Chapter 2

DG Method for the Navier-Stokes Equations

This chapter studies a DG finite element method for solving the transient and incompressible NSEs. We derive here optimal semi-discrete velocity and pressure error estimates in $L^\infty(\mathbf{L}^2)$ and $L^\infty(L^2)$-norms, respectively. Standard approximation results being insufficient, we establish new results for L^2-projection and modified Stokes operator, all within the context of suitable broken Sobolev spaces, and along with the standard duality arguments, we achieve our desired results. For sufficiently small data, uniform in time estimates are proved. Based on the backward Euler method, time discretization is carried out and fully discrete error estimates are derived. Finally, we conclude the chapter by conducting numerical experiments to verify our theoretical findings. This work has been published in [18].

2.1 Introduction

At the very outset, we recall the following momentum and continuity equations representing the fluid flow of incompressible NSEs

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$

along with the initial condition

$$\mathbf{u} = \mathbf{u}_0 \quad \text{for } t = 0, \tag{2.3}$$

and the boundary condition for $0 < t \leq T$

$$\mathbf{u} \;=\; \mathbf{0} \quad \text{on } \partial\Omega. \tag{2.4}$$

The usual normalization condition on the pressure is imposed, that is, $\int_\Omega p = 0$. Furthermore, we recall DG variational formulation on the discontinuous spaces $\mathbf{X}$ and M, as well as on the discrete discontinuous spaces $\mathbf{X}_h$ and M_h (or in $\mathbf{V}_h$) of (2.1)-(2.4). For the spaces $\mathbf{X}$ and M, the weak formulation for (2.1)-(2.4) is as follows : Find the pair $(\mathbf{u}(t), p(t)) \in \mathbf{X} \times M,\ t > 0$ such that

$$(\mathbf{u}_t(t), \mathbf{v}) + \nu\, a(\mathbf{u}(t),\, \mathbf{v}) + c^{\mathbf{u}(t)}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) + b(\mathbf{v},\, p(t)) = (\mathbf{f}(t),\, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{X}, \tag{2.5}$$

$$b(\mathbf{u}(t),\, q) = 0 \quad \forall\, q \in M, \tag{2.6}$$

$$(\mathbf{u}(0),\, \mathbf{v}) = (\mathbf{u}_0,\, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{X}. \tag{2.7}$$

Again, the semi-discrete DG formulation for (2.1)-(2.4) on $\mathbf{X}_h$ and M_h is: To find $(\mathbf{u}_h(t),\, p_h(t)) \in \mathbf{X}_h \times M_h,\, t > 0$ such that

$$(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h(t), \boldsymbol{\phi}_h) + c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t), \mathbf{u}_h(t), \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, p_h(t)) = (\mathbf{f}(t), \boldsymbol{\phi}_h), \tag{2.8}$$

$$b(\mathbf{u}_h(t), q_h) = 0, \quad \text{and} \quad (\mathbf{u}_h(0), \boldsymbol{\phi}_h) = (\mathbf{u}_0, \boldsymbol{\phi}_h), \tag{2.9}$$

for $(\boldsymbol{\phi}_h, q_h) \in (\mathbf{X}_h, M_h)$.
And an equivalent DG formulation corresponding to the scheme (2.8)–(2.9) on the space $\mathbf{V}_h$ is the following: Seek $\mathbf{u}_h(t) \in \mathbf{V}_h,\, t > 0$, such that

$$(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h(t), \boldsymbol{\phi}_h) + c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t), \mathbf{u}_h(t), \boldsymbol{\phi}_h) = (\mathbf{f}(t), \boldsymbol{\phi}_h),\ \forall\ \boldsymbol{\phi}_h \in \mathbf{V}_h. \tag{2.10}$$

Our attempt to study a DG method for the unsteady NSEs in fact has been preceded on numerous occasions, as has been discussed in the Introduction chapter. Rigorous analysis of DG methods for Stokes and NSEs can be attributed to Girault *et al.*, see [71, 72, 98, 147]. In [71, 147], a DG method has been formulated with nonoverlapping domain decomposition, with non-matching meshes at the interfaces in the latter work, for the steady state incompressible Stokes and NSEs. The extension to time-dependent NSEs can be found in [98], where an error analysis of a subgrid-scale linear eddy viscosity model combined with DG approximations has been worked out. And in [72], an operator-splitting scheme is used to decouple the pressure and convection terms, and DG method is applied for space discretization for the time-dependent incompressible NSEs. In all these work, error analysis involves working out energy norm estimates only for the velocity, although numerically, optimal L^2-error estimates have been presented.

In a recent work [92], several DG methods have been analyzed to solve a variational inequality from the stationary NSEs with a nonlinear slip boundary condition of friction type. Well-posedness of the discrete solutions has been shown, and energy error estimates have been derived. For several other works of similar nature, we refer to [38, 41, 116, 140, 153] and references therein. Numerical results for the discrete velocity in the majority of these works indicate optimal L^2 convergence rates. However, no analytical work can be found showcasing L^2-error estimates.

This chapter carries out *a priori* error analysis for a DG method applied to the unsteady incompressible NSEs. The main results presented in this chapter are the derivation of optimal $L^\infty(\mathbf{L}^2)$-norm error estimate for velocity and $L^\infty(L^2)$-norm error estimate for pressure. We would like to emphasize here that to the best of the authors knowledge, the optimal $L^\infty(L^2)$ error analysis of the discrete velocity for the unsteady incompressible NSEs is not available in the DG literature, although numerically it is present. Therefore, this work can be considered as a first attempt in that direction.

We would also like to point out that we treat here the SIPG case in details, keeping the other two cases in remarks. In the case of the NIPG or IIPG method, it has been shown numerically that the $\mathbf{L}^2$-norm velocity error convergence rate will be optimal only for an odd degree of polynomial approximation; for even degree, the result is sub-optimal, see [72, 146, 147]. However, for SIPG, the error estimate will not depend on the polynomial degree but only on the penalty parameter σ_e (see Section 1.4), which must be sufficiently large [146]. We, therefore, have analysed the SIPG method only, putting remarks wherever appropriate for the NIPG and IIPG methods, which lead to sub-optimal estimates as expected.

In the DG literature, the error analysis revolves around the operator $\mathbf{\Pi}_h$ (see Lemma 1.9), which is used to obtain optimal error estimates of the velocity in the energy norm and the pressure in L^2-norm. Using the duality argument, the optimal error estimate for the velocity of the steady NSEs can be obtained (see [71]). However this procedure fails in the case of unsteady NSEs, and we feel this is due to the lack of appropriate approximation operators and projection in the DG finite element set-up. We have made an attempt here to fill this gap by defining an L^2-projection $\mathbf{P}_h$ onto a suitable DG finite element space using the approximation operator $\mathbf{\Pi}_h$ and then by proving the requisite approximation properties. Next, we have introduced an approximation operator $\mathbf{S}_h$, a modified Stokes projection (modified compared to [86, eq. (4.52)] and see Section 2.2 for more details), again in the DG set-up. We have derived optimal approximation estimates with the help of the projection $\mathbf{P}_h$. The derivation of these estimates is technical due to the involvement of the broken Sobolev spaces. Armed with these approximation operators, we have achieved here, for $t > 0$ optimal L^2-norm error estimates for the velocity and the pressure. We would like to mention here that, this work extends the results of [86] to DG methods.

Finally, we do fully discrete analysis based on backward Euler method. Our analysis uses energy arguments and is based on the works of [69, 87].

Below, we summarize our main contributions obtained in this chapter:

- A modified version of the Stokes operator $\mathbf{S}_h$ on a DG finite element space is introduced, and its approximation properties are explored.
- Optimal $L^\infty(\mathbf{L}^2)$ and $L^\infty(L^2)$-norms error bounds for semi-discrete DG approximations to the velocity and pressure, respectively, are established.
- With the assumption that given data is small, uniform in time optimal error estimate for velocity is established.
- L^2 error estimates for fully discrete DG approximations to the velocity and pressure are derived when a backward Euler method for the time discretization is applied.

The outline of this chapter is as follows. Section 2.2 contains new approximation operators, for broken Sobolev spaces, and their estimates, apart from the standard projections. In Section 2.3, some auxiliary estimates are presented. Section 2.4 is devoted to the optimal $L^\infty(\mathbf{L}^2)$-norm error estimates of the velocity term, based on these new approximation properties. Uniform in time $t > 0$ error estimates are established under smallness conditions on the data. And in Section 2.5, optimal error estimates for the pressure are derived. In Section 2.6, a fully discrete scheme based

on the backward Euler method is formulated, and error estimates for the velocity and pressure are derived. Further, the numerical experiments are presented to support the theoretical results, and the outcome is analyzed in Section 2.7. Finally, Section 2.8 concludes this chapter by briefly summarizing the results.
Throughout this chapter, we will use C, $K(>0)$ as generic constants that depend on the given data, ν, α, K_1, K_2, C_2 but do not depend on h and Δt. Note that, K and C may grow algebraically with ν^{-1}. Further, the notations $K(t)$ and K_T will be used when they grow exponentially in time.

2.2 Approximation Operators

This section starts out by presenting a few approximation operations that will come in use later on in the analysis. As mentioned in the introduction, we feel the need for appropriate approximation operators on the broken Sobolev spaces, which would allow us to obtain an optimal $L^\infty(\mathbf{L}^2)$-norm error estimate for the discrete velocity, which are missing from the DG literature. Since we carry out our analysis for weakly divergence-free spaces, below, we derive new approximation properties for the space $\mathbf{J}_1 \cap \mathbf{H}^{r+1}(\Omega)$.

Lemma 2.1. *There exists an approximation operator $i_h : \boldsymbol{J}_1 \cap \boldsymbol{H}^{r+1}(\Omega) \to \boldsymbol{V}_h$, such that, the following approximation property holds true*

$$\|\boldsymbol{\phi} - i_h\boldsymbol{\phi}\| + h\|\boldsymbol{\phi} - i_h\boldsymbol{\phi}\|_\varepsilon \le Ch^{r+1}|\boldsymbol{\phi}|_{r+1}, \quad \forall \boldsymbol{\phi} \in \boldsymbol{J}_1 \cap \boldsymbol{H}^{r+1}(\Omega).$$

Proof. Lemma 1.9 implies there is an approximation operator $\boldsymbol{\Pi}_h : \mathbf{H}_0^1(\Omega) \to \mathbf{X}_h$ satisfying

$$b(\boldsymbol{\phi} - \boldsymbol{\Pi}_h\boldsymbol{\phi}, q_h) = 0, \quad \forall\, \boldsymbol{\phi} \in \mathbf{H}_0^1(\Omega),\ q_h \in M_h.$$

Restricting $\boldsymbol{\Pi}_h$ to $\mathbf{J}_1 \cap \mathbf{H}^{r+1}(\Omega)$, we observe that $b(\boldsymbol{\Pi}_h\boldsymbol{\phi}, q_h) = 0$, meaning $\boldsymbol{\Pi}_h\boldsymbol{\phi} \in \mathbf{V}_h$. We define this restriction as $i_h : \mathbf{J}_1 \cap \mathbf{H}^{r+1}(\Omega) \to \mathbf{V}_h$. From (1.28) and (1.29), we have

$$\|\boldsymbol{\phi} - i_h\boldsymbol{\phi}\| + h\|\boldsymbol{\phi} - i_h\boldsymbol{\phi}\|_\varepsilon = \|\boldsymbol{\phi} - \boldsymbol{\Pi}_h\boldsymbol{\phi}\| + h\|\boldsymbol{\phi} - \boldsymbol{\Pi}_h\boldsymbol{\phi}\|_\varepsilon \le Ch^{r+1}|\boldsymbol{\phi}|_{r+1}.$$

This concludes the proof. □

We now introduce an L^2- projection $\mathbf{P}_h : \mathbf{L}^2(\Omega) \to \mathbf{V}_h$ which satisfies for each $\boldsymbol{\phi} \in \mathbf{L}^2(\Omega)$

$$(\boldsymbol{\phi} - \mathbf{P}_h\boldsymbol{\phi}, \boldsymbol{\psi}_h) = 0, \quad \forall \boldsymbol{\psi}_h \in \mathbf{V}_h. \tag{2.11}$$

The following lemma is a consequence of Lemma 2.1.

Lemma 2.2. *There exists a positive constant* C*, independent of* h*, such that* $\boldsymbol{P}_h$ *satisfies the following approximation properties*

$$\|\boldsymbol{\phi}-\boldsymbol{P}_h\boldsymbol{\phi}\|+h\|\boldsymbol{\phi}-\boldsymbol{P}_h\boldsymbol{\phi}\|_\varepsilon \le Ch^{r+1}|\boldsymbol{\phi}|_{r+1}, \quad \forall\ \boldsymbol{\phi}\in \boldsymbol{J}_1\cap \boldsymbol{H}^{r+1}(\Omega).$$

Proof. Choose $\boldsymbol{\psi}_h = i_h\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}$ in (2.11) to find

$$(i_h\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}, i_h\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}) = (i_h\boldsymbol{\phi}-\boldsymbol{\phi}, i_h\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}).$$

Apply the Cauchy-Schwarz and Young's inequalities to obtain

$$\|i_h\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}\|\le \|\boldsymbol{\phi}-i_h\boldsymbol{\phi}\|.$$

Therefore, for $\boldsymbol{\phi}\in\mathbf{J}_1\cap\mathbf{H}^{r+1}(\Omega)$, from Lemma 2.1 and triangle inequality, we have

$$\|\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}\|\le \|\boldsymbol{\phi}-i_h\boldsymbol{\phi}\|+\|i_h\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}\|\le Ch^{r+1}|\boldsymbol{\phi}|_{r+1}. \tag{2.12}$$

Again, a use of triangle inequality, trace inequality (1.35), inverse inequality (1.38), (2.12) and Lemma 2.1 leads to

$$\begin{aligned}\|\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}\|_\varepsilon \le&\|\boldsymbol{\phi}-i_h\boldsymbol{\phi}\|_\varepsilon+\|i_h\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}\|_\varepsilon\\ \le&\|\boldsymbol{\phi}-i_h\boldsymbol{\phi}\|_\varepsilon+Ch^{-1}\|i_h\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}\|\\ \le&\|\boldsymbol{\phi}-i_h\boldsymbol{\phi}\|_\varepsilon+Ch^{-1}(\|\boldsymbol{\phi}-i_h\boldsymbol{\phi}\|+\|\boldsymbol{\phi}-\mathbf{P}_h\boldsymbol{\phi}\|)\le Ch^r|\boldsymbol{\phi}|_{r+1}.\end{aligned} \tag{2.13}$$

A combination of (2.12) and (2.13) completes the proof of this lemma. □

Before introducing our next projection, we take a pause here and look at the errors involving $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, for the L^2-projection $\mathbf{P}_h$ and for r_h (see Section 1.4), respectively. This will be useful in the estimate of the next projection. The proofs of these error estimates are contained in [98, Theorem 4.1]. However the estimate involving $a(\cdot,\cdot)$ has been done for an projection onto $\mathbf{X}_h$. Although ours is a projection onto $\mathbf{V}_h$, the proofs will remain the same, and in fact, our results are valid both for $\mathbf{X}_h$ and $\mathbf{V}_h$. We are reproducing the proofs below for the sake of completeness.

Lemma 2.3. *There exists a positive constant* C*, independent of* h*, such that for all* $\boldsymbol{u}\in\boldsymbol{H}^{r+1}(\Omega)\cap\boldsymbol{H}_0^1(\Omega)$*, we have*

$$|a(\boldsymbol{u}-\boldsymbol{P}_h\boldsymbol{u},\mathbf{v}_h)|\le Ch^r|\boldsymbol{u}|_{r+1}\|\mathbf{v}_h\|_\varepsilon, \quad \forall \mathbf{v}_h\in\boldsymbol{X}_h. \tag{2.14}$$

Proof. We expand the term $a(\mathbf{u}-\mathbf{P}_h\mathbf{u},\mathbf{v}_h)$ as follows:

$$\begin{aligned} a(\mathbf{u}-\mathbf{P}_h\mathbf{u},\mathbf{v}_h) &= \sum_{E\in\mathcal{E}_h}\int_E \nabla(\mathbf{u}-\mathbf{P}_h\mathbf{u}) : \nabla\mathbf{v}_h - \sum_{e\in\Gamma_h}\int_e \{\nabla(\mathbf{u}-\mathbf{P}_h\mathbf{u})\}\mathbf{n}_e\cdot[\mathbf{v}_h] \\ &\quad + \epsilon\sum_{e\in\Gamma_h}\int_e\{\nabla\mathbf{v}_h\}\mathbf{n}_e\cdot[\mathbf{u}-\mathbf{P}_h\mathbf{u}] + \sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\int_e[\mathbf{u}-\mathbf{P}_h\mathbf{u}]\cdot[\mathbf{v}_h] \\ &= S_1+S_2+S_3+S_4. \end{aligned} \tag{2.15}$$

Using the Cauchy-Schwarz inequality, the definition of $\|\cdot\|_\varepsilon$-norm, and Lemma 2.2, we obtain

$$|S_1| \le \sum_{E\in\mathcal{E}_h}\|\nabla(\mathbf{u}-\mathbf{P}_h\mathbf{u})\|_{L^2(E)}\|\nabla\mathbf{v}_h\|_{L^2(E)} \le Ch^r|\mathbf{u}|_{r+1}\|\mathbf{v}_h\|_\varepsilon. \tag{2.16}$$

To bound S_2, we will follow similar steps as in [71, Theorem 5.1]. If e is an edge that belongs to element E, then by using the Cauchy-Schwarz inequality and trace inequality (1.24), we have

$$\begin{aligned} &\left|\int_e\{\nabla(\mathbf{u}-\mathbf{P}_h\mathbf{u})\}\mathbf{n}_e\cdot[\mathbf{v}_h]\right| \\ \le& C\Big(\|\nabla(\mathbf{u}-\mathbf{P}_h\mathbf{u})\|_{L^2(E)} + h_E\|\nabla^2(\mathbf{u}-\mathbf{P}_h\mathbf{u})\|_{L^2(E)}\Big)\frac{\sigma_e^{1/2}}{|e|^{1/2}}\|[\mathbf{v}_h]\|_{L^2(e)}. \end{aligned} \tag{2.17}$$

Due to lack of a direct estimate of $\|\nabla^2(\mathbf{u}-\mathbf{P}_h\mathbf{u})\|_{L^2(E)}$, the standard Lagrange interpolant L_h, of degree r is used as an intermediary, and by using triangle inequality and inverse inequality (1.38), we obtain

$$\begin{aligned} \|\nabla^2(\mathbf{u}-\mathbf{P}_h\mathbf{u})\|_{L^2(E)} &\le \|\nabla^2(\mathbf{u}-L_h(\mathbf{u}))\|_{L^2(E)} + \|\nabla^2(L_h(\mathbf{u})-\mathbf{P}_h\mathbf{u})\|_{L^2(E)} \\ &\le \|\nabla^2(\mathbf{u}-L_h(\mathbf{u}))\|_{L^2(E)} + Ch_E^{-1}\|\nabla(L_h(\mathbf{u})-\mathbf{P}_h\mathbf{u})\|_{L^2(E)}. \end{aligned}$$

Again, a use of triangle inequality yields

$$\begin{aligned} \|\nabla^2(\mathbf{u}-\mathbf{P}_h\mathbf{u})\|_{L^2(E)} \le& \|\nabla^2(\mathbf{u}-L_h(\mathbf{u}))\|_{L^2(E)} \\ &+ Ch_E^{-1}\big(\|\nabla(\mathbf{u}-L_h(\mathbf{u})\|_{L^2(E)} + \|\nabla(\mathbf{u}-\mathbf{P}_h\mathbf{u})\|_{L^2(E)}\big). \end{aligned} \tag{2.18}$$

Applying (2.18) in (2.17) and using the Cauchy-Schwarz inequality, the bound for $|S_2|$ becomes

$$\begin{aligned} |S_2| \le C\Bigg(&\sum_{E\in\mathcal{E}_h}\Big(\|\nabla(\mathbf{u}-\mathbf{P}_h\mathbf{u})\|^2_{L^2(E)} + h_E^2\|\nabla^2(\mathbf{u}-L_h(\mathbf{u}))\|^2_{L^2(E)} \\ &+ \|\nabla(\mathbf{u}-L_h(\mathbf{u})\|^2_{L^2(E)}\Big)\Bigg)^{1/2} \times \left(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\mathbf{v}_h]\|^2_{L^2(e)}\right)^{1/2}. \end{aligned}$$

Note that, for any $\boldsymbol{\phi} \in \mathbf{X}$ and from the definition of $\|\cdot\|_\varepsilon$-norm, we have

$$\Big(\sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\boldsymbol{\phi}]\|^2_{L^2(e)} \Big)^{1/2} \leq \|\boldsymbol{\phi}\|_\varepsilon. \tag{2.19}$$

Now (2.19), Lemma 2.2, the standard approximation properties of L_h and the definition of $\|\cdot\|_\varepsilon$-norm yield

$$|S_2| \leq Ch^r |\mathbf{u}|_{r+1} \|\mathbf{v}_h\|_\varepsilon. \tag{2.20}$$

Furthermore, using trace inequality (1.36), (2.19), the definition of $\|\cdot\|_\varepsilon$-norm, Lemma 2.2 and the Cauchy-Schwarz inequality, we find

$$\begin{aligned} |S_3| \leq & C\Big(\sum_{e\in\Gamma_h} \frac{|e|}{\sigma_e} \|\{\nabla \mathbf{v}_h\}\|^2_{L^2(e)} \Big)^{1/2} \Big(\sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\mathbf{u} - \mathbf{P}_h\mathbf{u}]\|^2_{L^2(e)} \Big)^{1/2} \\ \leq & C\|\mathbf{v}_h\|_\varepsilon \|\mathbf{u} - \mathbf{P}_h\mathbf{u}\|_\varepsilon \leq Ch^r |\mathbf{u}|_{r+1} \|\mathbf{v}_h\|_\varepsilon. \end{aligned} \tag{2.21}$$

Using the Cauchy-Schwarz inequality and (2.19), S_4 is bounded by virtue of Lemma 2.2 as follows:

$$\begin{aligned} |S_4| \leq & \Big(\sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\mathbf{u} - \mathbf{P}_h\mathbf{u}]\|^2_{L^2(e)} \Big)^{1/2} \Big(\sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\mathbf{v}_h]\|^2_{L^2(e)} \Big)^{1/2} \\ & \leq Ch^r |\mathbf{u}|_{r+1} \|\mathbf{v}_h\|_\varepsilon. \end{aligned} \tag{2.22}$$

Collecting the bounds (2.16), (2.20), (2.21) and (2.22) in (2.15), we complete the rest of the proof. □

Lemma 2.4. *There is a positive constant C, independent of the mesh parameter h, such that for all $p \in H^r(\Omega)$, we have*

$$|b(\mathbf{v}_h, p - r_h(p))| \leq Ch^r |p|_r \|\mathbf{v}_h\|_\varepsilon, \quad \forall \mathbf{v}_h \in \boldsymbol{X}_h.$$

Proof. Since $\nabla \cdot \mathbf{v}_h \in \mathbb{P}_{r-1}(E)$, owing to (1.30), the term $b(\mathbf{v}_h, p - r_h(p))$ is reduced as

$$b(\mathbf{v}_h, p - r_h(p)) = \sum_{e\in\Gamma_h} \int_e \{p - r_h(p)\}[\mathbf{v}_h] \cdot \mathbf{n}_e.$$

We now use the Cauchy-Schwarz inequality, trace inequality (1.23) and the approximation result (1.31) to arrive at

$$|b(\mathbf{v}_h, p - r_h(p))| \leq C\Big(\sum_{E\in\mathcal{E}_h} \big(\|p - r_h p\|^2_{L^2(E)} + h_E^2 \|\nabla(p - r_h p)\|^2_{L^2(E)} \big) \Big)^{1/2}$$

$$\times \left(\sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\mathbf{v}_h]\|^2_{L^2(e)} \right)^{1/2}$$
$$\leq C h^r |p|_r \|\mathbf{v}_h\|_\varepsilon.$$

This completes the rest of the proof. □

We are now in a position to define the approximation operator, which we call as modified Stokes operator, $\mathbf{S}_h$, with $\mathbf{S}_h\mathbf{u} \in \mathbf{V}_h$, for the weak solution $\mathbf{u}$ of the problem (2.1)-(2.4), satisfying,

$$\nu\, a\,(\mathbf{u} - \mathbf{S}_h\mathbf{u},\, \boldsymbol{\phi}_h) = -b(\boldsymbol{\phi}_h,\, p), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h. \tag{2.23}$$

Below in Lemma 2.5, which is a DG extension of Lemma 4.7 in [86], we derive some approximation properties of the operator $\mathbf{S}_h$.

Lemma 2.5. *The term $\boldsymbol{u} - \boldsymbol{S}_h\boldsymbol{u}$ satisfies the following estimates for the SIPG case:*

$$\|\boldsymbol{u} - \boldsymbol{S}_h\boldsymbol{u}\|^2 + h^2\|\boldsymbol{u} - \boldsymbol{S}_h\boldsymbol{u}\|^2_\varepsilon \leq C h^{2r+2} \left(|\boldsymbol{u}|^2_{r+1} + |p|^2_r \right), \tag{2.24}$$

$$\|(\boldsymbol{u} - \boldsymbol{S}_h\boldsymbol{u})_t\|^2 + h^2\|(\boldsymbol{u} - \boldsymbol{S}_h\boldsymbol{u})_t\|^2_\varepsilon \leq C h^{2r+2} \left(|\boldsymbol{u}_t|^2_{r+1} + |p_t|^2_r \right), \tag{2.25}$$

where C is a positive constant independent of h.

