbol{\phi}_h)\big)$$
$$- \big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h) - l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h)\big)$$
$$+ \big(l^{\mathbf{U}_h^n}(\mathbf{u}_h^n, \mathbf{u}_H^n - \mathbf{u}_h^n, \boldsymbol{\phi}_h) - l^{\mathbf{u}_h^n}(\mathbf{u}_h^n, \mathbf{u}_H^n - \mathbf{u}_h^n, \boldsymbol{\phi}_h)\big). \tag{6.118}$$

Observe that, to estimate $b(\boldsymbol{\phi}_h, P^n - p_h^{*n})$, we need a bound for $\partial_t \mathbf{e}^n$. To do so, consider (6.107) with $\boldsymbol{\phi}_h$ replaced by $\partial_t \mathbf{e}^n$ and utilize Lemma 1.6 to find

$$\begin{aligned}
&\|\partial_t \mathbf{e}^n\|^2 + K_1\kappa\|\partial_t \mathbf{e}^n\|_\varepsilon^2 \le (\mathbf{u}_{ht}^{*n} - \partial_t \mathbf{u}_h^{*n},\, \partial_t \mathbf{e}^n) + \kappa\, a(\mathbf{u}_{ht}^{*n} - \partial_t \mathbf{u}_h^{*n},\, \partial_t \mathbf{e}^n) - \nu\, a(\mathbf{e}^n,\, \partial_t \mathbf{e}^n) \\
&- c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{e}^n,\, \partial_t \mathbf{e}^n) - c^{\mathbf{U}^n}(\mathbf{e}^n, \mathbf{U}_H^n, \partial_t \mathbf{e}^n) - c^{\mathbf{U}^n}(\mathbf{u}_h^{*n}, \mathbf{e}_H^n, \partial_t \mathbf{e}^n) - c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}_h^{*n}, \partial_t \mathbf{e}^n) \\
&+ c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{e}_h^n, \partial_t \mathbf{e}^n) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}_h^n, \partial_t \mathbf{e}^n) + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{e}_H^n, \partial_t \mathbf{e}^n) - c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{e}_h^n, \partial_t \mathbf{e}^n) \\
&+ c^{\mathbf{U}_h^n}(\mathbf{e}_h^n, \mathbf{u}_H^n - \mathbf{u}_h^n, \partial_t \mathbf{e}^n) - \big(l^{\mathbf{U}^n}(\mathbf{u}_h^{*n}, \mathbf{u}_H^n, \partial_t \mathbf{e}^n) - l^{\mathbf{u}_h^{*n}}(\mathbf{u}_h^{*n}, \mathbf{u}_H^n, \partial_t \mathbf{e}^n)\big) \\
&- \big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \partial_t \mathbf{e}^n) - l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \partial_t \mathbf{e}^n)\big) + \big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^n, \partial_t \mathbf{e}^n) - l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^n, \partial_t \mathbf{e}^n)\big) \\
&+ \big(l^{\mathbf{U}_h^n}(\mathbf{u}_h^n, \mathbf{u}_H^n - \mathbf{u}_h^n, \partial_t \mathbf{e}^n) - l^{\mathbf{u}_h^n}(\mathbf{u}_h^n, \mathbf{u}_H^n - \mathbf{u}_h^n, \partial_t \mathbf{e}^n)\big).
\end{aligned}$$

