Chapter 7

A Three-Step Two-Grid DG Method for the Oldroyd Model of Order One

In this chapter, we analyze a two-grid method combined with DG discretization for the equations of motion arising in Oldroyd model of order one. As in the previous chapter, that is, Chapter 6, we stick to the same algorithm. We discretize the time variable using the backward Euler method. Fully discrete optimal $\mathbf{L}^2$ and energy-norms error estimates for velocity and L^2-norm error estimates for pressure are derived for an appropriate choice of h and H. Finally, some numerical results are provided to validate the theoretical findings.

7.1 Introduction

At the very outset, let us revisit the variational problem for the Oldroyd model of order one in the spaces $\mathbf{X}$ and M from Chapter 4: Find the pair $(\mathbf{u}(t), p(t)) \in \mathbf{X} \times M, \quad t > 0$, such that

$$(\mathbf{u}_t(t), \boldsymbol{\phi}) + \mu\, a(\mathbf{u}(t), \boldsymbol{\phi}) + c^{\mathbf{u}(t)}(\mathbf{u}(t), \mathbf{u}(t), \boldsymbol{\phi}) + \int_0^t \beta(t-s) a(\mathbf{u}(s), \boldsymbol{\phi})\, ds + b(\boldsymbol{\phi}, p(t)) = (\mathbf{f}(t), \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathbf{X}, \tag{7.1}$$

$$b(\mathbf{u}(t), q) = 0 \quad \forall q \in M, \tag{7.2}$$

$$(\mathbf{u}(0), \boldsymbol{\phi}) = (\mathbf{u}_0, \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathbf{X}. \tag{7.3}$$

Utilizing the backward Euler method for temporal discretization, we present a DG two-grid algorithm for (7.1)-(7.3), which is stated as follows:

Step 1 (Nonlinear system on $\mathcal{E}_h$): Find $(\mathbf{U}_H^n, P_H^n)_{n\geq 1} \in \mathbf{X}_H \times M_H$ such that for all $(\boldsymbol{\phi}_H, q_H) \in \mathbf{X}_H \times M_H$ and for $\mathbf{U}_H^0 = \mathbf{P}_H \mathbf{u}_0$

$$\left.\begin{aligned} (\partial_t \mathbf{U}_H^n,\, \boldsymbol{\phi}_H) + \mu\, a\,(\mathbf{U}_H^n,\, \boldsymbol{\phi}_H) + a\,(q_r^n(\mathbf{U}_H),\, \boldsymbol{\phi}_H) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_H) \\ + b(\boldsymbol{\phi}_H,\, P_H^n) = (\mathbf{f}^n,\, \boldsymbol{\phi}_H), \\ b(\mathbf{U}_H^n,\, q_H) = 0. \end{aligned}\right\} \quad (7.4)$$

Step 2 (Update on $\mathcal{E}_h$ with one Newton iteration): Find $(\mathbf{U}_h^n, P_h^n)_{n\geq 1} \in \mathbf{X}_h \times M_h$ such that for all $(\boldsymbol{\phi}_h, q_h) \in \mathbf{X}_h \times M_h$ and for $\mathbf{U}_h^0 = \mathbf{P}_h \mathbf{u}_0$

$$\left.\begin{aligned} (\partial_t \mathbf{U}_h^n,\, \boldsymbol{\phi}_h) + \mu\, a\,(\mathbf{U}_h^n,\, \boldsymbol{\phi}_h) + a\,(q_r^n(\mathbf{U}_h),\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_h) \\ + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_h^n,\, \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h,\, P_h^n) = (\mathbf{f}^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_h), \\ b(\mathbf{U}_h^n,\, q_h) = 0. \end{aligned}\right\} \quad (7.5)$$

Step 3 (Correct on $\mathcal{E}_h$): Find $(\mathbf{U}_h^{*n}, P_h^{*n})_{n\geq 1} \in \mathbf{X}_h \times M_h$ such that for all $(\boldsymbol{\phi}_h, q_h) \in \mathbf{X}_h \times M_h$ and for $\mathbf{U}_h^{*0} = \mathbf{P}_h \mathbf{u}_0$

$$\left.\begin{aligned} (\partial_t \mathbf{U}_h^{*n},\, \boldsymbol{\phi}_h) + \mu\, a\,(\mathbf{U}_h^{*n},\, \boldsymbol{\phi}_h) + a\,(q_r^n(\mathbf{U}_h^*),\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^{*n}}(\mathbf{U}_h^{*n},\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_h) \\ + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_h^{*n},\, \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h,\, P_h^{*n}) = (\mathbf{f}^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_h^n,\, \boldsymbol{\phi}_h) \\ + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\, \mathbf{U}_H^n - \mathbf{U}_h^n,\, \boldsymbol{\phi}_h), \\ b(\mathbf{U}_h^{*n},\, q_h) = 0. \end{aligned}\right\} \quad (7.6)$$

We project the equations (7.4)-(7.6) in appropriate weakly divergence free spaces and the equations become:

Step 1 Seek $\mathbf{U}_H^n \in \mathbf{V}_H$ such that for all $\boldsymbol{\phi}_H \in \mathbf{V}_H$ and for $\mathbf{U}_H^0 = \mathbf{P}_H \mathbf{u}_0$

$$\begin{aligned} (\partial_t \mathbf{U}_H^n,\, \boldsymbol{\phi}_H) + \mu\, a\,(\mathbf{U}_H^n,\, \boldsymbol{\phi}_H) + a\,(q_r^n(\mathbf{U}_H),\, \boldsymbol{\phi}_H) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_H) \\ = (\mathbf{f}^n,\, \boldsymbol{\phi}_H). \end{aligned} \quad (7.7)$$

Step 2 Seek $\mathbf{U}_h^n \in \mathbf{V}_h$ such that for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ and for $\mathbf{U}_h^0 = \mathbf{P}_h \mathbf{u}_0$

$$\begin{aligned} (\partial_t \mathbf{U}_h^n,\, \boldsymbol{\phi}_h) + \mu\, a\,(\mathbf{U}_h^n,\, \boldsymbol{\phi}_h) + a\,(q_r^n(\mathbf{U}_h),\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_h^n,\, \boldsymbol{\phi}_h) \\ = (\mathbf{f}^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_h). \end{aligned} \quad (7.8)$$

Step 3 Seek $\mathbf{U}_h^{*n} \in \mathbf{V}_h$ such that for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ and for $\mathbf{U}_h^{*0} = \mathbf{P}_h \mathbf{u}_0$

$$\begin{aligned} (\partial_t \mathbf{U}_h^{*n},\, \boldsymbol{\phi}_h) + \mu\, a\,(\mathbf{U}_h^{*n},\, \boldsymbol{\phi}_h) + a\,(q_r^n(\mathbf{U}_h^*),\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^{*n}}(\mathbf{U}_h^{*n},\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_h) \quad (7.9) \\ + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_h^{*n},\, \boldsymbol{\phi}_h) = (\mathbf{f}^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_h^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\, \mathbf{U}_H^n - \mathbf{U}_h^n,\, \boldsymbol{\phi}_h). \end{aligned}$$

For the Oldroyd model of order one, as per our knowledge, only two works [23, 76] available in the literature which employed two-grid technique with CG approximations. In [76], a two-step fully discrete two-grid CG finite element approximation has been analyzed and optimal $L^\infty(\mathbf{H}^1)$-norm error estimate for velocity and $L^\infty(L^2)$-norm error estimate for the pressure with $h = \mathcal{O}(H^2 t^{-1/2})$ has been obtained. Nonetheless, $L^\infty(\mathbf{L}^2)$ error estimate was sub-optimal there. A three-step two-grid fully discrete CG scheme has been applied in [23], and optimal error estimates for velocity in $L^\infty(\mathbf{L}^2)$-norm when $h^2 = \mathcal{O}(H^{4-\theta} t^{-1/2})$, in $L^\infty(\mathbf{H}^1)$-norm when $h = \mathcal{O}(H^{3-\theta} t^{-1/2})$ and for pressure in $L^\infty(\mathbf{L}^2)$ norm with $h = \mathcal{O}(H^{3-\theta} t^{-1/2})$, for arbitrary small $\theta > 0$, are proved. Both of these works employ the backward Euler method for temporal discretization and non-smooth initial data, that is, $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$.

We would like to point out here that there is hardly any literature that studies a combination of DG method and two-grid technique for this model problem. If we proceed similar to Chapter 6 to derive the error estimates for velocity, we can reach up to the **Step 2** error

$$\|\mathbf{E}_h^n\|^2 \leq C(h^{2r+2} + H^{4r+2-2\theta} + \Delta t^2),$$

$$e^{-2\alpha t_m} \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{E}_h^n\|_\varepsilon^2 \leq C(h^{2r} + H^{4r+2-2\theta} + \Delta t^2),$$

where $\mathbf{E}_h^n = \mathbf{u}^n - \mathbf{U}_h^n$ and $\mathbf{U}_h^n =$ **Step 2** velocity approximation. If we proceed further to derive an optimal bound for $\|\mathbf{E}_h^n\|_\varepsilon$, we found difficulty in handling the nonlinear term due to the presence of upwinding and can only achieve sub-optimal estimates. For this reason, we have implemented the relation

$$K_1 \|\boldsymbol{\psi}_h\|_\varepsilon \leq \sup_{\boldsymbol{\phi}_h \in \mathbf{X}_h \setminus \{\mathbf{0}\}} \frac{a(\boldsymbol{\psi}_h, \boldsymbol{\phi}_h)}{\|\boldsymbol{\phi}_h\|_\varepsilon}, \quad \forall \boldsymbol{\psi}_h \in \mathbf{X}_h,$$

which is an immediate consequence of Lemma 1.6, and inf-sup condition from Lemma 1.8 to arrive at the combined optimal estimate for $\|\mathbf{E}_h^n\|_\varepsilon$ and fully discrete pressure error. Then, the estimates of **Step 2** will lead to the error estimates for **Step 3**. In this chapter, we have derived optimal fully discrete error estimates for velocity in $\mathbf{L}^2$-norm provided $h = \mathcal{O}\big(H^{\min\left(r+1-\theta, \frac{3r+2-2\theta}{r+1}\right)}\big)$, in energy norm for $h = \mathcal{O}(H^{\frac{3r+2-2\theta}{r}})$ and for pressure in L^2-norm provided $h = \mathcal{O}(H^{\frac{3r+2-2\theta}{r}})$. When $r = 1$, the largest between scaling between h and H for velocity in $\mathbf{L}^2$ and energy norms are $h = \mathcal{O}\big(H^{2-\theta}\big)$ and $h = \mathcal{O}(H^{5-2\theta})$, respectively, and for pressure in L^2-norm is $h = \mathcal{O}(H^{5-2\theta})$. Therefore, the results presented in this chapter is an improvement in scaling over the results presented in [23, 76] for energy norm velocity and L^2-norm pressure error estimates.

The following is a summary of the primary outcomes of this chapter:

- *A priori* bounds of the fully discrete two-grid DG solutions for each step.

- Optimal error bounds for the fully discrete two-grid DG velocity and pressure approximation for all three steps.
- Numerical experiments are carried out to show the performance of the scheme.

The rest of the part of this chapter comprises of the following sections: Section 7.2 deals with some new *a priori* results for the discrete solutions. In Section 7.3, the fully discrete **Step 2** error analysis is carried out and in Section 7.4, error estimates for **Step 3** solutions are derived. A few numerical experiments are given in Section 7.5 that are consistent with our theoretical findings. The chapter is concluded with a brief summary of the findings in Section 7.6.
Throughout this chapter, we will use C, $K(>0)$ as generic constants that depend on the given data, μ, α, γ, δ, K_1, K_2, C_2 but do not depend on h and Δt. Note that, K and C may grow algebraically with μ^{-1}. Further, the notations $K(t)$ and K_T will be used when they grow exponentially in time.