Proof. Since

$$\|\mathbf{u} - \mathbf{S}_h\mathbf{u}\|_\varepsilon \leq \|\mathbf{u} - \mathbf{P}_h\mathbf{u}\|_\varepsilon + \|\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}\|_\varepsilon, \tag{2.26}$$

it is sufficient to estimate $\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}$. In order to do that we choose

$$\boldsymbol{\phi}_h = \mathbf{P}_h(\mathbf{u} - \mathbf{S}_h\mathbf{u}) = \mathbf{u} - \mathbf{S}_h\mathbf{u} - (\mathbf{u} - \mathbf{P}_h\mathbf{u})$$

in (2.23) to observe that

$$\nu\, a(\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}, \mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}) = -\,\nu\, a(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u})$$
$$+\, b(\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}, r_h(p) - p). \tag{2.27}$$

From Lemma 2.3 and Young's inequality, we arrive at

$$\nu\big|a(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u})\big| \leq C\nu h^r |\mathbf{u}|_{r+1} \|\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}\|_\varepsilon$$
$$\leq \frac{K_1\nu}{4}\|\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}\|^2_\varepsilon + C h^{2r}|\mathbf{u}|^2_{r+1}. \tag{2.28}$$

We now apply Lemma 2.4 and Young's inequality to find

$$\begin{aligned}|b(\mathbf{P}_h\mathbf{u}-\mathbf{S}_h\mathbf{u},p-r_h(p))| \leq& Ch^r|p|_r\|\mathbf{P}_h\mathbf{u}-\mathbf{S}_h\mathbf{u}\|_\varepsilon\\ \leq&\frac{K_1\nu}{4}\|\mathbf{P}_h\mathbf{u}-\mathbf{S}_h\mathbf{u}\|_\varepsilon^2+Ch^{2r}|p|_r^2.\end{aligned} \tag{2.29}$$

By incorporating Lemma 1.6, (2.28) and (2.29) in (2.27), we obtain

$$\nu K_1\|\mathbf{P}_h\mathbf{u}-\mathbf{S}_h\mathbf{u}\|_\varepsilon^2 \leq Ch^{2r}\left(|\mathbf{u}|_{r+1}^2+|p|_r^2\right). \tag{2.30}$$

From (2.26) and Lemma 2.2, we now complete the energy norm estimate of $\mathbf{u}-\mathbf{S}_h\mathbf{u}$:

$$\|\mathbf{u}-\mathbf{S}_h\mathbf{u}\|_\varepsilon^2 \leq Ch^{2r}\left(|\mathbf{u}|_{r+1}^2+|p|_r^2\right). \tag{2.31}$$

For L^2-norm estimate, we employ the Aubin-Nitsche duality argument. For fixed h, let $\{\mathbf{w},q\}$ be the pair of unique solution of the steady Stokes problem stated as

$$-\nu\Delta\mathbf{w}+\nabla q=\mathbf{u}-\mathbf{S}_h\mathbf{u} \quad \text{in } \Omega, \quad \nabla\cdot\mathbf{w}=0 \quad \text{in } \Omega, \quad \mathbf{w}|_{\partial\Omega}=0. \tag{2.32}$$

satisfying the following regularity [86]:

$$\|\mathbf{w}\|_2+\|q\|_1 \leq C\|\mathbf{u}-\mathbf{S}_h\mathbf{u}\|. \tag{2.33}$$

Form L^2 inner product between (2.32) and $\mathbf{u}-\mathbf{S}_h\mathbf{u}$, and using the regularity of $\mathbf{w}$ and q, we obtain

$$\begin{aligned}\|\mathbf{u}-\mathbf{S}_h\mathbf{u}\|^2 =&\nu\sum_{E\in\mathcal{E}_h}\int_E\nabla\mathbf{w}:\nabla(\mathbf{u}-\mathbf{S}_h\mathbf{u})-\nu\sum_{E\in\mathcal{E}_h}\int_{\partial E}(\nabla\mathbf{w}\mathbf{n}_E)\cdot(\mathbf{u}-\mathbf{S}_h\mathbf{u})\\ &-\sum_{E\in\mathcal{E}_h}\int_E q\nabla\cdot(\mathbf{u}-\mathbf{S}_h\mathbf{u})+\sum_{E\in\mathcal{E}_h}\int_{\partial E}q\mathbf{n}_E\cdot(\mathbf{u}-\mathbf{S}_h\mathbf{u})\\ =&\nu\sum_{E\in\mathcal{E}_h}\int_E\nabla(\mathbf{u}-\mathbf{S}_h\mathbf{u}):\nabla\mathbf{w}-\nu\sum_{e\in\Gamma_h}\int_e\{\nabla\mathbf{w}\}\mathbf{n}_e\cdot[\mathbf{u}-\mathbf{S}_h\mathbf{u}]\\ &+b(\mathbf{u}-\mathbf{S}_h\mathbf{u},q).\end{aligned}$$

We then use (2.23) with $\mathbf{P}_h\mathbf{w}$ in place of $\boldsymbol{\phi}_h$ and noting that $[\mathbf{w}]\cdot\mathbf{n}_e=0$ on each interior edge to obtain

$$\begin{aligned}\|\mathbf{u}-\mathbf{S}_h\mathbf{u}\|^2 =&\nu\sum_{E\in\mathcal{E}_h}\int_E\nabla(\mathbf{u}-\mathbf{S}_h\mathbf{u}):\nabla(\mathbf{w}-\mathbf{P}_h\mathbf{w})\\ &+\nu\epsilon\sum_{e\in\Gamma_h}\int_e\{\nabla(\mathbf{w}-\mathbf{P}_h\mathbf{w})\}\mathbf{n}_e\cdot[\mathbf{u}-\mathbf{S}_h\mathbf{u}]\end{aligned}$$

$$
\begin{aligned}
&- \nu(1+\epsilon) \sum_{e\in\Gamma_h} \int_e \{\nabla \mathbf{w}\}\mathbf{n}_e \cdot [\mathbf{u} - \mathbf{S}_h\mathbf{u}] \\
&+ \nu \sum_{e\in\Gamma_h} \int_e \{\nabla(\mathbf{u} - \mathbf{S}_h\mathbf{u})\}\mathbf{n}_e \cdot [\mathbf{P}_h\mathbf{w} - \mathbf{w}] \\
&+ \nu \sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \int_e [\mathbf{u} - \mathbf{S}_h\mathbf{u}] \cdot [\mathbf{w} - \mathbf{P}_h\mathbf{w}] \\
&+ b(\mathbf{u} - \mathbf{S}_h\mathbf{u}, q) - b(\mathbf{P}_h\mathbf{w} - \mathbf{w}, p - r_h(p)). \qquad (2.34)
\end{aligned}
$$

Consider the SIPG form of $a(\cdot,\cdot)$ *i.e.* $\epsilon = -1$. Then the third term on the right hand side of (2.34) will vanish. Similar to the proof of Lemma 2.3, we bound the following terms and then use Lemma 2.2, (2.31) and (2.33) to find that

$$
\begin{aligned}
&\Big| \nu \sum_{E\in\mathcal{E}_h} \int_E \nabla(\mathbf{u} - \mathbf{S}_h\mathbf{u}) : \nabla(\mathbf{w} - \mathbf{P}_h\mathbf{w}) - \nu \sum_{e\in\Gamma_h} \int_e \{\nabla(\mathbf{w} - \mathbf{P}_h\mathbf{w})\}\mathbf{n}_e \cdot [\mathbf{u} - \mathbf{S}_h\mathbf{u}] \\
&+ \nu \sum_{e\in\Gamma_h} \int_e \{\nabla(\mathbf{u} - \mathbf{S}_h\mathbf{u})\}\mathbf{n}_e \cdot [\mathbf{P}_h\mathbf{w} - \mathbf{w}] + \nu \sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \int_e [\mathbf{u} - \mathbf{S}_h\mathbf{u}] \cdot [\mathbf{w} - \mathbf{P}_h\mathbf{w}] \Big| \\
&\le Ch\|\mathbf{w}\|_2 \|\mathbf{u} - \mathbf{S}_h\mathbf{u}\|_\varepsilon + Ch^{r+1}|\mathbf{u}|_{r+1}\|\mathbf{w}\|_2 \\
&\le \frac{1}{6}\|\mathbf{u} - \mathbf{S}_h\mathbf{u}\|^2 + Ch^{2r+2}\big(|\mathbf{u}|_{r+1}^2 + |p|_r^2\big). \qquad (2.35)
\end{aligned}
$$

And for the sixth term on the right-hand side of (2.34), by using the definition of $\mathbf{V}_h$, we have

$$
\begin{aligned}
b(\mathbf{u} - \mathbf{S}_h\mathbf{u}, q) =& b(\mathbf{u} - \mathbf{S}_h\mathbf{u} - \mathbf{P}_h\mathbf{u} + \mathbf{S}_h\mathbf{u}, q) + b(\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}, q) \\
=& b(\mathbf{u} - \mathbf{P}_h\mathbf{u}, q) + b(\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}, q - r_h(q)) \\
=& - \sum_{E\in\mathcal{E}_h} \int_E q\nabla \cdot (\mathbf{u} - \mathbf{P}_h\mathbf{u}) + \sum_{e\in\Gamma_h} \int_e \{q\}[\mathbf{u} - \mathbf{P}_h\mathbf{u}] \cdot \mathbf{n}_e \\
& + b(\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}, q - r_h(q)). \qquad (2.36)
\end{aligned}
$$

In addition, applying Green's theorem to the first term on the right hand side of (2.36) and regularity of q implies

$$
b\left(\mathbf{u} - \mathbf{S}_h\mathbf{u}, q\right) = \sum_{E\in\mathcal{E}_h} \int_E \nabla q \cdot (\mathbf{u} - \mathbf{P}_h\mathbf{u}) + b(\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u}, q - r_h(q)).
$$

From the Cauchy-Schwarz inequality, Young's inequality, Lemmas 2.2 and 1.5, (1.31), (2.33) and (2.30), we find

$$
|b(\mathbf{u} - \mathbf{S}_h\mathbf{u},\, q)| \le \Big| Ch^{r+1}\,|q|_1\,|\mathbf{u}|_{r+1} - \sum_{E\in\mathcal{E}_h} \int_E \nabla \cdot (\mathbf{P}_h\mathbf{u} - \mathbf{S}_h\mathbf{u})\,(q - r_h(q))
$$

$$
\begin{aligned}
&+\sum_{e\in\Gamma_h}\int_e\{q-r_h(q)\}[\mathbf{P}_h\mathbf{u}-\mathbf{S}_h\mathbf{u}]\cdot\mathbf{n}_e\Bigg| \\
\le& Ch^{r+1}|\mathbf{u}|_{r+1}\|\mathbf{u}-\mathbf{S}_h\mathbf{u}\|+Ch|q|_1\|\mathbf{P}_h\mathbf{u}-\mathbf{S}_h\mathbf{u}\|_\varepsilon \\
\le&\frac{1}{6}\|\mathbf{u}-\mathbf{S}_h\mathbf{u}\|^2+Ch^{2r+2}\left(|\mathbf{u}|^2_{r+1}+|p|^2_r\right). \qquad (2.37)
\end{aligned}
$$

Similarly, using the Cauchy-Schwarz inequality and Young's inequality, we arrive at

$$
|b(\mathbf{P}_h\mathbf{w}-\mathbf{w},p-r_h(p))|\le Ch^{r+1}|p|_r\|\mathbf{w}\|_2\le\frac{1}{6}\|\mathbf{u}-\mathbf{S}_h\mathbf{u}\|^2+Ch^{2r+2}|p|^2_r. \qquad (2.38)
$$

In view of (2.35), (2.37) and (2.38) in (2.34), we complete the estimate (2.24). Repeating the above set of arguments we arrive at the estimates (2.25) involving $(\mathbf{u}-\mathbf{S}_h\mathbf{u})_t$. The only differences are instead of the equation (2.23), we use the one obtained from differentiating in time, use $\boldsymbol{\phi}_h=\mathbf{P}_h(\mathbf{u}-\mathbf{S}_h\mathbf{u})_t$ in it and finally for the dual problem, we take the right hand side as $(\mathbf{u}-\mathbf{S}_h\mathbf{u})_t$. This completes the proof of Lemma 2.5. □

Remark 2.1. *In the case of NIPG formulation i.e. $\epsilon=1$, the third term on the right hand side of (2.34) is nonzero. And here we will lose a power of h. Using the Cauchy-Schwarz inequality, Young's inequality, trace inequality (1.24), and estimates (2.31) and (2.33), we can show that*

$$
\begin{aligned}
\left|\sum_{e\in\Gamma_h}\int_e\{\nabla\boldsymbol{w}\}\boldsymbol{n}_e\cdot[\boldsymbol{u}-\boldsymbol{S}_h\boldsymbol{u}]\right| \le& C\Bigg(\|\nabla\boldsymbol{w}\|^2+\sum_{E\in\mathcal{E}_h}h_E^2\|\nabla^2\boldsymbol{w}\|^2_{L^2(E)}\Bigg)^{1/2} \\
&\times\Bigg(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\boldsymbol{u}-\boldsymbol{S}_h\boldsymbol{u}]\|^2_{L^2(e)}\Bigg)^{1/2} \\
\le C\|\boldsymbol{w}\|_2\|\boldsymbol{u}-\boldsymbol{S}_h\boldsymbol{u}\|_\varepsilon \le& \frac{1}{6}\|\boldsymbol{u}-\boldsymbol{S}_h\boldsymbol{u}\|^2+Ch^{2r}\big(|\boldsymbol{u}|^2_{r+1}+|p|^2_r\big).
\end{aligned}
$$

Thus, for the NIPG case the estimates (2.24) and (2.25) become

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{S}_h\boldsymbol{u}(t)\|^2+\|\boldsymbol{u}-\boldsymbol{S}_h\boldsymbol{u}(t)\|^2_\varepsilon \le& Ch^{2r}\left(|\boldsymbol{u}|^2_{r+1}+|p|^2_r\right), \\
\|(\boldsymbol{u}-\boldsymbol{S}_h\boldsymbol{u})_t(t)\|^2+\|(\boldsymbol{u}-\boldsymbol{S}_h\boldsymbol{u})_t(t)\|^2_\varepsilon \le& Ch^{2r}\left(|\boldsymbol{u}_t|^2_{r+1}+|p_t|^2_r\right).
\end{aligned}
$$

Same can be said for the IIPG case i.e. $\epsilon=0$, as well.

2.3 Some Useful Estimates

In this section, we first present a Sobolev inequality for the functions of $\mathbf{X}_h$. Then, we concentrate on some estimates of the trilinear form $c(\cdot,\cdot,\cdot)$ and the upwind term $l(\cdot,\cdot,\cdot)$. Finally, we establish regularity results of semi-discrete velocity approximation.

To handle the nonlinear term, we need the L^4-norm estimate for the elements of $\mathbf{X}_h$, which we prove below. However, later on, a short and nice proof for $r \geq 1$ has appeared in a recent paper [101, Lemma 2.3].

Lemma 2.6. *When $\Omega \subset \mathbb{R}^2$ is convex, there exists a positive constant C that does not depend on h, such that*

$$\|\mathbf{v}_h\|_{L^4(\Omega)} \leq C\|\mathbf{v}_h\|^{1/2}\|\mathbf{v}_h\|_{\varepsilon}^{1/2}, \quad \forall \mathbf{v}_h \in \boldsymbol{X}_h.$$

Proof. When $r = 1$ and Ω is convex, from [72, Theorem 3.8] and for each $\mathbf{v}_h \in \mathbf{X}_h$, we have the following estimate:

$$\begin{aligned}\|\mathbf{v}_h\|_{L^4(\Omega)} \leq & C\|\mathbf{v}_h\|^{1/2}\|\mathbf{v}_h\|_{\varepsilon}^{1/2} + Ch^{1/2}\Big(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\mathbf{v}_h]\|_{L^2(e)}^2\Big)^{1/4}\|\mathbf{v}_h\|_{\varepsilon}^{1/2} \\ & + Ch^{1/2}\Big(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\mathbf{v}_h]\|_{L^2(e)}^2\Big)^{1/2}. \end{aligned} \tag{2.39}$$

Consider two elements E_1 and E_2 which share a common edge e. Therefore, using trace inequality (1.35), we obtain

$$\|[\mathbf{v}_h]\|_{L^2(e)}^2 \leq Ch_{E_1}^{-1}\|\mathbf{v}_h\|_{L^2(E_1)}^2 + Ch_{E_2}^{-1}\|\mathbf{v}_h\|_{L^2(E_2)}^2.$$

An application of the above inequality in the second term of the right-hand side of (2.39) leads to

$$h^{1/2}\Big(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\mathbf{v}_h]\|_{L^2(e)}^2\Big)^{1/4} \leq C\Big(\sum_{E\in\mathcal{E}_h}\|\mathbf{v}_h\|_{L^2(E)}^2\Big)^{1/4} \leq C\|\mathbf{v}_h\|^{1/2}.$$

In a similar manner, one can derive the following by applying the definition of $\|\cdot\|_{\varepsilon}$-norm as follows:

$$\begin{aligned} h^{1/2}\Big(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\mathbf{v}_h]\|_{L^2(e)}^2\Big)^{1/2} & \leq h^{1/2}\Big(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\mathbf{v}_h]\|_{L^2(e)}^2\Big)^{1/4}\Big(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\mathbf{v}_h]\|_{L^2(e)}^2\Big)^{1/4} \\ & \leq C\|\mathbf{v}_h\|^{1/2}\|\mathbf{v}_h\|_{\varepsilon}^{1/2}. \end{aligned}$$

Therefore, for $r = 1$, the desired estimate of this lemma follows by substituting the above estimates in (2.39).

For $r > 1$, the derivation of the L^4-estimate closely follow a combined technique proposed in [70, Section 4.1] and [72, Section 3]. Recall that, I_h is the Lagrange interpolation operator of degree one defined in every $E \in \mathcal{E}_h$ by $I_h(\mathbf{v}_h)|_E \in \mathbb{P}_1(E)$ and

$I_h(\mathbf{v}_h)(a_i) = \mathbf{v}_h(a_i)$, where a_i, $i = 1,2,3$, are the vertices of E. Let us denote the space of Crouzeix-Raviart elements of degree one by

$$CR^h = \big\{\mathbf{w}_h : \ \forall E \in \mathcal{E}_h, \ \mathbf{w}_h|_E \in \mathbb{P}_1(E), \ \forall e \in \Gamma_h, \quad \int_e [\mathbf{w}_h] = 0\big\}.$$

We now transform $\mathbf{v}_h$ into an element of CR^h by interpolating $\mathbf{v}_h$ with I_h and then convert $I_h(\mathbf{v}_h)$ into a function of CR^h with the following transformation

$$CR(\mathbf{v}_h) = I_h(\mathbf{v}_h) - \sum_{e\in\Gamma_h} \frac{1}{|e|}\Big(\int_e [I_h(\mathbf{v}_h)]\Big)\lambda_e, \tag{2.40}$$

where, given any $e \in \Gamma_h$, λ_e is the piecewise linear basis function defined as follows. Let the midpoint of e be denoted by $\mathbf{b}_e$, and let $E \in \mathcal{E}_h$ be an element with an edge e and $\mathbf{n}_e$ is the outward normal to E. Thus $\lambda_e(\mathbf{b}_e) = 1$, $\lambda_e(\mathbf{b}_{e'}) = 0$ if $e' \neq e$, $\lambda_e|_E \in \mathbb{P}_1(E)$ and $\lambda_e|_{E'} = 0 \,\forall\, E' \neq E$. Let $\hat{E}$ denote the reference element and given an element $E \in \mathcal{E}_h$, there exists an invertible affine map $F_E : \hat{E} \to E$. Let us denote, $\hat{\mathbf{v}}_h = \mathbf{v}_h \circ F_E$ and the gradient of $\hat{\mathbf{v}}_h$ on $\hat{E}$ by $\hat{\nabla}\hat{\mathbf{v}}_h$. Now,

$$\int_e [CR(\mathbf{v}_h)] = \int_e [I_h(\mathbf{v}_h)] - \frac{1}{|e|}\Big(\int_e [I_h(\mathbf{v}_h)]\Big)\int_e [\lambda_e] = 0,$$

which implies $CR(\mathbf{v}_h)$ belongs to CR^h.

(**a**): First of all, we show that

$$\|\mathbf{v}_h - CR(\mathbf{v}_h)\|_\varepsilon \leq C\|\mathbf{v}_h\|_\varepsilon. \tag{2.41}$$

From (2.40), we have

$$\|\mathbf{v}_h - CR(\mathbf{v}_h)\|_\varepsilon \leq \|\mathbf{v}_h - I_h(\mathbf{v}_h)\|_\varepsilon + \sum_{e\in\Gamma_h} \frac{1}{|e|}\Big(\int_e [I_h(\mathbf{v}_h)]\Big)\|\lambda_e\|_\varepsilon. \tag{2.42}$$

Following the proof of [70, Lemma 4.1] and applying (1.13), we can deduce

$$\|\nabla(\mathbf{v}_h - I_h(\mathbf{v}_h))\|_{L^2(E)} \leq C\|\nabla\mathbf{v}_h\|_{L^2(E)}, \tag{2.43}$$

$$\frac{1}{|e|}\|[\mathbf{v}_h - I_h(\mathbf{v}_h)]\|^2_{L^2(e)} \leq C\|\nabla\mathbf{v}_h\|^2_{L^2(E)}. \tag{2.44}$$

Combining (2.43)-(2.44) and using the definition of $\|\cdot\|_\varepsilon$-norm, we arrive at

$$\|\mathbf{v}_h - I_h(\mathbf{v}_h)\|_\varepsilon \leq C\Big(\sum_{E\in\mathcal{E}_h} \|\nabla\mathbf{v}_h\|^2_{L^2(E)}\Big)^{1/2} \leq C\|\mathbf{v}_h\|_\varepsilon. \tag{2.45}$$

Also, a use of (1.13) and the Cauchy-Schwarz inequality yields

$$\sum_{e\in\partial E} \frac{1}{|e|}\Big|\int_e [I_h(\mathbf{v}_h)]\Big|\|\nabla\lambda_e\|_{L^2(E)} \leq C\frac{|E|^{1/2}}{\rho_E}\sum_{e\in\partial E}\frac{1}{|e|^{1/2}}\|[I_h(\mathbf{v}_h)]\|_{L^2(e)}\|\hat{\nabla}\hat{\lambda}_e\|_{L^2(\hat{E})}$$

$$\leq C \sum_{e\in\partial E} \frac{1}{|e|^{1/2}} \|[I_h(\mathbf{v}_h)]\|_{L^2(e)}.$$

Thus, using triangle inequality and (2.44) in the above estimate, we arrive at

$$\begin{aligned}
\sum_{e\in\partial E} \frac{1}{|e|}\Big| \int_e [I_h(\mathbf{v}_h)]\Big| \|\nabla\lambda_e\|_{L^2(E)} \leq & C \sum_{e\in\partial E} \frac{1}{|e|^{1/2}} \|[I_h(\mathbf{v}_h)-\mathbf{v}_h]\|_{L^2(e)} \\
& + C \sum_{e\in\partial E} \frac{1}{|e|^{1/2}} \|[\mathbf{v}_h]\|_{L^2(e)} \\
\leq & C\|\nabla\mathbf{v}_h\|_{L^2(E)} + C \sum_{e\in\partial E} \frac{1}{|e|^{1/2}} \|[\mathbf{v}_h]\|_{L^2(e)}. \qquad (2.46)
\end{aligned}$$

For each $e \in \Gamma_h$, let Γ_e be the set of edges $\tilde{e} \in \Gamma_h$, so that $[\lambda_{\tilde{e}}]|_e$ is non-zero. Thus, a use of the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\sum_{e\in\Gamma_h} \frac{1}{|e|}\Big(\int_e [I_h(\mathbf{v}_h)]\Big) \|[\lambda_e]\|_{L^2(e)} \leq & |e|^{1/2} \sum_{\tilde{e}\in\Gamma_e} \frac{1}{|\tilde{e}|^{1/2}} \|[I_h(\mathbf{v}_h)]\|_{L^2(\tilde{e})} \|[\hat{\lambda}_{\tilde{e}}]\|_{L^2(\hat{e})} \\
\leq & |e|^{1/2} \sum_{\tilde{e}\in\Gamma_e} \frac{1}{|\tilde{e}|^{1/2}} \|[I_h(\mathbf{v}_h)]\|_{L^2(\tilde{e})}. \qquad (2.47)
\end{aligned}$$

Following (2.46)-(2.47) and using the definition of $\|\cdot\|_\varepsilon$-norm, one can show that

$$\sum_{e\in\Gamma_h} \frac{1}{|e|}\Big(\int_e [I_h(\mathbf{v}_h)]\Big) \|\lambda_e\|_\varepsilon \leq C\|\mathbf{v}_h\|_\varepsilon. \qquad (2.48)$$

Combining (2.45) and (2.48) in (2.42), we arrive at (2.41).

