Chapter 3

DG Method for the Kelvin-Voigt Model

A DG finite element approximation for the Kelvin-Voigt viscoelastic fluid flow equations, is proposed and analysed in this chapter. Based on the new *a priori* and regularity results for the semi-discrete solutions, well-posedness and consistency of the DG scheme are discussed. *A priori* error estimates of the semi-discrete DG approximations of the velocity and pressure, in $L^\infty(\mathbf{L}^2)$ and $L^\infty(L^2)$-norms, respectively, are then derived. Our proof relies on the standard elliptic duality argument and a modified Sobolev-Stokes operator defined on appropriate broken Sobolev spaces. For sufficiently small data, uniform in time error estimates are proved. Furthermore, backward Euler scheme is considered for a full discretization and optimal fully discrete error estimates are derived. Finally we work out numerical experiments to substantiate our theoretical findings. It is worth mentioning that the analysis here is the first of its kind for the Kelvin-Voigt model. This work has been published in [17] for $r = 1$.

3.1 Introduction

Let us recall, the Kelvin-Voigt viscoelastic fluid flow is modelled by the following momentum and continuity equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \kappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(x,t), \quad x\in\Omega,\ t>0, \tag{3.1}$$

$$\nabla\cdot\mathbf{u} = 0, \quad x\in\Omega,\ t>0, \tag{3.2}$$

where, $\kappa > 0$ is the retardation time. The velocity $\mathbf{u}$ further satisfies the initial and homogeneous Dirichlet boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \text{and} \qquad \mathbf{u} = \mathbf{0}, \quad \text{on } \partial\Omega,\ t \geq 0, \tag{3.3}$$

where $\partial\Omega$ represents the boundary of Ω.

As can be seen from Section 1.5.1, the literature for the problem (3.1)-(3.3) is confined to the finite element analysis for the CG methods. And, to the best of our knowledge, there is hardly any literature dedicated to the finite element analysis of DG methods applied to the Kelvin-Voigt equations of motion. This chapter can be considered as the first attempt in this direction. We mainly focus on deriving semi-discrete and fully discrete optimal error estimates for the SIPG method applied to the problem (3.1)-(3.3) as the NIPG and IIPG methods is known to provide sub-optimal error estimates which has been already discussed in Chapter 2. In the earlier chapters, we have mentioned that the Kelvin-Voigt model is a perturbation of the NSEs. Thus, in this chapter, we have followed the DG variational formulation for NSEs from Chapter 2 and defined a DG formulation for (3.1)-(3.3) on the discontinuous spaces $\mathbf{X}$ and M as: Find the pair $(\mathbf{u}(t), p(t)) \in \mathbf{X} \times M,\ t > 0$, such that

$$\begin{aligned}(\mathbf{u}_t(t), \boldsymbol{\phi}) &+ \kappa\, a(\mathbf{u}_t(t), \boldsymbol{\phi}) + \nu\, a(\mathbf{u}(t), \boldsymbol{\phi}) \\ &+ c^{\mathbf{u}(t)}(\mathbf{u}(t), \mathbf{u}(t), \boldsymbol{\phi}) + b(\boldsymbol{\phi}, p(t)) = (\mathbf{f}(t), \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathbf{X}, \end{aligned} \tag{3.4}$$

$$b(\mathbf{u}(t), q) = 0 \quad \forall q \in M, \tag{3.5}$$

$$(\mathbf{u}(0), \boldsymbol{\phi}) = (\mathbf{u}_0, \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathbf{X}. \tag{3.6}$$

The consistency proof of (3.4)-(3.6) can be done following the similar analysis as adopted in [98, Lemma 3.2] for the DG formulation of NSE.

Next, we define the semi-discrete DG variational formulation for the system of equations (3.1)-(3.3): For $t > 0$, find $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{X}_h \times M_h$ such that

$$\begin{aligned}(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) &+ \kappa\, a(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h(t), \boldsymbol{\phi}_h) \\ &+ c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t), \mathbf{u}_h(t), \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, p_h(t)) = (\mathbf{f}(t), \boldsymbol{\phi}_h),\ \ \forall \boldsymbol{\phi}_h \in \mathbf{X}_h \end{aligned} \tag{3.7}$$

$$b(\mathbf{u}_h(t), q_h) = 0,\ \ \forall q_h \in M_h \tag{3.8}$$

$$(\mathbf{u}_h(0), \boldsymbol{\phi}_h) = (\mathbf{u}_0, \boldsymbol{\phi}_h)\ \ \forall \boldsymbol{\phi}_h \in \mathbf{X}_h. \tag{3.9}$$

The equivalent DG formulation corresponding to the scheme (3.7)–(3.9) on the space $\mathbf{V}_h$ is the following: For $t > 0$, find $\mathbf{u}_h(t) \in \mathbf{V}_h$ such that

$$(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h(t), \boldsymbol{\phi}_h) + c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t), \mathbf{u}_h(t), \boldsymbol{\phi}_h) = (\mathbf{f}(t), \boldsymbol{\phi}_h), \ \forall \boldsymbol{\phi}_h \in \mathbf{V}_h. \quad (3.10)$$

In order to establish optimal semi-discrete error estimates related to the DG method, we have introduced a modified Sobolev-Stokes projection $\mathbf{S}_h^{so}$ (see Section 3.3) for broken Sobolev spaces. Optimal order estimates for $\mathbf{S}_h^{so}$ have been established then, based on the approximation properties of $\mathbf{P}_h$ (see Chapter 2). Although we have applied the ideas of [15], there are analytical differences and difficulties due to the DG formulation and difference in finite element spaces. For example, the analysis of the nonlinear term in the DG formulation needs a special kind of attention. Finally, the backward Euler method have been applied to discretize the time variable and optimal fully discrete error estimates are achieved.
The main ingredients in achieving the goals of the chapter are as follows:

1. We have defined a modified Sobolev-Stokes projection $\mathbf{S}_h^{so}$ in DG set up, which plays an essential role in deriving the semi-discrete error estimates. The optimal estimates for $\mathbf{S}_h^{so}$ are derived.

2. By means of the modified Sobolev-Stokes projection $\mathbf{S}_h^{so}$ and duality arguments, we have achieved optimal *a priori* error bounds for the semi-discrete DG approximations to the velocity in $L^\infty(\mathbf{L}^2)$-norm and pressure in $L^\infty(L^2)$-norm. These estimates are uniform in time for sufficiently small data.

3. Then the backward Euler scheme have been applied to the semi-discrete discontinuous Kelvin-Voigt model. Optimal error estimates have been derived for the fully discrete velocity and pressure.

4. Finally, we have provided numerical examples and analyze the outcomes to verify the theoretical results.

This chapter is divided into the following sections: The derivation of *a priori* and regularity bounds of the discrete solution are dealt with in Section 3.2. The modified Sobolev-Stokes operator and its properties, and optimal *a priori* error estimates for the velocity are represented in Section 3.3. The optimal *a priori* error estimates for the pressure are derived in Section 3.4. The backward Euler method for the discretization in the time direction is employed, and the fully discrete error estimates are obtained in Section 3.5. A few numerical examples are discussed, and the results are analyzed to verify the theoretical findings in Section 3.6. Finally, the main contributions of this chapter are summarized in Section 3.7.

Throughout this chapter, we will use C, $K(>0)$ as generic constants that depend on the given data, ν, κ, α, K_1, K_2, C_2 but do not depend on h and Δt. Note that, K and C may grow algebraically with ν^{-1}. Further, the notations $K(t)$ and K_T will be used when they grow exponentially in time.

3.2 *A priori* and Regularity Estimates

We start this section by presenting *a priori* and regularity bounds for $\mathbf{u}_h$ which will be used in deriving the existence and uniqueness of the semi-discrete solution and fully discrete error estimates.

Lemma 3.1. *Let $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. Then, there exists a positive constant C, such that, for each $t > 0$, the semi-discrete DG solution $\boldsymbol{u}_h(t)$, satisfies the following bounds:*

$$\sup_{0<t<\infty} \left(\|\boldsymbol{u}_h(t)\| + \|\boldsymbol{u}_h(t)\|_\varepsilon + \|\boldsymbol{u}_{ht}(t)\|_\varepsilon\right) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\left(\|\boldsymbol{u}_h(s)\|_\varepsilon^2 + \|\boldsymbol{u}_{ht}(s)\|_\varepsilon^2\right) ds \le C, \tag{3.11}$$

$$\sup_{0<t<\infty} \left(\|\boldsymbol{u}_{htt}(t)\| + \|\boldsymbol{u}_{htt}(t)\|_\varepsilon\right) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\left(\|\boldsymbol{u}_{hss}(s)\|^2 + \|\boldsymbol{u}_{hss}(s)\|_\varepsilon^2\right) ds \le C. \tag{3.12}$$

Moreover,

$$\limsup_{t\to\infty} \|\boldsymbol{u}_h(t)\|_\varepsilon \le \frac{C_2\|\mathbf{f}\|_{L^\infty(\boldsymbol{L}^2(\Omega))}}{K_1\nu}. \tag{3.13}$$

Proof. Choose $\boldsymbol{\phi}_h = \mathbf{u}_h$ in (3.10) and apply the coercivity result from Lemma 1.6, positivity of $c(\cdot,\cdot,\cdot)$ (1.19), estimate (1.14), the Cauchy-Schwarz inequality and Young's inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}_h\|^2 + \kappa\, a(\mathbf{u}_h,\mathbf{u}_h)\right) + \nu K_1\|\mathbf{u}_h\|_\varepsilon^2 \le \|\mathbf{f}\|\|\mathbf{u}_h\| \le \frac{\nu K_1}{2}\|\mathbf{u}_h\|_\varepsilon^2 + \frac{C_2^2}{2\nu K_1}\|\mathbf{f}\|^2. \tag{3.14}$$

A multiplication of (3.14) by $e^{2\alpha t}$, an integration from 0 to t, and an application of estimate (1.14), Lemmas 1.6 and 1.7, lead to

$$e^{2\alpha t}(\|\mathbf{u}_h(t)\|^2 + K_1\kappa\|\mathbf{u}_h(t)\|_\varepsilon^2) + (\nu K_1 - 2\alpha(C_2+\kappa K_2))\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\,ds$$
$$\leq \|\mathbf{u}_h(0)\|^2 + K_2\kappa\|\mathbf{u}_h(0)\|_\varepsilon^2 + C\int_0^t e^{2\alpha s}\|\mathbf{f}(s)\|^2\,ds. \quad (3.15)$$

Again, multiply (3.15) by $e^{-2\alpha t}$, use the fact that

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\,ds = \frac{1}{2\alpha}(1-e^{-2\alpha t})$$

and choose $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$ to obtain

$$\|\mathbf{u}_h(t)\|^2 + \|\mathbf{u}_h(t)\|_\varepsilon^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\,ds \leq C. \quad (3.16)$$

Again, multiply (3.14) by $e^{2\alpha t}$, integrate from 0 to t, and a use of Lemma 1.7 implies

$$e^{2\alpha t}\big(\|\mathbf{u}_h(t)\|^2 + \kappa\, a(\mathbf{u}_h(t),\mathbf{u}_h(t))\big) + \nu K_1\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\,ds$$
$$\leq \big(\|\mathbf{u}_h(0)\|^2 + \kappa K_2\|\mathbf{u}_h(0)\|_\varepsilon^2\big) + 2\alpha\int_0^t e^{2\alpha s}\big(\|\mathbf{u}_h(s)\|^2 + \kappa\, a(\mathbf{u}_h(s),\mathbf{u}_h(s))\big)\,ds$$
$$+ (e^{2\alpha t}-1)\frac{C_2^2\|\mathbf{f}\|_{L^\infty(\mathbf{L}^2(\Omega))}^2}{2\alpha\nu K_1}.$$

Multiply the above inequality by $e^{-2\alpha t}$, take limit supremum as $t\to\infty$ and noting that,

$$\nu K_1\limsup_{t\to\infty} e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\,ds = \frac{\nu K_1}{2\alpha}\limsup_{t\to\infty}\|\mathbf{u}_h(t)\|_\varepsilon^2,$$

we arrive at

$$\frac{\nu K_1}{2\alpha}\limsup_{t\to\infty}\|\mathbf{u}_h(t)\|_\varepsilon^2 \leq \frac{C_2^2\|\mathbf{f}\|_{L^\infty(\mathbf{L}^2(\Omega))}^2}{2\alpha\nu K_1}. \quad (3.17)$$

Next, differentiating (3.10) with respect to t, we obtain

$$(\mathbf{u}_{htt},\boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{htt},\boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_{ht},\boldsymbol{\phi}_h)$$
$$+ c^{\mathbf{u}_h}(\mathbf{u}_{ht},\mathbf{u}_h,\boldsymbol{\phi}_h) + c^{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{u}_{ht},\boldsymbol{\phi}_h) = (\mathbf{f}_t,\boldsymbol{\phi}_h),\ \forall\,\boldsymbol{\phi}_h\in\mathbf{V}_h. \quad (3.18)$$

Substitute $\boldsymbol{\phi}_h = \mathbf{u}_{ht}$ in (3.18), apply Lemma 1.6, the Cauchy-Schwarz and Young's inequalities, and the fact that $c^{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{u}_{ht},\mathbf{u}_{ht}) \geq 0$ from (1.19), we obtain

$$\frac{d}{dt}\big(\|\mathbf{u}_{ht}\|^2 + \kappa\, a(\mathbf{u}_{ht},\mathbf{u}_{ht})\big) + 2\nu K_1\|\mathbf{u}_{ht}\|_\varepsilon^2 \leq -2c^{\mathbf{u}_h}(\mathbf{u}_{ht},\mathbf{u}_h,\mathbf{u}_{ht})$$

$$+C\|\mathbf{f}_t\|^2+\frac{\nu K_1}{2}\|\mathbf{u}_{ht}\|_\varepsilon^2. \tag{3.19}$$

