Chapter 8

Conclusion and Future Work

In this chapter, we summarize the results derived in this thesis. Then we discuss possible extensions or new works in this direction.

8.1 Concluding Remarks

The primary goal of this thesis has been to study and analyse DG finite element method and two-grid technique combined with DG method for three incompressible fluid flows, namely the unsteady NSEs, the Kelvin-Voigt model, and the Oldroyd model of order one. The findings are subsequently supported through numerical computations. Due to the time-dependent nature of these flows, additional discretization of the time variable is required for numerical simulation. Thus, we have examined the backward Euler method, which is a first order accurate time-discrete scheme employed for temporal discretization.

In Chapter 2, we have studied a DG finite element method applied to the incompressible time-dependent NSEs. We have defined an L^2-projection and a modified Stokes projection onto a suitable DG finite element space, and proved their optimal approximation estimates. With the help of these two projections, we have derived the following semi-discrete optimal error estimates for the velocity approximation $\mathbf{u}_h$ and pressure approximation p_h

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\| + h\|(\mathbf{u}-\mathbf{u}_h)(t)\|_{\varepsilon} + h\|(p-p_h)(t)\| \leq K(t)h^{r+1},$$

where $K(t) = Ce^{Ct}$, and C is a positive constant independent of h and t. Under the smallness assumption on data, the velocity error estimates are shown to be uniform in time. Afterwards, to achieve complete discretization, the backward Euler method is used in the time direction and the following fully discrete error estimates of velocity and pressure are derived

$$\|\mathbf{u}^n - \mathbf{U}^n\| \leq K_T\,(h^{r+1} + \Delta t),$$
$$\|\mathbf{u}^n - \mathbf{U}^n\|_\varepsilon + \|p^n - P^n\| \leq K_T(h^r + \Delta t^{1/2}),$$

where $1 \leq n \leq M$, Δt is the time step, $K_T = Ce^{CT}$, T is the final time, and $(\mathbf{U}^n, P^n)$ is the fully discrete solution. Theoretical results are validated employing numerical experiments.

In Chapter 3, we have implemented a DG method to the Kelvin-Voigt equations of motion. For this model problem, we have constructed the semi-discrete DG formulation and obtained *a priori* estimates for the velocity approximation. Based on these bounds, well-posedness and consistency of the DG scheme are discussed. To establish error estimates, we have introduced a modified Sobolev-Stokes projection on appropriate DG spaces and proved the approximation properties. Then, by using duality arguments along with the approximation properties of Sobolev-Stokes projection, we have obtained the following semi-discrete optimal error estimates for the velocity and pressure

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| + h\|(\mathbf{u} - \mathbf{u}_h)(t)\|_\varepsilon + h\|(p - p_h)(t)\| \leq K(t)h^{r+1},$$

where $\mathbf{u}_h$ and p_h are the semi-discrete DG approximations of velocity and pressure, respectively. Furthermore, we have showed that the semi-discrete error estimates are uniform in time under the smallness condition on the data. Additionally, we have employed a backward Euler method for full discretization and have achieved optimal error bounds for the approximate solutions as

$$\|\mathbf{u}^n - \mathbf{U}^n\| \leq K_T(h^{r+1} + \Delta t),$$
$$\|\mathbf{u}^n - \mathbf{U}^n\|_\varepsilon + \|p^n - P^n\| \leq K_T(h^r + \Delta t),$$

where $(\mathbf{U}^n, P^n)$ is the fully discrete solution. Finally, we have conducted the numerical experiments and have shown that the outcomes verify the theoretical results. Also from our numerical results, we have observed that the scheme works well even for small values of ν and κ.

