Chapter 1

Introduction

1.1 Incompressible Flows

Incompressible fluids are fluids that maintain a constant density regardless of changes in pressure. These fluids do not significantly alter their volume under external forces, simplifying fluid mechanics equations. Mathematically speaking, the divergence of the velocity of the fluid is zero throughout the domain.

The governing equations of an incompressible fluid flow in a bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$ are given by the following system of differential equations:

$$\frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot \boldsymbol{\sigma} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}(x,t), \quad \nabla \cdot \mathbf{u} = 0, \quad \text{for } t > 0 \tag{1.1}$$

with appropriate initial and boundary conditions. Here, $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ik})_{1 \leq i,k \leq d}$ denotes the deviator of the stress tensor (also called the extra-stress tensor) with $tr\ \boldsymbol{\sigma} = 0$, $\mathbf{u} = (u_i)_{1 \leq i \leq d}$ represents the velocity vector, p is the pressure of the fluid and $\mathbf{f}$ is the external force.

In order to determine a fluid under consideration, we generally resort to the relationship between $\boldsymbol{\sigma}$ and $\mathbf{D}$, which is known as the rheological equation or equation of state. Here $\mathbf{D} = (\mathbf{D}_{ik}) = \frac{1}{2}(\mathbf{u}_{ix_k} + \mathbf{u}_{kx_i})$ is the tensor of deformation velocities. For example, a fluid with the rheological equation

$$\boldsymbol{\sigma} = 2\nu \mathbf{D} \tag{1.2}$$

represents the popular Navier-Stokes equations (NSEs), where $\nu > 0$ is the kinematic coefficient of viscosity.

1.2 Model Problems

For over a century, the Newtonian fluid model has stood as the primary model for understanding the behavior of viscous incompressible fluids, offering a means to describe the flow of most such fluids encountered in practical applications, especially at moderate velocities. Newtonian fluids adhere to Newton's law of viscosity, where the shear stress is directly proportional to the strain rate (rate of deformation). In these fluids, viscosity remains constant regardless of the applied stress.
As mentioned earlier, the relation (1.2) results in the NSEs, which a Newtonian fluid and by substituting it in (1.1), we can generate the equations of motion of the NSEs:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}(x,t), \tag{1.3}$$
$$\nabla \cdot \mathbf{u} = 0.$$

There are, however, some viscous incompressible fluids that do not follow the Newtonian defining equation (1.2) or the Newton's law of viscosity, and are classified as non-Newtonian fluids (variable viscosity based on the applied stress or strain rate). An important subclass of non-Newtonian fluids are the linear viscoelastic fluids that take into account the prehistory of the flow. The intermolecular interactions between the fluid particles are prominent in such fluids, and are governed by viscous and elastic forces. Polymeric fluids that are characterized by long chain of molecules are perhaps the most important class of viscoelastic fluids due to their vast presence in the chemical, pharmaceutical, food and oil industries. Molten plastics, engine oils, paints, ointments, gels and other biological fluids like egg white and blood are some examples of viscoelastic fluids.

New models or rheological equations are developed to describe such non-Newtonian fluids, and have been of interest to the research community since then. One such rheological relation, which characterizes a class of non-stationary linear viscoelastic fluids, namely, aqueous polymer solutions with a finite set of discretely distributed relaxation times $\{\lambda_l\},\ l = 1, 2, \ldots, L$ and with retardation times (delay times) $\{\kappa_m\},\ m = 1, 2, \ldots, M$, is given by [64, 94, 128, 173]

$$\left(1 + \sum_{l=1}^{L} \lambda_l \frac{\partial^l}{\partial t^l}\right) \boldsymbol{\sigma} = 2\nu \left(1 + \sum_{m=1}^{M} \kappa_m \nu^{-1} \frac{\partial^m}{\partial t^m}\right) \mathbf{D}. \tag{1.4}$$

The aforementioned equation results in three fluid models in its most basic form, particularly when considering a single-mode scenario. When setting $L = 1$ and $M = 0$, it yields the Maxwell fluid. In this case, after an instantaneous termination of the movement, the stress in the fluid do not instantly turn to zero, but decay like $e^{-\lambda_1^{-1}t}$ (where $\lambda_1 > 0$ represents the relaxation time).

Furthermore, for the conditions where $L = 0$ and $M = 1$, we arrive at the Kelvin-Voigt fluid. This type of fluid is characterized by the phenomenon that, after instantaneous removal of the stress, the velocity of the fluid does not vanish instantaneously but decays as $e^{-\kappa_1^{-1}t}$ (where κ_1 is referred to as the retardation time).

When both L and M are set to 1, the resulting fluid is known as Oldroyd fluid which exhibits both the characteristics, that is, after instantaneous cessation of motion, the stresses in the fluid do not vanish immediately, but die out like $e^{-\lambda_1^{-1}t}$, and after instantaneous removal of stresses, the velocity of the fluid does not vanish immediately, but dies out like $e^{-\kappa_1^{-1}t}$.

In a broader context, when $M = L+1$ $(L = 0, 1, 2, \dots)$ and $M = L$ $(L = 1, 2, \dots)$ in (1.4), we find two distinct linear viscoelastic fluid model which are Kelvin-Voigt model of order L and Oldroyd model of order L, respectively. In this thesis, we deal only with the cases $L = 0$ for Kelvin-Voigt model and $L = 1$ Oldroyd model, and will simply call them Kelvin-Voigt model and Oldroyd model of order one, respectively. In the middle of the twentieth century, the Kelvin-Voigt model in [126, 128] and Oldroyd model of order one in [125] were introduced and developed. These models are described by the following defining rheological equations:

$$\boldsymbol{\sigma} = 2\nu\left(1 + \kappa\nu^{-1}\frac{\partial}{\partial t}\right)\mathbf{D}, \tag{1.5}$$

for Kelvin-Voigt model ($L = 0$, $M = 1$) and

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\boldsymbol{\sigma} = 2\nu\left(1 + \kappa\nu^{-1}\frac{\partial}{\partial t}\right)\mathbf{D}, \tag{1.6}$$

for Oldroyd model of order one ($M = L = 1$). Here $\nu > 0$ is the kinematic coefficient of viscosity, $\lambda > 0$ is the relaxation time and $\kappa > 0$ is the retardation time with $\nu - \kappa\lambda^{-1} > 0$.

Using the relations (1.5) and (1.6) in (1.1), we find the equations of motion that represent the fluid flow of the Kelvin-Voigt model and the Oldroyd model of order one, respectively:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \kappa \Delta \mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(x,t), \ \ x \in \Omega, \ \ t > 0 \tag{1.7}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - \int_0^t \beta(t-\tau) \Delta \mathbf{u}(x,\tau)\, d\tau + \nabla p = \mathbf{f}(x,t), \ \ x \in \Omega, \ \ t > 0, \tag{1.8}$$

along with the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0, \ \ x \in \Omega, \ t > 0, \tag{1.9}$$

and initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \ \ \text{in } \Omega, \qquad \mathbf{u} = \mathbf{0}, \ \ \text{on } \partial\Omega, \ t \geq 0. \tag{1.10}$$

Here, $\partial\Omega$ denotes the boundary of Ω, $\mu = 2\kappa\lambda^{-1} > 0$, the kernel $\beta(t) = \gamma \exp(-\delta t)$, $\gamma = 2\lambda^{-1}(\nu - \kappa\lambda^{-1}) > 0$ and $\delta = \lambda^{-1} > 0$. We refer the reader to [94, 125, 126, 139] for additional details on these models.

It is worth mentioning here that the mathematical models of the Kelvin-Voigt and the Oldroyd model of order one can be considered as a smooth perturbation of NSEs. When the parameters $\kappa = 0$ (for Kelvin-Voigt) and $\gamma = 0$ (for Oldroyd model of order one), the two viscoelastic fluid systems reduce to the NSEs. As a result, the numerical schemes and associated results should be comparable for the three models. Considering NSEs as our pivot, we have mainly focused on the Kelvin-Voigt model and the Oldroyd model of order one in this thesis, working out the basic results for NSEs and establishing the main results for the two linear viscoelastic models.

Kelvin-Voigt and Oldroyd systems represent basic models for polymeric fluids, suspensions or biological fluids; non-Newtonian models, which are derived under the assumptions that the material can be viewed as a single stationary microscopic element with small stress and strain rates. Both these models find applications in the study of organic polymers and in various industries, like, food, oil and chemical industries. Apart from these, Kelvin-Voight model has been applied to explain the mechanism of diffuse axonal injury that are unexplained by traumatic brain injury models.

The above mentioned applications are motivation enough to study and understand these incompressible fluids (1.3), (1.7)-(1.10). However, as a system of differential equations, they are challenging from analysis point of view as well. These models are not only nonlinear but also a coupled system, the velocity being coupled with the pressure via the momentum equation. Also for large Reynolds number, the nonlinear convective term dominates the dissipiative term. Memory term, in case of less regular data, is another challenge. These present considerable difficulties in the finite element analysis and numerical simulations of these fluid models. This motivates us to apply and analyze the discontinuous Galerkin (DG) methods to these fluid systems.