Using (2.57) and Young's inequality, we can bound $2c^{\mathbf{u}_h}(\mathbf{u}_{ht},\mathbf{u}_h,\mathbf{u}_{ht})$ as follows:

$$|2c^{\mathbf{u}_h}(\mathbf{u}_{ht},\mathbf{u}_h,\mathbf{u}_{ht})|\le C\|\mathbf{u}_{ht}\|^{1/2}\|\mathbf{u}_{ht}\|_\varepsilon^{3/2}\|\mathbf{u}_h\|_\varepsilon\le\frac{\nu K_1}{2}\|\mathbf{u}_{ht}\|_\varepsilon^2+C\|\mathbf{u}_{ht}\|^2\|\mathbf{u}_h\|_\varepsilon^4.$$

Applying the above bound in (3.19), we arrive at

$$\frac{d}{dt}\big(\|\mathbf{u}_{ht}\|^2+\kappa\,a(\mathbf{u}_{ht},\mathbf{u}_{ht})\big)+\nu K_1\|\mathbf{u}_{ht}\|_\varepsilon^2\le C\|\mathbf{u}_{ht}\|^2\|\mathbf{u}_h\|_\varepsilon^4+C\|\mathbf{f}_t\|^2. \tag{3.20}$$

Multiply (3.20) by $e^{2\alpha t}$, integrate from 0 to t, and finally use Lemmas 1.6 and 1.7 to obtain

$$\begin{aligned}&e^{2\alpha t}(\|\mathbf{u}_{ht}(t)\|^2+\kappa K_1\|\mathbf{u}_{ht}(t)\|_\varepsilon^2)+(\nu K_1-2\alpha(C_2+\kappa K_2))\int_0^t e^{2\alpha s}\|\mathbf{u}_{ht}(s)\|_\varepsilon^2\,ds\\ &\le\|\mathbf{u}_{ht}(0)\|^2+K_2\kappa\|\mathbf{u}_{ht}(0)\|_\varepsilon^2+C\int_0^t e^{2\alpha s}\|\mathbf{u}_{hs}(s)\|^2\|\mathbf{u}_h(s)\|_\varepsilon^4\,ds+C\int_0^t e^{2\alpha s}\|\mathbf{f}_s(s)\|^2\,ds.\end{aligned}$$

Choosing $0<\alpha<\frac{\nu K_1}{2(C_2+\kappa K_2)}$, and applying (3.16), Gronwall's lemma and after a final multiplication by $e^{-2\alpha t}$, we obtain the estimate as

$$\|\mathbf{u}_{ht}(t)\|^2+\|\mathbf{u}_{ht}(t)\|_\varepsilon^2+e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_{hs}(s)\|_\varepsilon^2\,ds\le C. \tag{3.21}$$

Now, we substitute $\boldsymbol{\phi}_h=\mathbf{u}_{htt}$ in (3.18), use Lemma 1.6 and obtain

$$\begin{aligned}\|\mathbf{u}_{htt}\|^2+K_1\kappa\,\|\mathbf{u}_{htt}\|_\varepsilon^2\le&-\nu\,a(\mathbf{u}_{ht},\mathbf{u}_{htt})-c^{\mathbf{u}_h}(\mathbf{u}_{ht},\mathbf{u}_h,\mathbf{u}_{htt})-c^{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{u}_{ht},\mathbf{u}_{htt})\\&-(\mathbf{f}_t,\mathbf{u}_{htt}).\end{aligned} \tag{3.22}$$

Apply (2.55), Lemma 1.7 and Young's inequality to obtain

$$\|\mathbf{u}_{htt}\|^2+K_1\kappa\,\|\mathbf{u}_{htt}\|_\varepsilon^2\le C(\|\mathbf{u}_{ht}\|_\varepsilon^2+\|\mathbf{u}_{ht}\|_\varepsilon^2\|\mathbf{u}_h\|_\varepsilon^2+\|\mathbf{f}_t\|^2). \tag{3.23}$$

A use of (3.16), (3.21) in (3.23) yield

$$\|\mathbf{u}_{htt}\|^2+\|\mathbf{u}_{htt}\|_\varepsilon^2\le C. \tag{3.24}$$

Finally, multiply (3.24) by $e^{2\alpha t}$ and integrate with respect to time from 0 to t. Then multiply the resulting inequality by $e^{-2\alpha t}$ to arrive at

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_{hss}(s)\|^2+\|\mathbf{u}_{hss}(s)\|_\varepsilon^2)ds\le C. \tag{3.25}$$

A combination of (3.16), (3.17), (3.21), (3.24) and (3.25) completes the proof of Lemma 3.1. □

Now, the existence and uniqueness of the semi-discrete discontinuous Galerkin Kelvin-Voigt model (3.7)-(3.9) (or (3.10)) can be proved following the analysis in [98, Lemma 3.4], and using the results in (1.19), (3.11), Lemmas 1.6 and 1.8.

For deriving the optimal error estimates for semi-discrete discontinuous velocity and pressure approximations, we work on the weakly divergence free spaces. Below, we provide one of our main contributions, stating the optimal semi-discrete error estimates.

Theorem 3.1. *Let the assumption (**A2**) be satisfied and let* $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. *Furthermore, let the discrete initial velocity* $\boldsymbol{u}_h(0) \in \boldsymbol{V}_h$ *with* $\boldsymbol{u}_h(0) = \boldsymbol{P}_h\boldsymbol{u}_0$. *Then, there exists a positive constant* K, *independent of* h, *such that*

$$\|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\| + h\|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\|_{\varepsilon} + h\|(p-p_h)(t)\| \leq K(t)h^{r+1},$$

where $K(t)$ *grows exponentially in time.*

The sections 3.3 and 3.4 are devoted to the proof of Theorem 3.1.

3.3 DG Error Estimates for Velocity

This section deals with the optimal estimates of the velocity error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ in $\mathbf{L}^2$ and energy-norms for $t > 0$. We start by analysing the linearized error and therefore introduce the solution $\mathbf{v}_h \in \mathbf{V}_h$ of a DG approximation of a linearized Kelvin-Voigt problem, *that is*, $\mathbf{v}_h$ is the solution of

$$(\mathbf{v}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{v}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{v}_h, \boldsymbol{\phi}_h) = (\mathbf{f}, \boldsymbol{\phi}_h) - c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h. \tag{3.26}$$

With the help of $\mathbf{v}_h$, we split $\mathbf{e}$ into two parts as $\mathbf{e} = (\mathbf{u} - \mathbf{v}_h) + (\mathbf{v}_h - \mathbf{u}_h) = \boldsymbol{\xi} + \boldsymbol{\Theta}$. Observe that, $\boldsymbol{\xi}$ is the error committed by approximating a linearized Kelvin-Voigt problem and $\boldsymbol{\Theta}$ represents the error due to the presence of the non-linearity in problem (3.1). From the equations (3.26) and (3.4), we have the following equation in $\boldsymbol{\xi}$ as

$$(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \nu\, a(\boldsymbol{\xi}, \boldsymbol{\phi}_h) = -b(\boldsymbol{\phi}_h, p), \quad \boldsymbol{\phi}_h \in \mathbf{V}_h. \tag{3.27}$$

For deriving the optimal error estimates of $\boldsymbol{\xi}$ in $\mathbf{L}^2$ and energy-norms for $t > 0$, we introduce, as in [15], the following modified Sobolev-Stokes's projection $\mathbf{S}_h^{so}\mathbf{u} : [0, \infty) \to \mathbf{V}_h$ satisfying

$$\kappa\, a(\mathbf{u}_t - \mathbf{S}_h^{so}\mathbf{u}_t, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u} - \mathbf{S}_h^{so}\mathbf{u}, \boldsymbol{\phi}_h) = -b(\boldsymbol{\phi}_h, p) \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h, \tag{3.28}$$

where $\mathbf{S}_h^{so}\mathbf{u}(0) = \mathbf{P}_h\mathbf{u}_0$. In other words, given $(\mathbf{u}, p)$, find $\mathbf{S}_h^{so}\mathbf{u} : [0, \infty) \to \mathbf{V}_h$ satisfying (3.28). With $\mathbf{S}_h^{so}\mathbf{u}$ defined as above, we now split $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi} = \mathbf{u} - \mathbf{S}_h^{so}\mathbf{u} + \mathbf{S}_h^{so}\mathbf{u} - \mathbf{v}_h =: \boldsymbol{\zeta} + \boldsymbol{\rho}.$$

Using (3.28), we find the equation in $\boldsymbol{\zeta}$ to be

$$\kappa\, a(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) + \nu\, a(\boldsymbol{\zeta}, \boldsymbol{\phi}_h) = -b(\boldsymbol{\phi}_h, p) \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h. \tag{3.29}$$

Firstly, we will focus on deriving the estimates of $\boldsymbol{\zeta}$. Next, we will establish the estimates of $\boldsymbol{\rho}$. A combination of these estimates will result in the estimates of $\boldsymbol{\xi}$.

Lemma 3.2. *Under the assumptions of Theorem 3.1, and for $t > 0$, $\boldsymbol{\zeta}$ satisfies the following estimates:*

$$\|\boldsymbol{\zeta}(t)\|_\varepsilon^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\left(\|\boldsymbol{\zeta}(s)\|_\varepsilon^2 + \|\boldsymbol{\zeta}_s(s)\|_\varepsilon^2\right) ds \le Ch^{2r}.$$

Proof. Set $\boldsymbol{\phi}_h = \mathbf{P}_h\boldsymbol{\zeta} = \boldsymbol{\zeta} - (\mathbf{u} - \mathbf{P}_h\mathbf{u})$ in (3.29), use the definition of space $\mathbf{V}_h$ and obtain

$$\begin{aligned}\kappa\, a(\mathbf{P}_h\boldsymbol{\zeta}_t, \mathbf{P}_h\boldsymbol{\zeta}) + \nu\, a(\mathbf{P}_h\boldsymbol{\zeta}, \mathbf{P}_h\boldsymbol{\zeta}) = &- \kappa\, a(\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \mathbf{P}_h\boldsymbol{\zeta}) - \nu\, a(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{P}_h\boldsymbol{\zeta}) \\ &- b(\mathbf{P}_h\boldsymbol{\zeta}, p - r_h(p)).\end{aligned} \tag{3.30}$$

By a virtue of Lemma 2.3 and Young's inequality, we arrive at

$$\kappa\,|a(\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \mathbf{P}_h\boldsymbol{\zeta})| \le C\kappa h^r|\mathbf{u}_t|_{r+1}\|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon \le \frac{K_1\nu}{24}\|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon^2 + Ch^{2r}|\mathbf{u}_t|_{r+1}^2, \tag{3.31}$$

$$\nu\,|a(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{P}_h\boldsymbol{\zeta})| \le C\nu h^r|\mathbf{u}|_{r+1}\|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon \le \frac{K_1\nu}{6}\|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon^2 + Ch^{2r}|\mathbf{u}|_{r+1}^2. \tag{3.32}$$

Owing to Lemma 2.4 and Young's inequality, the term involving the pressure in (3.30) is reduced to

$$|b(\mathbf{P}_h\boldsymbol{\zeta}, p - r_h p)| \le Ch^r|p|_r\|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon \le \frac{K_1\nu}{6}\|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon^2 + Ch^{2r}|p|_r^2. \tag{3.33}$$

Apply (3.31)-(3.33) and the bound of Lemma 1.6 in (3.30). Then, multiply the resulting equation by $e^{2\alpha t}$, integrate from 0 to t, use Lemmas 1.7 and 1.6, and observe that $P_h\boldsymbol{\zeta}(0) = 0$, we obtain

$$\kappa\, K_1\, e^{2\alpha t}\|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon^2 + (K_1\nu - 2\alpha\kappa\, K_2)\int_0^t e^{2\alpha s}\|\mathbf{P}_h\boldsymbol{\zeta}(s)\|_\varepsilon^2 ds$$

$$\leq Ch^{2r} \int_0^t e^{2\alpha s} \left(|\mathbf{u}_s(s)|_{r+1}^2 + |\mathbf{u}(s)|_{r+1}^2 + |p(s)|_r^2 \right) ds. \tag{3.34}$$

Multiply (3.34) by $e^{-2\alpha t}$ and use assumption (**A2**) to complete the energy norm estimates of $\mathbf{P}_h \boldsymbol{\zeta}$ as

$$\|\mathbf{P}_h \boldsymbol{\zeta}(t)\|_\varepsilon^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\mathbf{P}_h \boldsymbol{\zeta}(s)\|_\varepsilon^2 ds \leq Ch^{2r}. \tag{3.35}$$

Since $\boldsymbol{\zeta} = \mathbf{u} - \mathbf{P}_h \mathbf{u} + \mathbf{P}_h \boldsymbol{\zeta}$, using the triangle inequality and the bounds in Lemma 2.2, (3.35), we arrive at

$$\|\boldsymbol{\zeta}(t)\|_\varepsilon^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\boldsymbol{\zeta}(s)\|_\varepsilon^2 ds \leq Ch^{2r}. \tag{3.36}$$

To derive the estimates of $\boldsymbol{\zeta}_t$ in energy norm, we substitute $\boldsymbol{\phi}_h = \mathbf{P}_h \boldsymbol{\zeta}_t$ in (3.29). Then, apply Young's inequality, (3.36), and Lemmas 1.6, 1.7, 2.2, 2.3, 2.4 and assumption (**A2**) to the resulting equation and arrive at

$$\|\mathbf{P}_h \boldsymbol{\zeta}_t(t)\|_\varepsilon^2 \leq Ch^{2r}. \tag{3.37}$$