In Chapter 4, we have defined a DG scheme for the Oldroyd model of order one and analyzed it. It has been proved that the proposed DG scheme is consistent. According to [75, Lemma 2.1], the integral term in the model problem has a positivity property that has been maintained for the SIPG case. With the help of this property, we have derived *a priori* bound of the semi-discrete solution for the SIPG case which is uniform in time. However, this positivity property does not hold true for the NIPG or IIPG case, but we derive uniform in time *a priori* bounds in these cases as well by analyzing long term behaviour of the solution. We then investigate the existence and uniqueness of semi-discrete DG solutions. We next apply the backward Euler method for time discretization. Since backward Euler method is a first order difference scheme, the right rectangle rule is chosen here to approximate the integral term. Positivity property of this quadrature rule in terms of the SIPG bilinear form has been established. The following optimal fully discrete error estimates have been achieved

$$\|\mathbf{u}^m - \mathbf{U}_h^m\|_\varepsilon^2 + e^{-2\alpha t_m}\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\|p^n - P_h^n\|^2 \le K_T(h^{2r} + \Delta t^2),$$

where $(\mathbf{U}_h^n, P_h^n)$ is the fully discrete solution. But the estimates for $\|\mathbf{u}^n - \mathbf{U}_h^n\|$ is sub-optimal, that is

$$\|\mathbf{u}^n - \mathbf{U}_h^n\| \le K_T(h^r + \Delta t).$$

To obtain an optimal bound for $\|\mathbf{u}^n - \mathbf{U}_h^n\|$, we have defined a modified Stokes-Volterra projection for DG spaces and derived its optimal estimates for SIPG case. With the help of this, we have obtained

$$\begin{aligned}\|\mathbf{u}^n - \mathbf{U}_h^n\| &\le K_T(h^{r+1} + \Delta t),\\ \|p^n - P_h^n\| &\le K_T(h^r + \Delta t).\end{aligned}$$

Moreover, several numerical experiments are conducted to support our theoretical results.

In Chapter 5, a two-step two-grid scheme combined with DG approximations is employed to the transient NSEs. In the first step, we solve the nonlinear system on a coarse mesh. Then, in the second step, we linearize the nonlinear system by using one Newton iteration around the coarse grid solution on a fine mesh and solve it. We establish optimal semi-discrete error estimates for the velocity and pressure in the second step as

$$\begin{aligned} e^{-2\alpha t}\int_0^t e^{2\alpha s}\|(\mathbf{u}-\mathbf{u}_h)(s)\|_\varepsilon^2\,ds &\le K(t)(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}),\\ \|(\mathbf{u}-\mathbf{u}_h)(t)\|_\varepsilon^2 + \|(p-p_h)(t)\|^2 &\le K(t)\big(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}\\ &\quad + h^{-1}H^{2r}(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2})\big),\end{aligned}$$

where h = fine mesh size, H = coarse mesh size and $(\mathbf{u}_h, p_h)$ is the semi-discrete solution in the second step. We have proved optimal velocity error estimates in energy norm when $h = \mathcal{O}(H^{\frac{r+1}{r}})$ and pressure error estimates in $L^\infty(L^2)$-norm when $h = \mathcal{O}(H^{\frac{r+1}{r}})$. We have further discretized the semi-discrete two-grid DG scheme in time, utilizing the backward Euler method, and derived *a priori* bounds for fully discrete solutions. The following fully discrete error estimates are derived

$$
\begin{aligned}
e^{-2\alpha t_M}\Delta t\sum_{n=1}^{M} e^{2\alpha t_n}\|\mathbf{u}^n-\mathbf{U}_h^n\|_\varepsilon^2 &\le K_T\big(h^{2r}+h^{2r/(r+1)}H^{2r}+H^{2r+2})\\
&\quad + h^{-1}H^{2r}(h^{2r}+h^{2r/(r+1)}H^{2r}+H^{2r+2})+\Delta t^2\big),\\
\|\mathbf{u}^n-\mathbf{U}_h^n\|_\varepsilon^2+\|p^n-P_h^n\|^2 &\le K_T\big(h^{2r}+h^{2r/(r+1)}H^{2r}+H^{2r+2}\\
&\quad + h^{-1}H^{2r}(h^{2r}+h^{2r/(r+1)}H^{2r}+H^{2r+2})+\Delta t\big),
\end{aligned}
$$

where $(\mathbf{U}_h^n, P_h^n)$ is the fully discrete solution of the second step. To validate the success of the proposed scheme, numerical results are provided. We have observed that the accuracy of the numerical solutions by the proposed two-grid DG method is quite close to that of the standard DG method. Furthermore, we have compared the computational times taken to compute the two-grid DG solution and the standard DG solution, and the results demonstrate that the proposed two-grid DG method requires significantly less computational time than the standard DG method.