(**b**): Secondly, we derive

$$\|\mathbf{v}_h - CR(\mathbf{v}_h)\|_{L^r(\Omega)} \leq Ch^{2/r}\|\mathbf{v}_h\|_\varepsilon, \quad r \in [2,\infty). \qquad (2.49)$$

For any element $E \in \mathcal{E}_h$ and $r \in [2,\infty)$, we can write

$$\|\mathbf{v}_h - CR(\mathbf{v}_h)\|_{L^r(E)} \leq \|\mathbf{v}_h - I_h(\mathbf{v}_h)\|_{L^r(E)} + \sum_{e\in\partial E} \frac{1}{|e|}\Big(\int_e [I_h(\mathbf{v}_h)]\Big) \|\lambda_e\|_{L^r(E)}.$$

Now, switching to the reference element $\hat{E}$ and using (1.13), we find

$$\begin{aligned}
\|\mathbf{v}_h - CR(\mathbf{v}_h)\|_{L^r(E)} \leq & C|E|^{1/r}\|\hat{\mathbf{v}}_h - \hat{I}_h(\hat{\mathbf{v}}_h)\|_{L^r(\hat{E})} \\
& + C|E|^{1/r} \sum_{e\in\partial E} \frac{1}{|e|}\Big(\int_e [I_h(\mathbf{v}_h)]\Big) \|\hat{\lambda}_e\|_{L^r(\hat{E})} \\
\leq & C|E|^{1/r}\|\hat{\nabla}\hat{\mathbf{v}}_h\|_{L^2(\hat{E})} + C|E|^{1/r} \sum_{e\in\partial E} \frac{1}{|e|}\Big(\int_e [I_h(\mathbf{v}_h)]\Big) \\
\leq & C|E|^{1/r}\|\nabla\mathbf{v}_h\|_{L^2(E)} + C|E|^{1/r} \sum_{e\in\partial E} \frac{1}{|e|^{1/2}} \|[I_h(\mathbf{v}_h)]\|_{L^2(e)}.
\end{aligned}$$

After summing over all elements $E \in \mathcal{E}_h$, applying equivalence of norms on $\mathbb{R}^n$, $n \geq 1$ and similar to the estimation technique (2.46), we find

$$\begin{aligned}\|\mathbf{v}_h - CR(\mathbf{v}_h)\|_{L^r(\Omega)} \leq & Ch^{2/r}\Big(\sum_{E\in\mathcal{E}_h}\|\nabla \mathbf{v}_h\|^2_{L^2(E)}\Big)^{1/2} + Ch^{2/r}\Big(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\mathbf{v}_h]\|^2_{L^2(e)}\Big)^{1/2} \\ \leq & Ch^{2/r}\|\mathbf{v}_h\|_{\varepsilon}.\end{aligned}$$

This completes the derivation of (2.49).

We now consider a function $\mathbf{v}(h) \in \mathbf{H}^1_0(\Omega)$ which satisfies

$$\int_\Omega \nabla \mathbf{v}(h) : \nabla \mathbf{w} = \sum_{E\in\mathcal{E}_h}\int_E \nabla CR(\mathbf{v}_h) : \nabla \mathbf{w}, \quad \forall \mathbf{w} \in \mathbf{H}^1_0(\Omega). \tag{2.50}$$

An application of (2.50), triangle inequality and (2.41) yields

$$\|\nabla \mathbf{v}(h)\| \leq \Big(\sum_{E\in\mathcal{E}_h}\|\nabla CR(\mathbf{v}_h)\|^2_{L^2(E)}\Big)^{1/2} \leq C\|\mathbf{v}_h\|_{\varepsilon}. \tag{2.51}$$

Next, following [72, Lemma 3.4] and using (2.49), for $r \geq 4$, one can derive

$$\|\mathbf{v}_h - \mathbf{v}(h)\|_{L^r(\Omega)} \leq Ch^{2/r}\|\mathbf{v}_h\|_{\varepsilon}. \tag{2.52}$$

Furthermore, when Ω is convex, using [72, Remark 3.6] and (2.49) for $r = 2$, we have

$$\|\mathbf{v}_h - \mathbf{v}(h)\| \leq Ch\|\mathbf{v}_h\|_{\varepsilon}. \tag{2.53}$$

Now, we are in a position to achieve our desired estimate. Using the fact that

$$\|\mathbf{v}(h)\|_{L^4(\Omega)} \leq C\|\mathbf{v}(h)\|^{1/2}\|\nabla \mathbf{v}(h)\|^{1/2}$$

and triangle inequality, we find

$$\|\mathbf{v}_h\|_{L^4(\Omega)} \leq \|\mathbf{v}_h - \mathbf{v}(h)\|_{L^4(\Omega)} + C\|\mathbf{v}(h)\|^{1/2}\|\nabla \mathbf{v}(h)\|^{1/2}.$$

Again, we use triangle inequality, (2.51), (2.52) and (2.53) to obtain

$$\|\mathbf{v}_h\|_{L^4(\Omega)} \leq Ch^{1/2}\|\mathbf{v}_h\|_{\varepsilon} + C\Big(h^{1/2}\|\mathbf{v}_h\|^{1/2}_{\varepsilon} + \|\mathbf{v}_h\|^{1/2}\Big)\|\mathbf{v}_h\|^{1/2}_{\varepsilon}.$$

Finally, an application of trace inequality (1.35) and inverse inequality (1.38) leads us to the desired estimate. □

Armed with the above estimate, we now present a few estimates of trilinear form $c(\cdot,\cdot,\cdot)$ and upwind term $l(\cdot,\cdot,\cdot)$ that would be useful in our later part.

Lemma 2.7. *There exists a positive constant C, which is independent of h, such that the following estimates hold true:*

$$|c^{\boldsymbol{\phi}_h}(\boldsymbol{w}_h,\, \boldsymbol{u},\, \mathbf{v}_h)| \leq C\|\boldsymbol{w}_h\|\|\boldsymbol{u}|_{1,4,\Omega}\|\mathbf{v}_h\|_{\varepsilon}, \quad \boldsymbol{u} \in \boldsymbol{W}^{1,4}(\Omega),\; \boldsymbol{w}_h \in \boldsymbol{V}_h,\; \boldsymbol{\phi}_h, \mathbf{v}_h \in \boldsymbol{X}_h, \tag{2.54}$$

$$|c^{\boldsymbol{w}}(\boldsymbol{w}_h,\, \boldsymbol{\phi}_h,\, \mathbf{v}_h)| \leq C\|\boldsymbol{w}_h\|_{\varepsilon}\|\boldsymbol{\phi}_h\|_{\varepsilon}\|\mathbf{v}_h\|_{\varepsilon}, \quad \boldsymbol{w} \in \boldsymbol{X},\; \boldsymbol{w}_h,\, \boldsymbol{\phi}_h,\, \mathbf{v}_h \in \boldsymbol{X}_h, \tag{2.55}$$

$$|\, c^{\boldsymbol{\Theta}}(\mathbf{v},\, \boldsymbol{w},\, \boldsymbol{\phi}_h)| \leq C\|\mathbf{v}\|_{\varepsilon}\|\boldsymbol{w}\|_{\varepsilon}\|\boldsymbol{\phi}_h\|_{\varepsilon}, \quad \boldsymbol{\Theta}, \mathbf{v}, \boldsymbol{w} \in \boldsymbol{X},\; \boldsymbol{\phi}_h \in \boldsymbol{X}_h, \tag{2.56}$$

$$|c^{\boldsymbol{w}}(\boldsymbol{w}_h,\, \boldsymbol{\phi}_h,\, \boldsymbol{w}_h)| \leq C\|\boldsymbol{w}_h\|^{1/2}\|\boldsymbol{w}_h\|_{\varepsilon}^{3/2}\|\boldsymbol{\phi}_h\|_{\varepsilon}, \quad \boldsymbol{w} \in \boldsymbol{X},\; \boldsymbol{w}_h,\, \boldsymbol{\phi}_h \in \boldsymbol{X}_h, \tag{2.57}$$

$$\begin{aligned} |\, c^{\boldsymbol{u}}(\boldsymbol{u},\, \boldsymbol{w},\, \boldsymbol{\phi}_h)| &\leq C\|\boldsymbol{u}\|_2(\|\boldsymbol{w}\| + h\|\boldsymbol{w}\|_{\varepsilon})\|\boldsymbol{\phi}_h\|_{\varepsilon}, \\ &\qquad \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega),\; \boldsymbol{w} \in \boldsymbol{X},\; \boldsymbol{\phi}_h \in \boldsymbol{X}_h, \end{aligned} \tag{2.58}$$

$$\begin{aligned} |\, c^{\mathbf{v}_h}(\boldsymbol{w},\, \boldsymbol{u},\, \boldsymbol{\phi}_h)| &\leq C\|\boldsymbol{u}\|_2(\|\boldsymbol{w}\| + h\|\boldsymbol{w}\|_{\varepsilon})\|\boldsymbol{\phi}_h\|_{\varepsilon}, \\ &\qquad \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega),\; \boldsymbol{w} \in \boldsymbol{X},\; \mathbf{v}_h, \boldsymbol{\phi}_h \in \boldsymbol{X}_h, \end{aligned} \tag{2.59}$$

$$\begin{aligned} &|l^{\boldsymbol{\Theta}_h}(\mathbf{v}_h,\, \boldsymbol{w},\, \boldsymbol{\phi}_h) - l^{\mathbf{v}_h}(\mathbf{v}_h,\, \boldsymbol{w},\, \boldsymbol{\phi}_h)| \\ &\qquad \leq C\|\boldsymbol{\Theta}_h - \mathbf{v}_h\|_{L^4(\Omega)}\|\boldsymbol{w}\|_{\varepsilon}\|\boldsymbol{\phi}_h\|_{L^4(\Omega)}, \quad \boldsymbol{w} \in \boldsymbol{X},\; \boldsymbol{\Theta}_h, \mathbf{v}_h, \boldsymbol{\phi}_h \in \boldsymbol{X}_h, \end{aligned} \tag{2.60}$$

$$\begin{aligned} &|l^{\boldsymbol{\Theta}_h}(\mathbf{v},\, \boldsymbol{w},\, \boldsymbol{\phi}_h) - l^{\mathbf{v}}(\mathbf{v},\, \boldsymbol{w},\, \boldsymbol{\phi}_h)| \\ &\qquad \leq C\|\boldsymbol{\Theta}_h - \mathbf{v}\|_{\varepsilon}\|\boldsymbol{w}\|_{\varepsilon}\|\boldsymbol{\phi}_h\|_{L^4(\Omega)}, \quad \boldsymbol{\Theta}_h, \boldsymbol{\phi}_h \in \boldsymbol{X}_h,\; \mathbf{v}, \boldsymbol{w} \in \boldsymbol{X}. \end{aligned} \tag{2.61}$$

Proof. The first two estimates (2.54) and (2.55) are proved in [72, Proposition 4.1]. For the third estimate, we use Hölder's inequality to find

$$\begin{aligned} &|c^{\boldsymbol{\Theta}}(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}_h)| \\ &= \Big| \sum_{E \in \mathcal{E}_h} \int_E (\mathbf{v} \cdot \nabla \mathbf{w}) \cdot \boldsymbol{\phi}_h + \sum_{E \in \mathcal{E}_h} \int_{\partial E_-} |\{\mathbf{v}\} \cdot \mathbf{n}_E|(\mathbf{w}^{int} - \mathbf{w}^{ext}) \cdot \boldsymbol{\phi}_h^{int} \\ &\quad + \frac{1}{2} \sum_{E \in \mathcal{E}_h} \int_E (\nabla \cdot \mathbf{v})\mathbf{w} \cdot \boldsymbol{\phi}_h - \frac{1}{2} \sum_{e \in \Gamma_h} \int_e [\mathbf{v}] \cdot \mathbf{n}_e \{\mathbf{w} \cdot \boldsymbol{\phi}_h\} \Big| \\ &\leq \sum_{E \in \mathcal{E}_h} \|\mathbf{v}\|_{L^4(E)}\|\nabla \mathbf{w}\|_{L^2(E)}\|\boldsymbol{\phi}_h\|_{L^4(E)} + \sum_{e \in \Gamma_h} \|\{\mathbf{v}\} \cdot \mathbf{n}_e\|_{L^4(e)}\|[\mathbf{w}]\|_{L^2(e)}\|\boldsymbol{\phi}_h\|_{L^4(e)} \\ &\quad + \frac{1}{2} \sum_{E \in \mathcal{E}_h} \|\nabla \mathbf{v}\|_{L^2(E)}\|\mathbf{w}\|_{L^4(E)}\|\boldsymbol{\phi}_h\|_{L^4(E)} + \frac{1}{2} \sum_{e \in \Gamma_h} \|[\mathbf{v}] \cdot \mathbf{n}_e\|_{L^2(e)}\|\{\mathbf{w} \cdot \boldsymbol{\phi}_h\}\|_{L^2(e)}. \end{aligned} \tag{2.62}$$

To bound the edge terms, we consider the elements E_1 and E_2 sharing e. Thus, using trace inequalities (1.25) and (1.37), we obtain

$$\|\{\mathbf{v}\} \cdot \mathbf{n}_e\|_{L^4(e)}\|[\mathbf{w}]\|_{L^2(e)}\|\boldsymbol{\phi}_h\|_{L^4(e)} \leq \frac{1}{2} \sum_{i,j=1}^{2} \|\mathbf{v} \cdot \mathbf{n}_e|_{E_i}\|_{L^4(e)}\|[\mathbf{w}]\|_{L^2(e)}\|\boldsymbol{\phi}_h|_{E_j}\|_{L^4(e)}$$

$$\leq C \sum_{i,j=1}^{2} \big(\|\mathbf{v}\|_{L^4(E_i)} + h_{E_i}^{1/2} \|\nabla \mathbf{v}\|_{L^2(E_i)}\big) |e|^{-1/2} \|[\mathbf{w}]\|_{L^2(e)} \|\boldsymbol{\phi}_h\|_{L^4(E_j)}. \quad (2.63)$$

In a similar fashion, one can derive the following bound

$$\|[\mathbf{v}]\cdot\mathbf{n}_e\|_{L^2(e)} \|\{\mathbf{w}\cdot\boldsymbol{\phi}_h\}\|_{L^2(e)} \\ \leq C \sum_{i,j=1}^{2} |e|^{-1/2} \, \|[\mathbf{v}]\|_{L^2(e)} \big(\|\mathbf{w}\|_{L^4(E_i)} + h_{E_i}^{1/2} \|\nabla \mathbf{w}\|_{L^2(E_i)}\big) \|\boldsymbol{\phi}_h\|_{L^4(E_j)}. \quad (2.64)$$

Applying (2.63)-(2.64) in (2.62), using Hölder's inequality and (1.14), we arrive at the estimate (2.56).

For the estimate (2.57), expand the term $c^{\mathbf{w}}(\mathbf{w}_h, \boldsymbol{\phi}_h, \mathbf{w}_h)$ as follows:

$$c^{\mathbf{w}}(\mathbf{w}_h, \boldsymbol{\phi}_h, \mathbf{w}_h) = \sum_{E\in\mathcal{E}_h} \int_E (\mathbf{w}_h \cdot \nabla\boldsymbol{\phi}_h)\cdot\mathbf{w}_h + \sum_{E\in\mathcal{E}_h} \int_{\partial E_-} |\{\mathbf{w}_h\}\cdot\mathbf{n}_E| (\boldsymbol{\phi}_h^{int} - \boldsymbol{\phi}_h^{ext})\cdot\mathbf{w}_h^{int} \\ + \frac{1}{2} \sum_{E\in\mathcal{E}_h} \int_E (\nabla\cdot\mathbf{w}_h)\boldsymbol{\phi}_h\cdot\mathbf{w}_h - \frac{1}{2} \sum_{e\in\Gamma_h} \int_e [\mathbf{w}_h]\cdot\mathbf{n}_e \{\boldsymbol{\phi}_h\cdot\mathbf{w}_h\} \\ = A_1 + A_2 + A_3 + A_4.$$

Using Hölder's inequality, estimate (1.14) and Lemma 2.6, we can bound A_1 as follows:

$$|A_1| \leq \sum_{E\in\mathcal{E}_h} \|\mathbf{w}_h\|_{L^4(E)} \|\nabla\boldsymbol{\phi}_h\|_{L^2(E)} \|\mathbf{w}_h\|_{L^4(E)} \leq C\|\mathbf{w}_h\|^{1/2} \|\mathbf{w}_h\|_{\varepsilon}^{3/2} \|\boldsymbol{\phi}_h\|_{\varepsilon}.$$

An application of Hölder's inequality yield

$$|A_2| \leq \sum_{e\in\Gamma_h} \|\{\mathbf{w}_h\}\cdot\mathbf{n}_e\|_{L^4(e)} \|[\boldsymbol{\phi}_h]\|_{L^2(e)} \|\mathbf{w}_h\|_{L^4(e)}.$$

Let E_1 and E_2 be the elements adjacent to e. Therefore, using (1.37), we arrive at

$$\|\{\mathbf{w}_h\}\cdot\mathbf{n}_e\|_{L^4(e)} \|[\boldsymbol{\phi}_h]\|_{L^2(e)} \|\mathbf{w}_h\|_{L^4(e)} \leq \frac{1}{2} \sum_{i,j=1}^{2} \|\mathbf{w}_h\cdot\mathbf{n}_e|_{E_i}\|_{L^4(e)} \|[\boldsymbol{\phi}_h]\|_{L^2(e)} \|\mathbf{w}_h|_{E_j}\|_{L^4(e)} \\ \leq C \sum_{i,j=1}^{2} \|\mathbf{w}_h\|_{L^4(E_i)} |e|^{-1/2} \|[\boldsymbol{\phi}_h]\|_{L^2(e)} \|\mathbf{w}_h\|_{L^4(E_j)}.$$

Again, use Hölder's inequality, (1.14) and Lemma 2.6 to find

$$|A_2| \leq \|\mathbf{w}_h\|_{L^4(\Omega)} \Big(\sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\boldsymbol{\phi}_h]\|_{L^2(e)}^2 \Big)^{1/2} \|\mathbf{w}_h\|_{L^4(\Omega)} \leq C\|\mathbf{w}_h\|_{L^4(\Omega)} \|\boldsymbol{\phi}_h\|_{\varepsilon} \|\mathbf{w}_h\|_{L^4(\Omega)} \\ \leq C\|\mathbf{w}_h\|^{1/2} \|\mathbf{w}_h\|_{\varepsilon}^{3/2} \|\boldsymbol{\phi}_h\|_{\varepsilon}.$$

Using Hölder's inequality, (1.14) and Lemma 2.6, A_3 can be bounded as

$$|A_3| \leq C \sum_{E\in\mathcal{E}_h} \|\nabla \mathbf{w}_h\|_{L^2(E)} \|\boldsymbol{\phi}_h\|_{L^4(E)} \|\mathbf{w}_h\|_{L^4(E)} \leq C\|\mathbf{w}_h\|^{1/2}\|\mathbf{w}_h\|_{\varepsilon}^{3/2}\|\boldsymbol{\phi}_h\|_{\varepsilon}.$$

Similar to A_2 and A_3, we can bound A_4 as follows

$$|A_4| \leq C\|\mathbf{w}_h\|^{1/2}\|\mathbf{w}_h\|_{\varepsilon}^{3/2}\|\boldsymbol{\phi}_h\|_{\varepsilon}.$$

Combining the bounds A_1, A_2, A_3, A_4, we arrive at the estimate (2.57).

For deriving the estimate (2.58), we follow (1.18) and rewrite $c^{\mathbf{u}}(\mathbf{u}, \mathbf{w}, \boldsymbol{\phi}_h)$ as follows

$$\begin{aligned} c^{\mathbf{u}}(\mathbf{u}, \mathbf{w}, \boldsymbol{\phi}_h) =& -\sum_{E\in\mathcal{E}_h} \int_E (\mathbf{u}\cdot\nabla\boldsymbol{\phi}_h)\cdot\mathbf{w} - \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E (\nabla\cdot\mathbf{u})\mathbf{w}\cdot\boldsymbol{\phi}_h \\ &+ \frac{1}{2}\sum_{e\in\Gamma_h}\int_e [\mathbf{u}]\cdot\mathbf{n}_e\{\mathbf{w}\cdot\boldsymbol{\phi}_h\} - \sum_{E\in\mathcal{E}_h}\int_{\partial E_-} |\mathbf{u}\cdot\mathbf{n}_E|\mathbf{w}^{ext}\cdot(\boldsymbol{\phi}_h^{int} - \boldsymbol{\phi}_h^{ext}) \\ &+ \int_{\Gamma_+} |\mathbf{u}\cdot\mathbf{n}|\mathbf{w}\cdot\boldsymbol{\phi}_h \\ =& \; A_5 + A_6 + A_7 + A_8 + A_9 \end{aligned} \tag{2.65}$$

Since $\mathbf{u}$ belongs to $\mathbf{H}_0^1(\Omega)$ and hence $\mathbf{u}$ is continuous, it is clear that $A_7 = A_9 = 0$. For A_5 and A_6, we use Hölder's inequality, Lemma 1.3, the estimate (1.14) and obtain

$$\begin{aligned} |A_5| + |A_6| \leq& \; C \,\|\, \mathbf{u}\,\|_{L^\infty(\Omega)} \Big(\sum_{E\in\mathcal{E}_h} \|\nabla\boldsymbol{\phi}_h\|_{L^2(E)}^2 \Big)^{1/2} \|\mathbf{w}\| \; + \; C \,\|\, \nabla\mathbf{u}\,\|_{L^4(\Omega)} \|\, \mathbf{w}\,\| \|\, \boldsymbol{\phi}_h\,\|_{L^4(\Omega)} \\ \leq& \; C \,\|\, \mathbf{u}\,\|_2 \|\, \mathbf{w}\,\| \|\, \boldsymbol{\phi}_h\,\|_{\varepsilon}. \end{aligned} \tag{2.66}$$

An application of trace inequality (1.23), Lemma 1.3 and Hölder's inequality leads to the following bound of A_8:

$$\begin{aligned} |A_8| &\leq C\|\mathbf{u}\|_{L^\infty(\Omega)} \sum_{e\in\Gamma_h} \|\mathbf{w}\|_{L^2(e)} |e|^{1/2-1/2} \|[\boldsymbol{\phi}_h]\|_{L^2(e)} \\ &\leq C\|\mathbf{u}\|_2 \Big(\sum_{E\in\mathcal{E}_h} (\|\mathbf{w}\|_{L^2(E)}^2 + h_E^2\|\nabla\mathbf{w}\|_{L^2(E)}^2) \Big)^{1/2} \Big(\sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\boldsymbol{\phi}_h]\|_{L^2(e)}^2 \Big)^{1/2} \\ &\leq C\|\mathbf{u}\|_2 (\|\mathbf{w}\| + h\|\mathbf{w}\|_{\varepsilon}) \|\boldsymbol{\phi}_h\|_{\varepsilon}. \end{aligned} \tag{2.67}$$

Combining the bounds (2.66) and (2.67) in (2.65), we establish (2.58).