A multiplication of (3.37) by $e^{2\alpha t}$, an integration from 0 to t with respect to time, and then again a multiplication by $e^{-2\alpha t}$ lead to

$$e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\mathbf{P}_h \boldsymbol{\zeta}_s(s)\|_\varepsilon^2 \, ds \leq Ch^{2r}. \tag{3.38}$$

Use triangle inequality and bounds of (3.37), (3.38), Lemma 2.2 yielding

$$\|\boldsymbol{\zeta}_t(t)\|_\varepsilon^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\boldsymbol{\zeta}_s(s)\|_\varepsilon^2 \, ds \leq Ch^{2r}. \tag{3.39}$$

Combining the estimates (3.36) and (3.39), we arrive at the desired result. □

Lemma 3.3. *Under the assumptions of Theorem 3.1, and for $t > 0$, $\boldsymbol{\zeta}$ satisfies the following estimates:*

$$\|\boldsymbol{\zeta}(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\boldsymbol{\zeta}(s)\|^2 \, ds \leq Ch^{2r+2}, \tag{3.40}$$

$$e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\boldsymbol{\zeta}_t(s)\|^2 \, ds \leq Ch^{2r+2}. \tag{3.41}$$

Proof. For the estimate of $\|\boldsymbol{\zeta}\|$, we apply the duality argument due to Aubin-Nitsche. Let $(\mathbf{v}, q) \in \mathbf{J}_1 \times L^2(\Omega)/\mathbb{R}$ be the pair of unique solution of the steady state Stokes system stated as

$$-\nu \Delta \mathbf{v} + \nabla q = \boldsymbol{\zeta} \quad \text{in } \Omega, \tag{3.42}$$

$$\nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v} = 0 \text{ on } \partial\Omega,$$

satisfying the following regularity:

$$\|\mathbf{v}\|_2 + \|q\|_1 \leq C\|\boldsymbol{\zeta}\|. \tag{3.43}$$

Forming L^2 inner product between (3.42) and $\boldsymbol{\zeta}$, and applying the regularity of $\mathbf{v}$ and q, we obtain

$$\begin{aligned}
\|\boldsymbol{\zeta}\|^2 =& \nu \sum_{E\in\mathcal{E}_h} \int_E \nabla\mathbf{v} : \nabla\boldsymbol{\zeta} - \nu \sum_{E\in\mathcal{E}_h} \int_{\partial E} (\nabla\mathbf{v}\mathbf{n}_E)\cdot\boldsymbol{\zeta} \\
& - \sum_{E\in\mathcal{E}_h} \int_E q\nabla\cdot\boldsymbol{\zeta} + \sum_{E\in\mathcal{E}_h} \int_{\partial E} q\mathbf{n}_E\cdot\boldsymbol{\zeta} \\
=& \nu \sum_{E\in\mathcal{E}_h} \int_E \nabla\boldsymbol{\zeta} : \nabla\mathbf{v} - \nu \sum_{e\in\Gamma_h} \int_e \{\nabla\mathbf{v}\}\mathbf{n}_e\cdot[\boldsymbol{\zeta}] + b(\boldsymbol{\zeta}, q).
\end{aligned}$$

Using (3.29) with $\mathbf{P}_h\mathbf{v}$ in place of $\boldsymbol{\phi}_h$, and observing $[\mathbf{v}]\cdot\mathbf{n}_e = 0$ on each interior edge and $b(\mathbf{v}, p - r_h(p)) = 0$, we find

$$\begin{aligned}
\|\boldsymbol{\zeta}(t)\|^2 =& \nu\, a(\boldsymbol{\zeta}, \mathbf{v} - \mathbf{P}_h\mathbf{v}) + \kappa\, a(\boldsymbol{\zeta}_t, \mathbf{v} - \mathbf{P}_h\mathbf{v}) + b(\boldsymbol{\zeta}, q) \\
& - b(\mathbf{P}_h\mathbf{v} - \mathbf{v}, p - r_h(p)) - \kappa\, a(\boldsymbol{\zeta}_t, \mathbf{v}).
\end{aligned} \tag{3.44}$$

For the last term of the above equality, we again form an L^2 inner product between (3.42) and $\boldsymbol{\zeta}_t$, use integration by parts and the fact that $[\mathbf{v}]\cdot\mathbf{n}_e = 0$ on each interior edge to derive the following:

$$\begin{aligned}
\|\boldsymbol{\zeta}(t)\|^2 =& \nu\, a(\boldsymbol{\zeta}, \mathbf{v} - \mathbf{P}_h\mathbf{v}) + \kappa\, a(\boldsymbol{\zeta}_t, \mathbf{v} - \mathbf{P}_h\mathbf{v}) + b(\boldsymbol{\zeta}, q) - b(\mathbf{P}_h\mathbf{v} - \mathbf{v}, p - r_h(p)) \\
& + \frac{\kappa}{\nu} b(\boldsymbol{\zeta}_t, q) - \frac{\kappa}{\nu}(\boldsymbol{\zeta}, \boldsymbol{\zeta}_t).
\end{aligned} \tag{3.45}$$

First and second terms on the right hand side of (3.45) can be estimated similar to Lemma 2.3. Then, we apply Lemma 2.2 and (3.43) to arrive at

$$\begin{aligned}
& |\nu\, a(\boldsymbol{\zeta}, \mathbf{v} - \mathbf{P}_h\mathbf{v}) + \kappa\, a(\boldsymbol{\zeta}_t, \mathbf{v} - \mathbf{P}_h\mathbf{v})| \\
& \leq Ch\|\mathbf{v}\|_2\|\boldsymbol{\zeta}\|_\varepsilon + Ch^{r+1}|\mathbf{u}|_{r+1}\|\mathbf{v}\|_2 + Ch\|\mathbf{v}\|_2\|\boldsymbol{\zeta}_t\|_\varepsilon + Ch^{r+1}|\mathbf{u}_t|_{r+1}\|\mathbf{v}\|_2 \\
& \leq \frac{1}{8}\|\boldsymbol{\zeta}\|^2 + Ch^2(\|\boldsymbol{\zeta}\|_\varepsilon^2 + \|\boldsymbol{\zeta}_t\|_\varepsilon^2) + Ch^{2r+2}(|\mathbf{u}|_{r+1}^2 + |\mathbf{u}_t|_{r+1}^2).
\end{aligned} \tag{3.46}$$

We can handle the third term on the right-hand side of (3.45) as

$$b(\boldsymbol{\zeta}, q) = b(\boldsymbol{\zeta} - \mathbf{P}_h\mathbf{u} + \mathbf{S}_h^{so}\mathbf{u}, q) + b(\mathbf{P}_h\boldsymbol{\zeta}, q) = b(\mathbf{u} - \mathbf{P}_h\mathbf{u}, q) + b(\mathbf{P}_h\boldsymbol{\zeta}, q - r_h(q))$$

$$= -\sum_{E\in\mathcal{E}_h}\int_E q\nabla\cdot(\mathbf{u}-\mathbf{P}_h\mathbf{u}) + \sum_{e\in\Gamma_h}\int_e \{q\}[\mathbf{u}-\mathbf{P}_h\mathbf{u}]\cdot\mathbf{n}_e + b(\mathbf{P}_h\boldsymbol{\zeta}, q-r_h(q)). \quad (3.47)$$

In addition, applying Green's theorem to the first term on the right hand side of (3.47) and using the fact that q is continuous, we obtain

$$b(\boldsymbol{\zeta}, q) = \sum_{E\in\mathcal{E}_h}\int_E \nabla q\cdot(\mathbf{u}-\mathbf{P}_h\mathbf{u}) + b(\mathbf{P}_h\boldsymbol{\zeta}, q-r_h(q)).$$

From the Cauchy-Schwarz and Young's inequalities, (1.31), (3.43), and Lemmas 1.5 and 2.2, we obtain

$$\begin{aligned}|b(\boldsymbol{\zeta}, q)| &\le \Big| Ch^{r+1}|q|_1|\mathbf{u}|_{r+1} - \sum_{E\in\mathcal{E}_h}\int_E(\nabla\cdot\mathbf{P}_h\boldsymbol{\zeta})(q-r_h(q)) \\ &\quad + \sum_{e\in\Gamma_h}\int_e\{q-r_h(q)\}[\mathbf{P}_h\boldsymbol{\zeta}]\cdot\mathbf{n}_e\Big| \\ &\le Ch^{r+1}|\mathbf{u}|_{r+1}\|\boldsymbol{\zeta}\| + Ch|q|_1\|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon \\ &\le \frac{1}{8}\|\boldsymbol{\zeta}\|^2 + Ch^2(h^{2r}|\mathbf{u}|^2_{r+1} + \|\mathbf{P}_h\boldsymbol{\zeta}\|^2_\varepsilon). \end{aligned} \quad (3.48)$$

Similar to (3.48), and using (1.31), (3.43), Lemmas 1.5 and 2.2, the 5th term on the right hand side of (3.45) can be bounded as follows:

$$\begin{aligned}|b(\boldsymbol{\zeta}_t, q)| &\le \Big| Ch^{r+1}|q|_1|\mathbf{u}_t|_{r+1} - \sum_{E\in\mathcal{E}_h}\int_E(\nabla\cdot\mathbf{P}_h\boldsymbol{\zeta}_t)(q-r_h(q)) \\ &\quad + \sum_{e\in\Gamma_h}\int_e\{q-r_h(q)\}[\mathbf{P}_h\boldsymbol{\zeta}_t]\cdot\mathbf{n}_e\Big| \\ &\le \frac{1}{8}\|\boldsymbol{\zeta}\|^2 + Ch^2(h^{2r}|\mathbf{u}_t|^2_{r+1} + \|\mathbf{P}_h\boldsymbol{\zeta}_t\|^2_\varepsilon). \end{aligned} \quad (3.49)$$

Apply the Cauchy-Schwarz inequality, Young's inequality, (1.31), (3.43) and Lemma 2.2 to arrive at

$$|b(\mathbf{P}_h\mathbf{v}-\mathbf{v}, p-r_h(p))| \le Ch^{r+1}|p|_r\|\mathbf{v}\|_2 \le \frac{1}{8}\|\boldsymbol{\zeta}\|^2 + Ch^{2r+2}|p|^2_r. \quad (3.50)$$

A use of (3.46) and (3.48)–(3.50) in (3.45) leads to

$$\begin{aligned}\frac{1}{2}\|\boldsymbol{\zeta}(t)\|^2 + \frac{\kappa}{2\nu}\frac{d}{dt}\|\boldsymbol{\zeta}(t)\|^2 \le& Ch^2(\|\boldsymbol{\zeta}\|^2_\varepsilon + \|\boldsymbol{\zeta}_t\|^2_\varepsilon + \|\mathbf{P}_h\boldsymbol{\zeta}\|^2_\varepsilon + \|\mathbf{P}_h\boldsymbol{\zeta}_t\|^2_\varepsilon) \\ &+ Ch^{2r+2}(|\mathbf{u}|^2_{r+1} + |\mathbf{u}_t|^2_{r+1} + |p|^2_r). \end{aligned} \quad (3.51)$$

A multiplication of (3.51) by $e^{2\alpha t}$ and an integration of the resulting equation with respect to time from 0 to t yield

$$e^{2\alpha t}\|\boldsymbol{\zeta}(t)\|^2 + \left(\frac{\nu-2\kappa\alpha}{\nu}\right)\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2 ds \le Ch^{2r+2}|\mathbf{u}_0|^2_{r+1}$$

$$+ Ch^2 \int_0^t e^{2\alpha s} \left(\|\boldsymbol{\zeta}(s)\|_\varepsilon^2 + \|\boldsymbol{\zeta}_t(s)\|_\varepsilon^2 + \|\mathbf{P}_h\boldsymbol{\zeta}(s)\|_\varepsilon^2 + \|\mathbf{P}_h\boldsymbol{\zeta}_t(s)\|_\varepsilon^2 \right) ds$$
$$+ Ch^{2r+2} \int_0^t e^{2\alpha s} (|\mathbf{u}(s)|_{r+1}^2 + |\mathbf{u}_t(s)|_{r+1}^2 + |p(s)|_r^2)\, ds. \tag{3.52}$$

Multiply (3.52) by $e^{-2\alpha t}$ and use (3.35), (3.36), (3.38), (3.39) with assumption (**A2**) to arrive at

$$\|\boldsymbol{\zeta}(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\boldsymbol{\zeta}(s)\|^2 ds \leq Ch^{2r+2}. \tag{3.53}$$

This completes the proof of (3.40) in Lemma 3.3.

Following the similar steps as involved in proving the $\mathbf{L}^2$ estimate of $\boldsymbol{\zeta}$ in (3.40), we arrive at the $\mathbf{L}^2$ estimate in (3.41) involving $\boldsymbol{\zeta}_t$. Only difference is in the dual problem, where the right hand side is changed to $\boldsymbol{\zeta}_t$. With the resulting $\mathbf{L}^2$ estimate of $\boldsymbol{\zeta}_t$, we conclude the proof of Lemma 3.3.

□

Below, in Lemma 3.4, we derive the bounds of $\boldsymbol{\rho}$.