7.2 *A priori* Bounds

In this section, we present *a priori* bounds for the discrete solutions of all three steps. First, we state *a priori* bound for **Step 1** discrete velocity approximation. Then, we move on to **Step 1** velocity error estimates, which play a crucial role in the derivation of **Step 2** and **Step 3** *a priori* estimates. Next, we will derive *a priori* bounds for **Step 2** and **Step 3** velocity approximations.

In Lemma 7.1 and Theorem 7.1, we present **Step 1** *a priori* estimates and error estimates, respectively.

Lemma 7.1. *Let* $0 < \alpha < \min\left(\delta, \frac{\mu K_1}{2C_2}\right)$ *and* $\boldsymbol{U}_H^0 = \boldsymbol{P}_H \boldsymbol{u}_0$. *Then, the solution* $\{\boldsymbol{U}_H^n\}_{n\geq 1}$ *of (7.7) satisfies the following estimate:*

$$\|\boldsymbol{U}_H^n\|^2 + e^{-2\alpha t_M}\Delta t \sum_{n=1}^{M} e^{2\alpha t_n}\|\boldsymbol{U}_H^n\|_\varepsilon^2 \leq C, \quad n = 0, 1, ..., M,$$

where C *is a positive constant.*

Since the proof of the above lemma is similar to Lemma 4.6, it is skipped. And the following theorem provides estimates for the **Step 1** error $\mathbf{E}_H^n = \mathbf{u}^n - \mathbf{U}_H^n$ which can be established following the proof techniques of Theorem 4.4 and Lemma 4.7.

Theorem 7.1. *Suppose the assumption (**A4**) holds true and let* $0 < \alpha < \min\left(\delta, \frac{\mu K_1}{2C_2}\right)$, *and choose* k_0 *small so that* $0 < \Delta t \le k_0$. *In addition, let the discrete initial velocity* $\boldsymbol{U}_H^0 \in \boldsymbol{V}_H$ *with* $\boldsymbol{U}_H^0 = \boldsymbol{P}_H \boldsymbol{u}_0$. *Then, there exists a constant* $K_T > 0$, *such that*

$$\|\mathbf{E}_H^n\| \le K_T(H^{r+1} + \Delta t), \quad \|\mathbf{E}_H^n\|_\varepsilon \le K_T(H^r + \Delta t).$$

Proof. For the estimate of $\|\mathbf{E}_H^n\|_\varepsilon$, we only have to modify the nonlinear terms of Lemma 4.7. In this case, we define $\boldsymbol{\zeta}_n = \mathbf{U}_H^n - (\mathbf{\Pi}_H(\mathbf{u}))^n$ and $\boldsymbol{\eta}_n = \mathbf{u}^n - (\mathbf{\Pi}_H(\mathbf{u}))^n$. Thus, rewriting the nonlinear terms of (4.50) in the following manner

$$\begin{aligned} c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \partial_t \boldsymbol{\zeta}_n) - c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \partial_t \boldsymbol{\zeta}_n) &= -c^{\mathbf{U}_H^n}(\mathbf{u}^n, \mathbf{E}_H^n, \partial_t \boldsymbol{\zeta}_n) - c^{\mathbf{U}_H^n}(\mathbf{E}_H^n, \mathbf{u}^n, \partial_t \boldsymbol{\zeta}_n) \\ &+ c^{\mathbf{U}_H^n}(\boldsymbol{\eta}_n, \mathbf{E}_H^n, \partial_t \boldsymbol{\zeta}_n) - c^{\mathbf{U}_H^n}(\boldsymbol{\zeta}_n, \boldsymbol{\eta}_n, \partial_t \boldsymbol{\zeta}_n) + c^{\mathbf{U}_H^n}(\boldsymbol{\zeta}_n, \boldsymbol{\zeta}_n, \partial_t \boldsymbol{\zeta}_n) \end{aligned}$$

and estimating them similar to Lemma 4.7, and proceeding in an identical way as in Lemma 4.7, we establish the estimate for $\|\mathbf{E}_H^n\|_\varepsilon$.

To prove the estimate for $\|\mathbf{E}_H^n\|$, we follow Lemma 4.11 and break the error as $\mathbf{E}_H^n = (\mathbf{u}^n - \mathbf{v}_H^n) + (\mathbf{v}_H^n - \mathbf{U}_H^n) := \boldsymbol{\xi}^n + \boldsymbol{\eta}^n$. The nonlinear terms of (4.98) are rewritten as

$$\begin{aligned} c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\eta}^n) - c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\eta}^n) &= -c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\eta}^n, \boldsymbol{\eta}^n) - c^{\mathbf{U}_H^n}(\boldsymbol{\xi}^n, \mathbf{u}^n, \boldsymbol{\eta}^n) \\ + c^{\mathbf{U}_H^n}(\boldsymbol{\xi}^n, \boldsymbol{\xi}^n, \boldsymbol{\eta}^n) + c^{\mathbf{U}_H^n}(\boldsymbol{\eta}^n, \boldsymbol{\xi}^n, \boldsymbol{\eta}^n) &- c^{\mathbf{U}_H^n}(\boldsymbol{\eta}^n, \mathbf{u}^n, \boldsymbol{\eta}^n) - c^{\mathbf{u}^n}(\mathbf{u}^n, \boldsymbol{\xi}^n, \boldsymbol{\eta}^n) \\ &+ l^{\mathbf{u}^n}(\mathbf{u}^n, \boldsymbol{\xi}^n, \boldsymbol{\eta}^n) - l^{\mathbf{U}_H^n}(\mathbf{u}^n, \boldsymbol{\xi}^n, \boldsymbol{\eta}^n). \end{aligned}$$

Therefore, proceed similar to Lemma 4.11 and Theorem 4.4, we arrive at the estimate for $\|\mathbf{E}_H^n\|$. □

Note that, utilizing triangle inequality, Theorem 7.1 and assumption (**A4**), we have

$$\|\mathbf{U}_H^n\|_\varepsilon \le \|\mathbf{E}_H^n\|_\varepsilon + \|\mathbf{u}^n\|_1 \le C, \quad n = 0, 1, ..., M. \tag{7.10}$$

In the next lemma, we state *a priori* bounds of **Step 2** solution $\mathbf{U}_h^n$.

Lemma 7.2. *Let* $0 < \alpha < \min\left(\delta, \frac{\mu K_1}{2C_2}\right)$. *Choose* k_0 *small so that* $0 < \Delta t \le k_0$ *and* $\boldsymbol{U}_h^0 = \boldsymbol{P}_h \boldsymbol{u}_0$. *Then, the solution* $\{\boldsymbol{U}_h^n\}_{n\ge1}$ *of (7.8) satisfies the following estimate:*

$$\|\boldsymbol{U}_h^m\|^2 + e^{-2\alpha t_m}\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\boldsymbol{U}_h^n\|_\varepsilon^2 \le K_T, \quad m = 1, ..., M.$$

Proof. Substitute $\boldsymbol{\phi}_h = \mathbf{U}_h^n$ in (7.8), and employ (1.14), (1.19), Lemma 1.6, the Cauchy-Schwarz inequality and Young's inequality to obtain

$$\frac{1}{2}\partial_t \|\mathbf{U}_h^n\|^2 + \mu K_1 \|\mathbf{U}_h^n\|_\varepsilon^2 + a\left(q_r^n(\mathbf{U}_h), \mathbf{U}_h^n\right) \le \frac{\mu K_1}{2}\|\mathbf{U}_h^n\|_\varepsilon^2$$

$$+C\|\mathbf{f}^n\|^2+|c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\,\mathbf{U}_H^n,\,\mathbf{U}_h^n)|+|c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\,\mathbf{U}_H^n,\,\mathbf{U}_h^n)|. \tag{7.11}$$

Now, (2.55), (2.57) and Young's inequality yield

$$\begin{aligned}
&|c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\,\mathbf{U}_H^n,\,\mathbf{U}_h^n)|+|c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\,\mathbf{U}_H^n,\,\mathbf{U}_h^n)|\\
&\le C\big(\|\mathbf{U}_h^n\|^{1/2}\|\mathbf{U}_h^n\|_\varepsilon^{3/2}\|\mathbf{U}_H^n\|_\varepsilon+\|\mathbf{U}_H^n\|_\varepsilon^2\|\mathbf{U}_h^n\|_\varepsilon\big)\\
&\le \frac{\mu K_1}{2}\|\mathbf{U}_h^n\|_\varepsilon^2+C\big(\|\mathbf{U}_h^n\|^2\|\mathbf{U}_H^n\|_\varepsilon^4+\|\mathbf{U}_H^n\|_\varepsilon^4\big).
\end{aligned}$$

Substitute the above bound in (7.11), then multiply by $\Delta t e^{2\alpha t_n}$, sum over $n=1$ to m, and using (1.14) and Lemma 4.5, we have

$$\begin{aligned}
e^{2\alpha t_m}\|\mathbf{U}_h^m\|^2+\bigg(\mu K_1-\frac{C_2(e^{2\alpha\Delta t}-1)}{\Delta t}\bigg)\Delta t\sum_{n=1}^m e^{2\alpha t_n}\|\mathbf{U}_h^n\|_\varepsilon^2 \le e^{2\alpha\Delta t}\|\mathbf{U}^0\|^2\\
+C\Delta t\sum_{n=1}^m e^{2\alpha t_n}\big(\|\mathbf{f}\|^2+\|\mathbf{U}_h^n\|^2\|\mathbf{U}_H^n\|_\varepsilon^4+\|\mathbf{U}_H^n\|_\varepsilon^4\big).
\end{aligned}$$

Choose α in such a way that

$$1+\frac{\mu K_1\Delta t}{C_2}\ge e^{2\alpha\Delta t}.$$

Finally, employ discrete Gronwall's inequality and (7.10), and multiply the resulting inequality through out by $e^{-2\alpha t_m}$ to arrive at our desired estimate. □

In the following lemma, we state *a priori* estimates of **Step 3** fully discrete solution $\mathbf{U}_h^{*n}$. These estimates can be obtained from (7.9) and similar to Lemma 7.2. Hence the proof is skipped.

Lemma 7.3. *Let* $0<\alpha<\min\Big(\delta,\,\frac{\mu K_1}{2C_2}\Big)$. *Choose* k_0 *small so that* $0<\Delta t\le k_0$ *and* $\boldsymbol{U}_h^{*0}=\boldsymbol{P}_h\boldsymbol{u}_0$. *Then, the solution* $\{\boldsymbol{U}_h^{*n}\}_{n\ge1}$ *of (7.9) satisfies the following estimate:*

$$\|\boldsymbol{U}_h^{*m}\|^2+e^{-2\alpha t_m}\Delta t\sum_{n=1}^m e^{2\alpha t_n}\|\boldsymbol{U}_h^{*n}\|_\varepsilon^2\le K_T,\quad m=1,...,M.$$

The existence and uniqueness of the fully discrete solutions to the discrete problems (7.5)(or (7.8)) and (7.6) (or 7.9) of **Step 2** and **Step 3**, respectively, can be achieved using (1.19), Lemmas 1.6, 1.8, 4.5, 4.6, and following similar steps as in [72].