1.3 DG Methods

DG methods belong to a class of numerical techniques employed for solving partial differential equations (PDEs). Their widespread adoption in diverse areas of computational science and engineering can be attributed to the numerous benefits they offer. Some key advantages of employing DG methods encompass: the capability to handle complicated geometry, their flexibility for adaptivity, locally varying polynomial order, higher order accuracy, and local mass conservation. For an extensive historical survey of several DG methods, we refer the reader to [47].

In this thesis, we mainly focus on the class of primal DG methods, namely variations of interior penalty DG methods. These methods are referred to as the symmetric interior penalty Galerkin (SIPG), non-symmetric interior penalty Galerkin (NIPG), and incomplete interior penalty Galerkin (IIPG) methods. All three methods contain a penalty term as well as a penalty parameter. The SIPG and IIPG methods converge if the penalty parameter is large enough. But the NIPG method converges for any non-negative values of the penalty parameter [146].

Unlike the NIPG and IIPG methods, the variational formulation of the SIPG method is symmetric and adjoint consistent. As a result, optimal theoretical and numerical convergence rates in L^2-norm can be found for the SIPG method. For the NIPG and IIPG methods, the numerical convergence rates in L^2-norm will be optimal only for an odd degree of polynomial approximation; for an even degree, the result is sub-optimal, see [146]. Therefore, we primarily analyze the SIPG method in this thesis and provide remarks for NIPG and IIPG.

Primary DG methods for the incompressible NSEs have been studied on numerous occasions, both theoretically and numerically. For rigorous theoretical works, we refer to [71, 72, 98, 147]. However, a notable void exists in establishing the optimal $\mathbf{L}^2$-norm error estimation for the velocity in the context of unsteady incompressible NSEs. Thus, in the beginning of this thesis, efforts are made to fill this theoretical gap, and to establish the optimal convergence order in the $\mathbf{L}^2$-norm of the velocity for unsteady NSEs. Additionally, to the best of our knowledge, DG methods have not previously been employed in the context of the Kelvin-Voigt model and the Oldroyd model of order one. Therefore, the next part of this thesis is dedicated to the development of DG schemes for both of these model problems and conduct a comprehensive analysis thereof.

We would like to note here that DG approaches have the potential to produce algebraic equations with a huge degrees of freedom, which can be computationally expensive and is a major obstacle in solving them. Also, nonlinearity itself is a big issue for large systems. To overcome this challenge, in the final part of the thesis, we have employed a combination of a cost-effective two-grid scheme and the DG method to the incompressible fluids (1.3), (1.7)-(1.10). The two-grid techniques are recognised as an efficient discretization approach for solving nonlinear problems. They could solve the system relatively inexpensively while maintaining a certain degree of accuracy. With these techniques, solving a nonlinear problem on the fine grid is reduced to solving the nonlinear problem on a coarse grid $\mathcal{E}_H$ with a mesh of size H and solving a linear problem on a fine grid $\mathcal{E}_h$ with a mesh of size h. Consequently, evaluating a nonlinear problem is not typically more difficult than handling one linear problem, since dim(finite element space on $\mathcal{E}_H$) $\ll$ dim(finite element space on $\mathcal{E}_h$) and comparatively less work is required to solve the nonlinear problem on the coarse mesh.

In this thesis, we have considered a two-step two-grid algorithm for transient NSEs where the linearization on the fine mesh is based on one Newton iteration around the coarse mesh solution. With this algorithm, we can only achieve optimal results for velocity in energy norm. An improved algorithm, containing an additional correction step, for the solutions of the second step on the fine mesh, has been applied to the Kelvin-Voigt and Oldroyd model of order one. This results in optimal $\mathbf{L}^2$-estimate of velocity in the third step.

All our numerical schemes have been accompanied by numerical compuations, carried out in MATLAB and FreeFEM++ [84], to validate our theoretical findings.

In the upcoming section, we acquaint ourselves with the necessary notations and preliminaries crucial for our analysis.

1.4 Notations and Preliminaries

This section offers an introduction to the notations and essential background information that will be employed in the forthcoming chapters.

Let Ω be a bounded and convex polygonal domain in $\mathbb{R}^d$, $d = 2, 3$ with boundary $\partial\Omega$. Further, let $\mathcal{D} \subseteq \Omega$. For $1 \leq p < \infty$, $L^p(\mathcal{D})$ denote the linear space of equivalence classes of measurable functions ϕ on $\mathcal{D}$ such that $\int_{\mathcal{D}} |\phi(x)|^p dx < \infty$ and associated norm is defined as

$$\|\phi\|_{L^p(\mathcal{D})} = \left(\int_{\mathcal{D}} |\phi(x)|^p dx \right)^{1/p}.$$

For $p = \infty$, $L^\infty(\mathcal{D})$ consists of measurable functions ϕ such that $\operatorname{ess\,sup}_{x\in\mathcal{D}} |\phi(x)| < \infty$ and associated norm is defined as

$$\|\phi\|_{L^\infty(\mathcal{D})} = \operatorname*{ess\,sup}_{x\in\mathcal{D}} |\phi(x)|.$$

Note that for $p = 2$, $L^2(\mathcal{D})$ is a Hilbert space. Our analysis relies on this space and its closed subspaces and on the following quotient space

$$L^2(\mathcal{D})/\mathbb{R} = \{\phi \in L^2(\mathcal{D}) : \int_{\mathcal{D}} \phi(x)\ dx = 0\}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi index with non-negative integers α_i, $i = 1, \ldots, d$ and its order is defined by $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Further, let us denote $D^\alpha\phi$ to be the α^{th} order partial (generalized/distributional) derivative of $\phi(x)$ with $x = (x_1, \ldots, x_d)$, that is,

$$D^\alpha\phi = \frac{D^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}.$$

We are now in a position to introduce the concept of Sobolev spaces which form an important tool in defining the variational formulations for the problems. For any given integer $m \geq 0$ and real number $p \geq 1$, the standard Sobolev space of order (m, p) on $\mathcal{D}$, denoted by $W^{m,p}(\mathcal{D})$, is defined as a linear space which consists of equivalence class of functions in $L^p(\mathcal{D})$ whose distributional derivatives up to and including order m are also in $L^p(\mathcal{D})$, that is,

$$W^{m,p}(\mathcal{D}) := \big\{\phi \in L^p(\mathcal{D}) : D^\alpha \phi \in L^p(\mathcal{D}),\ 0 \le |\alpha| \le m\big\}.$$

The associated norm of the space $W^{m,p}(\mathcal{D})$ is defined, for $1 \le p < \infty$, as

$$\|\phi\|_{m,p,\mathcal{D}} := \Bigg(\sum_{0\le|\alpha|\le m} \int_{\mathcal{D}} |D^\alpha \phi(x)|^p dx \Bigg)^{1/p} = \Bigg(\sum_{0\le|\alpha|\le m} \|D^\alpha \phi\|^p_{L^p(\mathcal{D})} \Bigg)^{1/p}.$$

When $p = \infty$, the norm on space $W^{m,\infty}(\mathcal{D})$ is defined as

$$\|\phi\|_{m,\infty,\mathcal{D}} := \max_{0\le|\alpha|\le m} \|D^\alpha \phi\|_{L^\infty(\mathcal{D})}.$$

For the case $p = 2$, these spaces are Hilbert spaces, denoted by $H^m(\mathcal{D})$, and are endowed with norm $\|\cdot\|_{m,2,\mathcal{D}} = \|\cdot\|_{m,\mathcal{D}}$. The natural inner product on the space $H^m(\mathcal{D})$ is defined by

$$(\phi, \psi)_{m,\mathcal{D}} := \sum_{0\le|\alpha|\le m} \int_{\mathcal{D}} D^\alpha \phi(x) D^\alpha \psi(x)\ dx, \quad \phi, \psi \in H^m(\mathcal{D}).$$

We also define the seminorm on $W^{m,p}(\mathcal{D})$, which consists of the L^p-norms of the highest order derivatives, as

$$|\phi|_{m,p,\mathcal{D}} = \Bigg(\sum_{|\alpha|=m} \int_{\mathcal{D}} |D^\alpha \phi(x)|^p dx \Bigg)^{1/p}.$$

The closure of $C_c^\infty(\mathcal{D})$, the space of infinitely differentiable functions with compact support, in $H^m(\mathcal{D})$, is denoted by $H_0^m(\mathcal{D})$. In other words, $H_0^m(\mathcal{D})$ is a subspace of $H^m(\mathcal{D})$ with elements vanishing on boundary in the sense of trace [2]. The dual space of $H_0^m(\mathcal{D})$ is defined as the completion of $C^\infty(\bar{\mathcal{D}})$ with respect to the norm

$$\|\phi\|_{-m,\mathcal{D}} := \sup \left\{ \frac{\langle \phi, \psi \rangle}{\|\psi\|_{m,\mathcal{D}}} : \psi \in H_0^m(\mathcal{D}), \|\psi\|_{m,\mathcal{D}} \ne 0 \right\},$$

and it is denoted by $H^{-m}(\mathcal{D})$.

If $\mathcal{D} = \Omega$ and $p = 2$, we simply drop the domain Ω from the notations of norm and semi-norm. In case of $H^m(\Omega)$, we denote them as $\|\cdot\|_m$ and $|\cdot|_m$, respectively. Further, for $p = 2$, the norm and inner-product for the space $L^2(\Omega)$, are denoted as $\|\cdot\|$ and $(\cdot, \cdot)$, respectively.