In Chapter 6, we have considered a three-step two-grid algorithm based on DG approximation for the Kelvin-Voigt model. The first step consists of solving the nonlinear system utilizing DG method for space dicretization on a coarse grid. Then, by employing the coarse grid solution and Newton's iteration type linearization, we solve the resulting system on a fine grid in the second step. A modified final solution is produced in the third step, which is a correction step for the solutions of the second step and is acquired by solving a different linear problem on the fine grid. *A priori* estimates of the semi-discrete solutions for all three steps have been shown. We have also derived some new interpolated Sobolev and trace inequalities, and based on these, semi-discrete optimal error estimates for the velocity and pressure have been obtained in the final step as

$$
\begin{aligned}
\|(\mathbf{u}-\mathbf{u}_h^*)(t)\| &\le K(t)(h^{r+1}+h^rH^{r+1-\theta}+H^{3r+2-2\theta}),\\
\|(\mathbf{u}-\mathbf{u}_h^*)(t)\|_\varepsilon+\|(p-p_h^*)(t)\| &\le K(t)(h^r+H^{3r+2-2\theta}),
\end{aligned}
$$

where $\theta > 0$ is arbitrarily small and $(\mathbf{u}_h^*, p_h^*)$ is the semi-discrete solution in the third step. Moreover, under a smallness condition on data, the above estimates are uniform in time. We have established that the largest scaling between H and h is $h = \mathcal{O}\big(H^{\min\left(r+1-\theta, \frac{3r+2-2\theta}{r+1}\right)}\big)$ for optimal velocity error estimate in $\mathbf{L}^2$-norm and it is $h = \mathcal{O}(H^{\frac{3r+2-2\theta}{r}})$ for velocity in energy norm and for pressure in L^2-norm. We have further discretized the two-grid DG model in time, using the backward Euler method and derived the *a priori* bounds for fully discrete solutions. In addition, the following fully discrete error estimates have been derived

$$\begin{aligned} \|\mathbf{u}^n - \mathbf{U}^n\| &\leq K_T(h^{r+1} + h^r H^{r+1-\theta} + H^{3r+2-2\theta} + \Delta t), \\ \|\mathbf{u}^n - \mathbf{U}^n\|_\varepsilon + \|p^n - P^n\| &\leq K_T(h^r + H^{3r+2-2\theta} + \Delta t), \end{aligned}$$

where $(\mathbf{U}^n, P^n)$ is the fully discrete solution in the third step. We have provided some numerical examples that validate our theoretical results. We have compared the computational times taken to compute the two-grid DG solution and the standard DG solution, which indicate that the two-grid DG solution reduce the computation time by around 50%. Additionally, as we refine the mesh more and more, the computational time gap increases between the solutions, namely the two-grid DG solution and the standard DG solution.

We have employed a two-grid DG method for the Oldroyd model of order one in Chapter 7. Here, we have employed the same algorithm as in Chapter 6, but for the Oldroyd model of order one. The resulting scheme is a fully discrete scheme, with the time discretization performed utilizing the backward Euler method. *A priori* bounds for the discrete solutions for all three steps have been derived. Furthermore, the following fully discrete optimal error estimates for the velocity and pressure have been established:

$$\begin{aligned} \|\mathbf{u}^n - \mathbf{U}_h^{*n}\| &\leq K_T(h^{r+1} + h^r H^{r+1-\theta} + H^{3r+2-2\theta} + \Delta t), \\ \|\mathbf{u}^n - \mathbf{U}_h^{*n}\|_\varepsilon + \|p^n - P_h^{*n}\| &\leq K_T(h^r + H^{3r+2-2\theta} + \Delta t), \end{aligned}$$

where $(\mathbf{U}_h^{*n}, P_h^{*n})$ is the fully discrete solution in the third step. Finally, we have verified our theoretical results by performing some numerical simulations.