Lemma 3.4. *Under the assumptions of Theorem 3.1, and for $t > 0$, the following estimates hold true:*

$$\|\boldsymbol{\rho}(t)\|^2 + h^2\|\boldsymbol{\rho}(t)\|_\varepsilon^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \left(\|\boldsymbol{\rho}(s)\|^2 + h^2\|\boldsymbol{\rho}(s)\|_\varepsilon^2 \right) ds \leq Ch^{2r+2}.$$

Proof. Subtract (3.29) from (3.27) and write the equation in $\boldsymbol{\rho}$ as

$$(\boldsymbol{\rho}_t, \boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\rho}_t, \boldsymbol{\phi}_h) + \nu\, a(\boldsymbol{\rho}, \boldsymbol{\phi}_h) = -(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h.$$

Substitute $\boldsymbol{\phi}_h = \boldsymbol{\rho}$ in the above equation and use Lemma 1.6 to obtain

$$\frac{1}{2}\frac{d}{dt} \left(\|\boldsymbol{\rho}\|^2 + \kappa\, a(\boldsymbol{\rho}, \boldsymbol{\rho}) \right) + \nu K_1 \|\boldsymbol{\rho}\|_\varepsilon^2 \leq -(\boldsymbol{\zeta}_t, \boldsymbol{\rho}). \tag{3.54}$$

Multiply (3.54) by $e^{2\alpha t}$, integrate the resulting inequality with respect to time from 0 to t, and use Lemmas 1.6, 1.7, the Cauchy-Schwarz inequality, Young's inequality and $\boldsymbol{\rho}(0) = 0$ to arrive at

$$e^{2\alpha t} \left(\|\boldsymbol{\rho}\|^2 + \kappa\, K_1 \|\boldsymbol{\rho}\|_\varepsilon^2 \right) + (\nu K_1 - 2\alpha(C_2 + \kappa K_2)) \int_0^t e^{2\alpha s} \|\boldsymbol{\rho}(s)\|_\varepsilon^2 ds$$
$$\leq C \int_0^t e^{2\alpha s} \|\boldsymbol{\zeta}_s(s)\|^2\, ds. \tag{3.55}$$

A multiplication of (3.55) by $e^{-2\alpha t}$ and a use of (3.41) in the resulting inequality yield

$$\|\boldsymbol{\rho}(t)\|^2 + \kappa K_1 \|\boldsymbol{\rho}(t)\|_\varepsilon^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\rho}(s)\|_\varepsilon^2 ds \le Ch^{2r+2}. \tag{3.56}$$

An application of (1.37), (1.38) and (3.56) leads to

$$h^2\|\boldsymbol{\rho}(t)\|_\varepsilon^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\rho}(s)\|^2 ds \le Ch^{2r+2}. \tag{3.57}$$

A combination of (3.56) and (3.57) concludes the proof of Lemma 3.4. □

Since $\boldsymbol{\xi} = \boldsymbol{\zeta} + (\mathbf{S}_h^{so}\mathbf{u} - \mathbf{v}_h) = \boldsymbol{\zeta} + \boldsymbol{\rho}$, we now apply Lemmas 3.2, 3.3 and 3.4 along with the triangle inequality to obtain the following estimates of $\boldsymbol{\xi}$.

$$\|\boldsymbol{\xi}(t)\|^2 + h^2\|\boldsymbol{\xi}(t)\|_\varepsilon^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\xi}(s)\|^2 ds \le Ch^{2r+2}, \quad t > 0. \tag{3.58}$$

The following lemma provides the estimates for $\boldsymbol{\Theta} = \mathbf{v}_h - \mathbf{u}_h$.

Lemma 3.5. *Under the assumptions of Theorem 3.1, and for $t > 0$, the following estimates hold true:*

$$\|\boldsymbol{\Theta}(t)\|^2 + \|\boldsymbol{\Theta}(t)\|_\varepsilon^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\Theta}(s)\|_\varepsilon^2\, ds \le K(t)h^{2r+2}.$$

Proof. From (3.10) and (3.26), we observe that

$$(\boldsymbol{\Theta}_t, \boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\Theta}_t, \boldsymbol{\phi}_h) + \nu\, a(\boldsymbol{\Theta}, \boldsymbol{\phi}_h) = -\big(c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) - c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h)\big) \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h.$$

Substitute $\boldsymbol{\phi}_h = \boldsymbol{\Theta}$ and use Lemma 1.6 to arrive at

$$\frac{1}{2}\frac{d}{dt}\big(\|\boldsymbol{\Theta}\|^2 + \kappa\, a(\boldsymbol{\Theta}, \boldsymbol{\Theta})\big) + \nu K_1\|\boldsymbol{\Theta}\|_\varepsilon^2 \le -\big(c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\Theta}) - c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\Theta})\big). \tag{3.59}$$

Since $\mathbf{u}$ is continuous, we have the following equality:

$$c^{\mathbf{u}_h}(\mathbf{u}, \mathbf{u}, \boldsymbol{\Theta}) = c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\Theta}).$$

Now, the nonlinear terms can be rewritten in the following way:

$$\begin{aligned} c^{\mathbf{u}_h}(\mathbf{u}, \mathbf{u}, \boldsymbol{\Theta}) - c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\Theta}) = {} & c^{\mathbf{u}_h}(\mathbf{u}_h, \boldsymbol{\Theta}, \boldsymbol{\Theta}) + c^{\mathbf{u}_h}(\boldsymbol{\Theta}, \mathbf{u}, \boldsymbol{\Theta}) - c^{\mathbf{u}_h}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\Theta}) \\ & - c^{\mathbf{u}_h}(\boldsymbol{\Theta}, \boldsymbol{\xi}, \boldsymbol{\Theta}) + c^{\mathbf{u}_h}(\boldsymbol{\xi}, \mathbf{u}, \boldsymbol{\Theta}) + c^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\Theta}) + l^{\mathbf{u}_h}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\Theta}) - l^{\mathbf{u}}(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\Theta}). \end{aligned}$$

Note that, the first term is non-negative due to (1.19) and is therefore dropped. We find that

$$c^{\mathbf{u}_h}(\mathbf{u}, \mathbf{u}, \boldsymbol{\Theta}) - c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\Theta}) \ge c^{\mathbf{u}_h}(\boldsymbol{\Theta}, \mathbf{u}, \boldsymbol{\Theta}) - c^{\mathbf{u}_h}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\Theta}) - c^{\mathbf{u}_h}(\boldsymbol{\Theta}, \boldsymbol{\xi}, \boldsymbol{\Theta})$$

$$+c^{\mathbf{u}_h}(\boldsymbol{\xi},\mathbf{u},\boldsymbol{\Theta})+c^{\mathbf{u}}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\Theta})+l^{\mathbf{u}_h}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\Theta})-l^{\mathbf{u}}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\Theta}). \quad (3.60)$$

Following an identical approach applied in Lemma 2.11 to bound the nonlinear terms, one can find the following bound for the terms on the right hand side of (3.60) as follows

$$\begin{aligned}|c^{\mathbf{u}_h}(\boldsymbol{\Theta},\mathbf{u},\boldsymbol{\Theta})|&+|c^{\mathbf{u}_h}(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\Theta})|+|c^{\mathbf{u}_h}(\boldsymbol{\Theta},\boldsymbol{\xi},\boldsymbol{\Theta})|+|c^{\mathbf{u}_h}(\boldsymbol{\xi},\mathbf{u},\boldsymbol{\Theta})|\\ &+|c^{\mathbf{u}}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\Theta})|+|l^{\mathbf{u}_h}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\Theta})-l^{\mathbf{u}}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\Theta})|\\ \leq \frac{K_1\nu}{2}\|\boldsymbol{\Theta}\|_\varepsilon^2&+C(\|\mathbf{u}\|_2^2+h^{-2}\|\boldsymbol{\xi}\|_\varepsilon^2)\|\boldsymbol{\Theta}\|^2+C\|\mathbf{u}\|_2^2(\|\boldsymbol{\xi}\|^2+h^2\|\boldsymbol{\xi}\|_\varepsilon^2)+C\|\boldsymbol{\xi}\|_\varepsilon^4. \quad (3.61)\end{aligned}$$

Substitute (3.61) in (3.60), and thereby in (3.59), and multiply the resulting inequality by $e^{2\alpha t}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left(e^{2\alpha t}(\|\boldsymbol{\Theta}\|^2+\kappa\, a(\boldsymbol{\Theta},\boldsymbol{\Theta}))\right)+\left(\frac{\nu K_1}{2}-\alpha(C_2+\kappa K_2)\right)e^{2\alpha t}\|\boldsymbol{\Theta}\|_\varepsilon^2 \leq$$
$$C(\|\mathbf{u}\|_2^2+h^{-2}\|\boldsymbol{\xi}\|_\varepsilon^2)e^{2\alpha t}(\|\boldsymbol{\Theta}\|^2+\kappa K_1\|\boldsymbol{\Theta}\|_\varepsilon^2)+Ce^{2\alpha t}\|\mathbf{u}\|_2^2(\|\boldsymbol{\xi}\|^2+h^2\|\boldsymbol{\xi}\|_\varepsilon^2)+Ce^{2\alpha t}\|\boldsymbol{\xi}\|_\varepsilon^4.$$

Integrate the above equation with respect to time from 0 to t and use the fact that $\boldsymbol{\Theta}(0)=\mathbf{0}$. Then Lemmas 1.6 and 1.7, Gronwall's inequality, (3.58) and assumption (**A2**) lead to the following estimates of $\boldsymbol{\Theta}$

$$e^{2\alpha t}(\|\boldsymbol{\Theta}(t)\|^2+\kappa K_1\|\boldsymbol{\Theta}(t)\|_\varepsilon^2)+\nu K_1\int_0^t e^{2\alpha s}\|\boldsymbol{\Theta}\|_\varepsilon^2\,ds \leq Ce^{Ct}h^{2r+2}e^{2\alpha t}.$$

Dividing throughout by $e^{2\alpha t}$ completes the proof. □

Proof of velocity error estimate in Theorem 3.1. A use of $\mathbf{e}=\boldsymbol{\xi}+\boldsymbol{\Theta}$, triangle's inequality, the estimates in (3.58) and Lemma 3.5 yields the desired result. □

Remark 3.1. *Under the following smallness assumption on the data*

$$N=\sup_{\mathbf{v}_h,\boldsymbol{w}_h,\boldsymbol{z}_h\in\boldsymbol{V}_h}\frac{c^{\boldsymbol{z}_h}(\boldsymbol{w}_h,\mathbf{v}_h,\boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_\varepsilon^2\|\mathbf{v}_h\|_\varepsilon}\quad \textit{and}\quad \frac{2NC_2}{K_1^2\nu^2}\|\mathbf{f}\|<1, \quad (3.62)$$

the bounds of Theorem 3.1 are uniform in time, that is,

$$\|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\|+h\|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\|_\varepsilon \leq Ch^{r+1},$$

where the constant $C>0$ is independent of h and time t.

To achieve this, we rewrite the nonlinear terms similar to (2.102) of Remark 2.4 as follows:

$$c^{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{u}_h,\boldsymbol{\Theta})-c^{\mathbf{u}_h}(\mathbf{u},\mathbf{u},\boldsymbol{\Theta})=-c^{\mathbf{u}_h}(\mathbf{u}_h,\boldsymbol{\Theta},\boldsymbol{\Theta})+c^{\mathbf{u}_h}(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\Theta})-c^{\mathbf{u}}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\Theta})$$

$$-c^{\mathbf{u}_h}(\boldsymbol{\xi},\mathbf{u},\boldsymbol{\Theta}) - c^{\mathbf{u}_h}(\boldsymbol{\Theta},\mathbf{v}_h,\boldsymbol{\Theta}) + l^{\mathbf{u}}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\Theta}) - l^{\mathbf{u}_h}(\mathbf{u},\boldsymbol{\xi},\boldsymbol{\Theta}).$$

In this case, we have to find a bound for $\|\mathbf{v}_h(t)\|_\varepsilon$ when $t \to \infty$, and have to consider (3.26) instead of (2.81). Therefore, proceeding in a similar manner as in Remark 2.4, and employing (1.14), (3.58), (3.13), (3.62), Lemmas 1.6 and 1.7, L'Hôpital's rule and assumption (**A2**), we find

$$\limsup_{t\to\infty} \|\boldsymbol{\Theta}(t)\|_\varepsilon \le Ch^{r+1},$$
$$\limsup_{t\to\infty} \|\boldsymbol{\Theta}(t)\| \le Ch^{r+1}.$$

Combining the above estimates and (3.58), we find that

$$\limsup_{t\to\infty} \big(\|\mathbf{u}(t)-\mathbf{u}_h(t)\| + h\|\mathbf{u}(t)-\mathbf{u}_h(t)\|_\varepsilon\big) \le Ch^{r+1}.$$

□

We would like to point out here that, the smallness condition (3.62) for the Kelvin-Voigt model can be compared to similar smallness assumption for Navier-Stokes equations (see [86, Remark 3.2]) and for Oldroyd model (see [83, Remark 1.1 and Theorem 1.1]). This is a restriction amounting to small solutions, needed to establish uniform in time error estimates. That is, under the smallness condition on data, the derived error estimates hold for all time.

3.4 DG Error Estimates for Pressure

This section presents the derivation of semi-discrete pressure error estimates. We begin by proving a lemma which is crucial for establishing these error estimates.