Next, we have a stability property for the L^2- projection $\mathbf{P}_h$ (see (2.11)) in $L^\infty(\Omega)$-norm which can be derived following the proof technique of [116, Lemma 6.1] and will be useful for future analysis:

$$\|\mathbf{P}_h\mathbf{v}\|_{L^\infty(\Omega)}\le C(|\mathbf{v}|_1+\|\mathbf{v}\|_{L^\infty(\Omega)}),\quad \forall\ \mathbf{v}\in\mathbf{J}_1\cap\mathbf{L}^\infty(\Omega). \tag{7.12}$$

7.3 Optimal Error Estimates in Step 2

This section deals with the derivation of the bounds of fully discrete velocity and pressure error in **Step 2**. Let us define $\mathbf{E}_h^n = \mathbf{u}^n - \mathbf{U}_h^n$ and $\mathbf{E}_h^{*n} = \mathbf{u}^n - \mathbf{U}_h^{*n}$. Below, we describe the error equations for **Step 2** and **Step 3**.

Error equation for **Step 2**: Consider equation (7.1) at $t = t_n$ and subtract it from (7.8), and for each $\boldsymbol{\phi}_h \in \mathbf{V}_h$,

$$
\begin{aligned}
(\partial_t \mathbf{E}_h^n, \boldsymbol{\phi}_h) + \mu\, a(\mathbf{E}_h^n, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{E}_h), \boldsymbol{\phi}_h) &= -(\mathbf{u}_t^n - \partial_t \mathbf{u}^n, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{u}), \boldsymbol{\phi}_h) \\
&- \int_0^{t_n} \beta(t_n - s) a(\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds - c^{\mathbf{U}_h^n}(\mathbf{E}_h^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h) \\
&- c^{\mathbf{U}_H^n}(\mathbf{E}_H^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) + \big(l^{\mathbf{U}_H^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)\big) \\
&+ \big(l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{U}_h^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)\big) - b(\boldsymbol{\phi}_h, p^n). \qquad (7.13)
\end{aligned}
$$

Error equation for **Step 3**: Consider equation (7.1) at $t = t_n$ and subtract it from (7.9), and for each $\boldsymbol{\phi}_h \in \mathbf{V}_h$,

$$
\begin{aligned}
(\partial_t \mathbf{E}_h^{*n}, \boldsymbol{\phi}_h) &+ \mu\, a(\mathbf{E}_h^{*n}, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{E}_h^*), \boldsymbol{\phi}_h) = -(\mathbf{u}_t^n - \partial_t \mathbf{u}^n, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{u}), \boldsymbol{\phi}_h) \\
&- \int_0^{t_n} \beta(t_n - s) a(\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds - c^{\mathbf{U}_h^{*n}}(\mathbf{E}_h^{*n}, \mathbf{U}_H^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{E}_h^{*n}, \boldsymbol{\phi}_h) \\
&- c^{\mathbf{U}_H^n}(\mathbf{E}_H^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{E}_h^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{E}_h^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) + \big(l^{\mathbf{U}_h^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) \\
&- l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{U}_h^{*n}}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{U}_H^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h) \\
&- l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h) - l^{\mathbf{U}_h^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h)\big) - b(\boldsymbol{\phi}_h, p^n). \qquad (7.14)
\end{aligned}
$$

To derive optimal error estimates of $\mathbf{E}_h^n$ and $\mathbf{E}_h^{*n}$ in $\mathbf{L}^2$ and energy-norms, we employ the modified Stokes-Volterra projection $\mathbf{S}_h^{vol}\mathbf{u}$ defined in (4.63). Let us split $\mathbf{E}_h^n$ and $\mathbf{E}_h^{*n}$ as

$$
\begin{aligned}
\mathbf{E}_h^n &:= (\mathbf{u}^n - \mathbf{S}_h^{vol}\mathbf{u}^n) + (\mathbf{S}_h^{vol}\mathbf{u}^n - \mathbf{U}_h^n) := \boldsymbol{\zeta}^n + \boldsymbol{\rho}^n, \qquad (7.15) \\
\mathbf{E}_h^{*n} &:= (\mathbf{u}^n - \mathbf{S}_h^{vol}\mathbf{u}^n) + (\mathbf{S}_h^{vol}\mathbf{u}^n - \mathbf{U}_h^{*n}) := \boldsymbol{\zeta}^n + \boldsymbol{\Theta}^n, \qquad (7.16)
\end{aligned}
$$

where $\boldsymbol{\zeta}^n = \mathbf{u}^n - \mathbf{S}_h^{vol}\mathbf{u}^n$, $\boldsymbol{\rho}^n = \mathbf{S}_h^{vol}\mathbf{u}^n - \mathbf{U}_h^n$ and $\boldsymbol{\Theta}^n = \mathbf{S}_h^{vol}\mathbf{u}^n - \mathbf{U}_h^{*n}$.

From the equations (4.63), (7.13) and (7.15), we arrive at the equation in $\boldsymbol{\rho}^n$ as

$$
\begin{aligned}
(\partial_t \boldsymbol{\rho}^n, \boldsymbol{\phi}_h) + \mu\, a(\boldsymbol{\rho}^n, \boldsymbol{\phi}_h) + a(q_r^n(\boldsymbol{\rho}), \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\rho}^n, \boldsymbol{\phi}_h) &= -(\mathbf{u}_t^n - \partial_t \mathbf{u}^n, \boldsymbol{\phi}_h) \\
- (\partial_t \boldsymbol{\zeta}^n, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{S}_h^{vol}\mathbf{u}), \boldsymbol{\phi}_h) &- \int_0^{t_n} \beta(t_n - s) a(\mathbf{S}_h^{vol}\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \\
- c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\boldsymbol{\zeta}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) &- c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\zeta}^n, \boldsymbol{\phi}_h) \\
- c^{\mathbf{U}_H^n}(\mathbf{E}_H^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) + \big(l^{\mathbf{U}_H^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) &- l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)\big)
\end{aligned}
$$

$$+\left(l^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\phi}_h)-l^{\mathbf{U}_h^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\phi}_h)\right). \tag{7.17}$$

Furthermore, using (4.63), (7.14) and (7.16), we obtain the equation in $\boldsymbol{\Theta}^n$ as

$$\begin{aligned}
&(\partial_t\boldsymbol{\Theta}^n,\boldsymbol{\phi}_h)+\mu\, a(\boldsymbol{\Theta}^n,\boldsymbol{\phi}_h)+a(q_r^n(\boldsymbol{\Theta}),\boldsymbol{\phi}_h)+c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\boldsymbol{\Theta}^n,\boldsymbol{\phi}_h)=-(\mathbf{u}_t^n-\partial_t\mathbf{u}^n,\boldsymbol{\phi}_h)\\
&-(\partial_t\boldsymbol{\zeta}^n,\boldsymbol{\phi}_h)+a(q_r^n(\mathbf{S}_h^{vol}\mathbf{u}),\boldsymbol{\phi}_h)-\int_0^{t_n}\beta(t_n-s)a(\mathbf{S}_h^{vol}\mathbf{u}(s),\boldsymbol{\phi}_h)\,ds\\
&-c^{\mathbf{U}_h^{*n}}(\boldsymbol{\Theta}^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)-c^{\mathbf{U}_h^{*n}}(\boldsymbol{\zeta}^n,\mathbf{U}_H^n,\boldsymbol{\phi}_h)-c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\boldsymbol{\zeta}^n,\boldsymbol{\phi}_h)\\
&-c^{\mathbf{U}_H^n}(\mathbf{E}_H^n,\mathbf{E}_h^n,\boldsymbol{\phi}_h)+c^{\mathbf{U}_h^n}(\mathbf{E}_h^n,\mathbf{E}_h^n,\boldsymbol{\phi}_h)-c^{\mathbf{U}_h^n}(\mathbf{E}_h^n,\mathbf{E}_H^n,\boldsymbol{\phi}_h)\\
&+\left(l^{\mathbf{U}_h^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\phi}_h)-l^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\phi}_h)\right)+\left(l^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\phi}_h)-l^{\mathbf{U}_h^{*n}}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\phi}_h)\right)\\
&+\left(l^{\mathbf{U}_H^n}(\mathbf{u}^n,\mathbf{E}_h^n,\boldsymbol{\phi}_h)-l^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{E}_h^n,\boldsymbol{\phi}_h)\right)+\left(l^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{E}_h^n,\boldsymbol{\phi}_h)-l^{\mathbf{U}_h^n}(\mathbf{u}^n,\mathbf{E}_h^n,\boldsymbol{\phi}_h)\right).
\end{aligned} \tag{7.18}$$

From (7.16), one can see that to derive optimal bounds for $\mathbf{E}_h^{*n}$, we need to bound $\boldsymbol{\Theta}^n$ in an optimal way. The bounds of $\boldsymbol{\Theta}^n$ depend on the bounds of $\boldsymbol{\zeta}^n$, $\mathbf{E}_H^n$, $\mathbf{E}_h^n$ that are present on the right side of (7.18). The estimates of $\boldsymbol{\zeta}^n$ and $\mathbf{E}_H^n$ are known from Lemma 4.8 and Theorem 7.1, respectively.

The next lemma states an estimate for $\boldsymbol{\rho}^n$.

Lemma 7.4. *Suppose the assumption (**A4**) holds true. Let* $0<\alpha<\min\left(\delta,\frac{\mu K_1}{2C_2}\right)$, $\boldsymbol{U}_h^0=\boldsymbol{P}_h\boldsymbol{u}_0$ *and* $\Delta t=\mathcal{O}(h^{r+1})$. *Then, the following holds true*

$$\|\boldsymbol{\rho}^n\|^2+e^{-2\alpha t_m}\Delta t\sum_{n=1}^{m}e^{2\alpha t_n}\|\boldsymbol{\rho}^n\|_\varepsilon^2\le K_T(h^{2r+2}+H^{4r+2-2\theta}+\Delta t^2).$$

Proof. Set $\boldsymbol{\phi}_h=\boldsymbol{\rho}^n$ in (7.17), and utilize Lemma 1.6 and (1.19) to arrive at

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\boldsymbol{\rho}^n\|^2+\mu K_1\|\boldsymbol{\rho}^n\|_\varepsilon^2+a(q_r^n(\boldsymbol{\rho}),\boldsymbol{\rho}^n)\le-(\mathbf{u}_t^n-\partial_t\mathbf{u}^n,\boldsymbol{\rho}^n)-(\partial_t\boldsymbol{\zeta}^n,\boldsymbol{\rho}^n)\\
&+a(q_r^n(\mathbf{S}_h^{vol}\mathbf{u}),\boldsymbol{\rho}^n)-\int_0^{t_n}\beta(t_n-s)a(\mathbf{S}_h^{vol}\mathbf{u}(s),\boldsymbol{\rho}^n)\,ds-c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n,\mathbf{U}_H^n,\boldsymbol{\rho}^n)\\
&-c^{\mathbf{U}_h^n}(\boldsymbol{\zeta}^n,\mathbf{U}_H^n,\boldsymbol{\rho}^n)-c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\boldsymbol{\zeta}^n,\boldsymbol{\rho}^n)-c^{\mathbf{U}_H^n}(\mathbf{E}_H^n,\mathbf{E}_H^n,\boldsymbol{\rho}^n)\\
&+\left(l^{\mathbf{U}_H^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\rho}^n)-l^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\rho}^n)\right)\\
&+\left(l^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\rho}^n)-l^{\mathbf{U}_h^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\rho}^n)\right).
\end{aligned} \tag{7.19}$$

The first, third and fourth terms on the right hand side of (7.19) can be estimated similar to H_1 of (4.41) and (4.97).