Furthermore, for the derivation of (2.59), we use $(\nabla\cdot\mathbf{w})u_i = \nabla\cdot(\mathbf{w}u_i) - \mathbf{w}\cdot\nabla u_i$ and from Green's formula, one can find

$$\int_E (\nabla\cdot\mathbf{w})\mathbf{u}\cdot\boldsymbol{\phi}_h = \int_E \nabla\cdot(\mathbf{u}\otimes\mathbf{w})\cdot\boldsymbol{\phi}_h - \int_E (\mathbf{w}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\phi}_h$$

$$= -\int_E (\mathbf{w}\cdot\nabla\boldsymbol{\phi}_h)\cdot\mathbf{u} + \int_{\partial E}(\mathbf{w}^{int}\cdot\mathbf{n}_E)\mathbf{u}^{int}\cdot\boldsymbol{\phi}_h^{int} - \int_E(\mathbf{w}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\phi}_h.$$

The above allows us to arrive at the following reformulation:

$$\begin{aligned}
c^{\mathbf{v}_h}(\mathbf{w},\mathbf{u},\boldsymbol{\phi}_h) =& \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E(\mathbf{w}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\phi}_h - \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E(\mathbf{w}\cdot\nabla\boldsymbol{\phi}_h)\cdot\mathbf{u}\\
&- \sum_{E\in\mathcal{E}_h}\int_{\partial E_-}|\{\mathbf{w}\}\cdot\mathbf{n}_E|(\mathbf{u}^{int}-\mathbf{u}^{ext})\cdot\boldsymbol{\phi}_h^{int} - \frac{1}{2}\sum_{e\in\Gamma_h}\int_e[\mathbf{w}]\cdot\mathbf{n}_e\{\mathbf{u}\cdot\boldsymbol{\phi}_h\}\\
&+ \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_{\partial E}(\mathbf{w}^{int}\cdot\mathbf{n}_E)\mathbf{u}^{int}\cdot\boldsymbol{\phi}_h^{int}\\
=& A_{10}+A_{11}+A_{12}+A_{13}+A_{14}. \qquad (2.68)
\end{aligned}$$

Since $\mathbf{u}$ is continuous, we have, $A_{12}=0$. The terms A_{10} and A_{11} are bounded using Hölder's inequality, (1.14) and Lemma 1.3 as follows:

$$|A_{10}| \le C\|\mathbf{w}\|\|\nabla\mathbf{u}\|_{L^4(\Omega)}\|\boldsymbol{\phi}_h\|_{L^4(\Omega)} \le C\|\mathbf{u}\|_2\|\mathbf{w}\|\|\boldsymbol{\phi}_h\|_\varepsilon, \qquad (2.69)$$

$$|A_{11}| \le C\|\mathbf{w}\|\|\boldsymbol{\phi}_h\|_\varepsilon\|\mathbf{u}\|_{L^\infty(\Omega)} \le C\|\mathbf{u}\|_2\|\mathbf{w}\|\|\boldsymbol{\phi}_h\|_\varepsilon. \qquad (2.70)$$

Next, switch the sum of A_{14} from elements to the edges. Then we consider this sum's contribution to any interior edge e. Let E_r and E_s be the two elements adjacent to e, with exterior normal $\mathbf{n}_r$ and $\mathbf{n}_s$, respectively. This implies

$$\int_e(\mathbf{w}|_{E_r}\cdot\mathbf{n}_r)\mathbf{u}|_{E_r}\cdot\boldsymbol{\phi}_h|_{E_r} + \int_e(\mathbf{w}|_{E_s}\cdot\mathbf{n}_s)\mathbf{u}|_{E_s}\cdot\boldsymbol{\phi}_h|_{E_s} = \int_e[(\mathbf{w}\cdot\mathbf{n}_e)\mathbf{u}\cdot\boldsymbol{\phi}_h].$$

Thus, by using the fact $[\mathbf{a}\cdot\mathbf{b}] = \{\mathbf{a}\}\cdot[\mathbf{b}] + [\mathbf{a}]\cdot\{\mathbf{b}\},\ \ \mathbf{a},\mathbf{b}\in\mathbb{R}^d$, the trace inequality (1.23), Lemma 1.3 and Hölder's inequality, we obtain

$$\begin{aligned}
A_{13}+A_{14} =& \frac{1}{2}\sum_{e\in\Gamma_h}\int_e\{\mathbf{w}\}\cdot\mathbf{n}_e[\mathbf{u}\cdot\boldsymbol{\phi}_h]\\
\le&\ C\|\,\mathbf{u}\,\|_{L^\infty(\Omega)}\sum_{e\in\Gamma_h}\frac{1}{|e|^{1/2}}\,\|[\boldsymbol{\phi}_h]\|_{L^2(e)}\,|e|^{1/2}\,\|\mathbf{w}\|_{L^2(e)}\\
\le&\ C\,\|\mathbf{u}\|_2\|\boldsymbol{\phi}_h\|_\varepsilon\Big(\sum_{E\in\mathcal{E}_h}\big(\|\mathbf{w}\|^2_{L^2(E)} + h_E^2\|\nabla\mathbf{w}\|^2_{L^2(E)}\big)\Big)^{1/2}\\
\le&\ C\|\mathbf{u}\|_2(\|\mathbf{w}\| + h\|\mathbf{w}\|_\varepsilon)\|\boldsymbol{\phi}_h\|_\varepsilon. \qquad (2.71)
\end{aligned}$$

Substitute the bounds (2.69)-(2.71) in (2.68) to arrive at estimate (2.59).

The proof of estimate (2.60) closely follows the analysis of [70, Proposition 4.10]. For any $\boldsymbol{\theta}_h\in\mathbf{X}_h$ and let $e\in\Gamma_h\setminus\partial\Omega$ be an edge adjacent to the elements E_1 and E_2 with $\mathbf{n}_e = \mathbf{n}_{E_1}$. The contribution of e to the term $l^{\boldsymbol{\theta}_h}(\mathbf{v}_h,\,\mathbf{w},\,\boldsymbol{\phi}_h)$ reduces to

$$\int_e(\{\mathbf{v}_h\}\cdot\mathbf{n}_e)[\mathbf{w}]\cdot\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}, \qquad (2.72)$$

where $\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}|_e = \boldsymbol{\phi}_h|_{E_1}$ if $\{\boldsymbol{\theta}_h\} \cdot \mathbf{n}_e < 0$, $\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}|_e = \boldsymbol{\phi}_h|_{E_2}$ if $\{\boldsymbol{\theta}_h\} \cdot \mathbf{n}_e > 0$, and $\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}|_e = \mathbf{0}$ if $\{\boldsymbol{\theta}_h\} \cdot \mathbf{n}_e = 0$. In a similar way, if $e \in \partial\Omega \cap E$, then we have $\mathbf{n}_e = \mathbf{n}_{\partial\Omega}$. Then, the contribution corresponding to e is

$$\int_e (\mathbf{v}_h \cdot \mathbf{n}_e)\mathbf{w} \cdot \boldsymbol{\phi}_h^{\boldsymbol{\theta}_h},$$

where $\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}|_e = \boldsymbol{\phi}_h|_E$ if $\boldsymbol{\theta}_h \cdot \mathbf{n}_e < 0$ and $\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}|_e = \mathbf{0}$ otherwise. Set $B = l^{\boldsymbol{\Theta}_h}(\mathbf{v}_h, \mathbf{w}, \boldsymbol{\phi}_h) - l^{\mathbf{v}_h}(\mathbf{v}_h, \mathbf{w}, \boldsymbol{\phi}_h)$. Then, following the above notation, B can be rewritten as

$$B = \sum_{e \in \Gamma_h} \int_e (\{\mathbf{v}_h\} \cdot \mathbf{n}_e)[\mathbf{w}] \cdot (\boldsymbol{\phi}_h^{\boldsymbol{\Theta}_h} - \boldsymbol{\phi}_h^{\mathbf{v}_h}).$$

The domain of integration can be partitioned as follows: $\Gamma_h = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, where

$$\mathcal{G}_1 = \{e : \{\boldsymbol{\Theta}_h\} \cdot \mathbf{n}_e \neq 0 \text{ and } \{\mathbf{v}_h\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\},$$
$$\mathcal{G}_2 = \{e : \{\boldsymbol{\Theta}_h\} \cdot \mathbf{n}_e = 0 \text{ and } \{\mathbf{v}_h\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\},$$
$$\mathcal{G}_3 = \Gamma_h \setminus (\mathcal{G}_1 \cup \mathcal{G}_2).$$

First we consider $\mathcal{G}_1$. For $e \in \mathcal{G}_1$, we decompose e into e_1 and e_2. e_1 is the part where $\{\boldsymbol{\Theta}_h\} \cdot \mathbf{n}_e$ and $\{\mathbf{v}_h\} \cdot \mathbf{n}_e$ have the same sign and e_2 part is for the opposite signs of $\{\boldsymbol{\Theta}_h\} \cdot \mathbf{n}_e$ and $\{\mathbf{v}_h\} \cdot \mathbf{n}_e$. On e_1, we then have $\boldsymbol{\phi}_h^{\boldsymbol{\Theta}_h} - \boldsymbol{\phi}_h^{\mathbf{v}_h} = 0$. On e_2, $\boldsymbol{\phi}_h^{\boldsymbol{\Theta}_h} - \boldsymbol{\phi}_h^{\mathbf{v}_h} = [\boldsymbol{\phi}_h]$, up to the sign. Using the fact of opposite signs, we can write

$$|\{\mathbf{v}_h\} \cdot \mathbf{n}_e| \leq |\{\boldsymbol{\Theta}_h - \mathbf{v}_h\} \cdot \mathbf{n}_e|.$$

Applying Hölder's inequality and trace inequality (1.37), we can deduce

$$\begin{aligned}
\left| \sum_{e \in \mathcal{G}_1} \int_e (\{\mathbf{v}_h\} \cdot \mathbf{n}_e)[\mathbf{w}] \cdot (\boldsymbol{\phi}_h^{\boldsymbol{\Theta}_h} - \boldsymbol{\phi}_h^{\mathbf{v}_h}) \right| &\leq \sum_{e \in \mathcal{G}_1} \|\{\boldsymbol{\Theta}_h - \mathbf{v}_h\}\|_{L^4(e)} \|[\mathbf{w}]\|_{L^2(e)} \|[\boldsymbol{\phi}_h]\|_{L^4(e)} \\
&\leq C \sum_{e \in \mathcal{G}_1} \sum_{i,j=1}^{2} \|\boldsymbol{\Theta}_h - \mathbf{v}_h\|_{L^4(E_i)} \sigma_e^{1/2} |e|^{-1/2} \|[\mathbf{w}]\|_{L^2(e)} \|\boldsymbol{\phi}_h\|_{L^4(E_j)} \\
&\leq C \|\boldsymbol{\Theta}_h - \mathbf{v}_h\|_{L^4(\Omega)} \|\mathbf{w}\|_{\varepsilon} \|\boldsymbol{\phi}_h\|_{L^4(\Omega)}.
\end{aligned} \tag{2.73}$$

Next, we consider $\mathcal{G}_2$. From the definition of $\mathcal{G}_2$, we have $|\{\mathbf{v}_h\} \cdot \mathbf{n}_e| \leq |\{\boldsymbol{\Theta}_h - \mathbf{v}_h\} \cdot \mathbf{n}_e|$. Therefore, in a similar fashion as above, we can show that

$$\left| \sum_{e \in \mathcal{G}_2} \int_e (\{\mathbf{v}_h\} \cdot \mathbf{n}_e)[\mathbf{w}] \cdot \boldsymbol{\phi}_h^{\mathbf{v}_h} \right| \leq C \|\boldsymbol{\Theta}_h - \mathbf{v}_h\|_{L^4(\Omega)} \|\mathbf{w}\|_{\varepsilon} \|\boldsymbol{\phi}_h\|_{L^4(\Omega)}. \tag{2.74}$$

There is zero contribution of $\mathcal{G}_3$ to B. The combination of the bounds (2.73)-(2.74) completes the proof of (2.60).

The derivation of estimate (2.61) is almost similar to the derivation of (2.60). In this case, the domain of integration is partitioned as: $\Gamma_h = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$, where

$$\begin{aligned}
\mathcal{P}_1 &= \{e : \{\mathbf{\Theta}_h\} \cdot \mathbf{n}_e \neq 0 \text{ and } \{\mathbf{v}\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\}, \\
\mathcal{P}_2 &= \{e : \{\mathbf{\Theta}_h\} \cdot \mathbf{n}_e = 0 \text{ and } \{\mathbf{v}\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\}, \\
\mathcal{P}_3 &= \Gamma_h \setminus (\mathcal{P}_1 \cup \mathcal{P}_2).
\end{aligned}$$

Thus, with a similar argument used in the derivation of (2.60), and employing Hölder's inequality, (1.14), (1.25) and (1.37), we arrive at

$$\begin{aligned}
&\left| \sum_{e \in \mathcal{P}_1} \int_e (\{\mathbf{v}\} \cdot \mathbf{n}_e)[\mathbf{w}] \cdot (\boldsymbol{\phi}_h^{\mathbf{\Theta}_h} - \boldsymbol{\phi}_h^{\mathbf{v}}) \right| \le \sum_{e \in \mathcal{P}_1} \|\{\mathbf{\Theta}_h - \mathbf{v}\}\|_{L^4(e)} \|[\mathbf{w}]\|_{L^2(e)} \|[\boldsymbol{\phi}_h]\|_{L^4(e)} \\
&\le C \sum_{e \in \mathcal{P}_1} \sum_{i,j=1}^{2} \left(\|\mathbf{\Theta}_h - \mathbf{v}\|_{L^4(E_i)} + h_{E_i}^{1/2} \|\nabla(\mathbf{\Theta}_h - \mathbf{v})\|_{L^2(E_i)} \right) \left(\frac{\sigma_e}{|e|} \right)^{1/2} \\
&\qquad \times \|[\mathbf{w}]\|_{L^2(e)} \|\boldsymbol{\phi}_h\|_{L^4(E_j)} \\
&\le C \|\mathbf{\Theta}_h - \mathbf{v}\|_{\varepsilon} \|\mathbf{w}\|_{\varepsilon} \|\boldsymbol{\phi}_h\|_{L^4(\Omega)}, \qquad (2.75)
\end{aligned}$$

and

$$\left| \sum_{e \in \mathcal{P}_2} \int_e (\{\mathbf{v}\} \cdot \mathbf{n}_e)[\mathbf{w}] \cdot \boldsymbol{\phi}_h^{\mathbf{v}} \right| \le C \|\mathbf{\Theta}_h - \mathbf{v}\|_{\varepsilon} \|\mathbf{w}\|_{\varepsilon} \|\boldsymbol{\phi}_h\|_{L^4(\Omega)}. \qquad (2.76)$$

With the bounds (2.75) and (2.76), we establish (2.61). This completes the rest of the proof. □

Before we proceed to the next section, we derive some regularity bounds for $\mathbf{u}_h$.

Lemma 2.8. *Let $0 < \alpha < \frac{\nu K_1}{2C_2}$. Then, for the semi-discrete DG velocity $\boldsymbol{u}_h(t)$, $t > 0$, the following holds true*

$$\|\boldsymbol{u}_h(t)\| + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\boldsymbol{u}_h(s)\|_{\varepsilon}^2 \, ds \le C. \qquad (2.77)$$

Moreover,

$$\limsup_{t \to \infty} \|\boldsymbol{u}_h(t)\|_{\varepsilon} \le \frac{C_2}{K_1 \nu} \|\mathbf{f}\|_{L^\infty(L^2(\Omega))}. \qquad (2.78)$$

Proof. Choose $\boldsymbol{\phi}_h = \mathbf{u}_h$ in (2.10), and apply Lemma 1.6, positivity property (1.19), estimate (1.14), the Cauchy-Schwarz inequality and Young's inequality to obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_h\|^2 + \nu K_1 \|\mathbf{u}_h\|_{\varepsilon}^2 \le \|\mathbf{f}\| \|\mathbf{u}_h\| \le \frac{\nu K_1}{2} \|\mathbf{u}_h\|_{\varepsilon}^2 + \frac{C_2^2}{2\nu K_1} \|\mathbf{f}\|^2. \qquad (2.79)$$

Multiplying (2.79) by $e^{2\alpha t}$, integrating from 0 to t and using (1.14), we find that

$$e^{2\alpha t}\|\mathbf{u}_h(t)\|^2 + (\nu K_1 - 2\alpha C_2)\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\,ds \le \|\mathbf{u}_h(0)\|^2 + C\int_0^t e^{2\alpha s}\|\mathbf{f}(s)\|^2\,ds. \tag{2.80}$$

Multiplying (2.80) by $e^{-2\alpha t}$ and using the fact that $e^{-2\alpha t}\int_0^t e^{2\alpha s}\,ds = \frac{1}{2\alpha}(1-e^{-2\alpha t})$ and choosing $\alpha < \frac{\nu K_1}{2C_2}$, we obtain (2.77). Again, Multiplying (2.79) by $e^{2\alpha t}$ and integrating from 0 to t, we obtain

$$e^{2\alpha t}\|\mathbf{u}_h(t)\|^2 + \nu K_1\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\,ds \le \|\mathbf{u}_h(0)\|^2 + 2\alpha\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|^2\,ds + (e^{2\alpha t}-1)\frac{C_2^2\|\mathbf{f}\|^2_{L^\infty(0,t;L^2(\Omega))}}{2\alpha\nu K_1}.$$

Now, multiplying the above inequality by $e^{-2\alpha t}$, taking limit supremum as $t \to \infty$ and noting that

$$\nu K_1 \limsup_{t\to\infty} e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\,ds = \frac{\nu K_1}{2\alpha}\limsup_{t\to\infty}\|\mathbf{u}_h(t)\|_\varepsilon^2,$$

we arrive at

$$\frac{\nu K_1}{2\alpha}\limsup_{t\to\infty}\|\mathbf{u}_h(t)\|_\varepsilon^2 \le \frac{C_2^2\|\mathbf{f}\|^2_{L^\infty(L^2(\Omega))}}{2\alpha\nu K_1},$$

which completes the proof of estimate (2.78). □

2.4 Error Estimates for Velocity

In this section, which is closely related to the Section 5 of [86], we derive the bounds of velocity error $\mathbf{e}(t) = \mathbf{u}(t) - \mathbf{u}_h(t)$, $t > 0$ in the energy and $\mathbf{L}^2$-norms for the DG set up. We start by analyzing the linearized error and therefore introduce the solution $\mathbf{v}_h \in \mathbf{V}_h$ of a DG approximation of a linearized (Stokes) problem, *that is*, $\mathbf{v}_h \in \mathbf{V}_h$ is the solution of

$$(\mathbf{v}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{v}_h, \boldsymbol{\phi}_h) = (\mathbf{f}, \boldsymbol{\phi}_h) - c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h. \tag{2.81}$$

With the help of $\mathbf{v}_h$, we split $\mathbf{e}$ into two parts as $\mathbf{e} = (\mathbf{u} - \mathbf{v}_h) + (\mathbf{v}_h - \mathbf{u}_h) = \boldsymbol{\xi} + \boldsymbol{\rho}$. Observe that $\boldsymbol{\xi}$ is the error committed by approximating a linearized (Stokes) problem and $\boldsymbol{\rho}$ represents the error due to the presence of the non-linearity in the problem (2.1). From (2.81) and (2.5), we have the equation of $\boldsymbol{\xi}$:

$$(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \nu\, a(\boldsymbol{\xi}, \boldsymbol{\phi}_h) = -b(\boldsymbol{\phi}_h, p), \quad \boldsymbol{\phi}_h \in \mathbf{V}_h. \tag{2.82}$$

Below, we establish $L^2(\mathbf{L}^2)$, $L^\infty(\mathbf{L}^2)$ estimates of $\boldsymbol{\xi}$ and $L^\infty(\mathbf{L}^2)$-estimates of $\boldsymbol{\rho}$ in Lemmas 2.9, 2.10 and 2.11, respectively. Proofs of these lemmas are similar to those of Lemmas 5.1, 5.2 and 5.4 of [86].

Lemma 2.9. *Suppose the assumption (**A1**) is satisfied and* $0 < \alpha < \frac{\nu K_1}{2C_2}$. *Further, let* $\mathbf{v}_h(t) \in \boldsymbol{V}_h$ *be a solution of (2.81) with initial condition* $\mathbf{v}_h(0) = \boldsymbol{P}_h \boldsymbol{u}_0$. *Then, for* $0 \le t < T$, *there exists a constant* $C > 0$, *such that, the following holds true*

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|^2\,ds \le Ch^{2r+2}.$$

Proof. Choosing $\boldsymbol{\phi}_h = \mathbf{P}_h\boldsymbol{\xi}$ in (2.82) and using Lemma 1.6, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\xi}\|^2 + \nu K_1\|\mathbf{P}_h\boldsymbol{\xi}\|_\varepsilon^2 &\le (\boldsymbol{\xi}_t, \mathbf{u} - \mathbf{P}_h\mathbf{u}) - \nu\, a(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{P}_h\boldsymbol{\xi}) \\ &\quad - b(\mathbf{P}_h\boldsymbol{\xi}, p - r_h(p)). \qquad (2.83)\end{aligned}$$

We first note that

$$(\boldsymbol{\xi}_t, \mathbf{u} - \mathbf{P}_h\mathbf{u}) = \frac{1}{2}\frac{d}{dt}\|\mathbf{u} - \mathbf{P}_h\mathbf{u}\|^2.$$

The term $b(\mathbf{P}_h\boldsymbol{\xi}, p - r_hp)$ can be bounded from Lemma 2.4 and Young's inequality as follows:

$$|b(\mathbf{P}_h\boldsymbol{\xi}, p - r_h(p))| \le \frac{K_1\nu}{6}\|\mathbf{P}_h\boldsymbol{\xi}\|_\varepsilon^2 + Ch^{2r}|p|_r^2.$$

Finally, the second term on the right hand side in (2.83) can be handled using Lemma 2.3 and Young's inequality as follows:

$$\nu|a(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{P}_h\boldsymbol{\xi})| \le \frac{K_1\nu}{6}\|\mathbf{P}_h\boldsymbol{\xi}\|_\varepsilon^2 + Ch^{2r}|\mathbf{u}|_{r+1}^2.$$

Combining all the above estimates in (2.83) and using triangle inequality with Lemma 2.2, we obtain

$$\frac{d}{dt}\|\boldsymbol{\xi}\|^2 + \nu K_1\|\boldsymbol{\xi}\|_\varepsilon^2 \le \frac{d}{dt}\|\mathbf{u} - \mathbf{P}_h\mathbf{u}\|^2 + Ch^{2r}(|\mathbf{u}|_{r+1}^2 + |p|_r^2). \qquad (2.84)$$

Multiplying (2.84) by $e^{2\alpha t}$ and using the L^p-estimate (1.14), we find

$$\begin{aligned}\frac{d}{dt}(e^{2\alpha t}\|\boldsymbol{\xi}\|^2) + (\nu K_1 - 2\alpha C_2)e^{2\alpha t}\|\boldsymbol{\xi}\|_\varepsilon^2 &\le \frac{d}{dt}(e^{2\alpha t}\|\mathbf{u} - \mathbf{P}_h\mathbf{u}\|^2) \\ &\quad + Ch^{2r}e^{2\alpha t}(|\mathbf{u}|_{r+1}^2 + |p|_r^2).\end{aligned}$$

By setting $0 < \alpha < \frac{\nu K_1}{2C_2}$, integrating from 0 to t and observing $\|\boldsymbol{\xi}(0)\|$ is of the order h^r, we obtain

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|_\varepsilon^2\,ds \le Ch^{2r}\int_0^t e^{2\alpha s}\big(|\mathbf{u}(s)|_{r+1}^2 + |p(s)|_r^2\big)\,ds. \tag{2.85}$$

To estimate L^2- norm error, we use the following duality argument [86]: For fixed $h > 0$ and $t \in (0,T)$, let $\mathbf{w}(s) \in \mathbf{J}_1$, $q(s) \in L^2(\Omega)/\mathbb{R}$, be the unique solution of the backward problem

$$\mathbf{w}_s + \nu\Delta\mathbf{w} - \nabla q = e^{2\alpha s}\boldsymbol{\xi}, \quad 0 \le s \le t, \tag{2.86}$$

with $\mathbf{w}(t) = 0$ satisfying

$$\int_0^t e^{-2\alpha s}(\|\Delta\mathbf{w}(s)\|^2 + \|\mathbf{w}_s(s)\|^2 + \|\nabla q(s)\|^2)\,ds \le C\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|^2\,ds. \tag{2.87}$$

Form L^2-inner product between (2.86) and $\boldsymbol{\xi}$ to obtain

$$\begin{aligned}
e^{2\alpha s}\|\boldsymbol{\xi}\|^2 =&(\boldsymbol{\xi},\mathbf{w}_s) - \sum_{E\in\mathcal{E}_h}\int_E \nu\nabla\boldsymbol{\xi} : \nabla\mathbf{w} + \sum_{E\in\mathcal{E}_h}\int_{\partial E}(\nu\nabla\mathbf{w}\mathbf{n}_E)\cdot\boldsymbol{\xi}\\
&+ \sum_{E\in\mathcal{E}_h}\int_E q\nabla\cdot\boldsymbol{\xi} - \sum_{E\in\mathcal{E}_h}\int_{\partial E} q\mathbf{n}_E\cdot\boldsymbol{\xi}\\
=&(\boldsymbol{\xi},\mathbf{w}_s) - \sum_{E\in\mathcal{E}_h}\int_E \nu\nabla\boldsymbol{\xi} : \nabla\mathbf{w} + \sum_{e\in\Gamma_h}\int_e \{\nu\nabla\mathbf{w}\}\mathbf{n}_e\cdot[\boldsymbol{\xi}] - b(\boldsymbol{\xi},q).
\end{aligned} \tag{2.88}$$

Using (2.82) with $\boldsymbol{\phi}_h = \mathbf{P}_h\mathbf{w}$ and (2.88), we obtain

$$\begin{aligned}
e^{2\alpha s}\|\boldsymbol{\xi}\|^2 =& (\boldsymbol{\xi},\mathbf{w}_s) + (\boldsymbol{\xi}_s,\mathbf{P}_h\mathbf{w}) - \sum_{E\in\mathcal{E}_h}\int_E \nu\nabla\boldsymbol{\xi} : \nabla(\mathbf{w}-\mathbf{P}_h\mathbf{w})\\
&+ (1+\epsilon)\sum_{e\in\Gamma_h}\int_e \{\nu\nabla\mathbf{w}\}\mathbf{n}_e\cdot[\boldsymbol{\xi}]\\
&- \epsilon\sum_{e\in\Gamma_h}\int_e \{\nu\nabla(\mathbf{w}-\mathbf{P}_h\mathbf{w})\}\mathbf{n}_e\cdot[\boldsymbol{\xi}] - \sum_{e\in\Gamma_h}\int_e \{\nu\nabla\boldsymbol{\xi}\}\mathbf{n}_e\cdot[\mathbf{P}_h\mathbf{w}]\\
&+ \nu\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\int_e [\boldsymbol{\xi}]\cdot[\mathbf{P}_h\mathbf{w}] - b(\boldsymbol{\xi},q-r_h(q)) + b(\mathbf{P}_h\mathbf{w},p_h-p).
\end{aligned} \tag{2.89}$$

