Chapter 4

DG Method for the Oldroyd Model of Order One

In this chapter, we analyze a DG finite element method for the equations of motion that arise in the Oldroyd model of order one. We investigate the existence and uniqueness of semi-discrete discontinuous solutions, as well as the consistency of the scheme. We derive new *a priori* and regularity results for the semi-discrete solution. We next apply the backward Euler method for time discretization and derive *a priori* estimates for the fully discrete solution. Optimal error estimates in L^∞-norm in time, and energy and $\mathbf{L}^2$-norms in space for the velocity, and $L^2(L^2)$ and $L^\infty(L^2)$-norms for the pressure are established for the fully discrete case. At the end, we conduct numerical experiments to support our theoretical results, and analyze the findings. Part of this work has been published in [143].

4.1 Introduction

For the sake of continuity, we first recall the time-dependent Oldroyd flow of order one, which is described by the following nonlinear integro-differential equations

$$\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \int_0^t \beta(t-\tau)\Delta\mathbf{u}(x,\tau)d\tau + \nabla p = \mathbf{f}(x,t), \quad x\in\Omega,\ t>0 \tag{4.1}$$

along with the continuity equation

$$\nabla\cdot\mathbf{u} = 0, \quad x\in\Omega,\ t>0, \tag{4.2}$$

and the initial-boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0, \ \text{ in } \Omega, \quad \mathbf{u} = \mathbf{0}, \ \text{ on } \partial\Omega,\ t\geq 0, \tag{4.3}$$

Here $\mu = 2\kappa\lambda^{-1}$, kernel $\beta(t) = \gamma \exp(-\delta t)$ and $\gamma = 2\lambda^{-1}(\nu - \kappa\lambda^{-1}) > 0$ with $\lambda > 0$ and $\kappa > 0$ are the relaxation and retardation time, respectively, $\delta = \lambda^{-1}$ and, $\nu > 0$ is the kinetic coefficient of viscosity.

We study here, a DG method for the Oldroyd model of order one, which is the first attempt in this direction to the best of authors' knowledge. The goal of this chapter is to formulate a DG scheme for the problem (4.1)-(4.3) and analyze it. For our DG scheme, we choose to discretize the equations (4.1)-(4.3) by mass conservative IP DG methods. The SIPG, NIPG and IIPG methods are considered for the space discretization. We first present the DG weak formulation of the model on the spaces $\mathbf{X}$ and M: Seek $(\mathbf{u}(t), p(t)) \in \mathbf{X} \times M$, $t > 0$, such that

$$(\mathbf{u}_t(t), \boldsymbol{\phi}) + \mu\, a(\mathbf{u}(t), \boldsymbol{\phi}) + c^{\mathbf{u}(t)}(\mathbf{u}(t), \mathbf{u}(t), \boldsymbol{\phi}) + \int_0^t \beta(t-s) a(\mathbf{u}(s), \boldsymbol{\phi})\, ds$$
$$+ b(\boldsymbol{\phi}, p(t)) = (\mathbf{f}(t), \boldsymbol{\phi}), \quad \forall\, \boldsymbol{\phi} \in \mathbf{X}, \tag{4.4}$$
$$b(\mathbf{u}(t), q) = 0, \quad \forall\, q \in M, \qquad (\mathbf{u}(0), \boldsymbol{\phi}) = (\mathbf{u}_0, \boldsymbol{\phi}), \quad \forall\, \boldsymbol{\phi} \in \mathbf{X}. \tag{4.5}$$

Below is the semi-discrete DG formulation for the equations (4.1)-(4.3) on the spaces $\mathbf{X}_h$ and M_h: For $t > 0$, we seek a pair of discontinuous functions $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{X}_h \times M_h$ satisfying

$$(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) + \mu\, a(\mathbf{u}_h(t), \boldsymbol{\phi}_h) + c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t), \mathbf{u}_h(t), \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \boldsymbol{\phi}_h)\, ds$$
$$+ b(\boldsymbol{\phi}_h, p_h(t)) = (\mathbf{f}(t), \boldsymbol{\phi}_h), \quad \forall\, \boldsymbol{\phi}_h \in \mathbf{X}_h, \tag{4.6}$$
$$b(\mathbf{u}_h(t), q_h) = 0, \quad \forall\, q_h \in M_h, \qquad (\mathbf{u}_h(0), \boldsymbol{\phi}_h) = (\mathbf{u}_0, \boldsymbol{\phi}_h), \quad \forall\, \boldsymbol{\phi}_h \in \mathbf{X}_h. \tag{4.7}$$