Applying (1.14), (4.65), the Cauchy-Schwarz inequality and assumption (**A4**), we arrive at

$$|(\partial_t\boldsymbol{\zeta}^n,\boldsymbol{\rho}^n)|\le\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|\boldsymbol{\zeta}_t(t)\|\|\boldsymbol{\rho}^n\|\,dt\le Ch^{r+1}\|\boldsymbol{\rho}^n\|_\varepsilon. \tag{7.20}$$

We now employ (2.57) to $c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{U}_H^n, \boldsymbol{\rho}^n)$ to find

$$|c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{U}_H^n, \boldsymbol{\rho}^n)| \le C\|\boldsymbol{\rho}^n\|^{1/2}\|\boldsymbol{\rho}^n\|_\varepsilon^{3/2}\|\mathbf{U}_H^n\|_\varepsilon. \tag{7.21}$$

Use the fact $\mathbf{U}_H^n = -\mathbf{E}_H^n + \mathbf{u}^n$ in the second argument of $c^{\mathbf{U}_h^n}(\boldsymbol{\zeta}^n, \mathbf{U}_H^n, \boldsymbol{\rho}^n)$ to arrive at

$$c^{\mathbf{U}_h^n}(\boldsymbol{\zeta}^n, \mathbf{U}_H^n, \boldsymbol{\rho}^n) = -c^{\mathbf{U}_h^n}(\boldsymbol{\zeta}^n, \mathbf{E}_H^n, \boldsymbol{\rho}^n) + c^{\mathbf{U}_h^n}(\boldsymbol{\zeta}^n, \mathbf{u}^n, \boldsymbol{\rho}^n). \tag{7.22}$$

An application of (6.36), (4.64), Theorem 7.1, assumption (**A4**), and the fact $h < H$ and $1-\theta > 0$ imply

$$\begin{aligned}|c^{\mathbf{U}_h^n}(\boldsymbol{\zeta}^n, \mathbf{E}_H^n, \boldsymbol{\rho}^n)| \le& C\|\boldsymbol{\zeta}^n\|^{1-\theta}\|\boldsymbol{\zeta}^n\|_\varepsilon^{\theta}\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon + Ch^{-\theta}\|\boldsymbol{\zeta}^n\|\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon \\ &+ Ch^{\frac{1-\theta}{2}}\|\boldsymbol{\zeta}^n\|^{\frac{1-\theta}{2}}\|\boldsymbol{\zeta}^n\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon \\ \le& Ch^{r+1-\theta}(H^r + \Delta t)\|\boldsymbol{\rho}\|_\varepsilon. \end{aligned}\tag{7.23}$$

From (2.59), (4.64) and assumption (**A4**), one can derive

$$|c^{\mathbf{U}_h^n}(\boldsymbol{\zeta}^n, \mathbf{u}^n, \boldsymbol{\rho}^n)| \le C\|\mathbf{u}^n\|_2(\|\boldsymbol{\zeta}^n\| + h\|\boldsymbol{\zeta}^n\|_\varepsilon)\|\boldsymbol{\rho}^n\|_\varepsilon \le Ch^{r+1}\|\boldsymbol{\rho}^n\|_\varepsilon.$$

Substitute the above two inequalities in (7.22), we arrive at

$$|c^{\mathbf{U}_h^n}(\boldsymbol{\zeta}^n, \mathbf{U}_H^n, \boldsymbol{\rho}^n)| \le C(h^{r+1} + h^{r+1-\theta}H^r + h^{r+1-\theta}\Delta t)\|\boldsymbol{\rho}^n\|_\varepsilon. \tag{7.24}$$

Again, we rewrite the seventh term on the right hand side of (7.19) as follows

$$\begin{aligned}c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\zeta}^n, \boldsymbol{\rho}^n) =& -c^{\mathbf{U}_H^n}(\mathbf{E}_H^n, \boldsymbol{\zeta}^n, \boldsymbol{\rho}^n) + c^{\mathbf{u}^n}(\mathbf{u}^n, \boldsymbol{\zeta}^n, \boldsymbol{\rho}^n) \\ &- \big(l^{\mathbf{u}^n}(\mathbf{u}^n, \boldsymbol{\zeta}^n, \boldsymbol{\rho}^n) - l^{\mathbf{U}_H^n}(\mathbf{u}^n, \boldsymbol{\zeta}^n, \boldsymbol{\rho}^n)\big). \end{aligned}\tag{7.25}$$

Employ (6.36), (4.64), Theorem 7.1, assumption (**A4**), and observe that $h < H$ and $1-\theta > 0$ to obtain

$$\begin{aligned}|c^{\mathbf{U}_H^n}(\mathbf{E}_H^n, \boldsymbol{\zeta}^n, \boldsymbol{\rho}^n)| \le& C\|\mathbf{E}_H^n\|^{1-\theta}\|\mathbf{E}_H^n\|_\varepsilon^{\theta}\|\boldsymbol{\zeta}^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon + CH^{-\theta}\|\mathbf{E}_H^n\|\|\boldsymbol{\zeta}^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon \\ &+ Ch^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon^{\frac{1+\theta}{2}}\|\boldsymbol{\zeta}^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon \\ \le Ch^r\big(H^{r+1-\theta} &+ H^{(r+1)(1-\theta)}\Delta t^{\theta} + H^{r\theta}\Delta t^{1-\theta} + H^{-\theta}\Delta t + h^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{1+\theta}{2}} \\ &+ h^{\frac{1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + \Delta t\big)\|\boldsymbol{\rho}^n\|_\varepsilon. \end{aligned}\tag{7.26}$$

With the help of (2.58), (4.64) and assumption (**A4**), we find

$$|c^{\mathbf{u}^n}(\mathbf{u}^n, \boldsymbol{\zeta}^n, \boldsymbol{\rho}^n)| \le C\|\mathbf{u}^n\|_2(\|\boldsymbol{\zeta}^n\| + h\|\boldsymbol{\zeta}^n\|_\varepsilon)\|\boldsymbol{\rho}^n\|_\varepsilon \le Ch^{r+1}\|\boldsymbol{\rho}^n\|_\varepsilon. \tag{7.27}$$

Using (4.64), Lemma 6.6, Theorem 7.1 and assumption (**A4**), one can obtain

$$|l^{\mathbf{u}^n}(\mathbf{u}^n, \boldsymbol{\zeta}^n, \boldsymbol{\rho}^n) - l^{\mathbf{U}_H^n}(\mathbf{u}^n, \boldsymbol{\zeta}^n, \boldsymbol{\rho}^n)| \le C\|\mathbf{E}_H^n\|^{1-\theta}\|\mathbf{E}_H^n\|_\varepsilon^{\theta}\|\boldsymbol{\zeta}^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon$$

$$
\begin{aligned}
&+ CH^{-\theta}\|\mathbf{E}_H^n\|\|\boldsymbol{\zeta}^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon + Ch^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon^{\frac{1+\theta}{2}}\|\boldsymbol{\zeta}^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon\\
\le& Ch^r\big(H^{r+1-\theta} + H^{(r+1)(1-\theta)}\Delta t^\theta + H^{r\theta}\Delta t^{1-\theta} + H^{-\theta}\Delta t\\
&+ h^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{1+\theta}{2}} + h^{\frac{1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + \Delta t\big)\|\boldsymbol{\rho}^n\|_\varepsilon. \qquad (7.28)
\end{aligned}
$$

Substituting (7.26)-(7.28) in (7.25), we arrive at

$$
\begin{aligned}
|c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\boldsymbol{\zeta}^n,\boldsymbol{\rho}^n)| \le& Ch^r\big(H^{r+1-\theta} + H^{(r+1)(1-\theta)}\Delta t^\theta + H^{r\theta}\Delta t^{1-\theta} + H^{-\theta}\Delta t\\
&+h^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{1+\theta}{2}} + h^{\frac{1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + \Delta t\big)\|\boldsymbol{\rho}^n\|_\varepsilon + Ch^{r+1}\|\boldsymbol{\rho}^n\|_\varepsilon. \qquad (7.29)
\end{aligned}
$$

A use of (6.36), Theorem 7.1 and assumption (**A4**), and notice that $h < H$ and $1-\theta > 0$ to find

$$
\begin{aligned}
|c^{\mathbf{U}_H^n}(\mathbf{E}_H^n,\mathbf{E}_H^n,\boldsymbol{\rho}^n)| \le& C\|\mathbf{E}_H^n\|^{1-\theta}\|\mathbf{E}_H^n\|_\varepsilon^{1+\theta}\|\boldsymbol{\rho}^n\|_\varepsilon + CH^{-\theta}\|\mathbf{E}_H^n\|\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon\\
&+ Ch^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon^{\frac{3+\theta}{2}}\|\boldsymbol{\rho}^n\|_\varepsilon\\
\le& C\big(H^{2r+1-\theta} + H^{(r+1)(1-\theta)}\Delta t^{1+\theta} + H^{r(1+\theta)}\Delta t^{1-\theta} + H^{r+1-\theta}\Delta t\\
&+ H^{r-\theta}\Delta t + H^{-\theta}\Delta t^2 + h^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{3+\theta}{2}}\\
&+ h^{\frac{1-\theta}{2}}H^{\frac{r(3+\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + \Delta t^2\big)\|\boldsymbol{\rho}^n\|_\varepsilon. \qquad (7.30)
\end{aligned}
$$

Employing Lemma 6.6 and Theorem 7.1, and similar to the above estimate, one can derive

$$
\begin{aligned}
&|l^{\mathbf{U}_H^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\rho}^n) - l^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\rho}^n)| \le C\|\mathbf{E}_H^n\|^{1-\theta}\|\mathbf{E}_H^n\|_\varepsilon^{1+\theta}\|\boldsymbol{\rho}^n\|_\varepsilon\\
&+ CH^{-\theta}\|\mathbf{E}_H^n\|\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon + Ch^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon^{\frac{3+\theta}{2}}\|\boldsymbol{\rho}^n\|_\varepsilon\\
\le& C\big(H^{2r+1-\theta} + H^{(r+1)(1-\theta)}\Delta t^{1+\theta} + H^{r(1+\theta)}\Delta t^{1-\theta} + H^{r+1-\theta}\Delta t + H^{r-\theta}\Delta t + H^{-\theta}\Delta t^2\\
&+ h^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{3+\theta}{2}} + h^{\frac{1-\theta}{2}}H^{\frac{r(3+\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + \Delta t^2\big)\|\boldsymbol{\rho}^n\|_\varepsilon. \qquad (7.31)
\end{aligned}
$$

Following the bounding technique of (6.62), and similar to the bounds (7.21) and (7.23), we can estimate

$$
\begin{aligned}
&|l^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\rho}^n) - l^{\mathbf{U}_h^n}(\mathbf{u}^n,\mathbf{E}_H^n,\boldsymbol{\rho}^n)|\\
&\le C\|\boldsymbol{\zeta}^n\|^{1-\theta}\|\boldsymbol{\zeta}^n\|_\varepsilon^{\theta}\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon + Ch^{-\theta}\|\boldsymbol{\zeta}^n\|\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon\\
&\quad + Ch^{\frac{1-\theta}{2}}\|\boldsymbol{\zeta}^n\|^{\frac{1-\theta}{2}}\|\boldsymbol{\zeta}^n\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\rho}^n\|_\varepsilon + C\|\boldsymbol{\rho}^n\|^{1/2}\|\boldsymbol{\rho}^n\|_\varepsilon^{3/2}\|\mathbf{U}_H^n\|_\varepsilon\\
&\le Ch^{r+1-\theta}(H^r + \Delta t)\|\boldsymbol{\rho}\|_\varepsilon + C\|\boldsymbol{\rho}^n\|^{1/2}\|\boldsymbol{\rho}^n\|_\varepsilon^{3/2}\|\mathbf{U}_H^n\|_\varepsilon. \qquad (7.32)
\end{aligned}
$$