For notational convenience, we denote the $\mathbb{R}^d$, $(d = 2, 3)$-valued function spaces by bold face letters such as

$$\mathbf{H}^m(\mathcal{D}) = [H^m(\mathcal{D})]^d, \quad \mathbf{H}_0^1(\mathcal{D}) = [H_0^1(\mathcal{D})]^d, \quad \text{and} \quad \mathbf{L}^p(\mathcal{D}) = [L^p(\mathcal{D})]^d.$$

Also for a given Banach space X with norm $\|\cdot\|_X$, let $L^p(0, T; X)$ be the space of all strongly measurable and p-th integrable X-valued functions $\psi : [0, T] \to X$ satisfying

$$\int_0^T \|\psi(t)\|_X^p dt < \infty, \quad 1 \leq p < \infty, \quad \text{and} \quad \operatorname*{ess\,sup}_{t\in[0,T]} \|\psi(t)\|_X < \infty, \quad p = \infty.$$

The norms for these spaces are defined as

$$\|\psi\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|\psi(t)\|_X^p dt \right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{t\in[0,T]} \|\psi(t)\|_X, & p = \infty. \end{cases}$$

There are other spaces that are useful for our analysis as well. For example, we consider divergence free subspace of usual solution space, since the fluid under consideration is incompressible, that is, the velocity vector is divergence free. We note here that the use of this space is limited to the analysis only and has not been considered for numerical computations. We now introduce below, the divergence free function space:

$$\mathbf{J}_1 = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{v} = 0\}$$

Also we use quotient spaces, since pressure is unique only up to a constant. Let $H^m(\Omega)/\mathbb{R}$ be the quotient space consisting of equivalence classes of elements of $H^m(\Omega)$ differing by constants and the associated norm is defined by $\|\cdot\|_{m/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|\cdot + c\|_m$, see [69].

We next list down a few standard inequalities:

(i) **Cauchy-Schwarz inequality:** The following inequality holds for all $\phi, \psi \in L^2(\Omega)$:

$$|(\phi, \psi)| \leq \|\phi\| \|\psi\|.$$

(ii) **Young's inequality:** For all $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for all $a, b \geq 0, \epsilon > 0$, the following inequality holds:

$$ab \leq \frac{\epsilon a^p}{p} + \frac{b^q}{q\epsilon^{q/p}}.$$

(iii) **Hölder's inequality:** For all $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for $\phi \in L^p(\Omega)$ and $\psi \in L^q(\Omega)$, the following inequality holds:

$$\int_\Omega \phi\psi \; dx \leq \|\phi\|_{L^p(\Omega)} \|\psi\|_{L^q(\Omega)}.$$

Next in our list is the standard Gronwall's lemma. For a proof, we refer to [77].

Lemma 1.1 (Classical Gronwall's lemma)**.** *Let g, h, y be three locally integrable non-negative functions on the time interval $[t_0, \infty)$ such that for all $t \geq t_0$ the following holds*

$$\frac{dy}{dt} \leq gy + h,$$

where $\frac{dy}{dt}$ is locally integrable. Then,

$$y(t) \leq y(t_0) \exp\Big(\int_{t_0}^t g(\tau) d\tau \Big) + \Big(\int_{t_0}^t h(s) \exp\Big(\int_s^t g(\tau) d\tau \Big) \; ds \Big).$$

However in our analysis, we often resort to a modified version which we present below. For a proof, we refer to [73].

Lemma 1.2 (Gronwall's Lemma)**.** *Let g, h, y be three locally integrable non-negative functions on the time interval $[t_0, \infty)$ such that for all $t \geq t_0$*

$$y(t) + G(t) \leq C + \int_{t_0}^t h(s) \; ds + \int_{t_0}^t g(s) y(s) \; ds,$$

where $G(t)$ is a non-negative function on $[t_0, \infty)$ and $C \geq 0$ is a constant. Then,

$$y(t) + G(t) \leq \Big(C + \int_{t_0}^t h(s) \exp\Big(- \int_{t_0}^s g(\tau) \; d\tau \Big) \; ds \Big) \exp\Big(\int_{t_0}^t g(s) \; ds \Big).$$

With all the above preparation, we now briefly look at the equations (1.3), (1.7)-(1.10) in continuous Galerkin (CG) framework. It is customary to study the models in a weaker form. For the sake of brevity, we only present the weak or variational formulation for the NSEs, that is, for the system (1.3), (1.9)-(1.10): Find a pair $(\mathbf{u}(t), p(t)) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$, $t > 0$, such that

$$(\mathbf{u}_t, \boldsymbol{\phi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\phi}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \boldsymbol{\phi}) - (p, \nabla \cdot \boldsymbol{\phi}) = (\mathbf{f}, \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathbf{H}_0^1(\Omega), \tag{1.11}$$

$$(\nabla \cdot \mathbf{u}, q) = 0 \quad \forall q \in L^2(\Omega), \tag{1.12}$$

with $\mathbf{u}(0) = \mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$, given $\mathbf{f} \in L^\infty(0, T; \mathbf{L}^2(\Omega))$ and $0 < t \leq T \leq \infty$.

Furthermore, the variational formulation for equations (1.7), (1.9)-(1.10), and (1.8), (1.9)-(1.10) will closely resemble that of (1.11)-(1.12). However, to accommodate the terms $-\kappa\Delta\mathbf{u}_t$ and $-\int_0^t \beta(t-s)\Delta\mathbf{u}(x,s)\,ds$ present in equations (1.7) and (1.8), additional terms must be included on the left-hand side of (1.11). These supplementary terms will be $\kappa\,(\nabla\mathbf{u}_t, \nabla\boldsymbol{\phi})$ and $\int_0^t \beta(t-s)(\nabla\mathbf{u}(s), \nabla\boldsymbol{\phi})\,ds$ for models (1.7) and (1.8), correspondingly.

For the well-posedness of the both weak and regular solutions for the model problems (1.3), (1.7)-(1.10), we refer to [10, 73, 160]. In our thesis, we assume the solution of each model to be sufficiently regular so as to obtain optimal error estimate for r^{th} order (piecewise) polynomial approximations. We present below the assumptions that we will employ in our later error analysis.

(**A0**). The exact solution of (1.3), (1.7)-(1.10) satisfies the following regularity assumptions:

$$
\begin{aligned}
\mathbf{u} &\in L^\infty(0,T;\mathbf{H}^{r+1}(\Omega)) \cap L^2(0,T;\mathbf{H}^{r+1}(\Omega)),\\
\mathbf{u}_t &\in L^2(0,T;\mathbf{H}^{r+1}(\Omega)) \cap L^\infty(0,T;\mathbf{H}^{r}(\Omega)),\\
p &\in L^\infty(0,T;H^{r}(\Omega)) \cap L^2(0,T;H^{r}(\Omega)),
\end{aligned}
$$

(**A1**). The exact solution of (1.3), (1.9)-(1.10) satisfies (**A0**) as well as the following regularity assumption:

$$p_t \in L^2(0,T;H^r(\Omega)).$$

(**A2**). The exact solution of (1.7), (1.9)-(1.10) satisfies (**A0**) as well as the following regularity assumption:

$$\mathbf{u}_t \in L^\infty(0,T;\mathbf{H}^{r+1}(\Omega)).$$

(**A3**). The exact solution of (1.8), (1.9)-(1.10) satisfies (**A1**) as well as the following regularity assumption:

$$\mathbf{u}_{tt} \in L^2(0,T;\mathbf{L}^2(\Omega)).$$

(**A4**). The exact solution of (1.8), (1.9)-(1.10) satisfies (**A1**) as well as the following regularity assumption:

$$\mathbf{u}_{tt} \in L^\infty(0,T;\mathbf{L}^2(\Omega)).$$

In order to bound the nonlinear convective term, we next present below the following well-known Sobolev inequalities, see [159, 161].

Lemma 1.3. *For any open set* $\Omega \subset \mathbb{R}^2$ *and for* $\mathbf{v} \in \boldsymbol{H}_0^1(\Omega)$

$$\|\mathbf{v}\|_{\boldsymbol{L}^4(\Omega)} \leq 2^{1/4}\|\mathbf{v}\|^{1/2}\|\nabla\mathbf{v}\|^{1/2}.$$

Moreover, when Ω *is bounded, then following estimates hold*

$$\|\mathbf{v}\|_{\boldsymbol{L}^\infty(\Omega)} \leq C\|\mathbf{v}\|^{1/2}\|\Delta\mathbf{v}\|^{1/2}, \quad \mathbf{v} \in \boldsymbol{H}^2(\Omega).$$

In the integro-parabolic system (1.8), the kernel β enjoys a positivity property which is crucial for the CG finite element analysis. We present below the positivity property, for a proof of which, we refer to [118].

Lemma 1.4. *For any* $\alpha > 0$ *and* $\psi \in L^2(0,t)$*, the following positive definite property holds for any* $t > 0$

$$\int_0^t \Big(\int_0^s e^{-\alpha(s-\tau)}\psi(\tau)d\tau\Big)\psi(s)ds \geq 0.$$

The problems (1.3), (1.7)-(1.10) is posed in an infinite dimensional function space and in finite element methods, we attempt a finite dimensional problem. This is achieved by discretizing the domain Ω into finitely many elements and then considering finite dimensional finite element spaces, where the problem is solved. When only space is discretized, time remaining continuous, we call it the semi-discrete case.