With an application of (1.14), Lemma 1.7, (2.55), (2.60), (2.138) and the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\|\partial_t \mathbf{e}^n\|^2 + K_1\kappa\|\partial_t \mathbf{e}^n\|_\varepsilon^2 \le C \Bigg(&\Delta t \sup_{0<t<\infty} \|\mathbf{u}_{htt}^*(t)\| + \Delta t \sup_{0<t<\infty} \|\mathbf{u}_{htt}^*(t)\|_\varepsilon + \|\mathbf{e}^n\|_\varepsilon \\
&+ \|\mathbf{U}_H^n\|_\varepsilon \|\mathbf{e}^n\|_\varepsilon + \|\mathbf{u}_h^{*n}\|_\varepsilon \|\mathbf{e}_H^n\|_\varepsilon + \|\mathbf{U}_H^n\|_\varepsilon \|\mathbf{e}_h^n\|_\varepsilon + \|\mathbf{e}_H^n\|_\varepsilon \|\mathbf{u}_h^n\|_\varepsilon \\
&+ \|\mathbf{U}_h^n\|_\varepsilon \|\mathbf{e}_H^n\|_\varepsilon + \|\mathbf{U}_h^n\|_\varepsilon \|\mathbf{e}_h^n\|_\varepsilon + \|\mathbf{e}_h^n\|_\varepsilon \|\mathbf{u}_H^n - \mathbf{u}_h^n\|_\varepsilon + \|\mathbf{u}_H^n\|_\varepsilon \|\mathbf{e}^n\|_\varepsilon \Bigg) \|\partial_t \mathbf{e}^n\|_\varepsilon.
\end{aligned}$$

A use of (6.10), (6.12), (6.21), (6.23), (6.108), and Lemmas 6.12, 6.13, 6.16 and 6.17, and Young's inequality leads to

$$\|\partial_t \mathbf{e}^n\|^2 + K_1\kappa\|\partial_t \mathbf{e}^n\|_\varepsilon^2 \le C\Delta t^2. \tag{6.119}$$

Apply Lemma 1.8 and bound the terms on the right hand side of (6.118) following the arguments in deriving (6.119) to find

$$\|P^n - p_h^{*n}\| \le C(\|\partial_t \mathbf{e}^n\| + \|\partial_t \mathbf{e}^n\|_\varepsilon) + C\Delta t.$$

An application of (6.119) completes the proof of this lemma. □

The following theorem presents the **Step 3** fully discrete pressure error estimate and this can be derived using Theorem 6.4 and Lemma 6.18.

Theorem 6.6. *Under the assumptions of Theorem 6.4 and Lemma 6.18, the following error estimate holds true:*

$$\|p^n - P^n\| \le K_T(h^r + H^{3r+2-2\theta} + \Delta t).$$

6.6 Numerical Experiments

In this section, a few numerical experiments are performed and the theoretical findings are confirmed. For space discretization $\mathbb{P}_r - \mathbb{P}_{r-1}$, $r = 1, 2$, DG finite elements are employed and for time discretization, backward Euler method is applied. We choose the domain $\Omega = [0, 1]^2$. We have considered here three examples, where the first two are computed on the time interval $[0, .1]$, the time step $\Delta t = \mathcal{O}(h^{r+1})$, $h = \mathcal{O}(H^2)$ for $r = 1$ and $h = \mathcal{O}(H^{8/3})$ for $r = 2$. And the third example is analyzed on the time interval $[0, 100]$.

Example 6.1. *Consider the Kelvin-Voigt model with exact solution* $(\boldsymbol{u}, p) = \big((u_1(x, y, t), u_2(x, y, t)), p(x, y, t)\big)$ *as*

$$\begin{aligned} u_1(x, y, t) =& t\, e^{-t^2} \sin(2\pi y)(1 - \cos(2\pi x)), \\ u_2(x, y, t) =& t\, e^{-t^2} \sin(2\pi x)(\cos(2\pi y) - 1), \\ p(x, y, t) =& 2\,\pi\, t\, e^{-t}(\cos(2\pi y) - \cos(2\pi x)). \end{aligned}$$

In Tables 6.1 and 6.2, we represent the computational errors and orders of convergence for the two-grid DG solution of (6.93)-(6.95) for $r = 1$ and 2 with viscositiy $\nu = 1$, respectively. Further, Tables 6.3 and 6.4 represent numerical errors and orders of convergence for $r = 1$ and 2, respectively, with $\nu = 1/100$. We set the penalty parameter $\sigma_e = 20$ and 40 for $r = 1$ and 2, respectively, and $\kappa = 10^{-2}$. We notice that the numerical outcomes of Tables 6.1-6.4 confirm the theoretically derived convergence orders, which is of $\mathcal{O}(h^{r+1})$ and $\mathcal{O}(h^r)$ in $\mathbf{L}^2$ and energy norms for velocity, and $\mathcal{O}(h^r)$ in L^2-norm for pressure, respectively.