Consider $\epsilon = -1$. Using the Cauchy-Schwarz inequality, (1.24), Lemmas 1.10 and 2.2, (1.31) and the fact that $[\mathbf{w}] = 0$, we easily obtain

$$\Big|\sum_{E\in\mathcal{E}_h}\int_E \nu\nabla\boldsymbol{\xi} : \nabla(\mathbf{w}-\mathbf{P}_h\mathbf{w}) - \sum_{e\in\Gamma_h}\int_e \{\nu\nabla(\mathbf{w}-\mathbf{P}_h\mathbf{w})\}\mathbf{n}_e\cdot[\boldsymbol{\xi}]$$
$$+ \sum_{e\in\Gamma_h}\int_e \{\nu\nabla\boldsymbol{\xi}\}\mathbf{n}_e\cdot[\mathbf{P}_h\mathbf{w}] - \nu\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\int_e [\boldsymbol{\xi}]\cdot[\mathbf{P}_h\mathbf{w}] + b(\boldsymbol{\xi},q-r_h(q))$$

$$- b(\mathbf{P}_h\mathbf{w}, r_h p - p)\Big|$$
$$\le Ch(\|\mathbf{w}\|_2 + \|q\|_1)\|\boldsymbol{\xi}\|_\varepsilon + Ch^{r+1}|p|_r\|\mathbf{w}\|_2.$$

Using the definition of $\mathbf{P}_h$, we rewrite

$$\begin{aligned}(\boldsymbol{\xi}, \mathbf{w}_s) + (\boldsymbol{\xi}_s, \mathbf{P}_h\mathbf{w}) &= \frac{d}{ds}(\boldsymbol{\xi}, \mathbf{w}) - (\boldsymbol{\xi}_s, \mathbf{w} - \mathbf{P}_h\mathbf{w}) \\ &= \frac{d}{ds}(\boldsymbol{\xi}, \mathbf{w}) - \frac{d}{ds}(\boldsymbol{\xi}, \mathbf{w} - \mathbf{P}_h\mathbf{w}) + (\mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{w}_s).\end{aligned}$$

Now from (2.89), using the Cauchy-Schwarz and Young's inequalities, we find

$$\begin{aligned}e^{2\alpha s}\|\boldsymbol{\xi}\|^2 \le \frac{d}{ds}(\boldsymbol{\xi}, \mathbf{P}_h\mathbf{w}) + \delta e^{-2\alpha s}(\|\mathbf{w}\|_2^2 + \|\mathbf{w}_s\|^2 + \|q\|_1^2) + C\delta^{-1}h^2 e^{2\alpha s}\|\boldsymbol{\xi}\|_\varepsilon \\ + C\delta^{-1}h^{2r+2}e^{2\alpha s}(|\mathbf{u}|_{r+1}^2 + |p|_r^2).\end{aligned} \tag{2.90}$$

On integrating (2.90) with respect to s from 0 to t and using (2.87), we obtain the following estimate

$$\begin{aligned}\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|^2 \le (\boldsymbol{\xi}(t), \mathbf{P}_h\mathbf{w}(t)) - (\boldsymbol{\xi}(0), \mathbf{P}_h\mathbf{w}(0)) + Ch^2\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|_\varepsilon^2 \\ + \delta\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|^2 + Ch^{2r+2}\int_0^t e^{2\alpha s}(|\mathbf{u}(s)|_{r+1}^2 + |p(s)|_r^2).\end{aligned}$$

Choosing δ appropriately, we then apply the estimate (2.85) and assumption (**A1**) to complete the rest of the proof. □

Remark 2.2. *For the NIPG case, similar to the Remark 2.1, the fourth term on the right hand side of (2.89) can be bounded as*

$$\Big|\sum_{e\in\Gamma_h}\int_e \{\nu\nabla \boldsymbol{w}\}\boldsymbol{n}_e \cdot [\boldsymbol{\xi}]\Big| \le C\|\boldsymbol{w}\|_2\|\boldsymbol{\xi}\|_\varepsilon$$

which implies that

$$\int_0^t e^{2\alpha s}(\|\boldsymbol{\xi}(s)\|^2 + \|\boldsymbol{\xi}(s)\|_\varepsilon^2)\, ds \le Ch^{2r}.$$

Thus, in the case of NIPG method, the estimate is sub-optimal. This also true for the IIPG method.

For optimal error estimates of $\boldsymbol{\xi}$ in $L^\infty(\mathbf{L}^2)$-norm, we decompose it as follows:

$$\boldsymbol{\xi} = (\mathbf{u} - \mathbf{S}_h\mathbf{u}) + (\mathbf{S}_h\mathbf{u} - \mathbf{v}_h) = \boldsymbol{\zeta} + \boldsymbol{\theta}.$$

Since the estimates of $\boldsymbol{\zeta} = \mathbf{u} - \mathbf{S}_h\mathbf{u}$ are known from Lemma 2.5, it is sufficient to estimate $\boldsymbol{\theta}$, which would allow us to draw the following conclusion.

Lemma 2.10. *Under the assumptions of Lemma 2.9,* $\boldsymbol{\xi}$ *satisfies the following estimates:*

$$\|\boldsymbol{\xi}(t)\| + h\|\boldsymbol{\xi}(t)\|_{\varepsilon} \leq Ch^{r+1}, \quad 0 \leq t \leq T.$$

Proof. From the equations (2.82) and (2.23), we obtain the equation in $\boldsymbol{\theta}$ as

$$(\boldsymbol{\theta}_t, \boldsymbol{\phi}_h) + \nu\, a(\boldsymbol{\theta}, \boldsymbol{\phi}_h) = -(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h.$$

Setting $\boldsymbol{\phi}_h = \boldsymbol{\theta}$ in the above equation and using Lemma 1.6, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\theta}\|^2 + \nu K_1 \|\boldsymbol{\theta}\|_{\varepsilon}^2 \leq -(\boldsymbol{\zeta}_t, \boldsymbol{\theta}).$$

Multiply by $e^{2\alpha t}$, integrate the resulting inequality with respect to time, and apply the Cauchy-Schwarz inequality and Young's inequality to obtain

$$\begin{aligned} e^{2\alpha t}\|\boldsymbol{\theta}(t)\|^2 + \nu K_1 \int_0^t e^{2\alpha s}\|\boldsymbol{\theta}(s)\|_{\varepsilon}^2 ds &\leq C\int_0^t e^{2\alpha s}\big(\|\boldsymbol{\theta}(s)\|^2 + \|\boldsymbol{\zeta}_s(s)\|^2\big)\, ds \\ &\leq C\int_0^t e^{2\alpha s}\big(\|\boldsymbol{\zeta}(s)\|^2 + \|\boldsymbol{\xi}(s)\|^2 + \|\boldsymbol{\zeta}_s(s)\|^2\big)\, ds. \end{aligned} \tag{2.91}$$

Using the estimates (2.24), (2.25), Lemma 2.9 and assumption (**A1**) in (2.91), we observe that

$$\|\boldsymbol{\theta}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\theta}(s)\|_{\varepsilon}^2\, ds \leq Ch^{2r+2}.$$

By using the inverse relation (1.38), we now conclude that

$$\|\boldsymbol{\theta}(t)\|^2 + h^2\|\boldsymbol{\theta}(t)\|_{\varepsilon}^2 \leq Ch^{2r+2}.$$

This along with (2.24) and assumption (**A1**) give us the desired result. □

Remark 2.3. *With the help of Remarks 2.1 and 2.2, for NIPG and IIPG cases, we can show that*

$$\| \boldsymbol{\xi}(t) \| + h \| \boldsymbol{\xi}(t) \|_{\varepsilon} \leq Ch^{r}, \quad 0 \leq t \leq T.$$

The following lemma provides the estimates for $\boldsymbol{\rho} = \mathbf{v}_h - \mathbf{u}_h$.

Lemma 2.11. *Under the assumptions of Lemma 2.9,* $\boldsymbol{\rho}$ *satisfies*

$$\| \boldsymbol{\rho}(t) \| + h \| \boldsymbol{\rho}(t) \|_{\varepsilon} \leq K(t)\, h^{r+1}, \quad 0 \leq t \leq T,$$

where $K(t) > 0$ *grows exponentially in time.*

Proof. From the equations (2.81) and (2.10), satisfied by $\mathbf{v}_h$ and $\mathbf{u}_h$, respectively, we obtain

$$(\boldsymbol{\rho}_t,\, \boldsymbol{\phi}_h) + \nu\, a\,(\boldsymbol{\rho},\, \boldsymbol{\phi}_h) = c^{\mathbf{u}_h}\,(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - c^{\mathbf{u}}\,(\mathbf{u},\, \mathbf{u},\, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h.$$

Setting $\boldsymbol{\phi}_h = \boldsymbol{\rho}$ and using Lemma 1.6, we find

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\rho}\|^2 + \nu K_1 \|\boldsymbol{\rho}\|_\varepsilon^2 \leq c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\rho}) - c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\rho}). \tag{2.92}$$

We first note that since $\mathbf{u}$ is continuous, we can rewrite

$$c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\rho}) = c^{\mathbf{u}_h}(\mathbf{u}, \mathbf{u}, \boldsymbol{\rho}).$$

Let us now rewrite the nonlinear terms as follows:

$$\begin{aligned} c^{\mathbf{u}_h}(\mathbf{u}, \mathbf{u}, \boldsymbol{\rho}) - c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\rho}) =\, & c^{\mathbf{u}_h}(\mathbf{u}_h, \boldsymbol{\rho}, \boldsymbol{\rho}) + c^{\mathbf{u}_h}(\boldsymbol{\rho}, \mathbf{u}, \boldsymbol{\rho}) - c^{\mathbf{u}_h}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\rho}) - c^{\mathbf{u}_h}(\boldsymbol{\rho}, \boldsymbol{\xi}, \boldsymbol{\rho}) \\ & + c^{\mathbf{u}_h}(\boldsymbol{\xi}, \mathbf{u}, \boldsymbol{\rho}) + c^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}) + l^{\mathbf{u}_h}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}) - l^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}) \\ \geq\, & c^{\mathbf{u}_h}(\boldsymbol{\rho}, \mathbf{u}, \boldsymbol{\rho}) - c^{\mathbf{u}_h}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\rho}) - c^{\mathbf{u}_h}(\boldsymbol{\rho}, \boldsymbol{\xi}, \boldsymbol{\rho}) + c^{\mathbf{u}_h}(\boldsymbol{\xi}, \mathbf{u}, \boldsymbol{\rho}) \\ & + c^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}) + l^{\mathbf{u}_h}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}) - l^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}). \end{aligned} \tag{2.93}$$

The first term is non-negative and is dropped, following (1.19). To bound the rest of the terms, we proceed as follows. A use of estimate (2.54), Young's inequality and Lemma 1.3 implies

$$|c^{\mathbf{u}_h}(\boldsymbol{\rho}, \mathbf{u}, \boldsymbol{\rho})| \leq C\|\boldsymbol{\rho}\||\mathbf{u}|_{1,4,\Omega}\|\boldsymbol{\rho}\|_\varepsilon \leq \frac{K_1\nu}{64}\|\boldsymbol{\rho}\|_\varepsilon^2 + C\|\mathbf{u}\|_2^2\|\boldsymbol{\rho}\|^2. \tag{2.94}$$

Next term is bounded using estimate (2.56) and Young's inequality as follows

$$|c^{\mathbf{u}_h}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\rho})| \leq \frac{K_1\nu}{16}\|\boldsymbol{\rho}\|_\varepsilon^2 + C\|\boldsymbol{\xi}\|_\varepsilon^4. \tag{2.95}$$

In a similar fashion, the third term on the right hand side of (2.93) is bounded using (2.56), inverse inequality from Lemma 1.10 and Young's inequality as

$$|c^{\mathbf{u}_h}(\boldsymbol{\rho}, \boldsymbol{\xi}, \boldsymbol{\rho})| \leq \frac{K_1\nu}{16}\|\boldsymbol{\rho}\|_\varepsilon^2 + Ch^{-2}\|\boldsymbol{\xi}\|_\varepsilon^2\|\boldsymbol{\rho}\|^2. \tag{2.96}$$

Employing (2.59) and Young's inequality, we bound the fourth nonlinear term on the right hand side of (2.93) as

$$|c^{\mathbf{u}_h}(\boldsymbol{\xi}, \mathbf{u}, \boldsymbol{\rho})| \leq \frac{K_1\nu}{64}\|\boldsymbol{\rho}\|_\varepsilon^2 + C\,\|\mathbf{u}\|_2^2(\|\boldsymbol{\xi}\|^2 + h^2\|\boldsymbol{\xi}\|_\varepsilon^2). \tag{2.97}$$

An application of (2.58) and Young's inequality leads to the bound of the fifth nonlinear term on the right hand side of (2.93):

$$|c^{\mathbf{u}}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\rho})| \leq \frac{K_1\nu}{64}\|\boldsymbol{\rho}\|_{\varepsilon}^2 + C\|\mathbf{u}\|_2^2(\|\boldsymbol{\xi}\|^2 + h^2\|\boldsymbol{\xi}\|_{\varepsilon}^2). \tag{2.98}$$

Finally, utilizing triangle inequality, (2.61), and inequalities (1.37), (1.38) and (1.14), we arrive at the following bound:

$$\begin{aligned} |l^{\mathbf{u}_h}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\rho}) - l^{\mathbf{u}}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\rho})| &\leq C\|\mathbf{u}-\mathbf{u}_h\|_{\varepsilon}\|\boldsymbol{\xi}\|_{\varepsilon}\|\boldsymbol{\rho}\|_{\varepsilon} \leq C(\|\boldsymbol{\xi}\|_{\varepsilon} + \|\boldsymbol{\rho}\|_{\varepsilon})\|\boldsymbol{\xi}\|_{\varepsilon}\|\boldsymbol{\rho}\|_{\varepsilon} \\ &\leq \frac{K_1\nu}{64}\|\boldsymbol{\rho}\|_{\varepsilon}^2 + Ch^{-2}\|\boldsymbol{\xi}\|_{\varepsilon}^2\|\boldsymbol{\rho}\|^2 + C\|\boldsymbol{\xi}\|_{\varepsilon}^4. \end{aligned} \tag{2.99}$$

Incorporating (2.94)-(2.99) in (2.93), and thereby in (2.92), and multiplying by $e^{2\alpha t}$ the resulting inequality, we observe that

$$\begin{aligned} \frac{d}{dt}(e^{2\alpha t}\|\boldsymbol{\rho}\|^2) + (\nu K_1 - 2C_2\alpha)e^{2\alpha t}\|\boldsymbol{\rho}\|_{\varepsilon}^2 &\leq C(\|\mathbf{u}\|_2^2 + h^{-2}\|\boldsymbol{\xi}\|_{\varepsilon}^2)e^{2\alpha t}\|\boldsymbol{\rho}\|^2 \\ &+ Ce^{2\alpha t}\|\mathbf{u}\|_2^2(\|\boldsymbol{\xi}\|^2 + h^2\|\boldsymbol{\xi}\|_{\varepsilon}^2) + Ce^{2\alpha t}\|\boldsymbol{\xi}\|_{\varepsilon}^4. \end{aligned} \tag{2.100}$$

Integrating (2.100) from 0 to t, using $\boldsymbol{\rho}(0) = \mathbf{0}$ and Gronwall's inequality, (2.85), Lemma 2.10 and assumption (**A1**) in the resulting expression, we arrive at

$$e^{2\alpha t}\|\boldsymbol{\rho}(t)\|^2 + \int_0^t e^{2\alpha s}\|\boldsymbol{\rho}(s)\|_{\varepsilon}^2\, ds \leq K(t)h^{2r+2}.$$

After multiplying the resulting inequality by $e^{-2\alpha t}$ and using the inverse relation (1.38), we obtain our desired estimate. This completes the proof. □

In Theorem 2.1 below, we state one of the main results of this chapter, related to the semi-discrete velocity error estimates.

Theorem 2.1. *Suppose the assumption (**A1**) holds true and let $0 < \alpha < \frac{\nu K_1}{2C_2}$. In addition, let the semi-discrete initial velocity $\boldsymbol{u}_h(0) \in \boldsymbol{V}_h$ with $\boldsymbol{u}_h(0) = \boldsymbol{P}_h\boldsymbol{u}_0$. Then, there exists a constant $K > 0$ such that for $0 < t \leq T$,*

$$\|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\| + h\|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\|_{\varepsilon} \leq K(t)h^{r+1},$$

where $K(t)$ grows exponentially in time.

Proof. The proof of Theorem 2.1 follows by virtue of the Lemmas 2.10 and 2.11. □

We would like to point out here that in the NIPG and IIPG cases, the estimates of $\boldsymbol{\rho}$ are sub-optimal as they involve the estimates of $\boldsymbol{\xi}$ in both energy and $\mathbf{L}^2$-norms, which are already shown as suboptimal in Remark 2.3. Since $\mathbf{e} = \boldsymbol{\xi} + \boldsymbol{\rho}$, we obtain the sub-optimal estimates of semi-discrete velocity error in the NIPG and IIPG cases.

Remark 2.4. *Under the smallness condition on the data, that is,*

$$N = \sup_{\mathbf{v}_h, \boldsymbol{w}_h, \boldsymbol{z}_h \in \boldsymbol{V}_h} \frac{c^{\boldsymbol{z}_h}(\boldsymbol{w}_h, \mathbf{v}_h, \boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_\varepsilon^2 \|\mathbf{v}_h\|_\varepsilon} \quad \text{and} \quad \frac{2NC_2}{K_1^2 \nu^2} \|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\Omega))} < 1, \tag{2.101}$$

the bounds of Theorem 2.1 are uniform in time, that is,

$$\| (\boldsymbol{u} - \boldsymbol{u}_h)(t) \| + h \| (\boldsymbol{u} - \boldsymbol{u}_h)(t) \|_\varepsilon \leq C h^{r+1},$$

where the constant $C > 0$ is independent of time t.

Proof. In order to derive the estimates, which are valid uniformly for all $t > 0$, let us first rewrite the nonlinear terms:

$$\begin{aligned} c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\rho}) - c^{\mathbf{u}_h}(\mathbf{u}, \mathbf{u}, \boldsymbol{\rho}) &= -c^{\mathbf{u}_h}(\mathbf{u}_h, \boldsymbol{\rho}, \boldsymbol{\rho}) + c^{\mathbf{u}_h}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\rho}) - c^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}) \\ &\quad - c^{\mathbf{u}_h}(\boldsymbol{\xi}, \mathbf{u}, \boldsymbol{\rho}) - c^{\mathbf{u}_h}(\boldsymbol{\rho}, \mathbf{v}_h, \boldsymbol{\rho}) + l^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}) - l^{\mathbf{u}_h}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}). \end{aligned} \tag{2.102}$$

From the proof of the Lemma 2.11, we can derive the bounds as

$$\begin{aligned} |c^{\mathbf{u}_h}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\rho}) + c^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}) + c^{\mathbf{u}_h}(\boldsymbol{\xi}, \mathbf{u}, \boldsymbol{\rho})| &\leq C\|\mathbf{u}\|_2(\|\boldsymbol{\xi}\| + h\|\boldsymbol{\xi}\|_\varepsilon)\|\boldsymbol{\rho}\|_\varepsilon \\ &\quad + C\|\boldsymbol{\xi}\|_\varepsilon^2 \|\boldsymbol{\rho}\|_\varepsilon. \end{aligned} \tag{2.103}$$

From the inequality (1.19), we have

$$c^{\mathbf{u}_h}(\mathbf{u}_h, \boldsymbol{\rho}, \boldsymbol{\rho}) \geq 0. \tag{2.104}$$

For the last three terms on the right hand side of (2.102), apply triangle inequality, follow the proof of (2.61), the fact $[\boldsymbol{\xi}] = [\mathbf{v}_h]$, the first part of the condition (2.101), (1.14) and Lemma 1.3 to derive

$$|c^{\mathbf{u}_h}(\boldsymbol{\rho}, \mathbf{v}_h, \boldsymbol{\rho})| \leq N\|\mathbf{v}_h\|_\varepsilon \|\boldsymbol{\rho}\|_\varepsilon^2, \tag{2.105}$$

$$|l^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho}) - l^{\mathbf{u}_h}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\rho})| \leq Ch\|\mathbf{u}\|_2 \|\boldsymbol{\xi}\|_\varepsilon \|\boldsymbol{\rho}\|_\varepsilon + C\|\boldsymbol{\xi}\|_\varepsilon^2 \|\boldsymbol{\rho}\|_\varepsilon + N\|\mathbf{v}_h\|_\varepsilon \|\boldsymbol{\rho}\|_\varepsilon^2. \tag{2.106}$$

Now, the proof of Lemma 2.11 is modified in the following manner: From (2.92), and combining (2.102)-(2.106), using Lemma 2.10 and assumption (**A1**), we obtain

$$\frac{d}{dt}\|\boldsymbol{\rho}\|^2 + 2(\nu K_1 - 2N\|\mathbf{v}_h\|_\varepsilon)\|\boldsymbol{\rho}\|_\varepsilon^2 \leq Ch^{r+1}\|\boldsymbol{\rho}\|_\varepsilon. \tag{2.107}$$

To derive a bound for $\|\mathbf{v}_h(t)\|_\varepsilon$ when $t \to \infty$, we rewrite (2.81) as follows

$$(\mathbf{v}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{v}_h, \boldsymbol{\phi}_h) + c^{\mathbf{u}}(\mathbf{u}, \mathbf{v}_h, \boldsymbol{\phi}_h) = (\mathbf{f}, \boldsymbol{\phi}_h) - c^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\phi}_h) \quad \forall\, \boldsymbol{\phi}_h \in \mathbf{V}_h.$$

Choose $\boldsymbol{\phi}_h = \mathbf{v}_h$ in the above equation, and apply Lemma 1.6, (1.19), (1.14), (2.58), the Cauchy-Schwarz and Young's inequalities to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}_h\|^2 + \nu K_1\|\mathbf{v}_h\|_\varepsilon^2 &\leq (C_2\|\mathbf{f}\| + C\|\mathbf{u}\|_2\|\boldsymbol{\xi}\| + Ch\|\mathbf{u}\|_2\|\boldsymbol{\xi}\|_\varepsilon)\|\mathbf{v}_h\|_\varepsilon \\ &\leq \frac{\nu K_1}{2}\|\mathbf{v}_h\|_\varepsilon^2 + \frac{C_2^2}{2\nu K_1}\|\mathbf{f}\|^2 + C\|\mathbf{u}\|_2^2\|\boldsymbol{\xi}\|^2 + Ch^2\|\mathbf{u}\|_2^2\|\boldsymbol{\xi}\|_\varepsilon^2.\end{aligned}$$

Multiply the above inequality by $e^{2\alpha t}$, integrate with respect to time from 0 to t, and employ Lemma 2.10 and assumption (**A1**) to arrive at

$$\begin{aligned}e^{2\alpha t}\|\mathbf{v}_h(t)\|^2 + \nu K_1\int_0^t e^{2\alpha s}\|\mathbf{v}_h(s)\|_\varepsilon^2\, ds &\leq \|\mathbf{v}_h(0)\|^2 + 2\alpha\int_0^t e^{2\alpha s}\|\mathbf{v}_h(s)\|^2\, ds \\ &\quad + (e^{2\alpha t}-1)\frac{C_2^2\|\mathbf{f}\|^2_{L^\infty(0,t;L^2(\Omega))}}{2\alpha\nu K_1} + \frac{C(e^{2\alpha t}-1)h^{2r+2}}{2\alpha}.\end{aligned}$$

Dividing both sides by $e^{2\alpha t}$, taking limit supremum as $t \to \infty$ and noting that

$$\nu K_1 \limsup_{t\to\infty} e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{v}_h(s)\|_\varepsilon^2\, ds = \frac{\nu K_1}{2\alpha}\limsup_{t\to\infty}\|\mathbf{v}_h(t)\|_\varepsilon^2,$$

we arrive at

$$\frac{\nu K_1}{2\alpha}\limsup_{t\to\infty}\|\mathbf{v}_h(t)\|_\varepsilon^2 \leq \frac{C_2^2\|\mathbf{f}\|^2_{L^\infty(0,\infty;L^2(\Omega))}}{2\alpha\nu K_1} + \frac{Ch^{2r+2}}{2\alpha}.$$

The above relation leads to

$$\limsup_{t\to\infty}\|\mathbf{v}_h(t)\|_\varepsilon^2 \leq \frac{C_2^2\|\mathbf{f}\|^2_{L^\infty(0,\infty;L^2(\Omega))}}{\nu^2 K_1^2} + Ch^{2r+2}. \tag{2.108}$$

Now, multiply (2.107) by $e^{2\alpha t}$ and integrate from 0 to t. After a final multiplication of the resulting equation by $e^{-2\alpha t}$, we arrive at

$$\begin{aligned}\|\boldsymbol{\rho}(t)\|^2 + 2e^{-2\alpha t}\int_0^t e^{2\alpha s}(\nu K_1 - 2N\|\mathbf{v}_h\|_\varepsilon)\|\boldsymbol{\rho}(s)\|_\varepsilon^2\, ds &\leq e^{-2\alpha t}\|\boldsymbol{\rho}(0)\|^2 \\ &\quad + 2\alpha e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\rho}(s)\|^2\, ds + Ch^{r+1}e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\rho}(s)\|_\varepsilon ds.\end{aligned}$$

Take $t \to \infty$, apply L'Hôpital's rule and use (2.108) to find

$$\begin{aligned}&\limsup_{t\to\infty}\|\boldsymbol{\rho}(t)\|^2 + \frac{1}{\alpha}\big(\nu K_1 - \frac{2NC_2}{K_1\nu}\|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\Omega))}\big)\limsup_{t\to\infty}\|\boldsymbol{\rho}(t)\|_\varepsilon^2 \\ &\leq \limsup_{t\to\infty}\|\boldsymbol{\rho}(t)\|^2 + \frac{Ch^{r+1}}{\alpha}\limsup_{t\to\infty}\|\boldsymbol{\rho}(t)\|_\varepsilon + \frac{Ch^{r+1}}{\alpha}\limsup_{t\to\infty}\|\boldsymbol{\rho}(t)\|_\varepsilon^2.\end{aligned} \tag{2.109}$$

Use the triangle inequality

$$\limsup_{t\to\infty} \|\boldsymbol{\rho}(t)\|_{\varepsilon} \leq \limsup_{t\to\infty} \|\mathbf{v}_h(t)\|_{\varepsilon} + \limsup_{t\to\infty} \|\mathbf{u}_h(t)\|_{\varepsilon},$$

(2.78), (2.108) and smallness condition (2.101) in (2.109) to find

$$\limsup_{t\to\infty} \|\boldsymbol{\rho}(t)\|_{\varepsilon} \leq Ch^{r+1}.$$

Therefore, from (1.14), we have

$$\limsup_{t\to\infty} \|\boldsymbol{\rho}(t)\| \leq Ch^{r+1}.$$

Together with the estimate of $\boldsymbol{\xi}$ from Lemma 2.10, we arrive at

$$\limsup_{t\to\infty} \big(\|\mathbf{u}(t) - \mathbf{u}_h(t)\| + h\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{\varepsilon}\big) \leq Ch^{r+1}.$$

□

2.5 Error Estimates for Pressure

In this section, we derive error estimates for the semi-discrete DG approximation of the pressure. This section is closely related to the Section 6 of [86]. Firstly, we establish some auxiliary estimates in Lemmas 2.12 and 2.14, which will be needed for proving our main result in Theorem 2.2. The proofs of these lemmas follow similar analytical ideas as applied in Lemmas 6.2 and 6.3 of [86].