Lemma 3.6. *Let the assumption (***A2***) be satisfied and let* $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. *Then, the error* $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$ *in approximating the velocity satisfies for* $t > 0$

$$\|\boldsymbol{e}_t(t)\| + \kappa\, \|\boldsymbol{e}_t(t)\|_\varepsilon \le K(t)h^r.$$

Proof. Subtract (3.10) from (3.4) to write the equation in error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ as

$$(\mathbf{e}_t,\boldsymbol{\phi}_h) + \kappa\, a(\mathbf{e}_t,\boldsymbol{\phi}_h) + \nu\, a(\mathbf{e},\boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h,p) = c^{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{u}_h,\boldsymbol{\phi}_h) - c^{\mathbf{u}}(\mathbf{u},\mathbf{u},\boldsymbol{\phi}_h), \quad (3.63)$$

for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$. Choose $\boldsymbol{\phi}_h = \mathbf{P}_h\mathbf{e}_t = \mathbf{e}_t - (\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t)$ in (3.63) and use Lemma 1.6 to observe that

$$\|\mathbf{e}_t\|^2 + \kappa\, K_1\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 \le (\mathbf{e}_t,\mathbf{u}_t - P_h\mathbf{u}_t) + \kappa\, a(\mathbf{P}_h\mathbf{u}_t - \mathbf{u}_t,\mathbf{P}_h\mathbf{e}_t) - \nu\, a(\mathbf{e},\mathbf{P}_h\mathbf{e}_t)$$

$$+c^{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{u}_h,\mathbf{P}_h\mathbf{e}_t)-c^{\mathbf{u}}(\mathbf{u},\mathbf{u},\mathbf{P}_h\mathbf{e}_t)-b(\mathbf{P}_h\mathbf{e}_t,p). \quad (3.64)$$

Since $\mathbf{u}$ has no jumps, we rewrite the nonlinear terms of (3.64) as

$$\begin{aligned} c^{\mathbf{u}_h}(\mathbf{u},\mathbf{u},\mathbf{P}_h\mathbf{e}_t)-c^{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{u}_h,\mathbf{P}_h\mathbf{e}_t) = &- c^{\mathbf{u}_h}(\mathbf{e},\mathbf{e},\mathbf{P}_h\mathbf{e}_t)+c^{\mathbf{u}_h}(\mathbf{e},\mathbf{u},\mathbf{P}_h\mathbf{e}_t) \\ &+ c^{\mathbf{u}_h}(\mathbf{u},\mathbf{e},\mathbf{P}_h\mathbf{e}_t). \quad (3.65) \end{aligned}$$

The nonlinear terms on the right hand side of (3.65) can be bounded by applying (2.56), Theorem 3.1, Young's inequality and assumption (**A2**) as

$$|c^{\mathbf{u}_h}(\mathbf{e},\mathbf{e},\mathbf{P}_h\mathbf{e}_t)| \le C\|\mathbf{e}\|_\varepsilon\|\mathbf{e}\|_\varepsilon\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon \le \frac{\kappa K_1}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + C\|\mathbf{e}\|_\varepsilon^2, \quad (3.66)$$

$$|c^{\mathbf{u}_h}(\mathbf{e},\mathbf{u},\mathbf{P}_h\mathbf{e}_t)| \le C\|\mathbf{e}\|_\varepsilon|\mathbf{u}|_1\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon \le \frac{\kappa K_1}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + C\|\mathbf{e}\|_\varepsilon^2, \quad (3.67)$$

$$|c^{\mathbf{u}_h}(\mathbf{u},\mathbf{e},\mathbf{P}_h\mathbf{e}_t)| \le C|\mathbf{u}|_1\|\mathbf{e}\|_\varepsilon\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon \le \frac{\kappa K_1}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + C\|\mathbf{e}\|_\varepsilon^2. \quad (3.68)$$

Using (3.66)-(3.68) in (3.65), we arrive at

$$|c^{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{u}_h,\mathbf{P}_h\mathbf{e}_t)-c^{\mathbf{u}_h}(\mathbf{u},\mathbf{u},\mathbf{P}_h\mathbf{e}_t)| \le \frac{\kappa K_1}{4}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + C\|\mathbf{e}\|_\varepsilon^2. \quad (3.69)$$

Applying the Cauchy-Schwarz and Young's inequalities, definition of space $\mathbf{V}_h$, (1.31) and Lemmas 2.2, 2.3 and 2.4, we bound the second, third and sixth terms on the right hand side of (3.64) as

$$\kappa\,|a(\mathbf{P}_h\mathbf{u}_t-\mathbf{u}_t,\mathbf{P}_h\mathbf{e}_t)| \le \frac{K_1\kappa}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + Ch^{2r}|\mathbf{u}_t|_{r+1}^2, \quad (3.70)$$

$$\nu\,|a(\mathbf{e},\mathbf{P}_h\mathbf{e}_t)| \le \frac{K_1\kappa}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + C\|\mathbf{e}\|_\varepsilon^2 + Ch^{2r}|\mathbf{u}|_{r+1}^2, \quad (3.71)$$

$$|b(\mathbf{P}_h\mathbf{e}_t,p))| = |b(\mathbf{P}_h\mathbf{e}_t,p-r_h(p)))| \le \frac{K_1\kappa}{12}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + Ch^{2r}|p|_r^2. \quad (3.72)$$

An application of the bounds from (3.69)-(3.72) in (3.64) leads to

$$\begin{aligned} \frac{1}{2}\left(\|\mathbf{e}_t\|^2 + K_1\,\kappa\,\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2\right) \le &\, C\|\mathbf{u}_t-\mathbf{P}_h\mathbf{u}_t\|^2 \\ &+ Ch^{2r}(|\mathbf{u}_t|_{r+1}^2 + |\mathbf{u}|_{r+1}^2 + |p|_r^2) + C\|\mathbf{e}\|_\varepsilon^2. \quad (3.73) \end{aligned}$$

Finally, a use of triangle inequality, (3.73), Lemma 2.2, Theorem 3.1, and assumption (**A2**) yield

$$\|\mathbf{e}_t\|^2 + K_1\,\kappa\,\|\mathbf{e}_t\|_\varepsilon^2 \le Ch^{2r}e^{CT}.$$

This completes the rest of the proof.

□

Proof of pressure error estimate in Theorem 3.1: A use of (3.4), (3.5), (3.7) and (3.8) leads to the following error equation for all $\mathbf{v}_h \in \mathbf{X}_h$ as

$$-b(\mathbf{v}_h, p_h - r_h(p)) = (\mathbf{u}_{ht} - \mathbf{u}_t, \mathbf{v}_h) + \kappa\, a(\mathbf{u}_{ht} - \mathbf{u}_t, \mathbf{v}_h) + \nu\, a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) + c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) - b(\mathbf{v}_h, p - r_h(p)). \tag{3.74}$$

Using the inf-sup condition stated in Lemma 1.8, there is $\mathbf{v}_h \in \mathbf{X}_h$ such that

$$b(\mathbf{v}_h, p_h - r_h(p)) = -\|p_h - r_h(p)\|^2, \quad \|\mathbf{v}_h\|_\varepsilon \le \frac{1}{\beta^*}\|p_h - r_h(p)\|. \tag{3.75}$$

A combination of (3.74) and (3.75) leads to

$$\|p_h - r_h(p)\|^2 = (\mathbf{u}_{ht} - \mathbf{u}_t, \mathbf{v}_h) + \kappa\, a(\mathbf{u}_{ht} - \mathbf{u}_t, \mathbf{v}_h) + \nu\, a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) + c^{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) - b(\mathbf{v}_h, p - r_h(p)). \tag{3.76}$$

Now, following the analysis used in Lemmas 3.5 and 3.6, we bound terms on the right hand side of (3.76) as

$$\|p_h - r_h(p)\|^2 \le C\big(\|\mathbf{u}_{ht} - \mathbf{u}_t\|^2 + \|\mathbf{u}_{ht} - \mathbf{u}_t\|_\varepsilon^2 + \|\mathbf{u}_h - \mathbf{u}\|_\varepsilon^2 + \|\mathbf{u}_h - \mathbf{u}\|^2 + h^{2r}(|\mathbf{u}|_{r+1}^2 + |p|_r^2 + |\mathbf{u}_t|_{r+1}^2)\big).$$

Using the triangle inequality and (1.31), we arrive at

$$\|p - p_h\|^2 \le C\big(\|\mathbf{u}_{ht} - \mathbf{u}_t\|^2 + \|\mathbf{u}_{ht} - \mathbf{u}_t\|_\varepsilon^2 + \|\mathbf{u}_h - \mathbf{u}\|_\varepsilon^2 + \|\mathbf{u}_h - \mathbf{u}\|^2 + h^{2r}(|\mathbf{u}|_{r+1}^2 + |p|_r^2 + |\mathbf{u}_t|_{r+1}^2)\big). \tag{3.77}$$

An application of Lemma 3.6, Theorem 3.1 and assumption (**A2**) in (3.77) leads to the desired pressure error estimate. This completes the proof. □

3.5 Fully Discrete Scheme and Error Estimates

For discretization in time variable of the semi-discrete DG Kelvin-Voigt system represented by (3.7)-(3.9), we employ the backward Euler scheme in this section. Now, the backward Euler approximations for (3.7)-(3.9) is defined as follows: Given $\mathbf{U}^0$, seek $(\mathbf{U}^n, P^n)_{n\ge1} \in \mathbf{X}_h \times M_h$, such that

$$(\partial_t \mathbf{U}^n, \mathbf{v}_h) + \kappa\, a(\partial_t \mathbf{U}^n, \mathbf{v}_h) + \nu\, a(\mathbf{U}^n, \mathbf{v}_h)$$

$$+\, c^{\mathbf{U}^{n-1}}(\mathbf{U}^{n-1}, \mathbf{U}^n, \mathbf{v}_h) + b(\mathbf{v}_h, P^n) = (\mathbf{f}^n, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{X}_h, \tag{3.78}$$

$$b(\mathbf{U}^n, q_h) = 0, \quad \forall q_h \in M_h, \tag{3.79}$$

where $\mathbf{U}^0 = \mathbf{u}_h(0) = \mathbf{P}_h \mathbf{u}_0$.

An equivalent formulation of (3.78)-(3.79) is defined as follows: For each $\mathbf{v}_h \in \mathbf{V}_h$, we find $\{\mathbf{U}^n\}_{n\geq 1} \in \mathbf{V}_h$, such that,

$$(\partial_t \mathbf{U}^n, \mathbf{v}_h) + \kappa\, a(\partial_t \mathbf{U}^n, \mathbf{v}_h) + \nu\, a(\mathbf{U}^n, \mathbf{v}_h) + c^{\mathbf{U}^{n-1}}(\mathbf{U}^{n-1}, \mathbf{U}^n, \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h), \tag{3.80}$$

where $\mathbf{U}^0 = \mathbf{u}_h(0) = \mathbf{P}_h \mathbf{u}_0$.

Next in Lemma 3.7, we present *a priori* estimates of backward Euler solution $\mathbf{U}^n$ of (3.80).

Lemma 3.7. *Let the assumption* (**A2**) *be satisfied and let* $0 < \alpha < \frac{\nu K_1}{2(C_2+\kappa K_2)}$. *Further, let* $\boldsymbol{U}^0 = \boldsymbol{P}_h \boldsymbol{u}_0$. *Then, there exists a positive constant* C, *such that the discrete solution* $\{\boldsymbol{U}^n\}_{n\geq 1}$ *of (3.80) satisfies*

$$\|\boldsymbol{U}^n\|^2 + \|\boldsymbol{U}^n\|_\varepsilon^2 + e^{-2\alpha t_M} \Delta t \sum_{n=1}^{M} e^{2\alpha t_n} \|\boldsymbol{U}^n\|_\varepsilon^2 \leq C, \quad n = 0, 1, ..., M,$$

Proof. Substitute $\mathbf{v}_h = \mathbf{U}^n$ in (3.80). Using

$$(\partial_t \mathbf{U}^n, \mathbf{U}^n) = \frac{1}{\Delta t}(\mathbf{U}^n - \mathbf{U}^{n-1}, \mathbf{U}^n) \geq \frac{1}{2\Delta t}(\|\mathbf{U}^n\|^2 - \|\mathbf{U}^{n-1}\|^2) = \frac{1}{2}\partial_t \|\mathbf{U}^n\|^2, \tag{3.81}$$

$$\begin{aligned} a(\partial_t \mathbf{U}^n, \mathbf{U}^n) &= \frac{1}{2}\left(\frac{1}{\Delta t} a(\mathbf{U}^n, \mathbf{U}^n) - \frac{1}{\Delta t} a(\mathbf{U}^{n-1}, \mathbf{U}^{n-1}) + \Delta t a(\partial_t \mathbf{U}^n, \partial_t \mathbf{U}^n)\right) \\ &\geq \frac{1}{2}\partial_t a(\mathbf{U}^n, \mathbf{U}^n), \end{aligned} \tag{3.82}$$

(1.19) and Lemma 1.6 in the resulting equation and then the Cauchy-Schwarz inequality, we arrive at

$$\partial_t \|\mathbf{U}^n\|^2 + \kappa\, \partial_t \left(a(\mathbf{U}^n, \mathbf{U}^n)\right) + 2\nu K_1 \|\mathbf{U}^n\|_\varepsilon^2 \leq 2\|\mathbf{f}^n\|\|\mathbf{U}^n\|. \tag{3.83}$$

Note that,

$$\begin{aligned} \sum_{n=1}^{m} \Delta t e^{2\alpha t_n} \partial_t \|\mathbf{U}^n\|^2 =& e^{2\alpha t_m} \|\mathbf{U}^m\|^2 - \sum_{n=1}^{m-1} e^{2\alpha t_n}(e^{2\alpha \Delta t} - 1)\|\mathbf{U}^n\|^2 \\ & - e^{2\alpha \Delta t}\|\mathbf{U}^0\|^2. \end{aligned} \tag{3.84}$$

Multiply (3.83) by $\Delta t e^{2\alpha t_n}$, sum over $n = 1$ to M, and use (1.14), (3.84), Lemmas 1.6 and 1.7 to obtain

$$e^{2\alpha t_M} \|\mathbf{U}^M\|^2 + K_1 \kappa\, e^{2\alpha t_M} \|\mathbf{U}^M\|_\varepsilon^2$$

$$+\left(\nu K_1-(C_2+\kappa\, K_2)\frac{(e^{2\alpha\Delta t}-1)}{\Delta t}\right)\Delta t\sum_{n=1}^{M}e^{2\alpha t_n}\|\mathbf{U}^n\|_{\varepsilon}^2$$
$$\leq e^{2\alpha\Delta t}\|\mathbf{U}^0\|^2+\kappa\, K_2e^{2\alpha\Delta t}\|\mathbf{U}^0\|_{\varepsilon}^2+\frac{2C_2^2}{\nu K_1}\Delta t\sum_{n=1}^{M}e^{2\alpha t_n}\|\mathbf{f}\|^2. \tag{3.85}$$

With α as $0<\alpha<\frac{\nu K_1}{2(C_2+\kappa K_2)}$ we have

$$1+\frac{\nu K_1\Delta t}{C_2+\kappa\, K_2}\geq e^{2\alpha\Delta t}.$$

On multiplying (3.85) by $e^{-2\alpha t_M}$ and applying assumption (**A2**), we establish our desired estimates. □

Using (1.19), and Lemmas 1.6, 1.8 and 3.7, the existence and uniqueness of the discrete solutions to the discrete problem (3.78)-(3.79) (or (3.80)) can be achieved following similar steps as in [72]. Below, we focus on the derivation of error estimates for the backward Euler method.