Replace (7.20), (7.21), (7.24), (7.29)-(7.32) in (7.19). Again, multiply the resulting inequality by $\Delta t e^{2\alpha t_n}$, sum over $1 \le n \le m \le M$, and utilize (1.14), Lemma 4.5, the fact $\boldsymbol{\rho}^0 = \mathbf{0}$, and Young's inequality to find

$$
e^{2\alpha t_m}\|\boldsymbol{\rho}^m\|^2 + \mu K_1\Delta t\sum_{n=1}^m e^{2\alpha t_n}\|\boldsymbol{\rho}^n\|_\varepsilon^2 \le C\Delta t\sum_{n=1}^{m-1} e^{2\alpha t_n}\|\boldsymbol{\rho}^n\|^2 + C\Delta t\sum_{n=1}^m e^{2\alpha t_n}\|\mathbf{U}_H^n\|_\varepsilon^4\|\boldsymbol{\rho}^n\|^2
$$

$$
\begin{aligned}
&+ C\Delta t^3 \sum_{n=1}^{m} \int_0^{t_n} e^{-2(\delta-\alpha)(t_n-s)} e^{2\alpha s} \big(\|\mathbf{S}_h^{vol}\mathbf{u}(s)\|_\varepsilon^2 + \|\mathbf{S}_h^{vol}\mathbf{u}_s(s)\|_\varepsilon^2\big)\, ds \\
&+ C\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \big(h^{2r+2} + h^{2r+2-2\theta} H^{2r} + h^{2r+2-2\theta}\Delta t^2 + h^{2r} H^{2r+2-2\theta} \\
&+ h^{2r} H^{(2r+2)(1-\theta)}\Delta t^{2\theta} + h^{2r} H^{2r\theta}\Delta t^{2-2\theta} + h^{2r} H^{-2\theta}\Delta t^2 \\
&+ h^{2r+1-\theta} H^{(r+1)(1-\theta)}\Delta t^{1+\theta} + h^{2r+1-\theta} H^{r(1+\theta)}\Delta t^{1-\theta} + h^{2r}\Delta t^2 + H^{4r+2-2\theta} \\
&+ H^{2(r+1)(1-\theta)}\Delta t^{2+2\theta} + H^{2r(1+\theta)}\Delta t^{2-2\theta} + H^{2r+2-2\theta}\Delta t^2 + H^{2r-2\theta}\Delta t^2 + H^{-2\theta}\Delta t^4 \\
&+ h^{1-\theta} H^{(r+1)(1-\theta)}\Delta t^{3+\theta} + h^{1-\theta} H^{r(3+\theta)}\Delta t^{1-\theta} + \Delta t^4\big).
\end{aligned}
$$

An application of discrete Gronwall's lemma, the fact $\Delta t = \mathcal{O}(h^{r+1})$, (7.10), and after a final multiplication by $e^{-2\alpha t_m}$ leads us to the desired estimate. □

Now, (4.64), (7.15) and Lemma 7.4 will follow the following **Step 2** velocity error estimates:

$$
\|\mathbf{E}_h^n\|^2 \leq C(h^{2r+2} + H^{4r+2-2\theta} + \Delta t^2), \tag{7.33}
$$

$$
e^{-2\alpha t_m}\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{E}_h^n\|_\varepsilon^2 \leq C(h^{2r} + H^{4r+2-2\theta} + \Delta t^2). \tag{7.34}
$$

The following lemma provides us **Step 2** pressure error estimate.

Lemma 7.5. *Suppose the assumptions of Lemma 7.4 hold true and let $\Delta t = \mathcal{O}(h^r)$. Then, the following holds*

$$
\|\mathbf{E}_h^n\|_\varepsilon + \|p^n - P_h^n\| \leq K_T(h^r + H^{2r+1-\theta} + \Delta t).
$$

Proof. Subtract (7.5) from the equation (7.1) with $t = t_n$ and for each $\boldsymbol{\phi}_h \in \mathbf{X}_h$:

$$
\begin{aligned}
&(\partial_t \mathbf{P}_h\mathbf{E}_h^n, \boldsymbol{\phi}_h) + \mu\, a(\mathbf{P}_h\mathbf{E}_h^n, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{P}_h\mathbf{E}_h), \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, r_h(p^n) - P_h^n) \\
&= -(\mathbf{u}_t^n - \partial_t\mathbf{u}^n, \boldsymbol{\phi}_h) - \big(\partial_t(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n), \boldsymbol{\phi}_h\big) - \mu\, a(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n, \boldsymbol{\phi}_h) \\
&- a(q_r^n(\mathbf{u} - \mathbf{P}_h\mathbf{u}), \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{u}), \boldsymbol{\phi}_h) - \int_0^{t_n} \beta(t_n - s) a(\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \\
&- b(\boldsymbol{\phi}_h, p^n - r_h(p^n)) - c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\rho}^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\boldsymbol{\zeta}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) \\
&- c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\zeta}^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_H^n}(\mathbf{E}_H^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) + \big(l^{\mathbf{U}_H^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)\big) \\
&+ \big(l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{U}_h^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)\big).
\end{aligned} \tag{7.35}
$$

The first seven terms on the right hand side of the above equality are bounded similar to Lemmas 4.8 and 4.12:

$$
|(\mathbf{u}_t^n - \partial_t\mathbf{u}^n, \boldsymbol{\phi}_h)| \leq C\Delta t \sup_{1\leq n\leq M} \|\mathbf{u}_{tt}^n\| \|\boldsymbol{\phi}_h\|_\varepsilon, \tag{7.36}
$$

$$|\big(\partial_t(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n), \boldsymbol{\phi}_h\big)| \le Ch^r \sup_{1\le n\le M} |\mathbf{u}_t^n|_r \|\boldsymbol{\phi}_h\|_\varepsilon, \tag{7.37}$$

$$|\mu\, a(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n, \boldsymbol{\phi}_h)| \le Ch^r |\mathbf{u}^n|_{r+1} \|\boldsymbol{\phi}_h\|_\varepsilon, \tag{7.38}$$

$$|a(q_r^n(\mathbf{u} - \mathbf{P}_h\mathbf{u}), \boldsymbol{\phi}_h| \le Ch^r \Delta t \sum_{i=1}^{n} \beta(t_n - t_i) |\mathbf{u}^i|_{r+1} \|\boldsymbol{\phi}_h\|_\varepsilon, \tag{7.39}$$

$$\left| a(q_r^n(\mathbf{u}), \boldsymbol{\phi}_h) - \int_0^{t_n} \beta(t_n - s) a(\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \right| \le C\Delta t \int_0^{t_n} \big(|\mathbf{u}(s)|_1 + h|\mathbf{u}(s)|_2 + |\mathbf{u}_s(s)|_1 + h|\mathbf{u}_s(s)|_2\big)\, ds\, \|\boldsymbol{\phi}_h\|_\varepsilon, \tag{7.40}$$

$$|b(\boldsymbol{\phi}_h, p^n - r_h(p^n))| \le Ch^r |p^n|_r \|\boldsymbol{\phi}_h\|_\varepsilon. \tag{7.41}$$

To bound $c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h)$, we rewrite

$$c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) = -c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{u}^n, \boldsymbol{\phi}_h). \tag{7.42}$$

Thus, $c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)$ is estimated using the form of $c(\cdot,\cdot,\cdot)$ presented in (6.38), and (1.14), (1.37), (1.39), (6.25), Theorem 7.1, Lemma 7.4 and Hölder's inequality :

$$\begin{aligned}
c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) =& \frac{1}{2}\sum_{E\in\mathcal{E}_h} \int_E (\boldsymbol{\rho}^n \cdot \nabla \mathbf{E}_H^n) \cdot \boldsymbol{\phi}_h - \frac{1}{2}\sum_{E\in\mathcal{E}_h} \int_E (\boldsymbol{\rho}^n \cdot \nabla \boldsymbol{\phi}_h) \cdot \mathbf{E}_H^n \\
&- \sum_{E\in\mathcal{E}_h} \int_{\partial E_-} |\{\boldsymbol{\rho}^n\}\cdot \mathbf{n}_E| (\mathbf{E}_H^{n,int} - \mathbf{E}_H^{n,ext}) \cdot \boldsymbol{\phi}_h^{int} + \frac{1}{2}\sum_{e\in\Gamma_h} \int_e \{\boldsymbol{\rho}^n\}\cdot \mathbf{n}_e \{\mathbf{E}_H^n\} \cdot [\boldsymbol{\phi}_h] \\
&+ \frac{1}{2}\sum_{e\in\Gamma_h} \int_e \{\boldsymbol{\rho}^n\}\cdot \mathbf{n}_e [\mathbf{E}_H^n] \cdot \{\boldsymbol{\phi}_h\} \\
\le& C\sum_{E\in\mathcal{E}_h} \|\boldsymbol{\rho}^n\|_{L^p(E)} \|\nabla \mathbf{E}_H^n\|_{L^2(E)} \|\boldsymbol{\phi}_h\|_{L^q(E)} + C\sum_{E\in\mathcal{E}_h} \|\boldsymbol{\rho}^n\|_{L^p(E)} \|\mathbf{E}_H^n\|_{L^q(E)} \|\nabla \boldsymbol{\phi}_h\|_{L^2(E)} \\
&+ C\Big(\sum_{E\in\mathcal{E}_h} \|\boldsymbol{\rho}^n\|_{L^p(E)}^p\Big)^{1/p} \Big(\sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\mathbf{E}_H^n]\|_{L^2(e)}^2\Big)^{1/2} \Big(\sum_{E\in\mathcal{E}_h} \|\boldsymbol{\phi}_h\|_{L^q(E)}^q\Big)^{1/q} \\
&+ C\Big(\sum_{E\in\mathcal{E}_h} \|\boldsymbol{\rho}^n\|_{L^p(E)}^p\Big)^{1/p} \Big(\sum_{E\in\mathcal{E}_h} \Big(\|\mathbf{E}_H^n\|_{L^q(E)} + h^{\frac{1}{q}} \|\mathbf{E}_H^n\|_{L^{2(q-1)}(E)}^{\frac{q-1}{q}} \|\nabla \mathbf{E}_H^n\|_{L^2(E)}^{\frac{1}{q}}\Big)^q\Big)^{1/q} \\
&\times \Big(\sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\boldsymbol{\phi}_h]\|_{L^2(e)}^2\Big)^{1/2} \\
\le& CH^{-\theta}(h^{r+1}H^r + h^{r+1}\Delta t + H^{3r+1-\theta} + H^{2r+1-\theta}\Delta t + H^r\Delta t + \Delta t^2)\|\boldsymbol{\phi}_h\|_\varepsilon,
\end{aligned} \tag{7.43}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and $p = \frac{2}{1-\theta}$. The estimate (2.54), Lemmas 1.3 and 7.4 yield

$$c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{u}^n, \boldsymbol{\phi}_h) \le C\|\boldsymbol{\rho}^n\| \|\mathbf{u}^n\|_2 \|\boldsymbol{\phi}_h\|_\varepsilon \le C(h^{r+1} + H^{2r+1-\theta} + \Delta t)\|\boldsymbol{\phi}_h\|_\varepsilon. \tag{7.44}$$

Substitute (7.43)-(7.44) in (7.42), we obtain

$$
\begin{aligned}
&|c^{\mathbf{U}_h^n}(\boldsymbol{\rho}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h)| \\
&\le CH^{-\theta}(h^{r+1}H^r + h^{r+1}\Delta t + H^{3r+1-\theta} + H^{2r+1-\theta}\Delta t + H^r\Delta t + \Delta t^2)\|\boldsymbol{\phi}_h\|_\varepsilon \\
&+ C(h^{r+1} + H^{2r+1-\theta} + \Delta t)\|\boldsymbol{\phi}_h\|_\varepsilon. \qquad (7.45)
\end{aligned}
$$