Table 6.1: Errors for two-grid DG approximations and order of convergence for Example 6.1 with $r = 1$ and $\nu = 1$.

h	$\Vert\mathbf{u}(t_M) - \mathbf{U}^M\Vert$	order	$\Vert\mathbf{u}(t_M) - \mathbf{U}^M\Vert_\varepsilon$	order	$\Vert p(t_M) - P^M\Vert$	order
1/4	2.8123×10^{-2}		1.6005×10^{-1}		1.2165×10^{-1}	
1/8	6.1009×10^{-3}	2.2046	1.3713×10^{-1}	0.2229	9.9415×10^{-2}	0.2912
1/16	1.3311×10^{-3}	2.1963	7.6041×10^{-2}	0.8507	6.8502×10^{-2}	0.5373
1/32	3.0528×10^{-4}	2.1244	3.9204×10^{-2}	0.9557	4.0196×10^{-2}	0.7691
1/64	7.1213×10^{-5}	2.0999	1.9691×10^{-2}	0.9934	2.1271×10^{-2}	0.9181

Table 6.2: Errors for two-grid DG approximations and order of convergence for Example 6.1 with $r = 2$ and $\nu = 1$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}^M\|$	order	$\|\mathbf{u}(t_M) - \mathbf{U}^M\|_\varepsilon$	order	$\|p(t_M) - P^M\|$	order
1/4	4.1952×10^{-3}		9.7518×10^{-2}		1.1138×10^{-1}	
1/8	5.5623×10^{-4}	2.9149	2.2689×10^{-2}	2.1036	3.2260×10^{-2}	1.7876
1/16	7.0121×10^{-5}	2.9877	5.2048×10^{-3}	2.1241	8.1395×10^{-3}	1.9867
1/32	8.4578×10^{-6}	3.0515	1.2441×10^{-3}	2.0646	2.0689×10^{-3}	1.9760

Table 6.3: Errors for two-grid DG approximations and order of convergence for Example 6.1 with $r = 1$ and $\nu = 1/100$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}^M\|$	order	$\|\mathbf{u}(t_M) - \mathbf{U}^M\|_\varepsilon$	order	$\|p(t_M) - P^M\|$	order
1/4	7.4361×10^{-2}		4.8762×10^{-1}		4.2109×10^{-2}	
1/8	2.4339×10^{-2}	1.6112	2.2497×10^{-1}	1.1159	3.7276×10^{-2}	0.1758
1/16	5.9670×10^{-3}	2.0281	1.0212×10^{-1}	1.1394	1.4008×10^{-2}	1.4119
1/32	1.4809×10^{-3}	2.0104	5.0580×10^{-2}	1.0137	6.0463×10^{-3}	1.2121
1/64	3.2607×10^{-4}	2.1832	2.5133×10^{-2}	1.0089	3.0008×10^{-3}	1.0107

Table 6.4: Errors for two-grid DG approximations and order of convergence for Example 6.1 with $r = 2$ and $\nu = 1/100$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}^M\|$	order	$\|\mathbf{u}(t_M) - \mathbf{U}^M\|_\varepsilon$	order	$\|p(t_M) - P^M\|$	order
1/4	1.9732×10^{-2}		2.5482×10^{-1}		1.0070×10^{-1}	
1/8	2.4634×10^{-3}	3.0018	4.6295×10^{-2}	2.4605	2.8521×10^{-2}	1.8200
1/16	3.3592×10^{-4}	2.8744	9.2606×10^{-3}	2.3216	7.1643×10^{-3}	1.9931
1/32	4.0441×10^{-5}	3.0542	2.1708×10^{-3}	2.0928	1.8493×10^{-3}	1.9538