We denote $\mathbf{e}_n=\mathbf{U}^n-\mathbf{u}_h(t_n)=\mathbf{U}^n-\mathbf{u}_h^n$, $n\in\mathbb{N}$, $1<n\leq M$. Now, consider the semidiscrete formulation (3.10) at $t=t_n$ and subtract it from (3.80) to arrive at

$$(\partial_t\mathbf{e}_n,\boldsymbol{\phi}_h)+\kappa\, a(\partial_t\mathbf{e}_n,\boldsymbol{\phi}_h)+\nu\, a(\mathbf{e}_n,\boldsymbol{\phi}_h)=(\mathbf{u}_{ht}^n,\boldsymbol{\phi}_h)-(\partial_t\mathbf{u}_h^n,\boldsymbol{\phi}_h)$$
$$+\kappa\, a(\mathbf{u}_{ht}^n,\boldsymbol{\phi}_h)-\kappa\, a(\partial_t\mathbf{u}_h^n,\boldsymbol{\phi}_h)+\Lambda_h(\boldsymbol{\phi}_h), \tag{3.86}$$

where $\Lambda_h(\boldsymbol{\phi}_h)=\Lambda_h^1(\boldsymbol{\phi}_h)+\Lambda_h^2(\boldsymbol{\phi}_h)$ with

$$\left.\begin{aligned}\Lambda_h^1(\boldsymbol{\phi}_h)&=c^{\mathbf{U}^{n-1}}(\mathbf{u}_h^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h)-c^{\mathbf{U}^{n-1}}(\mathbf{U}^{n-1},\mathbf{U}^n,\boldsymbol{\phi}_h),\\ \Lambda_h^2(\boldsymbol{\phi}_h)&=l^{\mathbf{u}_h^n}(\mathbf{u}_h^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h)-l^{\mathbf{U}^{n-1}}(\mathbf{u}_h^n,\mathbf{u}_h^n,\boldsymbol{\phi}_h).\end{aligned}\right\} \tag{3.87}$$

Lemma 3.8. *Under the hypotheses of Lemma 3.7 and Theorem 3.1, there exists a positive constant $K_T>0$, independent of h and Δt, such that, the following estimates hold true:*

$$(\|\boldsymbol{e}_n\|+\|\boldsymbol{e}_n\|_{\varepsilon})+\left(e^{-2\alpha t_M}\Delta t\sum_{n=1}^{M}e^{2\alpha t_n}\|\boldsymbol{e}_n\|_{\varepsilon}^2\right)^{1/2}\leq K_T\,\Delta t,$$

where K_T depends on T.

Proof. Choose $\boldsymbol{\phi}_h=\mathbf{e}_n$ in (3.86), and use (3.81), (3.82) and Lemma 1.6 to arrive at

$$\partial_t\big(\|\mathbf{e}_n\|^2+\kappa\, a(\mathbf{e}_n,\mathbf{e}_n)\big)+2K_1\nu\|\mathbf{e}_n\|_{\varepsilon}^2\leq 2(\mathbf{u}_{ht}^n,\mathbf{e}_n)-2(\partial_t\mathbf{u}_h^n,\mathbf{e}_n)$$
$$+2\kappa\, a(\mathbf{u}_{ht}^n,\mathbf{e}_n)-2\kappa\, a(\partial_t\mathbf{u}_h^n,\mathbf{e}_n)+2\Lambda_h(\mathbf{e}_n), \tag{3.88}$$

where $\Lambda_h(\mathbf{e}_n) = \Lambda_h^1(\mathbf{e}_n) + \Lambda_h^2(\mathbf{e}_n)$. Utilizing an identical technique applied in Lemma 2.17 to bound the terms $\Lambda_h^1(\mathbf{e}_n)$ and $\Lambda_h^2(\mathbf{e}_n)$, one can obtain

$$|\Lambda_h^1(\mathbf{e}_n)| \leq \frac{K_1\nu}{8}\|\mathbf{e}_n\|_\varepsilon^2 + C\|\mathbf{e}_{n-1}\|^2 + C\Delta t\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds, \tag{3.89}$$

$$\begin{aligned}|\Lambda_h^2(\mathbf{e}_n)| &\leq \frac{K_1\nu}{16}\|\mathbf{e}_n\|_\varepsilon^2 + \frac{K_1\nu}{16}\|\mathbf{e}_{n-1}\|_\varepsilon^2 + C\|\mathbf{u}_h^n\|_\varepsilon^4\|\mathbf{e}_{n-1}\|^2 \\ &\quad + C\Delta t\|\mathbf{u}_h^n\|_\varepsilon^2\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hs}(s)\|_\varepsilon^2\, ds. \end{aligned}\tag{3.90}$$

From (2.138), and utilizing the Cauchy-Schwarz, Young's inequalities and (1.14), we obtain

$$\begin{aligned}2(\mathbf{u}_{ht}^n, \mathbf{e}_n) - 2(\partial_t\mathbf{u}_h^n, \mathbf{e}_n) &\leq C\Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{htt}(t)\|^2\, dt\right)^{1/2}\|\mathbf{e}_n\|_\varepsilon \\ &\leq \frac{K_1\nu}{4}\|\mathbf{e}_n\|_\varepsilon^2 + C\Delta t\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{htt}(t)\|^2\, dt. \end{aligned}\tag{3.91}$$

Similarly, following the steps involved in bounding (3.91) and using Lemma 1.7, we arrive at

$$\begin{aligned}2\kappa\left(a(\mathbf{u}_{ht}^n, \mathbf{e}_n) - a(\partial_t\mathbf{u}_h^n, \mathbf{e}_n)\right) &\leq C\Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{htt}(t)\|_\varepsilon^2\, dt\right)^{1/2}\|\mathbf{e}_n\|_\varepsilon \\ &\leq \frac{K_1\nu}{4}\|\mathbf{e}_n\|_\varepsilon^2 + C\Delta t\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{htt}(t)\|_\varepsilon^2\, dt. \end{aligned}\tag{3.92}$$

Apply (3.89)-(3.92) in (3.88), multiply the resulting inequality by $\Delta t e^{2\alpha t_n}$ and sum over $n = 1$ to m $(\leq M)$, where $T = M\Delta t$. Then, using

$$\sum_{n=1}^{m} \Delta t e^{2\alpha t_n}\partial_t\|\mathbf{e}_n\|^2 = e^{2\alpha t_m}\|\mathbf{e}_m\|^2 - \sum_{n=1}^{m-1} e^{2\alpha t_n}(e^{2\alpha\Delta t} - 1)\|\mathbf{e}_n\|^2,$$

$$\sum_{n=1}^{m} \Delta t e^{2\alpha t_n}\partial_t a(\mathbf{e}_n, \mathbf{e}_n) = e^{2\alpha t_m}a(\mathbf{e}_m, \mathbf{e}_m) - \sum_{n=1}^{m-1} e^{2\alpha t_n}(e^{2\alpha\Delta t} - 1)a(\mathbf{e}_n, \mathbf{e}_n),$$

Lemmas 1.6 and 1.7, we arrive at

$$\begin{aligned}&e^{2\alpha t_m}(\|\mathbf{e}_m\|^2 + K_1\kappa\,\|\mathbf{e}_m\|_\varepsilon^2) + K_1\nu\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\|\mathbf{e}_n\|_\varepsilon^2 \\ &\quad\leq \sum_{n=1}^{m-1} e^{2\alpha t_n}(e^{2\alpha\Delta t} - 1)(\|\mathbf{e}_n\|^2 + K_2\kappa\|\mathbf{e}_n\|_\varepsilon^2) \\ &\qquad + C\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}(1 + \|\mathbf{u}_h^n\|_\varepsilon^4)\|\mathbf{e}_{n-1}\|^2 \\ &\qquad + C\Delta t^2\sum_{n=1}^{m} e^{2\alpha t_n}(1 + \|\mathbf{u}_h^n\|_\varepsilon^2)\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{ht}(t)\|_\varepsilon^2\, dt \end{aligned}$$

$$+ C\Delta t^2 \sum_{n=1}^{m} e^{2\alpha t_n} \int_{t_{n-1}}^{t_n} (\|\mathbf{u}_{htt}(t)\|^2 + \|\mathbf{u}_{htt}(t)\|_\varepsilon^2)\, dt. \tag{3.93}$$

Now the desired result is achieved by applying estimates (3.11) and (3.12), the fact $e^{2\alpha\Delta t} - 1 \leq C\Delta t$, and discrete Gronwall's lemma in (3.93). $\square$

Theorem 3.2. *Under the assumptions of Theorem 3.1 and Lemma 3.8, the following estimates hold true:*

$$\|\boldsymbol{u}(t_n) - \boldsymbol{U}^n\| \leq K_T(h^{r+1} + \Delta t),$$
$$\|\boldsymbol{u}(t_n) - \boldsymbol{U}^n\|_\varepsilon \leq K_T(h^r + \Delta t).$$

Proof. A combination of Theorem 3.1 and Lemma 3.8 leads to the desired result. $\square$

Lemma 3.9. *Under the hypotheses of Theorem 3.1 and Lemma 3.8, the error $\boldsymbol{e}_n = \boldsymbol{U}^n - \boldsymbol{u}_h^n$, satisfies*

$$\|\partial_t \boldsymbol{e}_n\| + \|\partial_t \mathbf{e}_n\|_\varepsilon \leq K_T \Delta t.$$

Proof. Rewrite the non-linear terms in (3.86) as

$$\begin{aligned}\Lambda_h(\boldsymbol{\phi}_h) = \Lambda_h^1(\boldsymbol{\phi}_h) + \Lambda_h^2(\boldsymbol{\phi}_h) = &- c^{\mathbf{U}^{n-1}}(\mathbf{u}_h^{n-1}, \mathbf{e}_n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \boldsymbol{\phi}_h) \\ &- c^{\mathbf{U}^{n-1}}(\mathbf{e}_{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h) + \Lambda_h^2(\boldsymbol{\phi}_h).\end{aligned} \tag{3.94}$$

A use of (2.55) yield

$$\begin{aligned}&|c^{\mathbf{U}^{n-1}}(\mathbf{u}_h^{n-1}, \mathbf{e}_n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}^{n-1}}(\mathbf{e}_{n-1}, \mathbf{U}^n, \boldsymbol{\phi}_h)| \\ &\leq C\|\mathbf{u}_h^{n-1}\|_\varepsilon \|\mathbf{e}_n\|_\varepsilon \|\boldsymbol{\phi}_h\|_\varepsilon + C\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_\varepsilon \|\mathbf{u}_h^n\|_\varepsilon \|\boldsymbol{\phi}_h\|_\varepsilon + C\|\mathbf{e}_{n-1}\|_\varepsilon \|\mathbf{U}^n\|_\varepsilon \|\boldsymbol{\phi}_h\|_\varepsilon.\end{aligned} \tag{3.95}$$

Further, using (1.14), (2.60) and triangle inequality, we arrive at

$$|\Lambda_h^2(\boldsymbol{\phi}_h)| \leq C\big(\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_\varepsilon + \|\mathbf{e}_{n-1}\|_\varepsilon\big)\|\mathbf{u}_h^n\|_\varepsilon \|\boldsymbol{\phi}_h\|_\varepsilon. \tag{3.96}$$

A combination of (3.95)-(3.96) in (3.94) yields

$$\begin{aligned}|\Lambda_h(\boldsymbol{\phi}_h)| \leq &C\big(\|\mathbf{u}_h^{n-1}\|_\varepsilon \|\mathbf{e}_n\|_\varepsilon + \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_\varepsilon \|\mathbf{u}_h^n\|_\varepsilon + \|\mathbf{e}_{n-1}\|_\varepsilon \|\mathbf{U}^n\|_\varepsilon \\ &+ \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_\varepsilon \|\mathbf{u}_h^n\|_\varepsilon + \|\mathbf{e}_{n-1}\|_\varepsilon \|\mathbf{u}_h^n\|_\varepsilon\big)\|\boldsymbol{\phi}_h\|_\varepsilon.\end{aligned} \tag{3.97}$$

A use of Lemma 1.7 leads to

$$|a(\mathbf{e}_n, \boldsymbol{\phi}_h)| \leq K_2 \|\mathbf{e}_n\|_\varepsilon \|\boldsymbol{\phi}_h\|_\varepsilon. \tag{3.98}$$

Apply (1.14), (2.138) and the Cauchy-Schwarz inequality to arrive at

$$(\mathbf{u}_{ht}^n, \boldsymbol{\phi}_h) - (\partial_t \mathbf{u}_h^n, \boldsymbol{\phi}_h) \leq C\Delta t^{1/2} \left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{htt}(s)\|^2 \, ds \right)^{1/2} \|\boldsymbol{\phi}_h\|_\varepsilon$$
$$\leq C \left(\sup_{0<t<\infty} \|\mathbf{u}_{htt}(t)\|^2 \right) \left(\int_{t_{n-1}}^{t_n} 1 ds \right)^{1/2} \Delta t^{1/2} \|\boldsymbol{\phi}_h\|_\varepsilon. \quad (3.99)$$