We next note that the integral term has positivity property of certain form (see Lemma 1.4) which has been preserved for the SIPG case, that is, for $\alpha > 0$ and $0 < t \leq t^*$,

$$\int_0^{t^*} \int_0^t exp(-\alpha(t-s)) a(\boldsymbol{\phi}_h(s), \boldsymbol{\phi}_h(t))\, ds\, dt \geq 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{X}_h.$$

The result has been proved in Lemma 4.2. With this property, we establish *a priori* estimates of the semi-discrete solutions for SIPG which are uniform in time. However, this does not hold true for the NIPG or IIPG case, but we derive uniform in time *a priori* bounds in these cases also by analyzing long term behaviour of the solution. We finally establish the existence of unique semi-discrete solutions, for all three cases.

For a temporal discretization of the semi-discrete DG formulation of the Oldroyd model of order one presented in equations (4.6)-(4.7), we further employ the backward

Euler method and analyze it. Since backward Euler method is a first order difference scheme, the right rectangle rule is chosen here to approximate the integral term

$$q_r^n(\boldsymbol{\psi}) = \Delta t \sum_{j=1}^{n} \beta_{n-j} \boldsymbol{\psi}^j \approx \int_0^{t_n} \beta(t_n - s)\boldsymbol{\psi}(s)\, ds, \tag{4.8}$$

where $\beta_{n-j} = \beta(t_n - t_j)$. Positivity of the quadrature rule has been shown in Lemma 4.5.

The fully discrete numerical scheme for the problem (4.6)-(4.7) is now presented below. For $1 \leq n \leq M$, given a function $\mathbf{U}_h^0$, seek $(\mathbf{U}_h^n, P_h^n) \in \mathbf{X}_h \times M_h$ such that for each $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times M_h$,

$$(\partial_t \mathbf{U}_h^n, \mathbf{v}_h) + \mu\, a(\mathbf{U}_h^n, \mathbf{v}_h) + a(q_r^n(\mathbf{U}_h), \mathbf{v}_h) + c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \mathbf{v}_h)$$
$$+ b(\mathbf{v}_h, P_h^n) = (\mathbf{f}^n, \mathbf{v}_h), \tag{4.9}$$
$$b(\mathbf{U}_h^n, q_h) = 0. \tag{4.10}$$

Note that $\mathbf{U}_h^0$ = local L^2 projection of $\mathbf{u}_0$ onto $\mathbf{X}_h$. *A priori* estimates of the fully discrete solution have been presented in Lemma 4.6 for all the three cases. The global well-posedness of the SIPG scheme has been discussed then. But in the other cases, namely, NIPG and IIPG, only local well-posedness has been established.

For error analysis, we have considered error involved in the fully discrete approximations rather than doing it in two steps, once for semi-discrete approximations, and the other for time discretization. Fully discrete optimal error estimates for the velocity in energy norm and pressure in $L^2(L^2)$ are shown. Additionally, optimal error estimates for the velocity and pressure in $L^\infty(\mathbf{L}^2)$ and $L^\infty(L^2)$-norms, respectively, have only been derived for the SIPG method as the NIPG and IIPG methods yield theoretical estimates that are sub-optimal (see Chapter 2 for the case of NSEs). Finally we carry out numerical computations to validate our results.

Below, we summarize our results:

- Positivity property of the kernel β associated with the integral operator in (4.1) for the SIPG discretization. Regularity results for the semi-discrete solution for SIPG, NIPG and IIPG cases. Existence and uniqueness of the discrete solution and consistency of the scheme.

- Positivity property of the right rectangle rule (4.8) in terms of the SIPG bilinear form $a(\cdot,\cdot)$ and regularity results for the fully discrete solution. Existence and uniqueness of the fully discrete solution.

- Optimal error estimates in energy-norm for fully discrete DG approximate solutions to the velocity for SIPG, NIPG and IIPG cases. Optimal $L^2(L^2)$-norm error estimate for fully discrete approximations to the pressure (for SIPG case).

- A new modified Stokes-Volterra projection $\mathbf{S}_h^{vol}$ for DG spaces, which plays an important role in deriving the fully discrete $L^\infty(\mathbf{L}^2)$-norm velocity error estimates related to the SIPG method. Optimal approximation estimates for $\mathbf{S}_h^{vol}$.

- Optimal error estimates for SIPG case in $L^\infty(\mathbf{L}^2)$ and $L^\infty(L^2)$ norms for fully discrete DG approximate solutions to the velocity and pressure, respectively.