Note that, (1.39), (7.12), Lemmas 1.3 and 2.2, Theorem 7.1, triangle inequality, and the fact $\Delta t = \mathcal{O}(H^{r+1})$ imply

$$
\begin{aligned}
\|\mathbf{U}_H^n\|_{L^\infty(\Omega)} \le & \|\mathbf{u}^n - \mathbf{P}_H\mathbf{u}^n\|_{L^\infty(\Omega)} + \|\mathbf{P}_H\mathbf{u}^n - \mathbf{U}_H^n\|_{L^\infty(\Omega)} + \|\mathbf{u}^n\|_{L^\infty(\Omega)} \\
\le & CH^{-1}\|\mathbf{P}_H\mathbf{u}^n - \mathbf{U}_H^n\| + C\|\mathbf{u}^n\|_2 \\
\le & CH^{-1}\|\mathbf{u}^n - \mathbf{P}_H\mathbf{u}^n\| + CH^{-1}\|\mathbf{E}_H^n\| + C\|\mathbf{u}^n\|_2 \le C. \qquad (7.46)
\end{aligned}
$$

Following (1.18) and utilizing the fact $\mathbf{u}^n \in \mathbf{H}_0^1(\Omega)$ and $\nabla\cdot\mathbf{u}^n = 0$, the nonlinear term $c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\rho}^n, \boldsymbol{\phi}_h)$ can be rewritten as

$$
\begin{aligned}
&c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\rho}^n, \boldsymbol{\phi}_h) = -\sum_{E\in\mathcal{E}_h}\int_E(\mathbf{U}_H^n\cdot\nabla\boldsymbol{\phi}_h)\cdot\boldsymbol{\rho}^n + \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E(\nabla\cdot\mathbf{E}_H^n)\boldsymbol{\rho}^n\cdot\boldsymbol{\phi}_h \\
&-\frac{1}{2}\sum_{e\in\Gamma_h}\int_e[\mathbf{E}_H^n]\cdot\mathbf{n}_e\{\boldsymbol{\rho}^n\cdot\boldsymbol{\phi}_h\} - \sum_{E\in\mathcal{E}_h}\int_{\partial E_-}|\{\mathbf{U}_H^n\}\cdot\mathbf{n}_E|\boldsymbol{\rho}^{n,ext}\cdot(\boldsymbol{\phi}_h^{int} - \boldsymbol{\phi}_h^{ext}) \\
&+\int_{\Gamma_+}|\mathbf{U}_H^n\cdot\mathbf{n}|\boldsymbol{\rho}^n\cdot\boldsymbol{\phi}_h.
\end{aligned}
$$

With the above reformulation and similar to (7.43), and applying (7.46), Theorem 7.1 and Lemma 7.4, we arrive at

$$
\begin{aligned}
|c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\rho}^n, \boldsymbol{\phi}_h)| \le & C\|\mathbf{U}_H^n\|_{L^\infty(\Omega)}\|\boldsymbol{\rho}^n\|\|\boldsymbol{\phi}_h\|_\varepsilon + CH^{-\theta}\|\boldsymbol{\rho}^n\|\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon \\
\le CH^{-\theta}(h^{r+1}H^r + h^{r+1}\Delta t + & H^{3r+1-\theta} + H^{2r+1-\theta}\Delta t + H^r\Delta t + \Delta t^2)\|\boldsymbol{\phi}_h\|_\varepsilon \\
& + C(h^{r+1} + H^{2r+1-\theta} + \Delta t)\|\boldsymbol{\phi}_h\|_\varepsilon. \qquad (7.47)
\end{aligned}
$$

The fourteenth term is bounded following the bounding approach of (7.32) but with a little modification:

$$
\begin{aligned}
|l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{U}_h^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)| \le & C\|\boldsymbol{\zeta}^n\|^{1-\theta}\|\boldsymbol{\zeta}^n\|_\varepsilon^\theta\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon \\
& + Ch^{-\theta}\|\boldsymbol{\zeta}^n\|\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon + Ch^{\frac{1-\theta}{2}}\|\boldsymbol{\zeta}^n\|^{\frac{1-\theta}{2}}\|\boldsymbol{\zeta}^n\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon \\
& + CH^{-\theta}\|\boldsymbol{\rho}^n\|\|\mathbf{E}_H^n\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon \\
\le Ch^{r+1-\theta}(H^r + \Delta t)\|\boldsymbol{\phi}_h\|_\varepsilon + & CH^{-\theta}(h^{r+1}H^r + h^{r+1}\Delta t + H^{3r+1-\theta} \\
& + H^{2r+1-\theta}\Delta t + H^r\Delta t + \Delta t^2)\|\boldsymbol{\phi}_h\|_\varepsilon. \qquad (7.48)
\end{aligned}
$$

The remaining terms on the right hand side of (7.35) can be bounded with an identical approach as Lemma 7.4.
Now, combine (7.36)-(7.41), (7.45), (7.47), (7.48) in (7.35) and use the fact $\Delta t = \mathcal{O}(h^r)$ to find

$$\begin{aligned}&(\partial_t \mathbf{P}_h \mathbf{E}_h^n, \boldsymbol{\phi}_h) + \mu\, a(\mathbf{P}_h \mathbf{E}_h^n, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{P}_h \mathbf{E}_h), \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, r_h(p^n) - P_h^n) \\ &\le C(h^{r+1} + H^{2r+1-\theta} + \Delta t)\|\boldsymbol{\phi}_h\|_\varepsilon.\end{aligned}$$

Now, Lemmas 1.6 and 1.8 lead to

$$\begin{aligned}\mu\, K_1\|\mathbf{P}_h \mathbf{E}_h^n\|_\varepsilon + K_1\|q_r^n(\mathbf{P}_h \mathbf{E}_h)\|_\varepsilon + \beta^*\|r_h(p^n) - P_h^n\| &\le \mu \sup_{\boldsymbol{\phi}_h \in \mathbf{X}_h\setminus\{\mathbf{0}\}} \frac{a(\mathbf{P}_h \mathbf{E}_h^n, \boldsymbol{\phi}_h)}{\|\boldsymbol{\phi}_h\|_\varepsilon} \\ &+ \sup_{\boldsymbol{\phi}_h \in \mathbf{X}_h\setminus\{\mathbf{0}\}} \frac{a(q_r^n(\mathbf{P}_h \mathbf{E}_h), \boldsymbol{\phi}_h)}{\|\boldsymbol{\phi}_h\|_\varepsilon} + \sup_{\boldsymbol{\phi}_h \in \mathbf{X}_h\setminus\{\mathbf{0}\}} \frac{b(\boldsymbol{\phi}_h, r_h(p^n) - P_h^n)}{\|\boldsymbol{\phi}_h\|_\varepsilon}.\end{aligned}$$

Finally, the above inequality, triangle inequality, (1.31) and Lemma 2.2 imply the desired estimate. □

7.4 Optimal Error Estimates in Step 3

This section presents optimal fully discrete error estimates for velocity and pressure in **Step 3**. The next lemma is an auxiliary result for the derivation of **Step 3** velocity error.

Lemma 7.6. *Suppose the assumptions of Lemma 7.3 hold true and let $\Delta t = \mathcal{O}(h^{r+1})$. Then, there holds:*

$$\|\boldsymbol{\Theta}^n\|^2 + e^{-2\alpha t_m}\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\boldsymbol{\Theta}^n\|_\varepsilon^2 \le K_T(h^{2r+2} + h^{2r}H^{2r+2-2\theta} + H^{6r+4-4\theta} + \Delta t^2).$$

Proof. First of all, we choose $\boldsymbol{\phi}_h = \boldsymbol{\Theta}^n$ in (7.18), and employ Lemma 1.6 and (1.19) to obtain

$$\begin{aligned}&\frac{1}{2}\partial_t\|\boldsymbol{\Theta}^n\|^2 + \mu K_1\|\boldsymbol{\Theta}^n\|_\varepsilon^2 + a(q_r^n(\boldsymbol{\Theta}), \boldsymbol{\Theta}^n) \le -(\mathbf{u}_t^n - \partial_t \mathbf{u}^n, \boldsymbol{\Theta}^n) - (\partial_t \boldsymbol{\zeta}^n, \boldsymbol{\Theta}^n) \\ &+ \left(a(q_r^n(\mathbf{S}_h^{vol}\mathbf{u}), \boldsymbol{\Theta}^n) - \int_0^{t_n} \beta(t_n - s) a(\mathbf{S}_h^{vol}\mathbf{u}(s), \boldsymbol{\Theta}^n)\, ds\right) - c^{\mathbf{U}_h^{*n}}(\boldsymbol{\Theta}^n, \mathbf{U}_H^n, \boldsymbol{\Theta}^n) \\ &- c^{\mathbf{U}_h^{*n}}(\boldsymbol{\zeta}^n, \mathbf{U}_H^n, \boldsymbol{\Theta}^n) - c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\zeta}^n, \boldsymbol{\Theta}^n) - c^{\mathbf{U}_H^n}(\mathbf{E}_H^n, \mathbf{E}_h^n, \boldsymbol{\Theta}^n) \\ &+ c^{\mathbf{U}_h^n}(\mathbf{E}_h^n, \mathbf{E}_h^n, \boldsymbol{\Theta}^n) - c^{\mathbf{U}_h^n}(\mathbf{E}_h^n, \mathbf{E}_H^n, \boldsymbol{\Theta}^n) \\ &+ \big(l^{\mathbf{U}_h^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\Theta}^n) - l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\Theta}^n)\big) + \big(l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\Theta}^n) - l^{\mathbf{U}_h^{*n}}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\Theta}^n)\big) \\ &+ \big(l^{\mathbf{U}_H^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\Theta}^n) - l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\Theta}^n)\big) + \big(l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\Theta}^n) - l^{\mathbf{U}_h^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\Theta}^n)\big)\end{aligned}$$

$$=M_1 + M_2 + \cdots + M_{13}. \tag{7.49}$$

Following the proof steps as in Lemma 7.4, we obtain

$$|M_1| \le C\Delta t^{1/2} \left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{tt}(t)\|^2\, dt\right)^{1/2} \|\mathbf{\Theta}^n\|_\varepsilon, \tag{7.50}$$

$$|M_2| \le Ch^{r+1}\|\mathbf{\Theta}\|_\varepsilon, \tag{7.51}$$

$$|M_3| \le C\Delta t\bigg(\int_0^{t_n} e^{-2\delta(t_n-s)}\big(\|\mathbf{S}_h^{vol}\mathbf{u}(s)\|_\varepsilon^2 + \|\mathbf{S}_h^{vol}\mathbf{u}_s(s)\|_\varepsilon^2\big)\, ds\bigg)^{1/2}\|\mathbf{\Theta}\|_\varepsilon, \tag{7.52}$$

$$|M_4| \le C\|\mathbf{\Theta}^n\|^{1/2}\|\mathbf{\Theta}^n\|_\varepsilon^{3/2}\|\mathbf{U}_H^n\|_\varepsilon, \tag{7.53}$$

$$|M_5| \le C(h^{r+1} + h^{r+1-\theta}H^r + h^{r+1-\theta}\Delta t)\|\mathbf{\Theta}^n\|_\varepsilon, \tag{7.54}$$

$$\begin{aligned}|M_6| \le& Ch^r\big(H^{r+1-\theta} + H^{(r+1)(1-\theta)}\Delta t^\theta + H^{r\theta}\Delta t^{1-\theta} + H^{-\theta}\Delta t \\ &+h^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{1+\theta}{2}} + h^{\frac{1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + \Delta t\big)\|\mathbf{\Theta}^n\|_\varepsilon + Ch^{r+1}\|\mathbf{\Theta}^n\|_\varepsilon.\end{aligned} \tag{7.55}$$