For spatial discretization, we consider a family of shape-regular triangulations $\mathcal{E}_h$ of $\bar{\Omega}$, which consists of triangles or tetrahedra E, according to the dimension, with h_E =diam(E) and $h = \max_{E\in\mathcal{E}_h} h_E$. Let ρ_E be the maximum diameter of the ball inscribed in E. By shape-regular we mean that there is a constant $\eta > 0$, independent of h, such that (see [46]),

$$\frac{h_E}{\rho_E} \leq \eta, \quad \forall E \in \mathcal{E}_h. \tag{1.13}$$

Let us denote the set of all edges (or faces in 3D) of the partition $\mathcal{E}_h$ by Γ_h. For any interior edge (or face in 3D) e shared by two elements E_m and E_n $(m < n)$; we associate with e a unit normal $\mathbf{n}_e$ such that it directs from the element E_m to the element E_n. Now, we define the average $\{\cdot\}$ and jump $[\cdot]$ of a discontinuous function w on edge (or face in 3D) e by

$$\{w\} = \frac{1}{2}\big((w|_{E_m})|_e + (w|_{E_n})|_e\big), \quad [w] = (w|_{E_m})|_e - (w|_{E_n})|_e.$$

For an edge (or face in 3D) e on the boundary $\partial\Omega$, the unit vector normal $\mathbf{n}_e$ coincides with the unit outward vector normal to $\partial\Omega$. In this case, the average and jump of w on e are defined to be equal to the trace of w on e.

For our subsequent analysis, we also require the following discontinuous spaces:

$$\mathbf{X} = \{\boldsymbol{\phi} \in \mathbf{L}^2(\Omega) : \ \forall E \in \mathcal{E}_h, \ \boldsymbol{\phi}|_E \in \mathbf{W}^{2,2d/(d+1)}(E)\},$$
$$M = \{q \in L^2(\Omega)/\mathbb{R} : \ \forall E \in \mathcal{E}_h, \ q|_E \in W^{1,2d/(d+1)}(E)\}.$$

Motivation for such spaces will follow soon !

The discontinuous space $\mathbf{X}$ is equipped with the following mesh dependent energy norm:

$$\|\boldsymbol{\phi}\|_\varepsilon = \left(\sum_{E\in\mathcal{E}_h} \|\nabla\boldsymbol{\phi}\|^2_{L^2(E)} + \sum_{e\in\Gamma_h} \frac{\sigma_e}{h_e} \|[\boldsymbol{\phi}]\|^2_{L^2(e)} \right)^{1/2} \quad \forall \boldsymbol{\phi} \in \mathbf{X},$$

where $h_e = \text{diam}(e)$ and σ_e will be referred to as penalty parameter, is a positive constant that is specified for each e.

Note that $h_e = |e|^{1/(d-1)}$. That is, when $d = 2$, $h_e = |e|$. Since we have carried out our analysis for $d = 2$, we use the notation $|e|$ instead of h_e in the following chapters. We have briefly discussed the case $d = 3$ in the remarks in this thesis whenever required.

The space M is equipped with the following norm:

$$\|q\|_M = \|q\|_{L^2(\Omega)/\mathbb{R}} \quad \forall q \in M.$$

For the functions in $\mathbf{X}$, we recall the L^p-norm bounds in terms of the energy norm $\|\cdot\|_\varepsilon$ ([116, Lemma 5.2]): For all $\boldsymbol{\phi} \in \mathbf{X}$, there is a positive constant C_p, independent of h, such that

$$\|\boldsymbol{\phi}\|_{L^p(\Omega)} \leq C_p \|\boldsymbol{\phi}\|_\varepsilon, \tag{1.14}$$

where $p \in [2, \infty)$ for $d = 2$ and $p \in [2, 6]$ for $d = 3$. That is, when $p = 2$, the constant of (1.14) is C_2.

To discretize the viscous and divergence terms, let us define the bilinear forms $a : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ and $b : \mathbf{X} \times M \to \mathbb{R}$ as follows:

$$a(\boldsymbol{\phi}, \mathbf{v}) = \sum_{E\in\mathcal{E}_h} \int_E \nabla\boldsymbol{\phi} : \nabla\mathbf{v}\, dx - \sum_{e\in\Gamma_h} \int_e \{\nabla\boldsymbol{\phi}\}\mathbf{n}_e \cdot [\mathbf{v}]\, ds \, + \epsilon \sum_{e\in\Gamma_h} \int_e \{\nabla\mathbf{v}\}\mathbf{n}_e \cdot [\boldsymbol{\phi}]\, ds$$
$$+ \sum_{e\in\Gamma_h} \frac{\sigma_e}{h_e} \int_e [\boldsymbol{\phi}] \cdot [\mathbf{v}]\, ds, \tag{1.15}$$

$$b(\boldsymbol{\phi}, q) = -\sum_{E\in\mathcal{E}_h}\int_E q\nabla\cdot\boldsymbol{\phi}\,dx + \sum_{e\in\Gamma_h}\int_e \{q\}[\boldsymbol{\phi}]\cdot\mathbf{n}_e\,ds, \tag{1.16}$$

where $\epsilon = -1$, 1 or 0. For $\epsilon = -1$, we have the SIPG method; for $\epsilon = 1$, we have the NIPG method, and for $\epsilon = 0$, we have the IIPG method. In the bilinear form $a(\cdot,\cdot)$, the last term of (1.15) is called penalty term. It is incorporated to weakly enforce the Dirichlet boundary condition inside the scheme, instead of being built into the finite element space. Furthermore, this term is useful to ensure coercivity of $a(\cdot,\cdot)$ in the finite element space. The development of this penalty term is discussed in Section 1.5.3.

For the discretization of the nonlinear term present in (1.3), (1.7) and (1.8), we utilize the upwind discretization proposed in [71]. To define this discretization, we also introduce a few notations. The notations $\mathbf{z}^{int}$ and $\mathbf{z}^{ext}$ denote the trace of the function $\mathbf{z}$ on the boundary of E coming from the interior and exterior of E, respectively. If the edge (or face in 3D) of E belongs to boundary $\partial\Omega$, the convention is identical with the definition of jump, *i.e.* we have $\mathbf{z}^{ext} = \mathbf{0}$. Let the vector $\mathbf{n}_E$ denote the unit outward normal to ∂E. We further introduce the inflow boundary of E with respect to a vector function $\mathbf{u}$ as

$$\partial E_- = \{\mathbf{x}\in\partial E :\ \{\mathbf{u}(\mathbf{x})\}\cdot\mathbf{n}_E < 0\}$$

With the above notations, we define for any $\mathbf{u},\mathbf{v},\mathbf{z},\mathbf{w}\in\mathbf{X}$,

$$\begin{aligned}c^{\mathbf{u}}(\mathbf{v},\mathbf{z},\mathbf{w}) = &\sum_{E\in\mathcal{E}_h}\left(\int_E(\mathbf{v}\cdot\nabla\mathbf{z})\cdot\mathbf{w}\,dx + \int_{\partial E_-}|\{\mathbf{v}\}\cdot\mathbf{n}_E|(\mathbf{z}^{int}-\mathbf{z}^{ext})\cdot\mathbf{w}^{int}\,ds\right)\\ &+\frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E(\nabla\cdot\mathbf{v})\mathbf{z}\cdot\mathbf{w}\,dx - \frac{1}{2}\sum_{e\in\Gamma_h}\int_e[\mathbf{v}]\cdot\mathbf{n}_e\{\mathbf{z}\cdot\mathbf{w}\}\,ds.\end{aligned} \tag{1.17}$$

Let Γ_+ denote the subset of the boundary $\partial\Omega$ where $\mathbf{v}\cdot\mathbf{n} > 0$. Then, the form $c(\cdot;\cdot,\cdot,\cdot)$ satisfies [71]

$$\begin{aligned}c^{\mathbf{v}}(\mathbf{v},\mathbf{z},\mathbf{w}) = &-\sum_{E\in\mathcal{E}_h}\left(\int_E(\mathbf{v}\cdot\nabla\mathbf{w})\cdot\mathbf{z}\,dx + \frac{1}{2}\int_E(\nabla\cdot\mathbf{v})\mathbf{z}\cdot\mathbf{w}\,dx\right)\\ &+\frac{1}{2}\sum_{e\in\Gamma_h}\int_e[\mathbf{v}]\cdot\mathbf{n}_e\{\mathbf{z}\cdot\mathbf{w}\}\,ds - \sum_{E\in\mathcal{E}_h}\int_{\partial E_-}|\{\mathbf{v}\}\cdot\mathbf{n}_E|\mathbf{z}^{ext}\cdot(\mathbf{w}^{int}-\mathbf{w}^{ext})\,ds\\ &+\int_{\Gamma_+}|\mathbf{v}\cdot\mathbf{n}|\mathbf{z}\cdot\mathbf{w}\,ds, \quad \forall\mathbf{v},\mathbf{z},\mathbf{w}\in\mathbf{X}.\end{aligned} \tag{1.18}$$