Example 6.2. *In this example, the choice of right-hand side source function* $\mathbf{f}$ *is made in such a manner that the exact solution* $(\boldsymbol{u}, p) = \big((u_1(x,y,t),\, u_2(x,y,t)),\, p(x,y,t)\big)$ *takes the following form:*

$$
\begin{aligned}
u_1(x,y,t) =& 2(x^2 - 2x^3 + x^4)(2y - 6y^2 + 4y^3)\cos(t),\\
u_2(x,y,t) =& -2(y^2 - 2y^3 + y^4)(2x - 6x^2 + 4x^3)\cos(t),\\
p(x,y,t) =& 10\cos(t)(3y^2 - 1).
\end{aligned}
$$

Tables 6.5 and 6.8 depict the error and convergence orders of the two-grid DG scheme for $r = 1$ and 2, respectively, with $\nu = 1/40$ and $\kappa = 10^{-3}$. The penalty parameter σ_e is chosen same as in Example 6.1. These results verify the derived theoretical results.

Additionally, we compute the approximate solutions by the standard DG scheme to better assess the performance of our two-grid DG scheme with the same fine mesh, σ_e, κ and Δt. Tables 6.6 and 6.9 represent the numerical error and convergence orders for $r = 1$ and 2, respectively, employed by the standard DG scheme. By comparing Table 6.5 with Table 6.6 and Table 6.8 with Table 6.9, we observe that the accuracy of the numerical solutions by the proposed two-grid DG method is quite close to that of the standard DG method. In Tables 6.7 and 6.10, we compare the computational times taken to compute the two-grid DG solution and the standard DG solution corresponding to $r = 1$ and 2, respectively. The tables demonstrate that the proposed two-grid DG method requires significantly less computational time than the standard DG method. Additionally, as we refine the mesh more and more, the computational time gap increases between both solutions, namely the two-grid DG solution and the standard DG solution.

Table 6.5: Errors for two-grid DG approximations and order of convergence for Example 6.2 with $r = 1$ and $\nu = 1/40$.

h	$\\|\mathbf{u}(t_M) - \mathbf{U}^M\\|$	order	$\\|\mathbf{u}(t_M) - \mathbf{U}^M\\|_\varepsilon$	order	$\\|p(t_M) - P^M\\|$	order
1/4	5.3827×10^{-1}		5.1546×10^{0}		5.9317×10^{-1}	
1/8	2.4591×10^{-1}	1.1301	4.4511×10^{0}	0.2117	3.1194×10^{-1}	0.9271
1/16	5.7934×10^{-2}	2.0856	2.1725×10^{0}	1.0347	9.7262×10^{-2}	1.6813
1/32	1.3499×10^{-2}	2.1015	1.0248×10^{0}	1.0839	4.2954×10^{-2}	1.1790
1/64	3.1882×10^{-3}	2.0820	4.8998×10^{-1}	1.0646	2.0681×10^{-2}	1.0544

Table 6.6: Errors for standard DG approximations and order of convergence for Example 6.2 with $r = 1$ and $\nu = 1/40$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}_{DG}^M\|$	order	$\|\mathbf{u}(t_M) - \mathbf{U}_{DG}^M\|_\varepsilon$	order	$\|p(t_M) - P_{DG}^M\|$	order
1/4	5.2667×10^{-1}		4.8993×10^{0}		5.8976×10^{-1}	
1/8	2.3369×10^{-1}	1.1722	4.3182×10^{0}	0.1821	3.0574×10^{-1}	0.9478
1/16	5.7869×10^{-2}	2.0137	2.1599×10^{0}	0.9994	9.6112×10^{-2}	1.6695
1/32	1.3476×10^{-2}	2.1023	1.0220×10^{0}	1.0794	4.2762×10^{-2}	1.1683
1/64	3.1863×10^{-3}	2.0805	4.8985×10^{-1}	1.0610	2.0636×10^{-2}	1.0511