Apply (6.36), Lemma 7.5 and Theorem 7.1 to find

$$\begin{aligned}|M_7| \le& C\|\mathbf{E}_H^n\|^{1-\theta}\|\mathbf{E}_H^n\|_\varepsilon^\theta\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{\Theta}^n\|_\varepsilon + CH^{-\theta}\|\mathbf{E}_H^n\|\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{\Theta}^n\|_\varepsilon \\ &+ Ch^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{\Theta}^n\|_\varepsilon \\ \le& C\big(h^rH^{r+1-\theta} + H^{3r+2-2\theta} + H^{r+1-\theta}\Delta t + h^rH^{r-r\theta+1-\theta}\Delta t^\theta + H^{3r-r\theta+2-2\theta}\Delta t^\theta \\ &+ H^{r-r\theta+1-\theta}\Delta t^{1+\theta} + h^rH^{r\theta}\Delta t^{1-\theta} + H^{2r+1+r\theta-\theta}\Delta t^{1-\theta} + H^{r\theta}\Delta t^{2-\theta} + h^r\Delta t \\ &+ H^{2r+1-\theta}\Delta t + \Delta t^2 + h^rH^{-\theta}\Delta t + H^{2r+1-2\theta}\Delta t + H^{-\theta}\Delta t^2 \\ &+ h^{\frac{2r+1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{1+\theta}{2}} + h^{\frac{2r+1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + h^{\frac{2r+1-\theta}{2}}\Delta t \\ &+ h^{\frac{1-\theta}{2}}H^{\frac{5r+3-r\theta-3\theta)}{2}}\Delta t^{\frac{1+\theta}{2}} + h^{\frac{1-\theta}{2}}H^{\frac{5r+2+r\theta-2\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + h^{\frac{1-\theta}{2}}H^{2r+1-\theta}\Delta t \\ &+ h^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{3+\theta}{2}} + h^{\frac{1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{3-\theta}{2}} \\ &+ h^{\frac{1-\theta}{2}}\Delta t^2\big)\|\mathbf{\Theta}^n\|_\varepsilon.\end{aligned} \tag{7.56}$$

Using (2.56) and Lemma 7.5, and noting that $r \ge 1$, one can find

$$\begin{aligned}|M_8| \le& C\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{\Theta}^n\|_\varepsilon \le C\big(h^{r+1} + h^rH^{2r+1-\theta} + H^{4r+2-2\theta} + h^r\Delta t \\ &+ H^{2r+1-\theta}\Delta t + \Delta t^2\big)\|\mathbf{\Theta}^n\|_\varepsilon.\end{aligned} \tag{7.57}$$

To handle M_9, let us rewrite it in the following manner

$$M_9 = -c^{\mathbf{E}_h^n}(\mathbf{E}_h^n, \mathbf{E}_H^n, \mathbf{\Theta}^n) + \big(l^{\mathbf{E}_h^n}(\mathbf{E}_h^n, \mathbf{E}_H^n, \mathbf{\Theta}^n) - l^{\mathbf{U}_h^n}(\mathbf{E}_h^n, \mathbf{E}_H^n, \mathbf{\Theta}^n)\big) = M_{91} + M_{92}.$$

Now, M_{91} is bounded employing (6.37), Lemma 7.5 and Theorem 7.1 as follows:

$$|M_{91}| \le C\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{E}_H^n\|^{1-\theta}\|\mathbf{E}_H^n\|_\varepsilon^\theta\|\mathbf{\Theta}^n\|_\varepsilon + CH^{-\theta}\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{E}_H^n\|\|\mathbf{\Theta}^n\|_\varepsilon$$

$$
\begin{aligned}
&+ Ch^{\frac{1-\theta}{2}}\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{E}_H^n\|^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{\Theta}^n\|_\varepsilon\\
\le& C\big(h^rH^{r+1-\theta} + H^{3r+2-2\theta} + H^{r+1-\theta}\Delta t + h^rH^{r-r\theta+1-\theta}\Delta t^\theta + H^{3r-r\theta+2-2\theta}\Delta t^\theta\\
&+ H^{r-r\theta+1-\theta}\Delta t^{1+\theta} + h^rH^{r\theta}\Delta t^{1-\theta} + H^{2r+1+r\theta-\theta}\Delta t^{1-\theta} + H^{r\theta}\Delta t^{2-\theta} + h^r\Delta t\\
&+ H^{2r+1-\theta}\Delta t + \Delta t^2 + h^rH^{-\theta}\Delta t + H^{2r+1-2\theta}\Delta t + H^{-\theta}\Delta t^2\\
&+ h^{\frac{2r+1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{1+\theta}{2}} + h^{\frac{2r+1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + h^{\frac{2r+1-\theta}{2}}\Delta t\\
&+ h^{\frac{1-\theta}{2}}H^{\frac{5r+3-r\theta-3\theta)}{2}}\Delta t^{\frac{1+\theta}{2}} + h^{\frac{1-\theta}{2}}H^{\frac{5r+2+r\theta-2\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + h^{\frac{1-\theta}{2}}H^{2r+1-\theta}\Delta t\\
&+ h^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{3+\theta}{2}} + h^{\frac{1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{3-\theta}{2}} + h^{\frac{1-\theta}{2}}\Delta t^2\big)\|\mathbf{\Theta}^n\|_\varepsilon. \qquad (7.58)
\end{aligned}
$$

To estimate M_{92} and M_{10}, we follow the technique involved in estimating Q_{72} and Q_8 (see Lemma 6.9). Thus, a use of Lemma 7.5 and Theorem 7.1 leads to

$$
\begin{aligned}
|M_{92}| + |M_{10}| \le& C\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{E}_H^n\|^{1-\theta}\|\mathbf{E}_H^n\|_\varepsilon^\theta\|\mathbf{\Theta}^n\|_\varepsilon + CH^{-\theta}\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{E}_H^n\|\|\mathbf{\Theta}^n\|_\varepsilon\\
&+ Ch^{\frac{1-\theta}{2}}\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{E}_H^n\|^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{\Theta}^n\|_\varepsilon\\
\le& C\big(h^rH^{r+1-\theta} + H^{3r+2-2\theta} + H^{r+1-\theta}\Delta t + h^rH^{r-r\theta+1-\theta}\Delta t^\theta\\
&+ H^{3r-r\theta+2-2\theta}\Delta t^\theta + H^{r-r\theta+1-\theta}\Delta t^{1+\theta} + h^rH^{r\theta}\Delta t^{1-\theta}\\
&+ H^{2r+1+r\theta-\theta}\Delta t^{1-\theta} + H^{r\theta}\Delta t^{2-\theta} + h^r\Delta t + H^{2r+1-\theta}\Delta t + \Delta t^2\\
&+ h^rH^{-\theta}\Delta t + H^{2r+1-2\theta}\Delta t + H^{-\theta}\Delta t^2 + h^{\frac{2r+1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{1+\theta}{2}}\\
&+ h^{\frac{2r+1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + h^{\frac{2r+1-\theta}{2}}\Delta t + h^{\frac{1-\theta}{2}}H^{\frac{5r+3-r\theta-3\theta)}{2}}\Delta t^{\frac{1+\theta}{2}}\\
&+ h^{\frac{1-\theta}{2}}H^{\frac{5r+2+r\theta-2\theta)}{2}}\Delta t^{\frac{1-\theta}{2}} + h^{\frac{1-\theta}{2}}H^{2r+1-\theta}\Delta t + h^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{3+\theta}{2}}\\
&+ h^{\frac{1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{3-\theta}{2}} + h^{\frac{1-\theta}{2}}\Delta t^2\big)\|\mathbf{\Theta}^n\|_\varepsilon. \qquad (7.59)
\end{aligned}
$$

M_{11} can be estimated similar to (7.32):

$$
|M_{11}| \le Ch^{r+1-\theta}(H^r + \Delta t)\|\mathbf{\Theta}^n\|_\varepsilon + C\|\mathbf{\Theta}^n\|^{1/2}\|\mathbf{\Theta}^n\|_\varepsilon^{3/2}\|\mathbf{U}_H^n\|_\varepsilon. \qquad (7.60)
$$

Furthermore, (2.61), Lemmas 6.6, 7.5 and Theorem 7.1 yield

$$
\begin{aligned}
|M_{12}| \le& C\|\mathbf{E}_H^n\|^{1-\theta}\|\mathbf{E}_H^n\|_\varepsilon^\theta\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{\Theta}^n\|_\varepsilon + CH^{-\theta}\|\mathbf{E}_H^n\|\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{\Theta}^n\|_\varepsilon\\
&+ Ch^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|^{\frac{1-\theta}{2}}\|\mathbf{E}_H^n\|_\varepsilon^{\frac{1+\theta}{2}}\|\mathbf{E}_h^n\|_\varepsilon\|\mathbf{\Theta}^n\|_\varepsilon\\
\le& C\big(h^rH^{r+1-\theta} + H^{3r+2-2\theta} + H^{r+1-\theta}\Delta t + h^rH^{r-r\theta+1-\theta}\Delta t^\theta + H^{3r-r\theta+2-2\theta}\Delta t^\theta\\
&+ H^{r-r\theta+1-\theta}\Delta t^{1+\theta} + h^rH^{r\theta}\Delta t^{1-\theta} + H^{2r+1+r\theta-\theta}\Delta t^{1-\theta} + H^{r\theta}\Delta t^{2-\theta}\\
&+ h^r\Delta t + H^{2r+1-\theta}\Delta t + \Delta t^2 + h^rH^{-\theta}\Delta t + H^{2r+1-2\theta}\Delta t\\
&+ H^{-\theta}\Delta t^2 + h^{\frac{2r+1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{1+\theta}{2}} + h^{\frac{2r+1-\theta}{2}}H^{\frac{r(1+\theta)}{2}}\Delta t^{\frac{1-\theta}{2}}\\
&+ h^{\frac{2r+1-\theta}{2}}\Delta t + h^{\frac{1-\theta}{2}}H^{\frac{5r+3-r\theta-3\theta)}{2}}\Delta t^{\frac{1+\theta}{2}} + h^{\frac{1-\theta}{2}}H^{\frac{5r+2+r\theta-2\theta)}{2}}\Delta t^{\frac{1-\theta}{2}}\\
&+ h^{\frac{1-\theta}{2}}H^{2r+1-\theta}\Delta t + h^{\frac{1-\theta}{2}}H^{\frac{(r+1)(1-\theta)}{2}}\Delta t^{\frac{3+\theta}{2}}
\end{aligned}
$$

$$+ h^{\frac{1-\theta}{2}} H^{\frac{r(1+\theta)}{2}} \Delta t^{\frac{3-\theta}{2}} + h^{\frac{1-\theta}{2}} \Delta t^2\big) \|\mathbf{\Theta}^n\|_\varepsilon. \tag{7.61}$$

and

$$|M_{13}| \leq C\|\mathbf{E}_h^n\|_\varepsilon \|\mathbf{E}_h^n\|_\varepsilon \|\mathbf{\Theta}^n\|_\varepsilon \leq C\big(h^{r+1} + h^r H^{2r+1-\theta} + H^{4r+2-2\theta} + h^r \Delta t + H^{2r+1-\theta}\Delta t + \Delta t^2\big)\|\mathbf{\Theta}^n\|_\varepsilon. \tag{7.62}$$

Substitute the bounds (7.50)-(7.62) in (7.49). Further, multiply the resulting inequality by $\Delta t e^{2\alpha t_n}$, sum over $1 \leq n \leq m \leq M$, and utilize Lemma 4.5, the fact $\mathbf{\Theta}^0 = \mathbf{0}$, $\Delta t = \mathcal{O}(h^{r+1})$ and Young's inequality to arrive at

$$e^{2\alpha t_m}\|\mathbf{\Theta}^m\|^2 + \mu K_1 \Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\mathbf{\Theta}^n\|_\varepsilon^2 \leq C\Delta t \sum_{n=1}^{m-1} e^{2\alpha t_n}\|\mathbf{\Theta}^n\|^2 + C\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\mathbf{U}_H^n\|_\varepsilon^4 \|\mathbf{\Theta}^n\|^2$$
$$+ C\Delta t^3 \sum_{n=1}^{m} \int_0^{t_n} e^{-2(\delta-\alpha)(t_n-s)} e^{2\alpha s}\big(\|\mathbf{S}_h^{vol}\mathbf{u}(s)\|_\varepsilon^2 + \|\mathbf{S}_h^{vol}\mathbf{u}_s(s)\|_\varepsilon^2\big)\, ds$$
$$+ C\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}(h^{2r+2} + h^{2r} H^{2r+2-2\theta} + H^{6r+4-4\theta} + \Delta t^2).$$

Finally, apply discrete Gronwall's lemma, (7.10), and after a final multiplication by $e^{-2\alpha t_m}$ completes the rest of the proof. □

An application of (4.64), (7.16) and Lemma 7.6 yield the following **Step 3** $L^\infty(\mathbf{L}^2)$-norm error estimate of the velocity which is stated in the next theorem.