For a particular choice $\mathbf{z} = \mathbf{w}$, we have

$$c^{\mathbf{v}}(\mathbf{v},\mathbf{w},\mathbf{w}) \geq 0, \quad \forall\mathbf{v},\mathbf{w}\in\mathbf{X}. \tag{1.19}$$

In the analysis, it will be useful to separate the upwind term from the form $c^{\mathbf{u}}(\mathbf{v},\mathbf{z},\mathbf{w})$. For $\mathbf{u},\mathbf{v},\mathbf{z},\mathbf{w}\in\mathbf{X}$, we now define:

$$
\begin{aligned}
d(\mathbf{v},\mathbf{z},\mathbf{w}) = &\sum_{E\in\mathcal{E}_h}\int_E(\mathbf{v}\cdot\nabla\mathbf{z})\cdot\mathbf{w}\,dx + \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E(\nabla\cdot\mathbf{v})\mathbf{z}\cdot\mathbf{w}\,dx \\
&-\frac{1}{2}\sum_{e\in\Gamma_h}\int_e[\mathbf{v}]\cdot\mathbf{n}_e\{\mathbf{z}\cdot\mathbf{w}\}\,ds.
\end{aligned}
$$

The upwind term is defined as follows:

$$
l^{\mathbf{u}}(\mathbf{v},\mathbf{z},\mathbf{w}) = \sum_{E\in\mathcal{E}_h}\int_{\partial E_-}|\{\mathbf{v}\}\cdot\mathbf{n}_E|(\mathbf{z}^{int}-\mathbf{z}^{ext})\cdot\mathbf{w}^{int}\,ds. \tag{1.20}
$$

It follows that

$$
c^{\mathbf{u}}(\mathbf{v},\mathbf{z},\mathbf{w}) = d(\mathbf{v},\mathbf{z},\mathbf{w}) + l^{\mathbf{u}}(\mathbf{v},\mathbf{z},\mathbf{w}).
$$

For the sake of simplicity, we will not use the notations dx and ds for spatial integration containing the terms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $c(\cdot,\cdot,\cdot)$ in the upcoming chapters, provided no confusion arises.

With the above forms of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, and $c(\cdot,\cdot,\cdot)$, we now consider the DG formulation of NSEs (1.3), (1.9)-(1.10) in the discontinuous spaces $\mathbf{X}$ and M [98]: Find the pair $(\mathbf{u}(t),p(t))\in\mathbf{X}\times M,\ t>0$ such that for all $(\mathbf{v},q)\in\mathbf{X}\times M$

$$
(\mathbf{u}_t(t),\mathbf{v}) + \nu\, a(\mathbf{u}(t),\,\mathbf{v}) + c^{\mathbf{u}(t)}(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) + b(\mathbf{v},\,p(t)) = (\mathbf{f}(t),\,\mathbf{v}), \tag{1.21}
$$

$$
b(\mathbf{u}(t),q) = 0, \tag{1.22}
$$

for $\mathbf{f}\in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{u}(0)=\mathbf{u}_0\in\mathbf{X}$. For deriving the consistency of the scheme defined in (1.21) - (1.22), one may refer to [98, Lemma 3.2].

The motivation behind choosing these special spaces $\mathbf{X}$ and M is to make sense of the traces that appear in the above DG formulation. For example, for any $\boldsymbol{\phi}\in\mathbf{X}$ and $q\in M$, the traces of $\boldsymbol{\phi}$ and q belong to $L^2(\partial E)$, and the trace of each component of $\nabla\boldsymbol{\phi}$ belongs to $L^2(\partial E)$ for all $E\in\mathcal{E}_h$ [116, page no. 1627]. This leads to a well-defined bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ defined in (1.15) and (1.16), respectively.

And the following observations make the trilinear form $c(\cdot,\cdot,\cdot)$, defined in (1.17), well-defined: For $d=2$, the trace of each function in $\mathbf{X}$ belongs to $W^{5/4,4/3}(\partial E)$ [26, page no. 316] and by Sobolev imbedding, $W^{5/4,4/3}(\partial E)\hookrightarrow L^3(\partial E)$ [65, Theorem 6]. Also, in the case of $d=3$, the functions of $\mathbf{X}$ are imbedded in $L^3(\partial E)$ [3, Theorem 5.36].

Finally, from Sobolev imbedding, we have $\mathbf{W}^{2,2d/(d+1)}(E) \hookrightarrow \mathbf{W}^{1,3}(E)$ [2, Section 5.4, page no. 97].

The above discussion justifies that the scheme (1.21)-(1.22) is well-posed. Furthermore, if Ω is a Lipschitz polygonal domain and $\mathbf{f} \in L^2(0,T;\mathbf{L}^{2d/(d+1)}(\Omega))$, the pair $(\mathbf{u},p)$ belongs to $L^2(0,T;\mathbf{W}^{2,2d/(d+1)}(\Omega)) \times L^2(0,T;W^{1,2d/(d+1)}(\Omega))$. For more details, we refer to [116, page no. 1627] and [71, page no. 55]. We would like to note here that, to derive optimal error estimates, we assume our solutions to be more regular.

We next recall some trace inequalities (for a proof, we refer to [146, Section 2.1.3], [96, Section 7]) to conduct our analysis .

Lemma 1.5. *For each E in $\mathcal{E}_h$ with h_E denoting the diameter, there is a positive constant C, independent of h_E, such that the following inequalities hold ($\mathbb{R}^d$, $d=2,3$):*

$$\|\mathbf{v}\|_{L^2(e)} \le C(h_E^{-1/2}\|\mathbf{v}\|_{L^2(E)} + h_E^{1/2}\|\nabla\mathbf{v}\|_{L^2(E)}) \quad \forall e \in \partial E,\ \forall \mathbf{v} \in \boldsymbol{H}^1(E), \tag{1.23}$$

$$\|\nabla\mathbf{v}\|_{L^2(e)} \le C(h_E^{-1/2}\|\nabla\mathbf{v}\|_{L^2(E)} + h_E^{1/2}\|\nabla^2\mathbf{v}\|_{L^2(E)}) \quad \forall e \in \partial E,\ \forall \mathbf{v} \in \boldsymbol{H}^2(E), \tag{1.24}$$

$$\|\mathbf{v}\|_{L^4(e)} \le C(h_E^{-1/4}\|\mathbf{v}\|_{L^4(E)} + h_E^{(3-d)/4}\|\nabla\mathbf{v}\|_{L^2(E)}) \quad \forall e \in \partial E,\ \forall \mathbf{v} \in \boldsymbol{H}^1(E). \tag{1.25}$$

We now introduce the discrete discontinuous spaces $\mathbf{X}_h \subset \mathbf{X}$ and $M_h \subset M$ to approximate velocity and pressure, respectively, as follows:

$$\mathbf{X}_h = \{\mathbf{w}_h \in \mathbf{L}^2(\Omega) \ : \ \ \mathbf{w}_h|_E \in (\mathbb{P}_r(E))^d, \ \ \forall E \in \mathcal{E}_h\},$$
$$M_h = \{q_h \in L^2(\Omega)/\mathbb{R} \ : \ \ q_h|_E \in \mathbb{P}_{r-1}(E), \ \ \forall E \in \mathcal{E}_h\},$$

where $r \ge 1$ is any integer and $\mathbb{P}_r(E)$ is the space of polynomials of degree less than or equal to r over E.

The following lemma states that the bilinear form $a(\cdot,\cdot)$ is coercive.

Lemma 1.6. *[146, page no. 38] If σ_e is large enough and $\epsilon \in \{-1,0\}$, then there exists a positive constant K_1, independent of h, such that the following holds true*

$$a(\mathbf{v}_h,\mathbf{v}_h) \ge K_1\|\mathbf{v}_h\|_\varepsilon^2, \quad \forall \mathbf{v}_h \in \boldsymbol{X}_h.$$

It is to be noted that the above lemma can be derived for the NIPG case ($\epsilon = 1$) with $K_1 = 1$ by using the definition of $\|\cdot\|_\varepsilon$-norm. The continuity of $a(\cdot,\cdot)$ in the discrete space $\mathbf{X}_h$ is stated in the next lemma.

Lemma 1.7. *[146, page no. 41] There is a positive constant K_2, independent of h, such that*

$$|a(\boldsymbol{w}_h, \mathbf{v}_h)| \leq K_2 \|\boldsymbol{w}_h\|_\varepsilon \|\mathbf{v}_h\|_\varepsilon, \quad \forall \boldsymbol{w}_h, \mathbf{v}_h \in \boldsymbol{X}_h.$$

We now state a uniform discrete inf-sup condition in Lemma 1.8 below.

Lemma 1.8. *[146, Theorem 6.8] There is a positive constant β^*, independent of h, such that*

$$\inf_{p_h \in M_h} \sup_{\boldsymbol{w}_h \in \widetilde{\widetilde{\boldsymbol{X}}}_h} \frac{b(\boldsymbol{w}_h, p_h)}{\|\boldsymbol{w}_h\|_\varepsilon \|p_h\|} \geq \beta^*,$$

where

$$\widetilde{\widetilde{\boldsymbol{X}}}_h = \Big\{ \boldsymbol{w}_h \in \boldsymbol{X}_h : \ \forall e \in \Gamma_h, \quad [\boldsymbol{w}_h]|_e \cdot \boldsymbol{n}_e = 0 \Big\}.$$

Let us recall some approximation properties for the discrete spaces $\mathbf{X}_h$ and M_h.