8.2 Future Plan

We want to continue working on the Kelvin-Voigt and Oldroyd model of order one in the future before moving on to coupled problems and other associated problems.

There are studies in the literature regarding approximations of the incompressible NSEs using local DG methods; these can be found in [48–50, 169]. We therefore plan to employ local DG methods for viscoelastic fluid flows, namely the Kelvin-Voigt model and the Oldroyd model of order one.

Unlike all known other DG methods, which result in a final system involving the degrees of freedom of the approximate velocity and pressure, the hybridizable DG method produces a final system involving the degrees of freedom of the approximate trace of the velocity and the mean of pressure. Since the approximate trace is defined on the element borders only and since the mean of pressure is a piece-constant function, the hybridizable DG method has significantly less globally coupled unknowns than other DG methods, especially for high-degree polynomial approximations, thereby allowing for a substantial reduction in the computational cost. We refer to [36, 99–101, 122, 145] for literature on approximations of the incompressible NSEs based on hybridizable DG methods. However there is hardy any work on hybridizable DG method for the Kelvin-Voigt model and the Oldroyd model of order one. This motivates us to analyse this hybrid method for the Kelvin-Voigt model and the Oldroyd model of order one.

For higher-order DG approximations, it is desirable to use higher-order time schemes. In [98], the Crank-Nicolson time discretization scheme in the context of IPDG for time-dependent NSEs, and in [162], implicit-explicit (IMEX) time discretization scheme in the context of local DG for the time-dependent Oseen and NSEs have been analyzed. We therefore plan to employ these high order time discretizations for the Kelvin-Voigt model and Oldroyd model order one in the context of IP DG, local DG and hybridizable DG methods, and study the stability and convergence analysis for these time schemes.

Apart from these, a pressure correction scheme (which belongs to the class of operator splitting schemes that decouple the non-linearity in the momentum equation from the incompressibility constraint) combined with the IPDG method has been applied for time-dependent NSEs in [116, 117]. For large-scale problems, these approaches are known to be computationally efficient. Thus, our plan is to extend this scheme to the Kelvin-Voigt model and the Oldroyd model of order one using IPDG, local DG and hybridizable DG approximations.

The filtration of fluids through porous media is an interesting research subject with many relevant applications [43, 59, 110, 121, 151]. To quote some examples, these phenomena occur in physiology when studying the filtration of blood through arterial vessel walls, in industrial processes involving, e.g., air or oil filters, in the environment concerning the percolation of waters of hydrological basins through rocks and sand. The modeling of such physical processes requires to consider different systems of partial differential equations in each subregion of the domain of interest. Typically, the motion of incompressible free fluids are described by the Navier-Stokes equations while Darcy equations are adopted to model the filtration process. These equations must be linked through suitably chosen conditions that describe the fluid flow across the surface of the porous media through which the filtration occurs.

The literature on the coupled Navier-Stokes/Darcy system in the CG framework is vast. For instance, the works [27, 28, 44, 60, 61, 88, 170, 182], and references therein have all explored the steady-state coupled equation. For the unsteady problem, we refer to [29, 33, 35, 42, 90, 91, 163, 171].

On the other hand, the DG literature for this coupled problem is very limited for both the steady and unsteady cases. The IP DG scheme has been studied for steady-state in [70] and for the time-dependent case in [34, 38]. A conservative hybridizable DG method for the steady and evolutionary Navier-Stokes/Darcy problems has been studied in [37] and [32], respectively. However, no work exists in the literature for local DG methods, and hence, there is a scope for analyzing this model in this direction. Furthermore, in [27, 89, 187], a two-grid or multi-grid method with CG approximations has been applied for the steady Navier-Stokes/Darcy model. The unsteady Navier-Stokes/Darcy problem has been analyzed by using the two-grid method in [113]. But there is no work on two-grid or multi-grid method combined with DG approximations. This motivates us to analyse two-grid/multi-grid method in combination with IP DG/local DG/hybridizable DG approximations for this coupled problem. Pressure correction DG schemes can also be applied to this problem.