Lemma 2.12. *Suppose the assumption (*$\mathbf{A1}$*) is satisfied and* $0 < \alpha < \frac{\nu K_1}{2C_2}$*. Then, there exists a constant* $K > 0$ *such that, the velocity error* $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$ *satisfies, for* $0 < t \leq T$*,*

$$\int_0^t e^{2\alpha s} \|\boldsymbol{e}_s(s)\|^2 \, ds \;\leq\; K(t) h^{2r}. \tag{2.110}$$

Proof. Let us denote $\boldsymbol{\chi} = \mathbf{S}_h\mathbf{u} - \mathbf{u}_h$. From the equations for $\mathbf{u}$, $\mathbf{u}_h$ and $\mathbf{S}_h\mathbf{u}$, that is, (2.5), (2.10) and (2.23), respectively, and for $\boldsymbol{\phi}_h \in \mathbf{V}_h$, we obtain

$$(\boldsymbol{\chi}_t, \boldsymbol{\phi}_h) + \nu\, a(\boldsymbol{\chi}, \boldsymbol{\phi}_h) + c^{\mathbf{u}}\,(\mathbf{u},\, \mathbf{u},\, \boldsymbol{\phi}_h) - c^{\mathbf{u}_h}\,(\mathbf{u}_h,\, \mathbf{u}_h,\, \boldsymbol{\phi}_h) = -(\boldsymbol{\zeta}_t,\, \boldsymbol{\phi}_h).$$

Choose $\boldsymbol{\phi}_h = \boldsymbol{\chi}_t$ in the above equality to obtain

$$\|\boldsymbol{\chi}_t\|^2 + \frac{\nu}{2}\frac{d}{dt}(a(\boldsymbol{\chi}, \boldsymbol{\chi})) + c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\chi}_t) - c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\chi}_t) = -(\boldsymbol{\zeta}_t, \boldsymbol{\chi}_t). \tag{2.111}$$

We can drop the superscripts from $c(\cdot,\cdot,\cdot)$ and the nonlinear terms can be rewritten as

$$c^{\mathbf{u}}\left(\mathbf{u},\,\mathbf{u},\,\boldsymbol{\chi}_t\right)-c^{\mathbf{u}_h}\left(\mathbf{u}_h,\,\mathbf{u}_h,\,\boldsymbol{\chi}_t\right)=-c\left(\mathbf{e},\,\mathbf{e},\,\boldsymbol{\chi}_t\right)+c\left(\mathbf{e},\,\mathbf{u},\,\boldsymbol{\chi}_t\right)+c\left(\mathbf{u},\,\mathbf{e},\,\boldsymbol{\chi}_t\right).$$

Use estimate (2.56), then (1.35) and (1.38) of Lemma 1.10, Theorem 2.1 and Young's inequality to bound $c(\mathbf{e},\mathbf{e},\boldsymbol{\chi}_t)$.

$$\begin{aligned}|c\left(\mathbf{e},\,\mathbf{e},\,\boldsymbol{\chi}_t\right)| &\le C\,\|\mathbf{e}\|_\varepsilon^2\,\|\boldsymbol{\chi}_t\|_\varepsilon \le Ch^{-1}\|\mathbf{e}\|_\varepsilon^2\,\|\boldsymbol{\chi}_t\| \le C\|\mathbf{e}\|_\varepsilon\,\|\boldsymbol{\chi}_t\|\\ &\le \frac{1}{6}\|\boldsymbol{\chi}_t\|^2+C\|\mathbf{e}\|_\varepsilon^2.\end{aligned} \tag{2.112}$$

Since $\mathbf{u}$ is continuous, Lemmas 1.3 and 1.10, Hölder's inequality, Young's inequality, and assumption (**A1**) yield

$$\begin{aligned}|c(\mathbf{e},\mathbf{u},\boldsymbol{\chi}_t)| &= \Big|\sum_{E\in\mathcal{E}_h}\int_E(\mathbf{e}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\chi}_t+\frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E(\nabla\cdot\mathbf{e})\mathbf{u}\cdot\boldsymbol{\chi}_t\\ &\quad -\frac{1}{2}\sum_{e\in\Gamma_h}\int_e[\mathbf{e}]\cdot\mathbf{n}_e\left\{\mathbf{u}\cdot\boldsymbol{\chi}_t\right\}\Big|\\ &\le \sum_{E\in\mathcal{E}_h}\|\mathbf{e}\|_{L^4(E)}\|\nabla\mathbf{u}\|_{L^4(E)}\|\,\boldsymbol{\chi}_t\|_{L^2(E)}+C\|\mathbf{u}\|_{L^\infty(\Omega)}\sum_{E\in\mathcal{E}_h}\|\nabla\mathbf{e}\|_{L^2(E)}\,\|\boldsymbol{\chi}_t\|_{L^2(E)}\\ &\quad +C\,\|\mathbf{u}\|_{L^\infty(\Omega)}\left(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\,\|[\mathbf{e}]\|_{L^2(e)}^2\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h}\|\boldsymbol{\chi}_t\|_{L^2(E)}^2\right)^{1/2}\\ &\le \frac{1}{6}\|\boldsymbol{\chi}_t\|^2+C\|\mathbf{e}\|_\varepsilon^2.\end{aligned} \tag{2.113}$$

Similarly, we can bound

$$|c(\mathbf{u},\mathbf{e},\boldsymbol{\chi}_t)| \le \frac{1}{6}\|\boldsymbol{\chi}_t\|^2+C\|\mathbf{e}\|_\varepsilon^2. \tag{2.114}$$

Apply (2.112)–(2.114) in (2.111) and multiply the resulting inequality by $e^{2\alpha t}$. Then, integrating from 0 to t with respect to time and applying Lemmas 1.6, 1.7, we obtain

$$\begin{aligned}\int_0^t e^{2\alpha s}\|\boldsymbol{\chi}_s(s)\|^2\,ds+\nu K_1e^{2\alpha t}\|\boldsymbol{\chi}(t)\|_\varepsilon^2 \le C\Big(&\|\boldsymbol{\chi}(0)\|_\varepsilon^2+\int_0^t e^{2\alpha s}\|\boldsymbol{\chi}(s)\|_\varepsilon^2\,ds\\ &+\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}_s(s)\|^2\,ds+\int_0^t e^{2\alpha s}\|\mathbf{e}(s)\|_\varepsilon^2\,ds\Big).\end{aligned} \tag{2.115}$$

Again, by using the estimate (2.24) and assumption (**A1**), we have

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\chi}(s)\|_\varepsilon^2\,ds \le Ce^{2\alpha t}h^{2r}+\int_0^t e^{2\alpha s}\|\mathbf{e}(s)\|_\varepsilon^2\,ds.$$

Using (2.25), assumption (**A1**) and Theorem 2.1 in (2.115), we obtain

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\chi}_s(s)\|^2\,ds \le K(t)h^{2r}.$$

Furthermore, a use of triangle inequality, estimate (2.25) and assumption (**A1**) leads to the desired estimate. □

For deriving the estimate of Lemma 2.14, we need an estimate for $\mathbf{u}_{ht}$, which is derived in the next lemma .

Lemma 2.13. *Under the assumptions of Lemma 2.12, the following estimate holds true*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{u}_{hs}(s)\|_\varepsilon^2\,ds \le K(t). \tag{2.116}$$

Proof. Differentiating (2.10) with respect to t, we obtain

$$(\mathbf{u}_{htt},\boldsymbol{\phi}_h)+\nu\, a(\mathbf{u}_{ht},\boldsymbol{\phi}_h)+c^{\mathbf{u}_h}(\mathbf{u}_{ht},\mathbf{u}_h,\boldsymbol{\phi}_h)+c^{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{u}_{ht},\boldsymbol{\phi}_h)=(\mathbf{f}_t,\boldsymbol{\phi}_h),\ \forall\,\boldsymbol{\phi}_h\in\mathbf{V}_h. \tag{2.117}$$

Take $\boldsymbol{\phi}_h=\mathbf{u}_{ht}$ in (2.117), and apply Lemma 1.6, (1.19), (1.14), the Cauchy-Schwarz inequality and Young's inequality to arrive at

$$\frac{d}{dt}\|\mathbf{u}_{ht}\|^2+\nu K_1\|\mathbf{u}_{ht}\|_\varepsilon^2 \le -2c^{\mathbf{u}_h}(\mathbf{u}_{ht},\mathbf{u}_h,\mathbf{u}_{ht})+C\|\mathbf{f}_t\|^2. \tag{2.118}$$

Using (2.57) and Young's inequality, we bound the nonlinear term in (2.118) as follows:

$$\begin{aligned}2|c^{\mathbf{u}_h}(\mathbf{u}_{ht},\mathbf{u}_h,\mathbf{u}_{ht})| &\le C\|\mathbf{u}_{ht}\|^{1/2}\|\mathbf{u}_{ht}\|_\varepsilon^{3/2}\|\mathbf{u}_h\|_\varepsilon\\ &\le \frac{\nu K_1}{2}\|\mathbf{u}_{ht}\|_\varepsilon^2+C\|\mathbf{u}_{ht}\|^2\|\mathbf{u}_h\|_\varepsilon^4.\end{aligned}$$

Incorporating this in (2.118), multiplying the resulting inequality by $e^{2\alpha t}$ and integrating from 0 to t, we obtain

$$\begin{aligned}e^{2\alpha t}\|\mathbf{u}_{ht}(t)\|^2+(\nu K_1-2\alpha C_2)\int_0^t e^{2\alpha s}\|\mathbf{u}_{ht}(s)\|_\varepsilon^2\,ds \le \|\mathbf{u}_{ht}(0)\|^2\\ +C\int_0^t e^{2\alpha s}\|\mathbf{u}_{ht}(s)\|^2\|\mathbf{u}_h(s)\|_\varepsilon^4\,ds+C\int_0^t e^{2\alpha s}\|\mathbf{f}_t(s)\|^2\,ds.\end{aligned}$$

Note that, using triangle inequality, Theorem 2.1 and assumption (**A1**), we find

$$\|\mathbf{u}_h\|_\varepsilon \le \|\mathbf{u}-\mathbf{u}_h\|_\varepsilon+\|\mathbf{u}\|_1 \le C. \tag{2.119}$$

Choosing $\alpha<\frac{\nu K_1}{2C_2}$, applying Gronwall's lemma, (2.77) and (2.119), and after a final multiplication by $e^{-2\alpha t}$, we obtain the estimate (2.116). □

Lemma 2.14. *Under the assumptions of Lemma 2.12, the error* $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$ *in approximating the velocity satisfies*

$$\|\boldsymbol{e}_t(t)\| \leq K(t)h^r, \ t > 0.$$

Proof. The error equation in $\mathbf{e}$ obtained from (2.5) and (2.10) is

$$(\mathbf{e}_t, \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{e}, \boldsymbol{\phi}_h) + c^{\mathbf{u}}\,(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) - c^{\mathbf{u}_h}\,(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) + b\,(\boldsymbol{\phi}_h, p) = 0, \tag{2.120}$$

for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$. Since $\mathbf{u}$ has no jumps, so for simplicity, the superscripts $\mathbf{u}$ and $\mathbf{u}_h$ in the nonlinear terms are dropped. Differentiate (2.120) with respect to t and choose $\boldsymbol{\phi}_h = \mathbf{P}_h\mathbf{e}_t$. A use of the definition of $\mathbf{P}_h$ and Lemma 1.6 leads to

$$\begin{aligned}\frac{d}{dt}\|\mathbf{e}_t\|^2 + 2\nu\, K_1\, \|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 \leq{}& \frac{d}{dt}\|\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t\|^2 + 2\nu\, a\,(\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \mathbf{P}_h\mathbf{e}_t)\\ &+ 2(c\,(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{P}_h\mathbf{e}_t) + c\,(\mathbf{u}_h, \mathbf{u}_{ht}, \mathbf{P}_h\mathbf{e}_t) - c\,(\mathbf{u}_t, \mathbf{u}, \mathbf{P}_h\mathbf{e}_t)\\ &- c\,(\mathbf{u}, \mathbf{u}_t, \mathbf{P}_h\mathbf{e}_t)) - 2b\,(\mathbf{P}_h\mathbf{e}_t, p_t).\end{aligned} \tag{2.121}$$

We rewrite the nonlinear terms as

$$\begin{aligned}&c(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{P}_h\mathbf{e}_t) + (c(\mathbf{u}_h, \mathbf{u}_{ht}, \mathbf{P}_h\mathbf{e}_t) - c(\mathbf{u}_t, \mathbf{u}, \mathbf{P}_h\mathbf{e}_t) - c(\mathbf{u}, \mathbf{u}_t, \mathbf{P}_h\mathbf{e}_t)\\ &= -c(\mathbf{u}_{ht}, \mathbf{e}, \mathbf{P}_h\mathbf{e}_t) - c(\mathbf{e}_t, \mathbf{u}, \mathbf{P}_h\mathbf{e}_t) - c(\mathbf{e}, \mathbf{u}_{ht}, \mathbf{P}_h\mathbf{e}_t) - c(\mathbf{u}, \mathbf{e}_t, \mathbf{P}_h\mathbf{e}_t).\end{aligned} \tag{2.122}$$

Using the Cauchy-Schwarz inequality, Young's inequality and Lemmas 1.5 and 1.10, the nonlinear terms on the right hand side of (2.122) can be bounded as in Lemma 2.11. Therefore, we have

$$|c(\mathbf{u}_{ht}, \mathbf{e}, \mathbf{P}_h\mathbf{e}_t)| \leq \frac{\nu K_1}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + C\|\mathbf{e}\|_\varepsilon^2\|\mathbf{u}_{ht}\|_\varepsilon^2, \tag{2.123}$$

$$|c(\mathbf{e}_t, \mathbf{u}, \mathbf{P}_h\mathbf{e}_t)| \leq \frac{\nu K_1}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + C(\|\mathbf{e}_t\|^2 + h^{2r}\|\mathbf{u}_t\|_{r+1}^2)\|\mathbf{u}\|_2^2, \tag{2.124}$$

$$|c(\mathbf{u}, \mathbf{e}_t, \mathbf{P}_h\mathbf{e}_t)| \leq \frac{\nu K_1}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + C(\|\mathbf{e}_t\|^2 + h^{2r}\|\mathbf{u}_t\|_{r+1}^2)\|\mathbf{u}\|_2^2, \tag{2.125}$$

$$|c(\mathbf{e}, \mathbf{u}_{ht}, \mathbf{P}_h\mathbf{e}_t)| \leq \frac{\nu K_1}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + C\|\mathbf{e}\|_\varepsilon^2\|\mathbf{u}_{ht}\|_\varepsilon^2. \tag{2.126}$$

The other terms on the right hand side of (2.121) is bounded as in Lemma 2.9:

$$2\nu|a(\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \mathbf{P}_h\mathbf{e}_t)| \leq \frac{K_1\nu}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + Ch^{2r}\|\mathbf{u}_t\|_{r+1}^2, \tag{2.127}$$

$$2|b(\mathbf{P}_h\mathbf{e}_t, p_t))| = 2|b(\mathbf{P}_h\mathbf{e}_t, p_t - r_h(p_t)))| \leq \frac{K_1\nu}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + Ch^{2r}\|p_t\|_r^2. \tag{2.128}$$

Using the bounds from (2.122)-(2.128) in (2.121) and applying assumption (**A1**), we obtain

$$\frac{d}{dt}\|\mathbf{e}_t\|^2 + \nu K_1 \|\mathbf{P}_h \mathbf{e}_t\|_\varepsilon^2 \le \frac{d}{dt}\|\mathbf{u}_t - \mathbf{P}_h \mathbf{u}_t\|^2 + C\|\mathbf{e}_t\|^2 + C\|\mathbf{e}\|_\varepsilon^2 \|\mathbf{u}_{ht}\|_\varepsilon^2 + Ch^{2r}\big(\|\mathbf{u}_t\|_{r+1}^2 + \|p_t\|_r^2\big). \tag{2.129}$$

Multiply (2.129) by $e^{2\alpha t}$ and integrate with respect to time, to write

$$\begin{aligned} e^{2\alpha t}\|\mathbf{e}_t(t)\|^2 + \nu K_1 \int_0^t e^{2\alpha s}\|\mathbf{P}_h \mathbf{e}_s(s)\|_\varepsilon^2\, ds &\le C\int_0^t e^{2\alpha s}\|\mathbf{e}_s(s)\|^2\, ds \\ &+ e^{2\alpha t}\|(\mathbf{u}_t - \mathbf{P}_h \mathbf{u}_t)(t)\|^2 + C\int_0^t e^{2\alpha s}\|\mathbf{e}(s)\|_\varepsilon^2 \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds \\ &+ Ch^{2r}\int_0^t e^{2\alpha s}\big(\|\mathbf{u}_s(s)\|_{r+1}^2 + \|p_s(s)\|_r^2\big)\, ds. \end{aligned}$$

Finally, from Lemmas 2.12 and 2.2, Theorem 2.1, (2.116) and assumption (**A1**), we conclude the rest of the proof. □

Below we present the semi-discrete DG error estimate for pressure, which has been obtained by following the identical proof approach of [86], but for DG and is the same as [86, eq. (3.10)].

Theorem 2.2. *Under the assumptions of Theorem 2.1, there exists a constant $K > 0$ such that, the following error estimates hold true:*

$$\|(p - p_h)(t)\| \le K(t)h^r, \quad 0 < t \le T,$$

Proof. From (2.5), (2.6), (2.8) and (2.9), we can write the error equation as follows:

$$\begin{aligned} b(\mathbf{v}_h,\, r_h(p) - p_h) &= (\mathbf{u}_{ht} - \mathbf{u}_t, \mathbf{v}_h) + \nu\, a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) \\ &+ c^{\mathbf{u}_h}(\mathbf{u}_h,\, \mathbf{u}_h,\, \mathbf{v}_h) - c^{\mathbf{u}}(\mathbf{u},\, \mathbf{u},\, \mathbf{v}_h) + b(\mathbf{v}_h,\, r_h(p) - p), \quad \forall \mathbf{v}_h \in \mathbf{X}_h. \end{aligned} \tag{2.130}$$

By virtue of the inf-sup condition presented in Lemma 1.8, there is $\mathbf{v}_h \in \mathbf{X}_h$ such that

$$b(\mathbf{v}_h, p_h - r_h(p)) = -\|p_h - r_h(p)\|^2, \quad \|\mathbf{v}_h\|_\varepsilon \le \frac{1}{\beta^*}\|p_h - r_h(p)\|. \tag{2.131}$$

Therefore, from (2.130), we obtain

$$\begin{aligned} \|p_h - r_h(p)\|^2 =& (\mathbf{u}_{ht} - \mathbf{u}_t, \mathbf{v}_h) + \nu\, a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - c(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) \\ &- b(\mathbf{v}_h, p - r_h(p)). \end{aligned} \tag{2.132}$$

The terms on the right hand side of (2.132) can be bounded as in Lemmas 2.9 and 2.11. Then, the equation (2.132) becomes

$$\|p_h - r_h(p)\|^2 \leq C\|\mathbf{u}_{ht} - \mathbf{u}_t\|^2 + C\|\mathbf{u}_h - \mathbf{u}\|_\varepsilon^2 + Ch^{2r}(|\mathbf{u}|_{r+1}^2 + |p|_r^2) + C\|\mathbf{u}_h - \mathbf{u}\|^2.$$

A use of the triangle inequality leads to

$$\|p - p_h\|^2 \leq C\,\|\mathbf{u}_{ht} - \mathbf{u}_t\|^2 + C\|\mathbf{u}_h - \mathbf{u}\|_\varepsilon^2 + Ch^{2r}(|\mathbf{u}|_{r+1}^2 + |p|_r^2) + C\|\mathbf{u}_h - \mathbf{u}\|^2.$$

Combining Lemma 2.14, Theorem 2.1 and assumption (**A1**) with the above inequality we obtain our desired pressure error estimate. □

We note that the pressure estimates in the NIPG and IIPG cases will be sub-optimal due to their dependence on velocity error estimates, which are already discussed as sub-optimal in Section 2.4.

2.6 Fully Discrete Error Estimates

For discretization in the time variable of the semi-discrete DG time dependent Navier-Stokes system represented by (2.8)-(2.9), we employ the backward Euler scheme in this section. Keeping in mind the notations presented, for time discretization, in the previous chapter, we proceed to describe the backward Euler scheme for the semi-discrete problem (2.8)-(2.9) as follows: Given $\mathbf{U}^0$, seek $(\mathbf{U}^n, P^n)_{n\geq 1} \in \mathbf{X}_h \times M_h$, such that

$$(\partial_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{U}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}}(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h)$$
$$+ b(\boldsymbol{\phi}_h, P^n) = (\mathbf{f}(t_n), \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{X}_h, \tag{2.133}$$
$$b(\mathbf{U}^n, q_h) = 0, \quad \forall\, q_h \in M_h. \tag{2.134}$$

Note that $\mathbf{U}^0 = \mathbf{u}_h(0) = \mathbf{P}_h \mathbf{u}_0$.
An equivalent formulation would be to find $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{V}_h$, for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$, such that

$$(\partial_t \mathbf{U}^n, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{U}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}}(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) = (\mathbf{f}(t_n), \boldsymbol{\phi}_h). \tag{2.135}$$

Below, we present *a priori* estimates of the fully discrete solution $\mathbf{U}^n$ of (2.135).

Lemma 2.15. *Let $0 < \alpha < \frac{\nu K_1}{2C_2}$. Further, let $\boldsymbol{U}^0 = \boldsymbol{P}_h \boldsymbol{u}_0$. Then, there exists a constant $C > 0$, such that, the solution $\{\boldsymbol{U}^n\}_{n\geq 1}$ of (2.135) satisfies the following a priori bounds:*

$$\|\boldsymbol{U}^n\|^2 + e^{-2\alpha t_M}\,\Delta t \sum_{n=1}^{M} e^{2\alpha t_n}\|\boldsymbol{U}^n\|_\varepsilon^2 \leq C, \quad n = 1, \cdots, M.$$

The lemma can be proved by choosing $\boldsymbol{\phi}_h = \mathbf{U}^n$ in (2.135) and using Lemma 1.6. Using (1.19) and Lemmas 1.6, 1.8, 2.15, the existence and uniqueness of the discrete solutions to the discrete problem (2.133)-(2.134) (or (2.135)) can be achieved following similar steps as in [72].