Lemma 1.9. *[38, Lemma 6.1] For the space $\boldsymbol{X}_h$, there exists an approximation operator $\boldsymbol{\Pi}_h : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{X}_h$, such that*

$$b(\boldsymbol{\Pi}_h \boldsymbol{\phi} - \boldsymbol{\phi}, q_h) = 0, \quad \forall q_h \in M_h, \quad \forall \boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega), \tag{1.26}$$

and for all E in $\mathcal{E}_h$, for all $\boldsymbol{\phi} \in \boldsymbol{W}^{s,p}(E) \cap \boldsymbol{H}_0^1(\Omega)$, $1 \leq p \leq \infty$, $1 \leq s \leq r+1$,

$$\|\boldsymbol{\phi} - \boldsymbol{\Pi}_h \boldsymbol{\phi}\|_{L^p(E)} + h_E \|\nabla(\boldsymbol{\phi} - \boldsymbol{\Pi}_h \boldsymbol{\phi})\|_{L^p(E)} \leq C h_E^s |\boldsymbol{\phi}|_{s,p,\Delta_E}, \tag{1.27}$$

where Δ_E is a macro element that contains E, and C is a positive constant independent of h and E. For $\boldsymbol{\phi} \in \boldsymbol{W}^{s,p}(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$, then bounds (1.27) yield the global estimates:

$$\|\boldsymbol{\phi} - \boldsymbol{\Pi}_h \boldsymbol{\phi}\|_{L^p(\Omega)} \leq C h^s |\boldsymbol{\phi}|_{s,p,\Omega}, \tag{1.28}$$

$$\|\boldsymbol{\phi} - \boldsymbol{\Pi}_h \boldsymbol{\phi}\|_\varepsilon \leq C h^{s-1} |\boldsymbol{\phi}|_s. \tag{1.29}$$

Moreover, for any $q \in L^2(\Omega)/\mathbb{R}$, there exists an approximation $r_h(q) \in M_h$ (see [71, page no. 61]), defined on each $E \in \mathcal{E}_h$ by

$$\int_E z_h (q - r_h(q)) = 0, \quad \forall z_h \in \mathbb{P}_{r-1}(E), \tag{1.30}$$

and satisfies the following approximation property for every $s \in [0, r]$:

$$\|q - r_h(q)\|_{L^2(E)} \leq C h_E^s |q|_{s,E}, \quad \forall q \in H^s(\Omega) \cap L^2(\Omega)/\mathbb{R}. \tag{1.31}$$

We further define the discrete divergence-free space $\mathbf{V}_h$, analogous to $\mathbf{J}_1$, as follows

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h : \ b(\mathbf{v}_h, q_h) = 0, \ \forall q_h \in M_h\}.$$

We now introduce the semi-discrete DG formulation for NSEs (1.3), (1.9)-(1.10). To find $(\mathbf{u}_h(t),\, p_h(t)) \in \mathbf{X}_h \times M_h$, $t>0$ such that

$$(\mathbf{u}_{ht}(t),\, \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{u}_h(t),\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_h(t)}\,(\mathbf{u}_h(t),\, \mathbf{u}_h(t),\, \boldsymbol{\phi}_h) + b\,(\boldsymbol{\phi}_h,\, p_h(t)) = (\mathbf{f}(t),\, \boldsymbol{\phi}_h), \tag{1.32}$$

$$b(\mathbf{u}_h(t),\, q_h) = 0, \quad \text{and} \quad (\mathbf{u}_h(0),\, \boldsymbol{\phi}_h) = (\mathbf{u}_0,\, \boldsymbol{\phi}_h), \tag{1.33}$$

for $(\boldsymbol{\phi}_h,\, q_h) \in (\mathbf{X}_h,\, M_h)$, given $\mathbf{f} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{u}_h(0) \in \mathbf{X}_h$.
And an equivalent DG formulation corresponding to the scheme (1.32)–(1.33) on the space $\mathbf{V}_h$ is the following: Given $\mathbf{f} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{u}_h(0) \in \mathbf{V}_h$, seek $\mathbf{u}_h(t) \in \mathbf{V}_h$, $t>0$, such that

$$(\mathbf{u}_{ht}(t),\, \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{u}_h(t),\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_h(t)}\,(\mathbf{u}_h(t),\, \mathbf{u}_h(t),\, \boldsymbol{\phi}_h) = (\mathbf{f}(t),\, \boldsymbol{\phi}_h),\ \forall\, \boldsymbol{\phi}_h \,\in\, \mathbf{V}_h. \tag{1.34}$$

From the coercivity result in Lemma 1.6, the positivity (1.19) and the inf-sup condition in Lemma 1.8, the existence and uniqueness of the discrete Navier-Stokes solution of (1.32)-(1.33) (or (1.34)) will follow easily, see [98] for details.
Before continuing, we state below some trace and inverse inequalities for the discrete function space $\mathbf{X}_h$.

Lemma 1.10. *[58, Section 1.4.3] There is a positive constant C, independent of h_E, such that for each E in $\mathcal{E}_h$ and $1 \le p \le \infty$, we have*

$$\|\mathbf{v}_h\|_{L^2(e)} \le C h_E^{-1/2} \|\mathbf{v}_h\|_{L^2(E)} \quad \forall e \in \partial E,\ \forall \mathbf{v}_h \in \boldsymbol{X}_h, \tag{1.35}$$

$$\|\nabla \mathbf{v}_h\|_{L^2(e)} \le C h_E^{-1/2} \|\nabla \mathbf{v}_h\|_{L^2(E)} \quad \forall e \in \partial E,\ \forall \mathbf{v}_h \in \boldsymbol{X}_h, \tag{1.36}$$

$$\|\mathbf{v}_h\|_{L^p(e)} \le C h_E^{-1/p} \|\mathbf{v}_h\|_{L^p(E)} \quad \forall e \in \partial E,\ \forall \mathbf{v}_h \in \boldsymbol{X}_h, \tag{1.37}$$

$$\|\nabla \mathbf{v}_h\|_{L^2(E)} \le C h_E^{-1} \|\mathbf{v}_h\|_{L^2(E)} \quad \forall \mathbf{v}_h \in \boldsymbol{X}_h, \tag{1.38}$$

$$\|\mathbf{v}_h\|_{L^p(E)} \le C h_E^{d(1/p-1/2)} \|\mathbf{v}_h\|_{L^2(E)} \quad \forall \mathbf{v}_h \in \boldsymbol{X}_h. \tag{1.39}$$

The semi-discrete formulation(s) mentioned above are still continuous in time and in a fully discrete scheme, we further discretize (it) in the temporal direction. For discretization in the time variable, we consider the first-order implicit backward Euler method, which, we feel, is sufficient to demonstrate the efficiency of the DG schemes. Higher-order time discretization schemes are technically more involved, and hence avoided to keep the presentation simple.

Let $\{t_n\}_{n=0}^{M}$ be a uniform partition of the time interval $[0, T]$ with the time step $\Delta t > 0$ and T as the final time. Therefore, $t_n = n\Delta t$, $1 \leq n \leq M$. We next define for a sequence $\{\boldsymbol{\phi}^n\}_{n\geq 0}$ the backward difference quotient

$$\partial_t \boldsymbol{\phi}^n = \frac{1}{\Delta t}(\boldsymbol{\phi}^n - \boldsymbol{\phi}^{n-1}), \quad n > 0.$$

For any continuous function $\boldsymbol{\phi}$, we write $\boldsymbol{\phi}(t_n)$ as $\boldsymbol{\phi}^n$.

We now describe below the fully discrete scheme based on backward Euler method for the semi-discrete NSEs (1.32)-(1.33) as follows: Find $(\mathbf{U}^n, P^n)_{n\geq 1} \in \mathbf{X}_h \times M_h$, such that

$$(\partial_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{U}^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}}\,(\mathbf{U}^{n-1},\, \mathbf{U}^n,\, \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h,\, P^n) = (\mathbf{f}^n, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{X}_h, \tag{1.40}$$

$$b(\mathbf{U}^n,\, q_h) = 0, \quad \forall\, q_h \in M_h. \tag{1.41}$$

We choose here $\mathbf{U}^0 = \mathbf{u}_h(0) \in \mathbf{X}_h$.

Equivalently, for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$, we find $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{V}_h$ such that

$$(\partial_t \mathbf{U}^n,\, \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{U}^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}}\,(\mathbf{U}^{n-1},\, \mathbf{U}^n,\, \boldsymbol{\phi}_h) = (\mathbf{f}^n,\, \boldsymbol{\phi}_h), \tag{1.42}$$

given $\mathbf{U}^0 = \mathbf{u}_h(0) \in \mathbf{V}_h$. Using (1.19) and Lemmas 1.6, 1.8 the existence and uniqueness of the discrete solutions to the discrete problem (1.40)-(1.41) (or (1.42)) can be achieved following similar steps as in [72].