Theorem 7.2. *Suppose the assumption (**A4**) holds true and let* $0 < \alpha < \min\left(\delta, \frac{\mu K_1}{2C_2}\right)$. *In addition, let the fully discrete* **Step 3** *initial velocity* $\boldsymbol{U}_h^{*0} \in \boldsymbol{V}_h$ *with* $\boldsymbol{U}_h^{*0} = \boldsymbol{P}_h \boldsymbol{u}_0$. *Then, there exists a constant* $K_T > 0$, *such that*

$$\|\mathbf{E}_h^{*n}\| \leq K_T(h^{r+1} + h^r H^{r+1-\theta} + H^{3r+2-2\theta} + \Delta t),$$

where K_T *depends on* T.

For **Step 3** error estimates of velocity and pressure in energy and L^2-norms, we have the following theorem.

Theorem 7.3. *Suppose the assumption (**A4**) holds true and let* $0 < \alpha < \min\left(\delta, \frac{\mu K_1}{2C_2}\right)$. *In addition, let the fully discrete* **Step 3** *initial velocity* $\boldsymbol{U}_h^{*0} \in \boldsymbol{V}_h$ *with* $\boldsymbol{U}_h^{*0} = \boldsymbol{P}_h \boldsymbol{u}_0$. *Then, there exists a constant* $K_T > 0$, *such that*

$$\|\mathbf{E}_h^{*n}\|_\varepsilon + \|p^n - P_h^{*n}\| \leq C(h^r + H^{3r+2-2\theta} + \Delta t).$$

Proof. Subtract (7.6) from the equation (7.1) with $t = t_n$ and for each $\boldsymbol{\phi}_h \in \mathbf{X}_h$:

$$\begin{aligned}
&(\partial_t \mathbf{P}_h \mathbf{E}_h^{*n}, \boldsymbol{\phi}_h) + \mu\, a(\mathbf{P}_h \mathbf{E}_h^{*n}, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{P}_h \mathbf{E}_h^{*}), \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, r_h(p^n) - P_h^{*n}) \\
&= -(\mathbf{u}_t^n - \partial_t \mathbf{u}^n, \boldsymbol{\phi}_h) - \big(\partial_t(\mathbf{u}^n - \mathbf{P}_h \mathbf{u}^n), \boldsymbol{\phi}_h\big) - \mu\, a(\mathbf{u}^n - \mathbf{P}_h \mathbf{u}^n, \boldsymbol{\phi}_h) \\
&- a(q_r^n(\mathbf{u} - \mathbf{P}_h \mathbf{u}), \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{u}), \boldsymbol{\phi}_h) - \int_0^{t_n} \beta(t_n - s) a(\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \\
&- b(\boldsymbol{\phi}_h, p^n - r_h(p^n)) - c^{\mathbf{U}_h^{*n}}(\boldsymbol{\Theta}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\Theta}^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^{*n}}(\boldsymbol{\zeta}^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) \\
&- c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \boldsymbol{\zeta}^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_H^n}(\mathbf{E}_H^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{E}_h^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{E}_h^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) \\
&+ \big(l^{\mathbf{U}_h^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{U}_h^{*n}}(\mathbf{u}^n, \mathbf{E}_H^n, \boldsymbol{\phi}_h)\big) \\
&+ \big(l^{\mathbf{U}_H^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h) - l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h) - l^{\mathbf{U}_h^n}(\mathbf{u}^n, \mathbf{E}_h^n, \boldsymbol{\phi}_h)\big).
\end{aligned}$$

The bounds for the terms on the right hand side of the above inequality are obtained following the same lines as the proof of Lemmas 7.5 and 7.6. Then proceed similar to the proof of Lemma 7.5, we complete the rest of the proof. □

7.5 Numerical experiments

A few numerical experiments are carried out and the theoretical results are validated in this section. To discretize the space, we utilize $\mathbb{P}_r - \mathbb{P}_{r-1}$, $r = 1, 2$, DG finite elements and for time discretization, backward Euler method is used. In this case, $\Omega = [0, 1]^2$ is chosen as the domain. To evaluate the performance of our two-grid DG scheme, we compute the approximate solutions using both the two-grid and standard DG schemes with the same fine mesh. Here, we have computed the solutions on the time interval $[0, .5]$ with the final time $T = .5$, and the time step $\Delta t = \mathcal{O}(h^{r+1})$.

Example 7.1. *Consider the Oldroyd model of order one with exact solution* $(\boldsymbol{u}, p) = \big((u_1(x, y, t), u_2(x, y, t)), p(x, y, t)\big)$ *as*

$$\begin{aligned}
u_1(x, y, t) &= 2(x^2 - 2x^3 + x^4)(2y - 6y^2 + 4y^3)\, te^{-t^2}, \\
u_2(x, y, t) &= -2(y^2 - 2y^3 + y^4)(2x - 6x^2 + 4x^3)\, te^{-t^2}, \\
p(x, y, t) &= 2(x - y)\, te^{-t}.
\end{aligned}$$

In Figures 7.1 and 7.2, we depict the errors of the two-grid DG scheme and the standard DG scheme for $r = 1$ and 2, respectively, with $\mu = \{0.1, 0.01\}$. In Figure 7.1, the parameters are $\gamma = 0.01$, $\delta = 0.1$, $\sigma_e = 10$, $h = \mathcal{O}(H^2)$ (only for two-grid DG). For the Figure 7.2, we choose $\gamma = 0.001$, $\delta = 0.1$, $\sigma_e = 20$, $h = \mathcal{O}(H^{8/3})$ (only for two-grid DG). Figures 7.1 and 7.2 indicate that the accuracy of the numerical solutions produced with the proposed two-grid DG method is comparable to that of the standard DG method, and also validate the theoretical results that were derived.

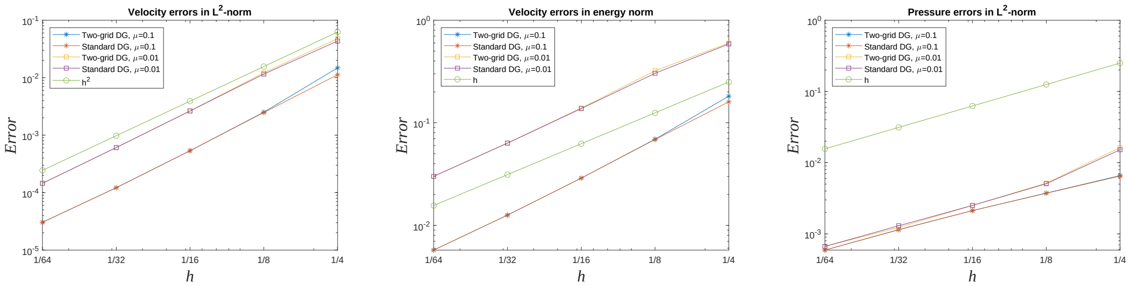

Figure 7.1: Velocity and pressure errors using $\mathbb{P}_1 - \mathbb{P}_0$ element for example 7.1.

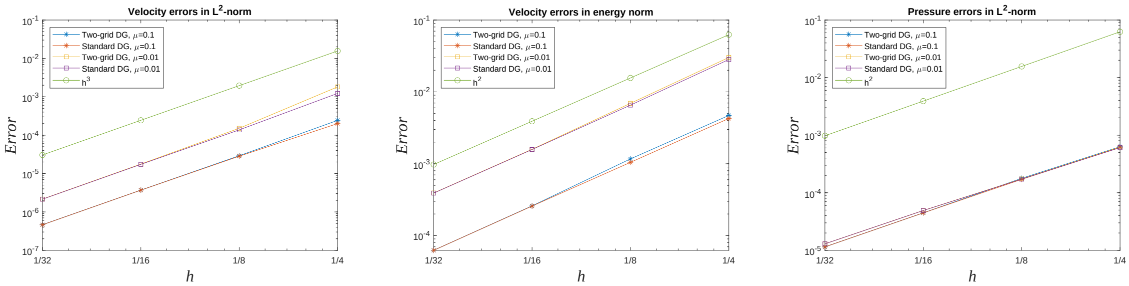

Figure 7.2: Velocity and pressure errors using $\mathbb{P}_2 - \mathbb{P}_1$ element for example 7.1.

In Tables 7.1 and 7.2, we compare the computational times taken to compute the two-grid DG solution and the standard DG solution corresponding to $r = 1$ and 2, respectively, and for $\mu = 0.1$. The tables show that, compared to the standard DG method, the proposed two-grid DG method takes a significant reduction in computational time. Besides, the computational time difference between the two solutions (the direct DG solution and the two-grid DG solution) grows as we refine the mesh further.

Table 7.1: Comparison of computational time (in Seconds) between “standard DG solution” and the solution obtained by the ”two-grid DG method” for Example 7.1 with $r = 1$.

h	Two-grid DG solution	Standard DG solution
1/4	0.40	0.47
1/8	3.05	4.03
1/16	34.70	61.78
1/32	338.05	665.96
1/64	4644.87	10869.23

Table 7.2: Comparison of computational time (in Seconds) between “standard DG solution” and the solution obtained by the ”two-grid DG method” for Example 7.1 with $r = 2$.

h	Two-grid DG solution	Standard DG solution
1/4	9.66	11.411
1/8	65.13	99.74
1/16	880.23	1514.16
1/32	13792.43	25516.93

7.6 Conclusion

In this chapter, a three-step two-grid algorithm for the DG approximation of the Oldroyd model of order one is presented and examined. The resulting scheme is a fully discrete scheme with the time discretization performed utilizing the backward Euler method. We have proved that the largest scaling between the fine mesh size h and coarse mesh size H are $h = \mathcal{O}\big(H^{\min\big(r+1-\theta,\frac{3r+2-2\theta}{r+1}\big)}\big)$ and $h = \mathcal{O}(H^{\frac{3r+2-2\theta}{r}})$ for velocity in $\mathbf{L}^2$-norm and in energy norm, respectively. It is $h = \mathcal{O}(H^{\frac{3r+2-2\theta}{r}})$ for the pressure approximation in L^2-norm, for arbitrary small $\theta > 0$. The accuracy of the method is demonstrated by the final presentation of numerical results.