Table 6.7: Comparison of computational time between "standard DG solution" and the solution obtained by the "two-grid DG method" for Example 6.2 with $r = 1$.

h	Two-grid DG solution (in Seconds)	Standard DG solution (in Seconds)
1/4	0.11	0.12
1/8	1.31	1.70
1/16	16.74	26.79
1/32	238.68	434.40
1/64	3477.48	7337.49

Table 6.8: Errors for two-grid DG approximations and order of convergence for Example 6.2 with $r = 2$ and $\nu = 1/40$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}^M\|$	order	$\|\mathbf{u}(t_M) - \mathbf{U}^M\|_\varepsilon$	order	$\|p(t_M) - P^M\|$	order
1/4	3.1700×10^{-2}		5.9339×10^{-1}		3.1545×10^{-1}	
1/8	4.2622×10^{-3}	2.8947	1.6380×10^{-1}	1.8569	7.8565×10^{-2}	2.0054
1/16	5.4807×10^{-4}	2.9591	4.0639×10^{-2}	2.0110	1.9606×10^{-2}	2.0025
1/32	6.7881×10^{-5}	3.0132	1.0122×10^{-2}	2.0052	4.8975×10^{-3}	2.0012

Table 6.9: Errors for standard DG approximations and order of convergence for Example 6.2 with $r = 2$ and $\nu = 1/40$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}_{DG}^M\|$	order	$\|\mathbf{u}(t_M) - \mathbf{U}_{DG}^M\|_\varepsilon$	order	$\|p(t_M) - P_{DG}^M\|$	order
1/4	2.9187×10^{-2}		5.8205×10^{-1}		3.1542×10^{-1}	
1/8	4.1899×10^{-3}	2.8003	1.6117×10^{-1}	1.8525	7.8567×10^{-2}	2.0052
1/16	5.3754×10^{-4}	2.9624	4.0506×10^{-2}	1.9923	1.9606×10^{-2}	2.0026
1/32	6.7753×10^{-5}	2.9880	1.0093×10^{-2}	2.0046	4.8966×10^{-3}	2.0014

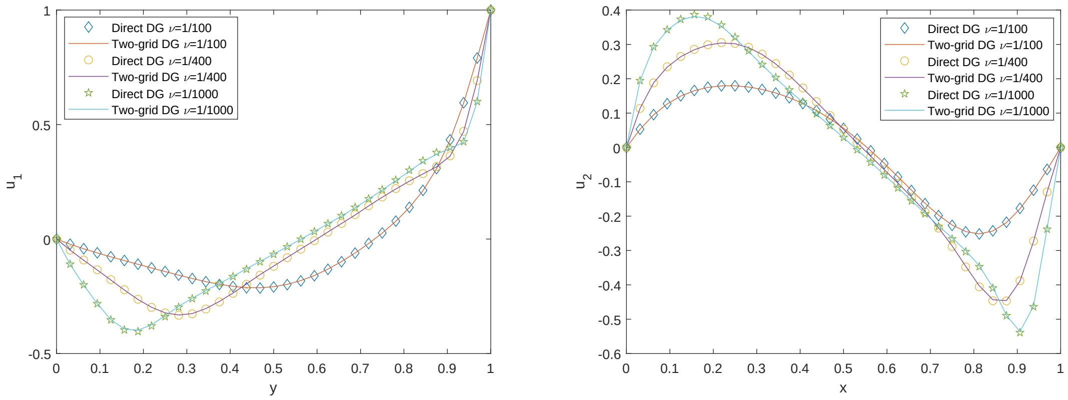
Table 6.10: Comparison of computational time between "standard DG solution" and the solution obtained by the "two-grid DG method" for Example 6.2 with $r = 2$.