Following the similar analysis as in (3.99), we obtain

$$\kappa \left(a(\mathbf{u}_{ht}^n, \boldsymbol{\phi}_h) - a(\partial_t \mathbf{u}_h^n, \boldsymbol{\phi}_h) \right)$$
$$\leq C \left(\sup_{0<t<\infty} \|\mathbf{u}_{htt}(t)\|_\varepsilon^2 \right) \left(\int_{t_{n-1}}^{t_n} 1 ds \right)^{1/2} \Delta t^{1/2} \|\boldsymbol{\phi}_h\|_\varepsilon. \quad (3.100)$$

Substitute $\boldsymbol{\phi}_h = \partial_t \mathbf{e}_n$ in (3.86) and use (3.97)–(3.100) with $\boldsymbol{\phi}_h$ replaced by $\partial_t \mathbf{e}_n$. Then, apply Young's inequality, and estimates from (3.11), (3.12) and Lemmas 1.6, 3.7, 3.8 to arrive at the desired result. This completes the rest of the proof. □

Lemma 3.10. *Under the hypotheses of Theorem 3.1 and Lemma 3.8, for* $n = 1, 2, \cdots,$ M*, the following holds true*

$$\|P^n - p_h^n\| \leq K_T \Delta t.$$

Proof. Subtract (3.7) from (3.78) to arrive at

$$b(\boldsymbol{\phi}_h, P^n - p_h^n) = (\mathbf{u}_{ht}^n, \boldsymbol{\phi}_h) - (\partial_t \mathbf{u}_h^n, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}^n, \boldsymbol{\phi}_h) - \kappa\, a(\partial_t \mathbf{u}_h^n, \boldsymbol{\phi}_h)$$
$$- (\partial_t \mathbf{e}_n, \boldsymbol{\phi}_h) - \kappa\, a(\partial_t \mathbf{e}_n, \boldsymbol{\phi}_h) - \nu\, a(\mathbf{e}_n, \boldsymbol{\phi}_h) + \Lambda_h(\boldsymbol{\phi}_h),$$

for all $\boldsymbol{\phi}_h \in \mathbf{X}_h$. Performing the steps required to prove Lemma 3.9 and applying Lemma 1.8, we obtain

$$\|P^n - p_h^n\| \leq C(\|\partial_t \mathbf{e}_n\| + \|\partial_t \mathbf{e}_n\|_\varepsilon + \|\mathbf{e}_n\|_\varepsilon + \Delta t).$$

Finally, an application of the Lemmas 3.8 and 3.9 concludes the proof. □

The following error estimate on the pressure is easily derived from Lemma 3.10 and Theorem 3.1.

Theorem 3.3. *Under the assumptions of Theorem 3.1 and Lemma 3.10, the following hold true:*

$$\|p(t_n) - P^n\| \leq K_T(h^r + \Delta t).$$

Remark 3.2. *Similar to Remark 2.5, the optimal order convergence rates derived in this chapter can be extended to the 3D case.*

3.6 Numerical Experiments

In this section, we present numerical examples to verify the theoretical results stated in the Theorems 3.2 and 3.3. We have used the mixed finite element spaces $\mathbb{P}_r - \mathbb{P}_{r-1},\ r = 1, 2$ for the space discretization and a backward Euler method for the time discretization. The domain $\Omega = [0, 1] \times [0, 1]$ is chosen here. We have considered here four examples where the first three examples are computed in time interval $[0, 1]$ and the fourth one in time interval $[0, 75]$. In all the cases, time step is $\Delta t = \mathcal{O}(h^{r+1})$.

Example 3.1. *In our first example, the right hand side function* $\mathbf{f}$ *is chosen in such a way that the exact solution* $(\boldsymbol{u}, p) = ((u_1, u_2), p)$ *is*

$$u_1 = 2x^2(x-1)^2 y(y-1)(2y-1)\cos(t) \qquad p = 2(x-y)\cos(t),$$
$$u_2 = -2x(x-1)(2x-1)y^2(y-1)^2\cos(t).$$

In Tables 3.1 and 3.2, we present the errors and convergence rates of the approximate velocity and pressure obtained by using the discontinuous mixed finite element spaces $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$, respectively, with the retardation time $\kappa = 10^{-2}$ and kinematic viscosity $\nu = 1$. The penalty parameter is chosen as $\sigma_e = 10$ for $r = 1$ and $\sigma_e = 20$ for $r = 2$. The numerical results in Tables 3.1 and 3.2 support the theoretical derivations in Theorems 3.2 and 3.3.

Table 3.1: Errors and rates of convergence of velocity and pressure using discontinuous $\mathbb{P}_1$–$\mathbb{P}_0$ element for Example 3.1.

h	$\\|\mathbf{u}(T) - \mathbf{U}^M\\|_\varepsilon$	Rate	$\\|\mathbf{u}(T) - \mathbf{U}^M\\|$	Rate	$\\|p(T) - P^M\\|$	Rate
1/4	3.3927×10^{-2}		2.4193×10^{-3}		1.0667×10^{-2}	
1/8	1.4413×10^{-2}	1.2350	5.3378×10^{-4}	2.1802	7.0680×10^{-3}	0.5937
1/16	6.1566×10^{-3}	1.2271	1.2089×10^{-4}	2.1424	4.1982×10^{-3}	0.7515
1/32	2.7335×10^{-3}	1.1713	2.8341×10^{-5}	2.0928	2.3113×10^{-3}	0.8610
1/64	1.2676×10^{-3}	1.1086	6.8138×10^{-6}	2.0563	1.2137×10^{-3}	0.9292

Table 3.2: Errors and rates of convergence of velocity and pressure DG approximations using $\mathbb{P}_2$–$\mathbb{P}_1$ element for Example 3.1.

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	3.1902×10^{-3}		1.4918×10^{-4}		1.4342×10^{-3}	
1/8	7.4906×10^{-4}	2.0904	1.9706×10^{-5}	2.9203	4.1075×10^{-4}	1.8040
1/16	1.7576×10^{-4}	2.0914	2.4316×10^{-6}	3.0186	1.1433×10^{-4}	1.8450
1/32	4.2589×10^{-5}	2.0450	2.9815×10^{-7}	3.0278	3.0616×10^{-5}	1.9008

Example 3.2. *In this example, we take the force term* $\mathbf{f}$ *resulting in the following solutions*

$$u_1 = 2\pi \sin^2(\pi x) \sin(\pi y) \cos(\pi y) \cos(t) \qquad p = 10 \cos(\pi x) \cos(\pi y) \cos(t),$$
$$u_2 = -2\pi \sin(\pi x) \cos(\pi x) \sin^2(\pi y) \cos(t).$$

In Tables 3.3-3.10, we have shown the errors and rates of convergence for velocity and pressure approximations based on discontinuous $\mathbb{P}_1 - \mathbb{P}_0$ mixed finite element space for different values of κ and ν, respectively. The Tables 3.3-3.6 depict the numerical results for different values of $\kappa = \{1, 10^{-3}, 10^{-6}, 10^{-9}\}$ with the choice of $\nu = 1$ and $\sigma_e = 10$. The numerical convergence rates in the tables validate the theoretical findings obtained in Theorems 3.2 and 3.3. Moreover, it can be inferred that the numerical results still hold for small values of κ. The Tables 3.7-3.10 represent the errors and convergence rates for different values of $\nu = \{1, 1/100, 1/1000, 1/10000\}$ with $\kappa = 1$ and the corresponding penalty parameters are chosen as $\sigma_e = \{10, 20, 50, 200\}$. The numerical outcomes depicted in the tables verify the derived theoretical results. We can therefore conclude that the scheme is robust with repsect to the retardation time and the viscosity. Further, we have presented CPU times for obtaining the results of Tables 3.7-3.10 in Table 3.11. The table shows that for the proposed DG scheme, the CPU time increases as we go for smaller ν, from $\nu = 1$ to 1/10000. This is because as we go for smaller and smaller values of ν, more iterative steps are needed for the nonlinear solver to achieve the desired accuracy, resulting in increased CPU time.

Table 3.3: Errors and rates of convergence of velocity and pressure using $\mathbb{P}_1$–$\mathbb{P}_0$ discontinuous finite element for Example 3.2 with $\kappa = 1$.

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	1.5402×10^{0}		9.4053×10^{-2}		7.9248×10^{-1}	
1/8	8.3518×10^{-1}	0.8830	3.6571×10^{-2}	1.3627	4.6779×10^{-1}	0.7605
1/16	4.0990×10^{-1}	1.0268	1.1250×10^{-2}	1.7007	3.0020×10^{-1}	0.6399
1/32	1.9845×10^{-1}	1.0464	3.0411×10^{-3}	1.8872	1.7352×10^{-1}	0.7907
1/64	9.7912×10^{-2}	1.0192	7.8060×10^{-4}	1.9619	9.1903×10^{-2}	0.9169

Table 3.4: Errors and rates of convergence of velocity and pressure using $\mathbb{P}_1$–$\mathbb{P}_0$ discontinuous finite element for Example 3.2 with $\kappa = 10^{-3}$.

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	2.8738×10^{0}		1.9019×10^{-1}		1.1188×10^{0}	
1/8	1.5151×10^{0}	0.9235	7.2358×10^{-2}	1.3942	7.6840×10^{-1}	0.5420
1/16	7.1620×10^{-1}	1.0810	2.2403×10^{-2}	1.6914	5.2929×10^{-1}	0.5377
1/32	3.4390×10^{-1}	1.0583	6.0853×10^{-3}	1.8803	3.1001×10^{-1}	0.7717
1/64	1.6943×10^{-1}	1.0212	1.5656×10^{-3}	1.9585	1.6448×10^{-1}	0.9143

Table 3.5: Errors and rates of convergence of velocity and pressure using $\mathbb{P}_1$–$\mathbb{P}_0$ discontinuous finite element for Example 3.2 with $\kappa = 10^{-6}$.

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	2.8733×10^{0}		1.9012×10^{-1}		1.1201×10^{0}	
1/8	1.5150×10^{0}	0.9234	7.2334×10^{-2}	1.3942	7.6949×10^{-1}	0.5416
1/16	7.1618×10^{-1}	1.0809	2.2397×10^{-2}	1.6913	5.3010×10^{-1}	0.5376
1/32	3.4389×10^{-1}	1.0583	6.0837×10^{-3}	1.8802	3.1049×10^{-1}	0.7716
1/64	1.6943×10^{-1}	1.0212	1.5652×10^{-3}	1.9585	1.6474×10^{-1}	0.9143

Table 3.6: Errors and rates of convergence of velocity and pressure using $\mathbb{P}_1$–$\mathbb{P}_0$ discontinuous finite element for Example 3.2 with $\kappa = 10^{-9}$.

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	2.8733×10^{0}		1.9012×10^{-1}		1.1201×10^{0}	
1/8	1.5150×10^{0}	0.9234	7.2334×10^{-2}	1.3942	7.6949×10^{-1}	0.5416
1/16	7.1618×10^{-1}	1.0809	2.2397×10^{-2}	1.6913	5.3010×10^{-1}	0.5376
1/32	3.4389×10^{-1}	1.0583	6.0837×10^{-3}	1.8802	3.1050×10^{-1}	0.7716
1/64	1.6943×10^{-1}	1.0212	1.5652×10^{-3}	1.9585	1.6474×10^{-1}	0.9143

Table 3.7: Errors and rates of convergence of velocity and pressure using $\mathbb{P}_1$–$\mathbb{P}_0$ discontinuous finite element for Example 3.2 with $\nu = 1$.

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	1.5402×10^{0}		9.4053×10^{-2}		7.9248×10^{-1}	
1/8	8.3518×10^{-1}	0.8830	3.6571×10^{-2}	1.3627	4.6779×10^{-1}	0.7605
1/16	4.0990×10^{-1}	1.0268	1.1250×10^{-2}	1.7007	3.0020×10^{-1}	0.6399
1/32	1.9845×10^{-1}	1.0464	3.0411×10^{-3}	1.8872	1.7352×10^{-1}	0.7907
1/64	9.7912×10^{-2}	1.0192	7.8060×10^{-4}	1.9619	9.1903×10^{-2}	0.9169

Table 3.8: Errors and rates of convergence of velocity and pressure using $\mathbb{P}_1$–$\mathbb{P}_0$ discontinuous finite element for Example 3.2 with $\nu = 1/100$.

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	2.1245×10^{0}		1.7552×10^{-1}		2.7807×10^{0}	
1/8	1.1761×10^{0}	0.8531	6.9141×10^{-2}	1.3440	2.5762×10^{0}	0.1102
1/16	5.8002×10^{-1}	1.0198	2.3090×10^{-2}	1.5822	1.9189×10^{0}	0.4249
1/32	2.8034×10^{-1}	1.0488	6.5192×10^{-3}	1.8245	1.1279×10^{0}	0.7666
1/64	1.3849×10^{-1}	1.0173	1.6976×10^{-3}	1.9411	5.9625×10^{-1}	0.9196

Table 3.9: Errors and rates of convergence of velocity and pressure using $\mathbb{P}_1$–$\mathbb{P}_0$ discontinuous finite element for Example 3.2 with $\nu = 1/1000$.