We next discuss the error estimates of the backward Euler method. Set $\mathbf{e}_n = \mathbf{U}^n - \mathbf{u}_h^n$, for fixed $n \in \mathbb{N}$, $1 \leq n \leq M$. Considering the semi-discrete scheme (2.10) at $t = t_n$ and subtracting from (2.135), we arrive at

$$(\partial_t \mathbf{e}_n,\, \boldsymbol{\phi}_h) + \nu\; a\,(\mathbf{e}_n,\, \boldsymbol{\phi}_h) = (\mathbf{u}_{ht}^n,\, \boldsymbol{\phi}_h) - (\partial_t \mathbf{u}_h^n,\, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h, \tag{2.136}$$

where $\Lambda_h(\boldsymbol{\phi}_h) = \Lambda_h^1(\boldsymbol{\phi}_h) + \Lambda_h^2(\boldsymbol{\phi}_h)$ with

$$\begin{cases} \Lambda_h^1(\boldsymbol{\phi}_h) = c^{\mathbf{U}^{n-1}}(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}^{n-1}}(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h), \\ \Lambda_h^2(\boldsymbol{\phi}_h) = l^{\mathbf{u}_h^n}(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - l^{\mathbf{U}^{n-1}}(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h). \end{cases} \tag{2.137}$$

Using Taylor's expansion, we find that

$$(\mathbf{u}_{ht}^n,\, \boldsymbol{\phi}_h) - (\partial_t \mathbf{u}_h^n,\, \boldsymbol{\phi}_h) = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} (s - t_{n-1})\,(\mathbf{u}_{hss}(s),\, \boldsymbol{\phi}_h)\, ds. \tag{2.138}$$

Lemma 2.16. *Suppose the assumption (**A1**) is hold true and $0 < \alpha < \frac{\nu K_1}{2C_2}$. Then, for the semi-discrete DG velocity $\boldsymbol{u}_h(t)$, $t > 0$, the following holds true*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\,\|\boldsymbol{u}_{hss}(s)\|_{-1,h}^2\, ds \leq C,$$

where

$$\|\boldsymbol{u}_{htt}\|_{-1,h} = \sup\left\{\frac{\langle \boldsymbol{u}_{htt}, \boldsymbol{\phi}_h\rangle}{\|\boldsymbol{\phi}_h\|_\varepsilon},\ \boldsymbol{\phi}_h \in \boldsymbol{X}_h, \boldsymbol{\phi}_h \neq 0\right\}.$$

Proof. We begin by rewriting the (2.117) as follows:

$$(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) = -\nu\, a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) - c^{\mathbf{u}_h}(\mathbf{u}_{ht}, \mathbf{u}_h, \boldsymbol{\phi}_h) - c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_{ht}, \boldsymbol{\phi}_h) - (\mathbf{f}_t, \boldsymbol{\phi}_h),\ \forall\, \boldsymbol{\phi}_h \in \mathbf{V}_h.$$

Using Lemma 1.7, (2.55) and the Cauchy-Schwarz inequality, we obtain

$$(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) \leq C\|\mathbf{u}_{ht}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon + C\|\mathbf{u}_{ht}\|_\varepsilon\|\mathbf{u}_h\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon + C\|\mathbf{f}_t\|\|\boldsymbol{\phi}_h\|_\varepsilon.$$

Furthermore, using (2.119), we have

$$\|\mathbf{u}_{htt}\|_{-1,h}^2 \leq C\|\mathbf{u}_{ht}\|_{\varepsilon}^2 + C\|\mathbf{f}_t\|^2.$$

Multiply the above inequality by $e^{2\alpha t}$ and integrate from 0 to t. Then again multiply by $e^{-2\alpha t}$ and use (2.116) to obtain the desired estimate. □

Lemma 2.17. *Suppose the assumptions of Theorem 2.1 and Lemma 2.15 are satisfied. Then, there exists a constant $K_T = K_T(T) > 0$ such that the following estimates hold true:*

$$\|\boldsymbol{e}_n\| + \left(\nu K_1 e^{-2\alpha t_M} \Delta t \sum_{n=1}^{M} e^{2\alpha t_n} \|\boldsymbol{e}_n\|_{\varepsilon}^2\right)^{1/2} \leq K_T \Delta t,$$

Proof. We put $\boldsymbol{\phi}_h = \mathbf{e}_n$ in error equation (2.136). With the observation

$$(\partial_t \mathbf{e}_n, \mathbf{e}_n) \geq \frac{1}{2}\partial_t\|\mathbf{e}_n\|^2,$$

and a use of Lemma 1.6 yields

$$\partial_t\|\mathbf{e}_n\|^2 + 2K_1\,\nu\,\|\mathbf{e}_n\|_{\varepsilon}^2 \leq 2(\mathbf{u}_{ht}^n, \mathbf{e}_n) - 2(\partial_t \mathbf{u}_h^n, \mathbf{e}_n) + 2\Lambda_h(\mathbf{e}_n), \tag{2.139}$$

where $\Lambda_h(\mathbf{e}_n) = \Lambda_h^1(\mathbf{e}_n) + \Lambda_h^2(\mathbf{e}_n)$. Drop the superscripts from nonlinear terms of $\Lambda_h^1(\mathbf{e}_n)$ in (2.137) and rewrite it as

$$\begin{aligned}\Lambda_h^1(\mathbf{e}_n) = &-c(\mathbf{U}^{n-1}, \mathbf{e}_n, \mathbf{e}_n) - c(\mathbf{e}_{n-1}, \mathbf{u}^n, \mathbf{e}_n) - c(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}^n - \mathbf{u}_h^n, \mathbf{e}_n)\\ &+ c(\mathbf{e}_{n-1}, \mathbf{u}^n - \mathbf{u}_h^n, \mathbf{e}_n) + c(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}^n, \mathbf{e}_n).\end{aligned} \tag{2.140}$$

From (1.19), we have $c(\mathbf{U}^{n-1}, \mathbf{e}_n, \mathbf{e}_n) \geq 0$. A use of (2.54), Young's inequality, Lemma 1.3 and assumption (**A1**) leads to the bound for the second term as

$$|c(\mathbf{e}_{n-1}, \mathbf{u}^n, \mathbf{e}_n)| \leq C\|\mathbf{u}^n\|_2\|\mathbf{e}_{n-1}\|\|\mathbf{e}_n\|_{\varepsilon} \leq C\|\mathbf{e}_{n-1}\|^2 + \frac{K_1\nu}{8}\|\mathbf{e}_n\|_{\varepsilon}^2. \tag{2.141}$$

The fourth term on the right hand side of (2.140) is bounded using the same techniques as in Lemma 2.11 for estimating $c^{\mathbf{u}_h}(\boldsymbol{\rho}, \boldsymbol{\xi}, \boldsymbol{\rho})$. An application of Young's inequality and Theorem 2.1 leads to

$$|c(\mathbf{e}_{n-1}, \mathbf{u}^n - \mathbf{u}_h^n, \mathbf{e}_n)| \leq C\|\mathbf{e}_{n-1}\|\|\mathbf{e}_n\|_{\varepsilon} \leq C\|\mathbf{e}_{n-1}\|^2 + \frac{K_1\nu}{8}\|\mathbf{e}_n\|_{\varepsilon}^2. \tag{2.142}$$

A use of $\mathbf{u}_h^n - \mathbf{u}_h^{n-1} = \displaystyle\int_{t_{n-1}}^{t_n} \mathbf{u}_{hs}(s)\, ds$ and the Hölder, Young inequalities, (1.14), Lemmas 1.5, 1.10, Theorem 2.1 in the third term on the right hand side of (2.140) yield

$$\begin{aligned}
|c\,(\mathbf{u}_h^n - \mathbf{u}_h^{n-1},\, \mathbf{u}^n - \mathbf{u}_h^n,\, \mathbf{e}_n)| &\leq \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon\, \|\mathbf{e}_n\|_\varepsilon\, ds \\
&\leq \frac{K_1\nu}{8}\|\mathbf{e}_n\|_\varepsilon^2 \,+\, C\Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds. \qquad (2.143)
\end{aligned}$$

Apply the form of $c(\cdot,\,\cdot,\,\cdot)$ presented in (1.17), (1.14), Lemmas 1.3 and 1.10 along with the regularity of $\mathbf{u}^n$ to arrive at

$$\begin{aligned}
|c\,(\mathbf{u}_h^n - \mathbf{u}_h^{n-1},\, \mathbf{u}^n,\, \mathbf{e}_n)| &\leq \Delta t^{1/2}\Big(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|^2 ds\Big)^{1/2} \|\nabla \mathbf{u}^n\|_{L^4(\Omega)}\|\mathbf{e}_n\|_{L^4(\Omega)} \\
&\quad + C\Delta t^{1/2}\Big(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2 ds\Big)^{1/2} \|\mathbf{u}\|_{L^\infty([0,T]\times\Omega)}\|\mathbf{e}_n\| \\
&\quad + C\,\|\mathbf{u}\|_{L^\infty([0,T]\times\Omega)} \int_{t_{n-1}}^{t_n} \Big(\sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|}\|[\mathbf{u}_{hs}(s)]\|_{L^2(e)}^2\Big)^{1/2} \Big(\sum_{E\in\mathcal{E}_h} \|\mathbf{e}_n\|_{L^2(E)}^2\Big)^{1/2} ds \\
&\leq \frac{K_1\nu}{8}\|\mathbf{e}_n\|_\varepsilon^2 + C\Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds. \qquad (2.144)
\end{aligned}$$

Combining the estimates (2.141)-(2.144) in (2.140), we obtain

$$|\Lambda_h^1(\mathbf{e}_n)| \leq \frac{K_1\nu}{8}\|\mathbf{e}_n\|_\varepsilon^2 + C\|\mathbf{e}_{n-1}\|^2 + C\Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds. \qquad (2.145)$$

Apply (2.137), and use the estimate (2.60) and triangle inequality to obtain

$$\begin{aligned}
|\Lambda_h^2(\mathbf{e}_n)| \leq & C\|\mathbf{u}_h^n - \mathbf{U}^{n-1}\|_{L^4(\Omega)}\|\mathbf{u}_h^n\|_\varepsilon\|\mathbf{e}_n\|_{L^4(\Omega)} \\
\leq & C\big(\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{L^4(\Omega)}\|\mathbf{u}_h^n\|_\varepsilon\|\mathbf{e}_n\|_{L^4(\Omega)} + \|\mathbf{e}_{n-1}\|_{L^4(\Omega)}\|\mathbf{u}_h^n\|_\varepsilon\|\mathbf{e}_n\|_{L^4(\Omega)}\big).
\end{aligned}$$

Then, an application of (1.14), (2.119), Lemma 2.6 and Young's inequality lead to

$$\begin{aligned}
|\Lambda_h^2(\mathbf{e}_n)| \leq & C\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_\varepsilon\|\mathbf{u}_h^n\|_\varepsilon\|\mathbf{e}_n\|_\varepsilon + C\|\mathbf{e}_{n-1}\|^{1/2}\|\mathbf{e}_{n-1}\|_\varepsilon^{1/2}\|\mathbf{u}_h^n\|_\varepsilon\|\mathbf{e}_n\|_\varepsilon \\
\leq & \frac{K_1\nu}{8}\|\mathbf{e}_n\|_\varepsilon^2 + \frac{K_1\nu}{8}\|\mathbf{e}_{n-1}\|_\varepsilon^2 + C\|\mathbf{e}_{n-1}\|^2 + C\Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds. \qquad (2.146)
\end{aligned}$$

From (2.138), we have

$$\begin{aligned}
2\big((\mathbf{u}_{ht}^n, \mathbf{e}_n) - (\partial_t \mathbf{u}_h^n, \mathbf{e}_n)\big) &\leq C\,\Delta t^{1/2}\,\Big(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}(s)\|_{-1,h}^2\, ds\Big)^{1/2} \|\mathbf{e}_n\|_\varepsilon \\
&\leq \frac{K_1\nu}{64}\|\mathbf{e}_n\|_\varepsilon^2 \,+\, C\Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}(s)\|_{-1,h}^2\, ds. \qquad (2.147)
\end{aligned}$$

Combine (2.145)-(2.147), multiply (2.139) by $\Delta t e^{2\alpha n\Delta t}$, sum the resulting inequality from $n = 1$ to $m\,(\leq M)$ and observe that

$$\sum_{n=1}^{m} \Delta t e^{2\alpha n\Delta t}\partial_t \|\mathbf{e}_n\|^2 = e^{2\alpha m\Delta t}\|\mathbf{e}_m\|^2 - \sum_{n=1}^{m-1} e^{2\alpha n\Delta t}(e^{2\alpha\Delta t} - 1)\|\mathbf{e}_m\|^2$$

to obtain

$$\begin{aligned} e^{2\alpha m\Delta t}\|\mathbf{e}_m\|^2 + K_1\nu\Delta t\sum_{n=1}^{m} e^{2\alpha n\Delta t}\|\mathbf{e}_n\|_\varepsilon^2 &\leq \sum_{n=1}^{m-1} e^{2\alpha n\Delta t}(e^{2\alpha\Delta t} - 1)\|\mathbf{e}_n\|^2 \\ &+ C\Delta t\sum_{n=1}^{m} e^{2\alpha n\Delta t}\|\mathbf{e}_{n-1}\|^2 + C\Delta t^2\sum_{n=1}^{m} e^{2\alpha n\Delta t}\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hs}(s)\|_\varepsilon^2\,ds \\ &+ C\Delta t^2\sum_{n=1}^{m} e^{2\alpha n\Delta t}\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hss}(s)\|_{-1,h}^2\,ds. \end{aligned} \tag{2.148}$$

We bound the terms involving $\mathbf{u}_h$ using Lemmas 2.13 and 2.16. Observe that

$$\begin{aligned} C\Delta t^2\sum_{n=1}^{m} e^{2\alpha n\Delta t}\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hs}(s)\|_\varepsilon^2\,ds &= C\Delta t^2\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} e^{2\alpha(t_n-s)}e^{2\alpha s}\|\mathbf{u}_{hs}(s)\|_\varepsilon^2\,ds \\ &\leq C\Delta t^2 e^{2\alpha\Delta t}\int_0^{t_m} e^{2\alpha s}\|\mathbf{u}_{hs}(s)\|_\varepsilon^2\,ds \leq C\Delta t^2 e^{2\alpha(m+1)\Delta t}. \end{aligned} \tag{2.149}$$

In a similar manner as in (2.149), we can bound

$$C\Delta t^2\sum_{n=1}^{m} e^{2\alpha n\Delta t}\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hss}(s)\|_{-1,h}^2\,ds \leq C\Delta t^2 e^{2\alpha(m+1)\Delta t}. \tag{2.150}$$

Applying (2.149) and (2.150) in (2.148) and using the fact $e^{2\alpha\Delta t} - 1 \leq C(\alpha)\Delta t$, we obtain

$$e^{2\alpha m\Delta t}\|\mathbf{e}_m\|^2 + K_1\nu\Delta t\sum_{n=1}^{m} e^{2\alpha n\Delta t}\|\mathbf{e}_n\|_\varepsilon^2 \leq C\Delta t\sum_{n=1}^{m-1} e^{2\alpha n\Delta t}\|\mathbf{e}_n\|^2 + C\Delta t^2 e^{2\alpha(m+1)\Delta t}.$$

Now the desired result is achieved by applying discrete Gronwall's lemma. □

Theorem 2.3. *Suppose the assumptions of Theorem 2.1 and Lemma 2.17 are satisfied. Then, the following estimates hold true:*

$$\|\boldsymbol{u}(t_n) - \boldsymbol{U}^n\| \leq K_T\,(h^{r+1} + \Delta t),$$

where $K_T > 0$ depends on T.

Proof. Combine Theorem 2.1 with Lemma 2.17 to complete the proof. □

Lemma 2.18. *Let the assumptions of Lemma 2.17 be satisfied. Then, the fully discrete velocity error* $\boldsymbol{e}_n = \boldsymbol{U}^n - \boldsymbol{u}_h^n$ *satisfies*

$$e^{-2\alpha t_m}\,\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\partial_t \boldsymbol{e}_n\|^2 \,+\, \nu\, K_1\,\|\boldsymbol{e}_m\|_\varepsilon^2 \,\le\, K_T\,\Delta t.$$

Proof. In (2.136), we choose $\boldsymbol{\phi}_h = \partial_t \mathbf{e}_n$ to obtain

$$\|\partial_t \mathbf{e}_n\|^2 + \nu\, a(\mathbf{e}_n, \partial_t \mathbf{e}_n) = (\mathbf{u}_{ht}^n, \partial_t \mathbf{e}_n) - (\partial_t \mathbf{u}_h^n, \partial_t \mathbf{e}_n) + \Lambda_h(\partial_t \mathbf{e}_n). \tag{2.151}$$

We now drop the superscripts for the nonlinear terms in $\Lambda_h^1(\partial_t \mathbf{e}_n)$ and rewrite $\Lambda_h^1(\partial_t \mathbf{e}_n)$ as

$$\begin{aligned}\Lambda_h^1(\partial_t \mathbf{e}_n) &= -c\,(\mathbf{u}_h^{n-1} - \mathbf{u}^{n-1},\, \mathbf{e}_n,\, \partial_t \mathbf{e}_n) - c\,(\mathbf{u}^{n-1},\, \mathbf{e}_n,\, \partial_t \mathbf{e}_n)\\ &\quad + c\,(\mathbf{u}_h^n - \mathbf{u}_h^{n-1},\, \mathbf{u}_h^n - \mathbf{u}^n,\, \partial_t \mathbf{e}_n) + c\,(\mathbf{u}_h^n - \mathbf{u}_h^{n-1},\, \mathbf{u}^n,\, \partial_t \mathbf{e}_n)\\ &\quad - c\,(\mathbf{e}_{n-1},\, \mathbf{u}_h^n - \mathbf{u}^n,\, \partial_t \mathbf{e}_n) - c\,(\mathbf{e}_{n-1},\, \mathbf{u}^n,\, \partial_t \mathbf{e}_n) - c\,(\mathbf{e}_{n-1},\, \mathbf{e}_n,\, \partial_t \mathbf{e}_n).\end{aligned} \tag{2.152}$$

The L^p bound (1.14), Lemmas 1.3, 1.5 and 1.10, Theorem 2.1 and Young's inequality give the bounds for the nonlinear terms in the right hand side of (2.152) except the last term, as in Lemma 2.17.

$$|c\,(\mathbf{u}_h^{n-1} - \mathbf{u}^{n-1},\, \mathbf{e}_n,\, \partial_t \mathbf{e}_n)| \le \frac{1}{16}\|\partial_t \mathbf{e}_n\|^2 + C\,\|\mathbf{e}_n\|_\varepsilon^2, \tag{2.153}$$

$$|c(\mathbf{u}^{n-1},\, \mathbf{e}_n,\, \partial_t \mathbf{e}_n)| \le \frac{1}{16}\|\partial_t \mathbf{e}_n\|^2 + C\|\mathbf{e}_n\|_\varepsilon^2, \tag{2.154}$$

$$\begin{aligned}|c(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n - \mathbf{u}^n, \partial_t \mathbf{e}_n) + c(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}^n, \partial_t \mathbf{e}_n)| &\le \frac{1}{16}\|\partial_t \mathbf{e}_n\|^2\\ &\quad + C\Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds,\end{aligned} \tag{2.155}$$

$$|c(\mathbf{e}_{n-1}, \mathbf{u}_h^n - \mathbf{u}^n, \partial_t \mathbf{e}_n) + c(\mathbf{e}_{n-1}, \mathbf{u}^n, \partial_t \mathbf{e}_n)| \le \frac{1}{16}\|\partial_t \mathbf{e}_n\|^2 + C\|\mathbf{e}_{n-1}\|_\varepsilon^2. \tag{2.156}$$

The last term in the right hand side of (2.152) can be rewritten as

$$-c(\mathbf{e}_{n-1}, \mathbf{e}_n, \partial_t \mathbf{e}_n) = -\frac{1}{\Delta t} c(\mathbf{e}_{n-1}, \mathbf{e}_n, \mathbf{e}_n) + \frac{1}{\Delta t} c(\mathbf{e}_{n-1}, \mathbf{e}_n, \mathbf{e}_{n-1}).$$

The estimate (2.55) yields

$$\begin{aligned}&\frac{1}{\Delta t}|c(\mathbf{e}_{n-1}, \mathbf{e}_n, \mathbf{e}_n)| + \frac{1}{\Delta t}|c(\mathbf{e}_{n-1}, \mathbf{e}_n, \mathbf{e}_{n-1})|\\ &\le \frac{C}{\Delta t}(\|\mathbf{e}_{n-1}\|_\varepsilon^2 \|\mathbf{e}_n\|_\varepsilon + \|\mathbf{e}_{n-1}\|_\varepsilon \|\mathbf{e}_n\|_\varepsilon^2).\end{aligned} \tag{2.157}$$

Collecting the bounds (2.153)-(2.157) in (2.152) and using Young's inequality, we arrive at

$$|\Lambda_h^1(\partial_t \mathbf{e}_n)| \leq \frac{1}{4}\|\partial_t \mathbf{e}_n\|^2 + C(\|\mathbf{e}_n\|_\varepsilon^2 + \|\mathbf{e}_{n-1}\|_\varepsilon^2) + \frac{C}{\Delta t}(\|\mathbf{e}_n\|_\varepsilon^2 + \|\mathbf{e}_{n-1}\|_\varepsilon^2)$$
$$+ \frac{C}{\Delta t}\|\mathbf{e}_{n-1}\|_\varepsilon^2\|\mathbf{e}_n\|_\varepsilon^2 + C\Delta t\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds. \tag{2.158}$$

Since $\mathbf{u}^n$ has zero jump, the term of (2.72) would be zero, with $\mathbf{w}$ replaced by $\mathbf{u}^n$. And therefore the difference of the upwind terms in (2.60) is zero. Subtracting this zero term with $\mathbf{u}^n$ from $\Lambda_h^2(\partial_t \mathbf{e}_n)$ and applying (2.60) yields

$$|\Lambda_h^2(\partial_t \mathbf{e}_n)| \;\leq\; C\|\mathbf{u}_h^n - \mathbf{U}^{n-1}\|_{L^4(\Omega)}\; \|\mathbf{u}_h^n - \mathbf{u}^n\|_\varepsilon\; \|\partial_t \mathbf{e}_n\|_{L^4(\Omega)}.$$

Using estimate (1.14), Lemma 1.10 and Theorem 2.1, we obtain

$$|\Lambda_h^2(\partial_t \mathbf{e}_n)| \leq C\|\mathbf{u}_h^n - \mathbf{U}^{n-1}\|_\varepsilon\; \|\mathbf{u}_h^n - \mathbf{u}^n\|_\varepsilon\; \left(\frac{1}{\min\limits_{E\in\mathcal{E}_h} h_E^{1/2}}\|\partial_t \mathbf{e}_n\|\right)$$
$$\leq C\|\mathbf{u}_h^n - \mathbf{U}^{n-1}\|_\varepsilon\|\partial_t \mathbf{e}_n\|.$$

Again, a use of the triangle inequality and Young's inequality yield

$$|\Lambda_h^2(\partial_t \mathbf{e}_n)| \leq C\big(\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_\varepsilon + \|\mathbf{e}_{n-1}\|_\varepsilon\big)\|\partial_t \mathbf{e}_n\|$$
$$\leq \frac{1}{4}\|\partial_t \mathbf{e}_n\|^2 + C\Delta t\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds + C\|\mathbf{e}_{n-1}\|_\varepsilon^2. \tag{2.159}$$

From (2.138), we have

$$(\mathbf{u}_{ht}^n, \partial_t \mathbf{e}_n) - (\partial_t \mathbf{u}_h^n, \partial_t \mathbf{e}_n) \leq C\Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}(s)\|_{-1,h}^2\, ds\right)^{1/2} \|\partial_t \mathbf{e}_n\|_\varepsilon$$
$$\leq C\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}(s)\|_{-1,h}^2\, ds + \frac{C}{\Delta t}\|\mathbf{e}_n - \mathbf{e}_{n-1}\|_\varepsilon^2. \tag{2.160}$$

Since $a(\cdot,\cdot)$ is symmetric, one can obtain

$$a(\mathbf{e}_n, \partial_t \mathbf{e}_n) = \frac{1}{2}\left(\frac{1}{\Delta t}a(\mathbf{e}_n, \mathbf{e}_n) - \frac{1}{\Delta t}a(\mathbf{e}_{n-1}, \mathbf{e}_{n-1}) + \Delta t a(\partial_t \mathbf{e}_n, \partial_t \mathbf{e}_n)\right). \tag{2.161}$$

Again,

$$\sum_{n=1}^{m} e^{2\alpha t_n}\Big(a(\mathbf{e}_n, \mathbf{e}_n) - a(\mathbf{e}_{n-1}, \mathbf{e}_{n-1})\Big)$$
$$= e^{2\alpha t_m}a(\mathbf{e}_m, \mathbf{e}_m) - \sum_{n=1}^{m-1} e^{2\alpha t_n}(e^{2\alpha\Delta t} - 1)a(\mathbf{e}_n, \mathbf{e}_n). \tag{2.162}$$

Combining (2.158)–(2.162), multiply (2.151) by $\Delta t e^{2\alpha t_n}$, sum over $n = 1$ to $m\,(\le M)$ and using Lemma 1.6, we obtain

$$\begin{aligned}
\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\partial_t \mathbf{e}_n\|^2 + \nu K_1 e^{2\alpha t_m} \|\mathbf{e}_m\|_\varepsilon^2 \le\; & C\,\Delta t \sum_{n=1}^{m-1} e^{2\alpha t_n} a(\mathbf{e}_n, \mathbf{e}_n) \\
+ C\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{e}_n\|_\varepsilon^2 + C\,\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{e}_{n-1}\|_\varepsilon^2 &+ \frac{C}{\Delta t}\,\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{e}_n\|_\varepsilon^2 \|\mathbf{e}_{n-1}\|_\varepsilon^2 \\
+ \frac{C}{\Delta t} \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} (\|\mathbf{e}_n\|_\varepsilon^2 + \|\mathbf{e}_{n-1}\|_\varepsilon^2) &+ C\,\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}(s)\|_{-1,h}^2\, ds \\
+ C\Delta t^2 \sum_{n=1}^{m} e^{2\alpha t_n} \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds. & \qquad (2.163)
\end{aligned}$$