Below, we state the discrete version of Cauchy-Schwarz inequality, which will be used in our later analysis.

Cauchy-Schwarz inequality: For a finite pair of positive real numbers $\{\phi_j, \psi_j\}_{j=1}^{n}$, the following holds

$$\sum_{j=1}^{n} \phi_j \psi_j \leq \Big(\sum_{j=1}^{n} \phi_j^2\Big)^{1/2} \Big(\sum_{j=1}^{n} \psi_j^2\Big)^{1/2}.$$

We present below without proof the following version of discrete Gronwall's Lemma. The proof can be found in [81, 133].

Lemma 1.11. *Let $\{a_n\}$ and $\{d_n\}$ be finite sequences of nonnegative real numbers and $\{b_n\}$ be a nondecreasing real finite sequence satisfying*

$$a_n \leq b_n + \sum_{i=0}^{n-1} d_i a_i, \quad \forall n \geq 0,$$

Then,

$$a_n \le b_n \exp\Big(\sum_{i=0}^{n-1} d_i\Big), \quad \forall n \ge 0.$$

But for our subsequent analysis, we use more general version of the discrete Gronwall's Lemma, which is simply a reproduction of Lemma 5.1 from [87].

Lemma 1.12. *Let $\Delta t, B$ and a_i, b_i, c_i, d_i, for $i \in \mathbb{N}$, be non-negative numbers such that*

$$a_n + \Delta t \sum_{i=1}^{n} b_i \le B + \Delta t \sum_{i=1}^{n} c_i + \Delta t \sum_{i=1}^{m} d_i a_i, \ \ n \ge 1, \tag{1.43}$$

for $m = n$ or $n-1$. Then,

$$a_n + \Delta t \sum_{i=1}^{n} b_i \le \Big\{B + \Delta t \sum_{i=1}^{n} c_i\Big\} exp\Big(\Delta t \sum_{i=1}^{m} \gamma_i d_i\Big), \tag{1.44}$$

where $\gamma_i = 1$ when $m = n-1$ and $\gamma_i = (1 - \Delta t d_i)^{-1}$, $\Delta t d_i < 1$ when $m = n$.

1.5 A Brief Literature Review

In this section, we will mainly concern ourselves to the development of (primal) DG methods in the context of Stokes and NSEs, since DG methods have not been employed for the two linear viscoelastic models (Kelvin-Voigt model and Oldroyd model of order one) on earlier occasions. For these two models, we will limit ourselve to a brief literature survey of finite element methods applied to them.

We refrain ourselves from the audacity of looking into the vast literature of NSEs, and instead, begin this section with the perturbed models.

1.5.1 Kelvin-Voigt Model

Early work on the Kelvin-Voight model revolves around the study of solvability, existence of global attractor and long-term dynamics, and can be seen in the works of Oskolkov, Shadiev, Kotsiolis, Kalantarov *et. al.* [95, 97, 127, 129, 131, 132]. A detailed discussion can be found in [10, 15] and the literature, referred therein.

Later part of the work on the model (1.7), (1.9)-(1.10) is devoted to the development of numerical solutions in the CG finite element framework. For example, semi-discrete CG finite element analysis is carried out in [15, 137], and time discrete schemes based on first- and second-order accurate backward difference schemes can be found in [13, 14, 136, 138]. In these work, the model has been analysed for both zero and non-zero forcing term. Regularity of the continuous and discrete solutions has been derived. Exponential decay property of the weak solution has been established when the force function is zero, which has been shown to be preserved by the semi-discrete solution as well. Furthermore, optimal error estimates have been carried out in both semi- and fully-discrete set up. These results have been show to be independent of inverse powers of κ in the latter work.

Apart from these, there are several other finite element based works, namely, asymptotic analysis [104, 105], pressure projection method [184], weak Galerkin method [63], Euler implicit/explicit time discrete scheme with the scalar auxiliary variable approach [178], two-grid method [11], multilevel space-time method [183], modified characteristic projection method [155], to name a few.

1.5.2 Oldroyd Model of Order One

For literature related to Oldroyd model based on the analysis of existence and uniqueness of solutions, asymptotic analysis and dynamical system (or long time solution behavior), we refer to the works of Oskolkov, Kotsiolis, Karazeeva, Sobolevski [4, 97, 102, 103, 129, 130, 156]. For more detailed description, see [73, 75] and references therein.

Semi-discrete CG finite element approximations for the problem (1.8), (1.9)-(1.10) have been studied in [75, 83, 134].

Temporal discretization schemes can be found in the works of [24, 80, 135, 165], where the backward Euler method (for time discretization) with semi-group theoretic approach in [135], backward Euler method with smooth initial data (that is, $\mathbf{u}_0 \in \mathbf{H}^2(\Omega)$) in [165], backward Euler method with non-smooth initial data (that is, $\mathbf{u}_0 \in \mathbf{J}_1$) in [24], and a second-order Crank-Nicolson extrapolation technique in [80] have been utilized.

In addition to these, there are works based on large-time numerical schemes and asymptotic analysis [82, 167], stability analysis of various implicit/explicit fully discrete schemes [78, 79, 185], projection methods [115, 154, 188], stabilized methods [168, 186], modified characteristics finite element method [181], two-grid method [23] and penalty methods [25, 164, 166] that have been carried out for this model. A few work on the stochastic model and its coupled version with Cahn-Hilliard can be seen in [56, 119, 120].

1.5.3 DG Methods

Despite the appearance of DG methods in the 1970s and their existence in many forms since then, their significant development has primarily occurred in the past two decades. Notably, these methods have been independently developed for elliptic and parabolic problems, as well as for hyperbolic problems.

The utilization of the DG method was first introduced by Reed and Hill in 1973 [144] as a tool to solve the steady-state neutron transport equation. Lesaint and Raviart conducted the initial numerical analysis of the DG method in 1974 [112], where they derived error estimates in the L^2-norm on a general triangulated grid, and later on, Johnson and Pitkäranta in 1986 [93] improved these estimates. In subsequent years, Caussignac and Touzani [30, 31] further improved the method for approximating the three-dimensional boundary-layer equations for incompressible steady-state linear and non-linear fluid flows.

Chavent and Salzano [40] expanded the application of DG methods to deal with the time-dependent hyperbolic nonlinear scalar conservation law. Their approach involved employing linear finite elements for spatial discretization and the forward Euler method for time discretization. The resulting scheme showed stability, although subject to a highly restrictive Courant-Friedrichs-Lewy stability condition. The resolution of this issue was achieved by the use of a slope limiter in the work of Chavent and Cockburn [39].

For elliptic and parabolic problems, initial development of DG methods can be attributed to various researchers, including Douglas and Dupont [62], Baker [19], Wheeler [172], and Arnold [5]. The techniques developed in those work were usually referred to as the interior penalty (IP) Galerkin method. In the IP methods, the Dirichlet boundary condition is enforced weakly inside the scheme by the utilization of a boundary penalty, instead of to its direct incorporation into the finite element space. In the year 1973, Babuška [7] employed a penalty method to weakly enforce the Dirichlet boundary condition in the variational formulation for the solution of the Poisson problem, specifically when dealing with homogeneous boundary conditions of Dirichlet type. However, the scheme represented there was inconsistent. In 1971, Nitsche [123] developed a different penalty method that aimed to maintain the consistency of the formulation and was applied to the Poisson's equation with generic Dirichlet boundary conditions. In order to achieve an approximation for elliptic and parabolic problems that lies between C^0 and C^1 , Douglas and Dupont [62] applied a penalization on the jump in the normal derivative of continuous approximate solution. Similar type of penalty technique is employed for the biharmonic equation by Babuška and Zlámal [8], and Baker [19]. The IP DG methods of Wheeler [172] and Arnold [5] replace the necessity of continuity in the conforming finite element approximate solutions by the interior penalty and these methods are usually referred to as the SIPG method. The SIPG method exhibits both symmetry and adjoint consistency in its scheme. However, the choice of the penalty parameter in this method relies on the bounds of the coefficients of the problem, as well as several constants associated with the inverse properties and these constants are not explicitly known. In order to deal with the difficulty, Oden *et al.* [124] introduced a new type of DG methods which is a modification of the global element method [55, 85] and is based on non-symmetric weak formulation. An extension of [124] that incorporates both the interior and boundary penalty is the NIPG method, which was proposed by Rivière *et al.* [149]. Rivière *et al.* further studied this method in [150]. The IIPG approach, introduced and studied by Dawson *et al.* in 2004 [53], is a simplified IP DG method that is less complicated to implement than SIPG and NIPG methods. Sun and Wheeler [157, 158] extended this technique further for transport problems in a porous media. The work conducted by Arnold *et al.* [6] provides a unified study of DG approaches employed for second-order elliptic problems.

The first DG related work for the incompressible Stokes equations can be credited to Baker *et al.* [20], where pointwise divergence-free discontinuous polynomials for velocity approximation and continuous piecewise polynomial functions for pressure approximation have been employed. However, a rigorous framework has appeared much later in the work of Girault *et al.* [71], where DG methods has been formulated and analyzed with nonoverlapping domain decomposition for the steady incompressible Stokes problem. We refer to [9, 111, 114, 152] and the references cited therein for other noteworthy studies related to incompressible Stokes equations.