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	3.0968×10^{0}		3.1054×10^{-1}		9.3942×10^{0}	
1/8	2.0223×10^{0}	0.6147	2.0393×10^{-1}	0.6067	11.5269×10^{0}	-0.2951
1/16	9.9427×10^{-1}	1.0243	9.2334×10^{-2}	1.1431	9.7804×10^{0}	0.2370
1/32	3.9501×10^{-1}	1.3317	3.0098×10^{-2}	1.6171	6.1873×10^{0}	0.6605
1/64	1.6088×10^{-1}	1.2958	8.2414×10^{-3}	1.8687	3.3562×10^{0}	0.8824

Table 3.10: Errors and rates of convergence of velocity and pressure using $\mathbb{P}_1$–$\mathbb{P}_0$ discontinuous finite element for Example 3.2 with $\nu = 1/10000$.

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	3.9021×10^{0}		4.2416×10^{-1}		13.8802×10^{0}	
1/8	2.7820×10^{0}	0.4881	3.1569×10^{-1}	0.4260	19.0729×10^{0}	-0.4585
1/16	1.4817×10^{0}	0.9088	1.6018×10^{-1}	0.9787	17.6551×10^{0}	0.1114
1/32	5.8599×10^{-1}	1.3383	5.7335×10^{-2}	1.4822	11.9890×10^{0}	0.5583
1/64	2.0647×10^{-1}	1.5049	1.6400×10^{-2}	1.8056	6.7234×10^{0}	0.8344

Table 3.11: CPU time (s) for Example 3.2 with $\nu = \{1, 1/100, 1/1000, 1/10000\}$.

h	$\nu = 1$	$\nu = 1/100$	$\nu = 1/1000$	$\nu = 1/10000$
1/4	0.72	0.88	2.35	3.75
1/8	8.18	10.29	16.35	20.62
1/16	86.47	130.39	151.15	191.50
1/32	925.98	978.47	1285.22	1457.76
1/64	16250.16	18240.89	18673.18	19685.24

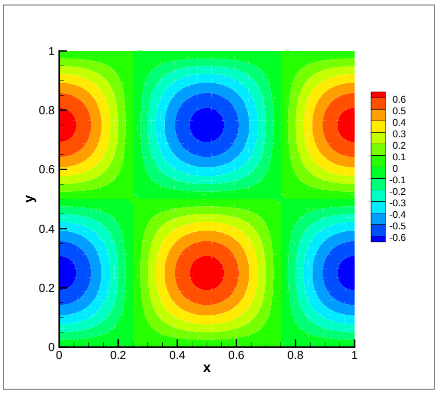
Example 3.3 (Taylor-Green vortex)**.** *Another widely used test case for the verification of numerical methods is the Taylor-Green vortex problem. The analytical unsteady solution is* $(\boldsymbol{u}, p) = ((u_1, u_2), p)$, *where*

$$u_1 = -\cos(2\pi x)\sin(2\pi y)e^{\frac{-8\pi^2\nu t}{1+8\pi^2\kappa}}, \qquad p = -\frac{1}{4}(\cos(4\pi x) + \cos(4\pi y))e^{\frac{-16\pi^2\nu t}{1+8\pi^2\kappa}},$$

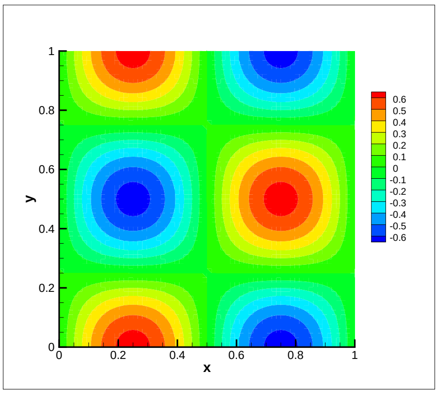
$$u_2 = \sin(2\pi x)\cos(2\pi y)e^{\frac{-8\pi^2\nu t}{1+8\pi^2\kappa}}.$$

The initial condition is obtained from the above exact solution.

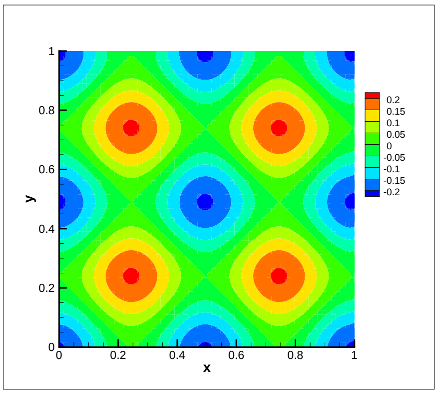
Here, the contours of exact velocity components and pressure magnitudes are presented in Figure 3.1 and the contours of $\mathbb{P}_1 - \mathbb{P}_0$ DG approximate velocity components and pressure magnitudes are shown in Figure 3.2 with $\kappa = 10^{-2}$, $\nu = 1/100$, $\sigma_e = 20$ and final time $T = 1.0$. A plot comparison of contours between the exact and DG approximate solutions validates the theoretical findings. Further, the numerical convergence results are shown in Tables 3.12, 3.13 and 3.14 for the cases $\nu = 1/100$, $1/1000$ and $1/10000$, respectively. Note that, for the cases $\nu = 1/1000$ and $1/10000$, $\sigma_e = 50$ and 200, respectively. We observe from the results in tables that the optimal convergence rates are achieved for this benchmark test problem verifying the theoretical results in Theorems 3.2 and 3.3.



(a) u_1-velocity contour

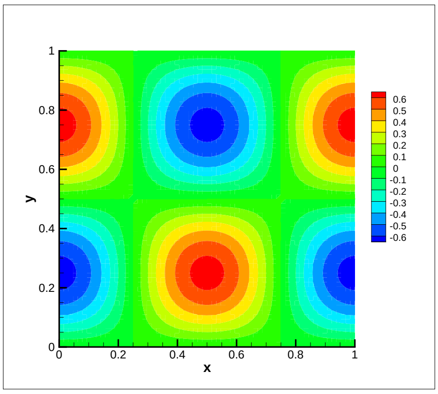


(b) u_2-velocity contour

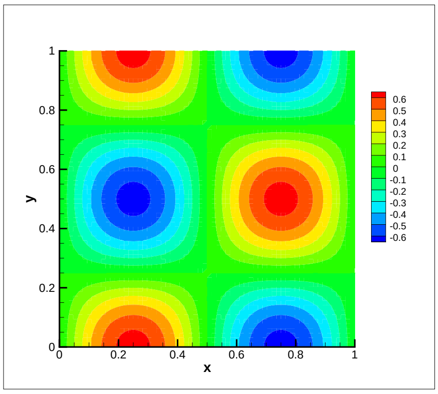


(c) Pressure contour

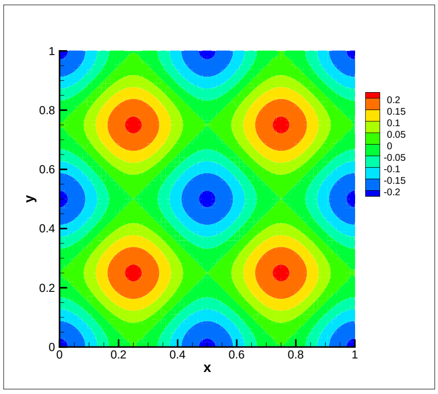
Figure 3.1: Velocity components and pressure plots for exact solution of Example 3.3.



(a) u_1-velocity contour



(b) u_2-velocity contour



(c) Pressure contour

Figure 3.2: Velocity components and pressure plots of Example 3.3 for DG method with $\mathbb{P}_1 - \mathbb{P}_0$ element.

Table 3.12: Errors and rates of convergence of velocity and pressure DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ element for Example 3.3 with $\nu = 1/100$.

h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	1.95622		1.8842×10^{-1}		5.1839×10^{-1}	
1/8	1.10616	0.8225	6.7070×10^{-2}	1.4902	3.6098×10^{-1}	0.5220
1/16	0.54554	1.0198	1.8027×10^{-2}	1.8954	2.0458×10^{-1}	0.8192
1/32	0.26317	1.0516	4.4449×10^{-3}	2.0199	9.6212×10^{-2}	1.0884
1/64	0.12927	1.0255	1.0842×10^{-3}	2.0354	4.4230×10^{-2}	1.1211

Table 3.13: Errors and rates of convergence of velocity and pressure DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ element for Example 3.3 with $\nu = 1/1000$.

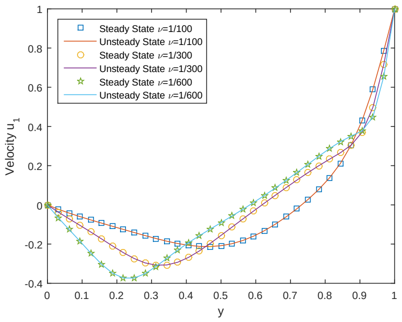
h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	4.05548		4.3236×10^{-1}		1.0029×10^{0}	
1/8	2.16577	0.9049	1.9918×10^{-1}	1.1181	6.5306×10^{-1}	0.6190
1/16	0.93863	1.2062	6.5539×10^{-2}	1.6037	3.7808×10^{-1}	0.7885
1/32	0.39069	1.2645	1.8138×10^{-2}	1.8532	1.7453×10^{-1}	1.1152
1/64	0.17551	1.1544	4.6031×10^{-3}	1.9783	7.8598×10^{-2}	1.1509

Table 3.14: Errors and rates of convergence of velocity and pressure DG approximations using $\mathbb{P}_1$–$\mathbb{P}_0$ element for Example 3.3 with $\nu = 1/10000$.

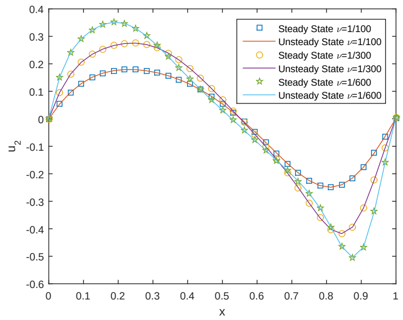
h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate	$\|p(T) - P^M\|$	Rate
1/4	5.98427		9.7215×10^{-1}		2.5750×10^{0}	
1/8	3.23232	0.8886	4.6190×10^{-1}	1.0735	1.5929×10^{0}	0.6929
1/16	1.42203	1.1846	1.5501×10^{-1}	1.5751	8.2872×10^{-1}	0.9426
1/32	0.61274	1.2145	4.3481×10^{-2}	1.8339	3.5883×10^{-1}	1.2075
1/64	0.28118	1.1237	1.1169×10^{-2}	1.9608	1.6003×10^{-1}	1.1649

Example 3.4 (Benchmark Problem)**.** *This example considers the lid-driven cavity flow on a two-dimensional unit square* $[0,1] \times [0,1]$. *No forces are acting on the body. The lid of the cavity is moving in the horizontal direction with a non-zero velocity* $(u_1, u_2) = (1, 0)$. *The no-slip boundary conditions are applied to other parts of the cavity boundaries.*

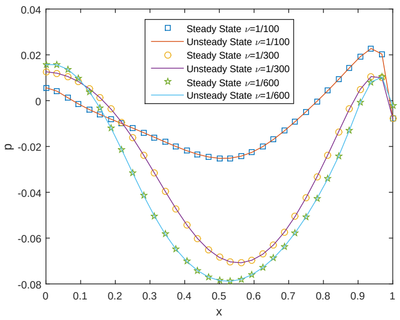
For the space discretization, we employ $\mathbb{P}_1 - \mathbb{P}_0$ mixed finite element space and for the time discretization backward Euler method. We choose the lines $(0.5, y)$ and $(x, 0.5)$ for numerical simulations. Figures 3.3a and 3.3b plot the fully discrete backward Euler and steady-state velocity approximations of (3.1)-(3.3), whereas Figures 3.3c and 3.3d represent the graphs of unsteady and steady states pressure approximations for viscosities $\nu = \{1/100, 1/300, 1/600\}$, retardation times $\kappa = 0.1 \times \nu$, final time $T = 75$, mesh size $h = 1/32$, time step $\Delta t = \mathbb{O}(h^2)$ and penalty parameter $\sigma_e = 40$. The graphs depict that the time-dependent Kelvin-Voigt solution converges to its steady-state solutions for a considerable large time.



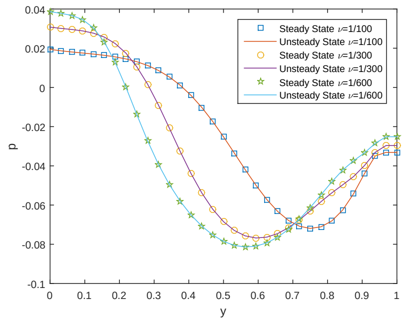
(a) First component of velocity for $x = 0.5$ line



(b) Second component of velocity for $y = 0.5$ line



(c) Pressure for $y = 0.5$ line



(d) Pressure for $x = 0.5$ line

Figure 3.3: Velocity components and pressure for lid driven cavity flow.

3.7 Conclusion

In this chapter, we have applied the SIPG method to the Kelvin-Voigt equations of motion represented by (3.1)-(3.3), which is the first work in this direction. We have defined the semi-discrete DG formulation to (3.1)-(3.3) and have derived *a priori* bounds to the velocity approximation. In order to establish error estimates, we have introduced a modified Sobolev-Stokes projection $\mathbf{S}_h^{so}$ on appropriate DG spaces and proved the approximation properties. Then, by using duality arguments along with the approximation properties of $\mathbf{P}_h$ and $\mathbf{S}_h^{so}$, we have obtained optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$ and pressure in $L^\infty(L^2)$-norms. Moreover, under the smallness assumption on the data, we have shown that the semi-discrete error estimates are uniform in time. Furthermore, we have employed a backward Euler method for full discretization and have achieved optimal convergence rates for the approximate solution. Finally, we have conducted the numerical experiments and have shown that the outcomes verify the theoretical results. Also from our numerical results we have observed that the scheme works well even for small values of ν and κ.