Using Lemmas 1.7, 2.13, 2.16 and 2.17 in (2.163), we obtain

$$\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\partial_t \mathbf{e}_n\|^2 + \nu K_1 e^{2\alpha t_m} \|\mathbf{e}_m\|_\varepsilon^2 \le \frac{C}{\Delta t} \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{e}_n\|_\varepsilon^2 \|\mathbf{e}_{n-1}\|_\varepsilon^2 + C \Delta t e^{2\alpha t_{m+1}}.$$

Finally, a use of the discrete Gronwall's inequality and Lemma 2.17 give us the desired estimate. This completes the proof. □

Lemma 2.19. *Suppose the assumptions of Lemma 2.17 are satisfied. Then, the following estimates hold true:*

$$\|\partial_t \boldsymbol{e}_n\|_{-1,h} \le K_T \Delta t^{1/2}.$$

Proof. The non-linear terms in the error equation (2.136) can be rewritten as

$$\begin{aligned}
\Lambda_h(\boldsymbol{\phi}_h) &= \Lambda_h^1(\boldsymbol{\phi}_h) + \Lambda_h^2(\boldsymbol{\phi}_h) \\
&= - c^{\mathbf{U}^{n-1}} (\mathbf{u}_h^{n-1}, \mathbf{e}_n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}} (\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n - \mathbf{u}^n, \boldsymbol{\phi}_h) \\
&\quad + c^{\mathbf{U}^{n-1}} (\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}^{n-1}} (\mathbf{e}_{n-1}, \mathbf{e}_n, \boldsymbol{\phi}_h) \\
&\quad - c^{\mathbf{U}^{n-1}} (\mathbf{e}_{n-1}, \mathbf{u}_h^n, \boldsymbol{\phi}_h) + \Lambda_h^2(\boldsymbol{\phi}_h). \qquad (2.164)
\end{aligned}$$

Using estimate (2.55), we obtain

$$\begin{aligned}
&|c^{\mathbf{U}^{n-1}} (\mathbf{u}_h^{n-1}, \mathbf{e}_n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}} (\mathbf{e}_{n-1}, \mathbf{e}_n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}} (\mathbf{e}_{n-1}, \mathbf{u}_h^n, \boldsymbol{\phi}_h)| \\
&\le C\, \|\mathbf{u}_h^{n-1}\|_\varepsilon \|\mathbf{e}_n\|_\varepsilon \|\boldsymbol{\phi}_h\|_\varepsilon + C \|\mathbf{e}_{n-1}\|_\varepsilon \|\mathbf{e}_n\|_\varepsilon \|\boldsymbol{\phi}_h\|_\varepsilon + C\, \|\mathbf{e}_{n-1}\|_\varepsilon \|\mathbf{u}_h^n\|_\varepsilon \|\boldsymbol{\phi}_h\|_\varepsilon. \quad (2.165)
\end{aligned}$$

Again, by using Theorem 2.1, estimate (2.54) and Lemma 1.3, one can obtain the bounds similar to Lemma 2.17 as follows:

$$|c^{\mathbf{U}^{n-1}}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n - \mathbf{u}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}^n, \boldsymbol{\phi}_h)|$$

$$\leq C \, \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\| \|\boldsymbol{\phi}_h\|_\varepsilon. \tag{2.166}$$

Furthermore, using similar techniques as in Lemma 2.18, we find that

$$|\Lambda_h^2(\boldsymbol{\phi}_h)| \leq C(\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_\varepsilon \|\mathbf{u}_h^n\|_\varepsilon + \|\mathbf{e}_{n-1}\|_\varepsilon \|\mathbf{u}_h^n\|_\varepsilon)\|\boldsymbol{\phi}_h\|_\varepsilon. \tag{2.167}$$

Now, Lemma 1.7 yields

$$|a(\mathbf{e}_n, \boldsymbol{\phi}_h)| \leq C\|\mathbf{e}_n\|_\varepsilon \|\boldsymbol{\phi}_h\|_\varepsilon. \tag{2.168}$$

Applying (2.138), the Cauchy-Schwarz inequality, Young's inequality and Lemma 2.16, we arrive at

$$\begin{aligned}(\mathbf{u}_{ht}^n, \boldsymbol{\phi}_h) - (\partial_t \mathbf{u}_h^n, \boldsymbol{\phi}_h) \leq \; & C \, \Delta t^{1/2} \left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}(s)\|_{-1,h}^2 \, ds \right)^{1/2} \|\boldsymbol{\phi}_h\|_\varepsilon \\ \leq \; & C \Delta t^{1/2} \|\boldsymbol{\phi}_h\|_\varepsilon. \end{aligned} \tag{2.169}$$

Combining all the bounds (2.164)–(2.169) in (2.136), using the definition of $\| \cdot \|_{-1,h}$ and finally using (2.119), Lemmas 2.8 and 2.18, we obtain our desired result. This completes the rest of the proof. □

Lemma 2.20. *Suppose the hypotheses of Lemma 2.17 are satisfied. Then, the following estimates hold true:*

$$\|P^n - p_h(t_n)\| \leq \; K_T \, \Delta t^{1/2}, \; 1 \leq n \leq M.$$

Proof. Subtract (2.8) from (2.133) to find

$$\begin{aligned} b(\boldsymbol{\phi}_h, P^n - p_h^n) =& (\mathbf{u}_{ht}^n, \boldsymbol{\phi}_h) - (\partial_t \mathbf{u}_h^n, \boldsymbol{\phi}_h) - (\partial_t \mathbf{e}_n, \boldsymbol{\phi}_h) - \nu \, a(\mathbf{e}_n, \boldsymbol{\phi}_h) \\ & + c^{\mathbf{u}_h^n}(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}^{n-1}}(\mathbf{U}^{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{X}_h. \end{aligned} \tag{2.170}$$

Using Lemma 1.8 and bounding the terms on the right hand side of (2.170) following the steps involved in the proof of Lemma 2.19, we arrive at

$$\|P^n - p_h^n\| \leq C\|\partial_t \mathbf{e}_n\|_{-1,h} + C\|\mathbf{e}_n\|_\varepsilon + C\Delta t^{1/2}.$$

Finally, we complete the rest of the proof by applying the Lemmas 2.18 and 2.19. □

The following error estimate for the pressure is easily derived from Lemma 2.20 and Theorem 2.2.

Theorem 2.4. *Suppose the assumptions of Theorem 2.2 and Lemma 2.20 are satisfied. Then, the following estimates hold true:*

$$\|p^n - P^n\| \leq K_T(h^r + \Delta t^{1/2}),$$

where K_T depends on T.

Remark 2.5. *The optimal results derived in this chapter can be extended to the 3D case. The analysis would vary regarding trace inequalities [58, 96], inverse inequalities [58], Sobolev embeddings [107], etc. The main difference in the 3D case is the estimate of the nonlinear term. The existing L^p bounds for 3D case [107]*

$$\|\boldsymbol{\phi}\|_{L^p(\Omega)} \leq C_p\|\boldsymbol{\phi}\|_\varepsilon, \quad \forall \boldsymbol{\phi} \in \boldsymbol{X}_h, \quad p \in [2,6]$$

can be used to estimate the L^6-norm but is not sufficient for the L^3-norm estimate. A modified estimate for the functions in discrete space $\boldsymbol{X}_h$, that is, (see [101, eq. (23)])

$$\|\boldsymbol{\phi}\|_{L^3(\Omega)} \leq C\|\boldsymbol{\phi}\|^{1/2}\|\boldsymbol{\phi}\|_\varepsilon^{1/2}, \quad \forall \boldsymbol{\phi} \in \boldsymbol{X}_h$$

will lead to the same semi-discrete and fully discrete convergence rates of velocity and pressure as in the 2D case.

2.7 Numerical Experiments

In this section, we conduct a few numerical experiments to validate the theoretical results stated in Theorems 2.3 and 2.4. The numerical results reproduce results that already exist in the literature. We included them in our work for completeness. We discretize in the space direction by means of the mixed finite element spaces $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$, and discretization in the time direction is obtained by employing a backward Euler method. The domain $\Omega = [0,1] \times [0,1]$ and the time step $\Delta t = \mathcal{O}(h^{r+1})$ $(r = 1, 2)$ are chosen here. We have considered three examples and all of them are computed with time intervals $[0, 1]$.

Example 2.1. *Consider the transient NSEs (2.1)-(2.4) with exact solution $(\boldsymbol{u}, p) = ((u_1, u_2), p)$ as*

$$\begin{aligned} u_1 &= 2x^2(x-1)^2 y(y-1)(2y-1)e^t, \qquad p = 2(x-y)e^t, \\ u_2 &= -2x(x-1)(2x-1)y^2(y-1)^2 e^t. \end{aligned}$$

In Tables 2.1 and 2.2, we depict the fully discrete errors and convergence rates for the mixed finite element spaces $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$, respectively, with viscosity $\nu = 1$. In Tables 2.3 and 2.4, we represent the numerical results for the mixed finite element spaces $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$, respectively, with viscosity $\nu = 1/10$. Tables 2.5 and 2.6 represent the numerical results for the mixed finite element spaces $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$, respectively, with viscosity $\nu = 1/100$, and Tables 2.7 and 2.8 are for the mixed finite element spaces $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$, respectively, with viscosity $\nu = 1/1000$. For Tables 2.1-2.8, the penalty parameter is chosen as $\sigma_e = 50$. It is worth noticing that the numerical results of Tables 2.1-2.8 support the theoretical convergence rates proved in Theorems 2.3 and 2.4.

Table 2.1: Errors and convergence rates of DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ finite element for Example 2.1 ($\nu = 1$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	6.2539×10^{-2}		5.7822×10^{-3}		1.5885×10^{-1}	
1/8	3.9839×10^{-2}	0.6506	3.9495×10^{-3}	0.5500	1.6543×10^{-1}	-0.0586
1/16	1.8768×10^{-2}	1.0859	1.7199×10^{-3}	1.1994	1.2821×10^{-1}	0.3677
1/32	7.5888×10^{-3}	1.3063	5.5009×10^{-4}	1.6446	7.9055×10^{-2}	0.6976
1/64	3.2342×10^{-3}	1.2304	1.5047×10^{-4}	1.8702	4.2956×10^{-3}	0.8810
1/128	1.5216×10^{-3}	1.0879	3.8856×10^{-5}	1.9533	2.2185×10^{-3}	0.9533

Table 2.2: Errors and convergence rates of DG approximations using $\mathbb{P}_2$–$\mathbb{P}_1$ finite element for Example 2.1 ($\nu = 1$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	1.8905×10^{-2}		1.1509×10^{-3}		5.5945×10^{-2}	
1/8	4.3667×10^{-3}	2.1141	1.2261×10^{-4}	3.2306	1.6099×10^{-2}	1.7970
1/16	9.6177×10^{-4}	2.1827	1.1971×10^{-5}	3.3564	4.2294×10^{-3}	1.9284
1/32	2.2560×10^{-4}	2.0919	1.3053×10^{-6}	3.1970	1.0736×10^{-3}	1.9779

Table 2.3: Errors and convergence rates of DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ finite element for Example 2.1 ($\nu = 1/10$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	1.5691×10^{-1}		1.2479×10^{-2}		7.1783×10^{-2}	
1/8	7.1694×10^{-2}	1.1300	7.2256×10^{-3}	0.7883	4.6913×10^{-2}	0.6136
1/16	3.2350×10^{-2}	1.1480	2.7143×10^{-3}	1.4125	2.7718×10^{-2}	0.7591
1/32	1.4573×10^{-2}	1.1504	8.4103×10^{-4}	1.6904	1.5183×10^{-2}	0.8683
1/64	6.8472×10^{-3}	1.0897	2.2792×10^{-4}	1.8836	7.9215×10^{-3}	0.9386
1/128	3.3314×10^{-3}	1.0393	5.8652×10^{-5}	1.9583	4.0369×10^{-3}	0.9725

Table 2.4: Errors and convergence rates of DG approximations using $\mathbb{P}_2$–$\mathbb{P}_1$ finite element for Example 2.1 ($\nu = 1/10$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	4.2688×10^{-2}		3.3297×10^{-3}		5.7177×10^{-3}	
1/8	1.0160×10^{-2}	2.0708	3.7152×10^{-4}	3.1639	1.6144×10^{-3}	1.8244
1/16	2.2498×10^{-3}	2.1751	3.8666×10^{-5}	3.2643	4.2307×10^{-4}	1.9320
1/32	5.2776×10^{-4}	2.0919	4.4856×10^{-6}	3.1077	1.0736×10^{-4}	1.9783

Table 2.5: Errors and convergence rates of DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ finite element for Example 2.1 ($\nu = 1/100$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	1.4067×10^{0}		1.0953×10^{-1}		7.6389×10^{-2}	
1/8	6.1611×10^{-1}	1.1911	2.4106×10^{-2}	2.1839	4.4502×10^{-2}	0.7795
1/16	2.7425×10^{-1}	1.1677	5.3894×10^{-3}	2.1611	2.4692×10^{-2}	0.8498
1/32	1.2733×10^{-1}	1.1070	1.2621×10^{-3}	2.0943	1.3037×10^{-2}	0.9215
1/64	6.1066×10^{-2}	1.0601	3.0476×10^{-4}	2.0501	6.7001×10^{-3}	0.9603
1/128	2.9868×10^{-2}	1.0318	7.5028×10^{-5}	2.0221	3.3965×10^{-3}	0.9801

Table 2.6: Errors and convergence rates of DG approximations using $\mathbb{P}_2$–$\mathbb{P}_1$ finite element for Example 2.1 ($\nu = 1/100$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	7.6821×10^{-2}		5.2191×10^{-3}		6.1674×10^{-4}	
1/8	1.9599×10^{-2}	1.9707	7.0828×10^{-4}	2.8814	1.6318×10^{-4}	1.9181
1/16	4.4009×10^{-3}	2.1549	8.3562×10^{-5}	3.0834	4.2363×10^{-5}	1.9456
1/32	1.0326×10^{-3}	2.0915	1.0101×10^{-5}	3.0483	1.0739×10^{-5}	1.9798

Table 2.7: Errors and convergence rates of DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ finite element for Example 2.1 ($\nu = 1/1000$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	9.2020×10^{0}		6.2784×10^{-1}		2.0811×10^{-1}	
1/8	5.4485×10^{0}	0.7560	2.0503×10^{-1}	1.6145	6.2463×10^{-2}	1.7363
1/16	2.6725×10^{0}	1.0276	5.1025×10^{-2}	2.0065	2.6400×10^{-2}	1.2424
1/32	1.2665×10^{0}	1.0773	1.2003×10^{-2}	2.0878	1.3222×10^{-2}	0.9976
1/64	6.0925×10^{-1}	1.0557	2.8714×10^{-3}	2.0635	6.7133×10^{-3}	0.9778
1/128	2.9817×10^{-1}	1.0308	7.0076×10^{-4}	2.0347	3.3930×10^{-3}	0.9844

Table 2.8: Errors and convergence rates of DG approximations using $\mathbb{P}_2$–$\mathbb{P}_1$ finite element for Example 2.1 ($\nu = 1/1000$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	8.4772×10^{-2}		7.4192×10^{-3}		7.3153×10^{-5}	
1/8	2.3096×10^{-2}	1.8759	9.9196×10^{-4}	2.9029	1.7204×10^{-5}	2.0881
1/16	5.4881×10^{-3}	2.0733	1.2225×10^{-4}	3.0204	4.2842×10^{-6}	2.0056
1/32	1.2984×10^{-3}	2.0795	1.4908×10^{-5}	3.0357	1.0778×10^{-6}	1.9909

Example 2.2. *In this example, the choice of right-hand side source function* $\mathbf{f}$ *is made in such a manner that the exact solution* $(\mathbf{u}, p) = ((u_1, u_2), p)$ *takes the following form:*

$$u_1 = \sin(2\pi(x-t))\sin(2\pi(y-t)), \qquad p = sin(2\pi(x-t))\cos(2\pi(y-t)),$$
$$u_2 = \cos(2\pi(x-t))\cos(2\pi(y-t)).$$

We have shown the fully discrete errors and convergence rates in Tables 2.9-2.14 for the approximate velocity and pressure. Tables 2.9 and 2.10 are based on $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$ mixed finite element spaces, respectively, with $\nu = 1$, and Tables 2.11 and 2.12 are based on $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$ mixed finite element spaces, respectively, with $\nu = 1/10$. Tables 2.13 and 2.14 are based on the numerical results for the mixed finite element spaces $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$, respectively, with viscosity $\nu = 1/100$. It can be noted that for Tables 2.9-2.14, $\sigma_e = 50$. The numerical outcomes depicted in the tables verify the derived theoretical results.

Table 2.9: Errors and convergence rates of DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ finite element space for Example 2.2 ($\nu = 1$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	3.2574×10^0		2.5555×10^{-1}		18.1219×10^0	
1/8	2.3538×10^0	0.4687	1.8313×10^{-1}	0.4807	11.7638×10^0	0.6233
1/16	1.1564×10^0	1.0253	8.7605×10^{-2}	1.0637	8.0837×10^0	0.5412
1/32	4.6251×10^{-1}	1.3221	2.9031×10^{-2}	1.5934	4.7996×10^0	0.7520
1/64	1.9130×10^{-1}	1.2736	7.9618×10^{-3}	1.8664	2.5576×10^0	0.9081
1/128	8.8534×10^{-2}	1.1115	2.0442×10^{-3}	1.9615	1.3060×10^0	0.9695

Table 2.10: Errors and convergence rates of DG approximations using $\mathbb{P}_2$–$\mathbb{P}_1$ finite element space for Example 2.2 ($\nu = 1$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	1.2165×10^0		7.4256×10^{-2}		3.4015×10^0	
1/8	2.6504×10^{-1}	2.1984	8.3078×10^{-3}	3.1599	9.0585×10^{-1}	1.9088
1/16	5.4251×10^{-2}	2.2885	8.0436×10^{-4}	3.3685	2.2941×10^{-1}	1.9812
1/32	1.2332×10^{-2}	2.1372	8.5524×10^{-5}	3.2334	5.7646×10^{-2}	1.9926

Table 2.11: Errors and convergence rates of DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ finite element for Example 2.2 ($\nu = 1/10$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	4.9395×10^0		4.5985×10^{-1}		2.0448×10^0	
1/8	3.5060×10^0	0.4945	3.3428×10^{-1}	0.4601	1.3675×10^0	0.5803
1/16	1.6701×10^0	1.0699	1.4395×10^{-1}	1.2154	8.6774×10^{-1}	0.6562
1/32	6.6862×10^{-1}	1.3207	4.4712×10^{-2}	1.6869	4.9209×10^{-1}	0.8183
1/64	2.8437×10^{-1}	1.2334	1.1953×10^{-2}	1.9032	2.5768×10^{-1}	0.9333
1/128	1.3398×10^{-1}	1.0857	3.0441×10^{-3}	1.9734	1.3090×10^{-1}	0.9770

Table 2.12: Errors and convergence rates of DG approximations using $\mathbb{P}_2$–$\mathbb{P}_1$ finite element for Example 2.2 ($\nu = 1/10$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	1.9587×10^0		1.7700×10^{-1}		6.1914×10^{-1}	
1/8	3.9841×10^{-1}	2.2975	2.9831×10^{-2}	2.5688	1.2164×10^{-1}	2.3475
1/16	6.5895×10^{-2}	2.5960	3.7340×10^{-3}	2.9980	2.7611×10^{-2}	2.1393
1/32	1.3091×10^{-2}	2.3315	4.5668×10^{-4}	3.0314	6.7838×10^{-3}	2.0251

Table 2.13: Errors and convergence rates of DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ finite element space for Example 2.2 ($\nu = 1/100$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	7.2896×10^0		9.9750×10^{-1}		4.8542×10^{-1}	
1/8	3.9428×10^0	0.8866	4.1141×10^{-1}	1.2777	2.8460×10^{-1}	0.7702
1/16	1.6268×10^0	1.2772	1.2148×10^{-1}	1.7598	1.1640×10^{-1}	1.2898
1/32	6.8318×10^{-1}	1.2517	3.1488×10^{-2}	1.9479	5.4180×10^{-2}	1.1032
1/64	3.1570×10^{-1}	1.1137	7.9203×10^{-3}	1.9912	2.6673×10^{-2}	1.0223
1/128	1.5417×10^{-1}	1.0340	1.9763×10^{-3}	2.0027	1.3298×10^{-2}	1.0041

Table 2.14: Errors and convergence rates of DG approximations using $\mathbb{P}_2$–$\mathbb{P}_1$ finite element space for Example 2.2 ($\nu = 1/100$)

h	$\Vert\mathbf{u}(T) - \mathbf{U}^M\Vert_\varepsilon$	Rate	$\Vert\mathbf{u}(T) - \mathbf{U}^M\Vert$	Rate	$\Vert p(T) - P^M\Vert$	Rate
1/4	2.0132×10^0		3.4900×10^{-1}		2.5990×10^{-1}	
1/8	4.1700×10^{-1}	2.2713	4.6990×10^{-2}	2.8928	5.8091×10^{-2}	2.1615
1/16	8.4263×10^{-2}	2.3070	5.8522×10^{-3}	3.0053	1.4389×10^{-2}	2.0133
1/32	1.8693×10^{-2}	2.1723	7.2406×10^{-4}	3.0148	3.6225×10^{-3}	1.9899

Example 2.3 (Taylor-Green vortex)**.** *Another widely used test case for the verification of numerical methods for the incompressible NSEs is the TaylorGreen vortex problem. The analytical unsteady solution is* $(\boldsymbol{u}, p) = ((u_1, u_2), p)$*, where*

$$u_1 = \cos(2\pi x)\sin(2\pi y)e^{-8\pi^2 \nu t}, \qquad p = -\frac{1}{4}(\cos(4\pi x) + \cos(4\pi y))e^{-16\pi^2 \nu t},$$
$$u_2 = -\sin(2\pi x)\cos(2\pi y)e^{-8\pi^2 \nu t}.$$

The initial condition is obtained from the above exact solution.

We have considered $\mathbb{P}_1-\mathbb{P}_0$ and $\mathbb{P}_2-\mathbb{P}_1$ DG cases with $\nu = 1/100$ and $\sigma_e = 100$. The numerical convergence results are shown in Tables 2.15 and 2.16 for the cases $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$, respectively. We find that the optimal convergence rates are achieved for this important nontrivial test problem with periodic boundary conditions.

Table 2.15: Errors and convergence rates of DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ finite element space for Example 2.3 ($\nu = 1/100$)

h	$\Vert\mathbf{u}(T) - \mathbf{U}^M\Vert_\varepsilon$	Rate	$\Vert\mathbf{u}(T) - \mathbf{U}^M\Vert$	Rate	$\Vert p(T) - P^M\Vert$	Rate
1/4	2.7601×10^0		3.3217×10^{-1}		9.1378×10^{-2}	
1/8	1.6546×10^0	–	1.8703×10^{-1}	–	9.5355×10^{-2}	–
1/16	7.6371×10^{-1}	1.1154	7.5382×10^{-2}	1.3109	6.9945×10^{-2}	0.4470
1/32	2.8430×10^{-1}	1.4256	2.2804×10^{-2}	1.7248	4.3044×10^{-2}	0.7004
1/64	1.0942×10^{-1}	1.3774	5.9644×10^{-3}	1.9348	2.3518×10^{-2}	0.8720
1/128	5.1111×10^{-2}	1.0981	1.4877×10^{-3}	2.0032	1.2366×10^{-2}	0.9272

Table 2.16: Errors and convergence rates of DG approximations using $\mathbb{P}_2$–$\mathbb{P}_1$ finite element space for Example 2.3 ($\nu = 1/100$)

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	1.4875×10^{0}		6.2938×10^{-2}		4.1092×10^{-2}	
1/8	2.7512×10^{-1}	2.4348	6.6431×10^{-3}	3.2440	1.1988×10^{-2}	1.7772
1/16	6.7602×10^{-2}	2.0249	8.1619×10^{-4}	3.0248	2.9784×10^{-3}	2.0090
1/32	1.6961×10^{-2}	1.9948	1.0191×10^{-4}	3.0015	7.3905×10^{-4}	2.0108

2.8 Conclusion

This chapter discusses a DG finite element method for the incompressible time dependent NSEs. With the help of L^2-projection and modified Stokes operator on appropriate discontinuous spaces, semi-discrete optimal error estimates are derived for the velocity in $L^\infty(\mathbf{L}^2)$ norm and for the pressure in $L^\infty(L^2)$ norm. Under the smallness assumption on data, the error estimates are shown to be uniform in time. Then, in the time direction, a first order accurate backward Euler scheme is employed to achieve complete discretization. And fully discrete discontinuous error estimates of velocity and pressure are derived. Finally, with the help of numerical experiments, theoretical results are confirmed.