The steady incompressible Stokes and NSEs have been studied by Girault *et al.*

[71] where they have formulated a DG method with nonoverlapping domain decomposition, with approximations of order $r = 1, 2$ or 3. The authors have established a discrete inf-sup condition and then have demonstrated optimal energy norm estimates for the velocity and L^2-norm estimates for the pressure. As a follow-up to the work in [71], the authors in [147] have subdivided the domain into subdomains with non-matching meshes at the interfaces, unlike in [71], where a matching condition was adopted. They have also proved an improved inf-sup condition by considering the Raviart-Thomas interpolant and shown optimal error estimates in energy norm and L^2-norm for velocity and pressure, respectively. The extension to time-dependent NSEs can be found in [98], where an error analysis of a subgrid-scale linear eddy viscosity model combined with DG approximations has been worked out. Optimal semi-discrete error estimates of the velocity and pressure with respect to the grid size have been derived with improved robustness in terms of Reynolds number. Then two fully discrete schemes, which are first and second-order in time, respectively, have been analyzed, and optimal error estimates of velocity have been established. By employing an operator-splitting scheme to decouple the pressure and convection terms and using discontinuous finite elements in the space discretization for the time-dependent incompressible NSEs, Girault *et al.* in [72] have established optimal energy norm error estimates for velocity and sub-optimal error estimates for pressure in $L^2(L^2)$-norm.
Same order polynomial approximation for the velocity and the pressure in a DG method for the steady-state incompressible NSEs has been studied by Cockburn *et al.* [50]. They have obtained optimal order of convergence for the velocity approximation in the DG norm and suboptimal order convergence for pressure in L^2-norm. Discontinuous in time and conforming in space high order DG scheme has been studied for Stokes and NSEs, and fully-discrete error estimates are established by Chrysafinos and Walkington [45]. Pietro and Ern [57] have introduced some discrete functional analysis tools for DG spaces. Discrete Sobolev embeddings on the DG space that are counterpart of the continuous Sobolev embeddings and a compactness result for bounded sequences in the appropriate DG-norm are established by them. Together with these tools the convergence of DG approximations of the steady incompressible NSEs have also shown. A penalty-free DG methods for steady-state incompressible NSEs can be found in [148]. Four different kinds of DG methods (interior penalty

DG methods, local DG, DG of Brezzi and DG of Bassi) have been applied to solve the steady NSEs with a nonlinear slip boundary condition of friction type by Jing *et al.* [92]. Optimal error estimates have been derived for the velocity in a broken $\mathbf{H}^1$-norm and for the pressure in an L^2-norm where linear elements for the velocity approximation and piecewise constants for the pressure approximation are employed. Recently, a pressure correction scheme combined with the DG method has been applied for time-dependent NSEs in [116, 117]. We refer to [41, 140, 153] as well as the references therein, for various other types of DG methods in discretizing the incompressible NSEs, which are mainly focused on numerical experiments. For works related to the approximations of the incompressible NSEs based on local and hybridizable DG methods, we refer to [48–50, 169] and [36, 99, 122, 141, 145], respectively. Both the lists mention some notable work, and by no means exhaustive.

1.5.4 Two-Grid DG Methods

In the context of the CG finite element framework, two-grid methods were introduced initially for elliptic problems by Xu [174, 175]. Later on, for the steady-state NSEs, this technique is further investigated by Layton *et al.*, Dai *et al.* and Girault *et al.* [52, 67, 108, 109]. Girault *et al.* have extended the two-grid analysis for the transient NSEs in [68]. A further extension of this analysis to the fully discrete scheme can be found in [1, 54]. Two-grid methods in CG settings have recently been studied for time-dependent NSEs in [12, 74].

For the Kelvin-Voigt model with forcing term $\mathbf{f} = \mathbf{0}$, the two-grid technique has been applied by Bajpai *et al.* [11, 16] with classical finite element approximation for spatial discretization. The Crank-Nicolson scheme and a second order accurate backward difference scheme for time discretization have been employed in [11] and [16], respectively. Optimal velocity and pressure error estimates have been derived under certain scaling between h and H.

For the Oldroyd model of order one, work related to the two-grid method, in CG setting, can be found in [23, 76]. In [76], a two step fully discrete two-grid CG finite element approximation has been analysed and optimal $\mathbf{H}^1$-norm error for velocity and L^2-norm error for the pressure with $h = \mathcal{O}(H^2 t^{-1/2})$ has been obtained. But, $\mathbf{L}^2$-norm error estimate for velocity is sub-optimal there. A three step two-grid fully discrete CG scheme has been applied in [23], and optimal error estimates for velocity in $\mathbf{L}^2$-norm when $h^2 = \mathcal{O}(H^{4-\theta}t^{-1/2})$, in $\mathbf{H}^1$-norm when $h = \mathcal{O}(H^{3-\theta}t^{-1/2})$ and for pressure in L^2-norm with $h = \mathcal{O}(H^{3-\theta}t^{-1/2})$, for arbitrary small $\theta > 0$, are proved.

The combination of two-grid technique with DG approximation have been analyzed for quasi-linear elliptic problems of non monotone type by Bi and Ginting [21] and *a priori* error estimates in broken H^1-norm of the proposed scheme have been established. Later on, Congreve *et al.* [51] have developed *a priori* and *a posteriori* error analysis of the two-grid hp-DG method for the second-order quasi-linear elliptic problems of strongly monotone type. For strongly nonlinear elliptic problems, Bi *et al.* [22] have derived pointwise error estimates of the DG methods and analyzed two-grid DG discretizations along with the derivation of mesh-dependent energy norm error estimates. Recently, the two-grid algorithms of DG methods for solving the mildly nonlinear second-order elliptic problems have been applied by Zhong *et al.* [189] which is different from the ones in [22] and simpler than the ones in [21, 51]. For non-linear parabolic problems, Yang [176] has proposed a semi-discrete two-grid DG scheme and obtained error estimates in broken H^1-norm. This work has been extended to the fully discrete cases in [177, 180], where fully discrete broken H^1 and L^2-norm error estimates are shown. Also, in [179], a two-grid DG algorithm for the nonlinear Sobolev equations has been presented with optimal convergence estimates. In [190], Zhong *et al.* have introduced a DG two-grid algorithm for the convection-diffusion-reaction equations and established the corresponding error estimates. However, to the best of our knowledge, no such work are available for steady and unsteady NSEs.

1.6 Chapter-wise Outline of the Thesis

The thesis comprises of eight chapters which have been organized as follows.
In Chapter 2, we apply a DG finite element method to the transient and incompressible NSEs. An L^2-projection and a modified Stokes operator are introduced, and their optimal approximation properties are derived. With the help of these new estimates, optimal semi-discrete velocity and pressure error estimates are presented. Also, for sufficiently small data, uniform in time estimates are shown. Time discretization is carried out based on the backward Euler method and fully discrete error estimates are

derived. Numerical experiments are given to verify our theoretical findings.
In Chapter 3, we analyze DG finite element approximations for the Kelvin-Voigt model. *A priori* and regularity results for the semi-discrete solutions, and well-posedness and consistency of the DG scheme are discussed. For optimal L^2-norm error estimates of velocity, we have defined a modified Sobolev-Stokes projection, and optimal error estimates of semi-discrete velocity and pressure approximations are derived. Uniform in time error estimates are proved for sufficiently small data. Furthermore, backward Euler scheme is considered for a full discretization and optimal fully discrete error estimates are derived. Numerical experiments are presented in support of the theoretical results.
Chapter 4 deals with a DG finite element method for the equations of motion that arise in the Oldroyd model of order one. We establish new *a priori* bounds for the semi-discrete solutions. And we study the existence and uniqueness of the semi-discrete DG solutions, as well as the consistency of the scheme. A fully discrete scheme is introduced based on first order backward Euler method. New *a priori* bounds for the fully-discrete solutions are presented. Optimal L^2 error estimates for the fully discrete solutions are then shown based on newly introduced modified Stokes-Volterra projection. We conduct numerical experiments to prove the validity of our theoretical results, and analyze the findings.

In Chapter 5, we apply a two steps two-grid scheme to the DG formulation of the transient NSEs. We establish optimal error estimates of the two-grid DG approximations for the velocity and pressure for an appropriate choice of coarse and fine mesh parameters. We further discretize the two-grid DG model in time, using the backward Euler method, and derive the fully discrete error estimates. Numerical results are presented to confirm the efficiency of the proposed scheme.

In Chapter 6, we consider a three steps two-grid algorithm based on DG approximation for the Kelvin-Voigt model. *A priori* and regularity bounds of the semi-discrete two-grid DG solutions for second and third steps are established. With the help of newly derived interpolated Sobolev and trace inequalities, optimal semi-discrete and fully discrete error estimates for velocity and pressure are shown. Also, for sufficiently small data, uniform in time error estimates are presented. We conduct numerical simulations to substantiate our theoretical findings and establish the time efficiency of this method.

Chapter 7 deals with a three steps two-grid technique combined with DG approximations for the Oldroyd model of order one. *A priori* estimates for the fully discrete solutions are proved. We establish optimal error estimates for fully discrete solutions. We present numerical results to validate the theoretical results.

Finally, in our last chapter, Chapter 8, we summarize our findings and present a plan for future.