h	Two-grid DG solution (in Seconds)	Standard DG solution (in Seconds)
1/4	2.68	3.22
1/8	39.69	53.98
1/16	766.87	1250.63
1/32	21374.86	38261.20

Example 6.3 (Benchmark Problem)**.** *In this case, the lid-driven cavity flow on the computational domain* $[0,1]^2$ *is examined. The velocity at the top of the boundary* $\boldsymbol{u} = (1,0)$*, is what majorly drives the flow of fluid. Other portions of the cavity boundaries are subject to the no-slip boundary conditions. On the body, no forces are acting i.e.,* $\mathbf{f} = (0,0)$*.*

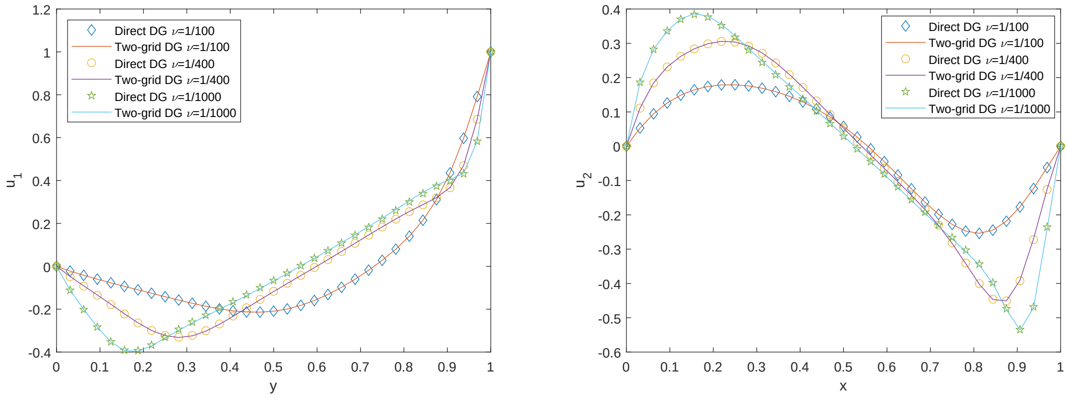
We perform a comparison along $(x, 0.5)$ and $(0.5, y)$ lines for velocity components for both two-grid DG and standard DG schemes with $r = 1$ and 2. For the time discretization backward Euler method is employed with $\Delta t = 0.01$ and final time $T = 100$. For the sake of simplicity, numerical simulations of the standard DG and two-grid DG methods are conducted with the uniform mesh sizes $h = 1/32$ and $H = 1/16$ (only for two-grid DG) to present the stability and accuracy of our method. For this test case, we choose different $\nu = \{1/100, 1/400, 1/1000\}$, $\kappa = 10\nu$ and the penalty parameter is $\sigma_e = 40$.

The comparisons of the horizontal velocity component at $x = 0.5$ and the vertical velocity component at $y = 0.5$ are shown in Figure 6.1 for $r = 1$ and in Figure 6.2 for $r = 2$ to indicate that the standard DG and two-grid DG methods produce similar numerical solutions.



(a) First velocity component along $x = 0.5$ (b) Second velocity component along $y = 0.5$

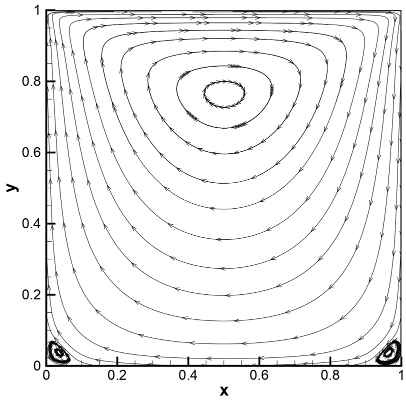
Figure 6.1: Velocity profiles through the geometric center for Example 6.3 with $r = 1$.



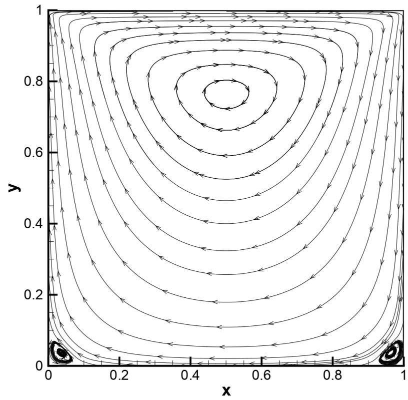
(a) First velocity component along $x = 0.5$ (b) Second velocity component along $y = 0.5$

Figure 6.2: Velocity profiles through the geometric center for Example 6.3 with $r = 2$.

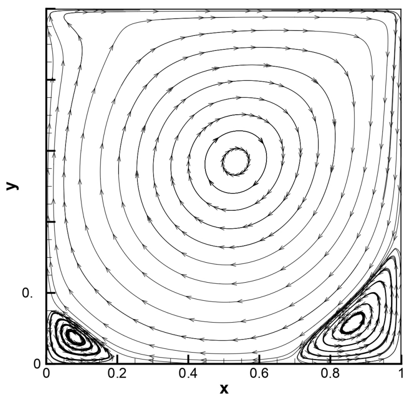
If $\kappa = 0$, the Kelvin-Voigt model transforms into the well-known NSEs. In Figure 6.3, we depict for various ν and κ the streamline plots of the NSEs and our model problem at final time $T = 20$ for $r = 2$. From these graphs, we observe that the the swirls situated in the corners are almost the same with $\nu = 1$ and present a huge difference for $\nu = 1/1000$ with different κ. This implies that the term $-\kappa\Delta\mathbf{u}_t$ has a small effect on the flow field with large ν and show an effect to stabilize the velocity field for small ν.



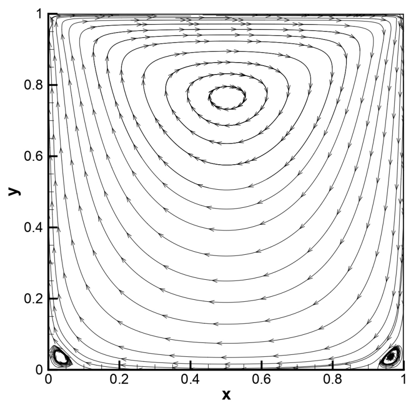
(a) Streamlines for $\nu = 1$ and $\kappa = 0$.



(b) Streamlines for $\nu = 1$ and $\kappa = 10$.



(c) Streamlines for $\nu = 1/1000$ and $\kappa = 0$.



(d) Streamlines for $\nu = 1/1000$ and $\kappa = 10$.

Figure 6.3: Streamlines for NSEs (left column) and model problem (right column).

6.7 Conclusion

This chapter proposes and analyses a three-step two-grid technique for the DG approximation of the Kelvin-Voigt viscoelastic model. Optimal order convergence estimates of the semi-discrete velocity approximation in $\mathbf{L}^2$-norm when $h = \mathcal{O}\big(H^{\min\left(r+1-\theta,\frac{3r+2-2\theta}{r+1}\right)}\big)$, in energy norm for $h = \mathcal{O}(H^{\frac{3r+2-2\theta}{r}})$ and pressure approximation in L^2-norm when $h = \mathcal{O}(H^{\frac{3r+2-2\theta}{r}})$ for $t > 0$ are proved. And with a smallness condition on data, these estimates are shown uniform with time. A complete discretization of the semi-discrete two-grid model is achieved by applying a backward Euler method in the time direction. Fully discrete error estimates are derived with optimal order of convergence. Finally, numerical results are depicted to show the effectiveness of the scheme.