This chapter is arranged as follows: In Section 4.2, we present the consistency of the DG scheme and *a priori* bounds of the semi-discrete solution. A fully discrete scheme can be found in Section 4.3. Fully discrete error estimates for the velocity in energy norm and pressure in $L^2(L^2)$-norm are presented there. Section 4.4 deals with the fully discrete optimal error estimate in $L^\infty(\mathbf{L}^2)$-norm for velocity, whereas optimal $L^\infty(L^2)$-norm estimate for pressure is presented in Section 4.5. Numerical experiments are conducted in Section 4.6, and the results are analyzed. Finally, Section 4.7 concludes this chapter by summarizing the results briefly.
Throughout this chapter, we will use C, $K(>0)$ as generic constants that depend on the given data, μ, α, γ, δ, K_1, K_2, C_2 but do not depend on h and Δt. Note that, K and C may grow algebraically with μ^{-1}. Further, the notations $K(t)$ and K_T will be used when they grow exponentially in time.

4.2 Consistency and *a priori* Bounds

We begin this section by discussing about the consistency of the scheme (4.4)-(4.5). This is followed by *a priori* bounds of the semi-discrete solutions.
We present the lemma below stating the consistency of the scheme.

Lemma 4.1. *Let $(\boldsymbol{u}, p)$ be the solution of (4.1)-(4.3). Then $(\boldsymbol{u}, p)$ satisfies (4.4)-(4.5).*

Proof. Due to (4.2), (4.3) and the fact that $[\mathbf{u}]\cdot\mathbf{n}_e = 0$ on every edge e, equation (4.5) is directly satisfied. Now, multiply (4.1) by $\boldsymbol{\phi}\in\mathbf{X}$ and integrate over each mesh element $E\in\mathcal{E}_h$. By using Green's formula and summing over all elements E, we obtain

$$\begin{aligned}\sum_{E\in\mathcal{E}_h}\int_E \mathbf{u}_t\cdot\boldsymbol{\phi} + \mu\sum_{E\in\mathcal{E}_h}\int_E \nabla\mathbf{u}:\nabla\boldsymbol{\phi} - \mu\sum_{e\in\Gamma_h}\int_e[\nabla\mathbf{u}\mathbf{n}_e\cdot\boldsymbol{\phi}] + \sum_{E\in\mathcal{E}_h}\int_E \mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi}\\ + \int_0^t\beta(t-\tau)\Big(\sum_{E\in\mathcal{E}_h}\int_E\nabla\mathbf{u}(\tau):\nabla\boldsymbol{\phi} - \sum_{e\in\Gamma_h}\int_e[\nabla\mathbf{u}(\tau)\mathbf{n}_e\cdot\boldsymbol{\phi}]\Big)d\tau\\ - \sum_{E\in\mathcal{E}_h}\int_E p\nabla\cdot\boldsymbol{\phi} + \sum_{e\in\Gamma_h}\int_e[p\boldsymbol{\phi}\cdot\mathbf{n}_e] = \int_\Omega \mathbf{f}\cdot\boldsymbol{\phi}.\end{aligned}$$

Note that, $[\nabla\mathbf{u}\mathbf{n}_e\cdot\boldsymbol{\phi}] = \{\nabla\mathbf{u}\}\mathbf{n}_e\cdot[\boldsymbol{\phi}] + [\nabla\mathbf{u}]\mathbf{n}_e\cdot\{\boldsymbol{\phi}\}$. Thus, the regularity of $\mathbf{u}$ implies

$$\begin{aligned}&\mu\sum_{E\in\mathcal{E}_h}\int_E\nabla\mathbf{u}:\nabla\boldsymbol{\phi} - \mu\sum_{e\in\Gamma_h}\int_e[\nabla\mathbf{u}\mathbf{n}_e\cdot\boldsymbol{\phi}]\\ =&\mu\sum_{E\in\mathcal{E}_h}\int_E\nabla\mathbf{u}:\nabla\boldsymbol{\phi} - \mu\sum_{e\in\Gamma_h}\int_e\{\nabla\mathbf{u}\}\mathbf{n}_e\cdot[\boldsymbol{\phi}] = \mu\, a(\mathbf{u},\boldsymbol{\phi}).\end{aligned}$$

An application of similar arguments as above, we can find

$$\begin{aligned}&\int_0^t\beta(t-\tau)\Big(\sum_{E\in\mathcal{E}_h}\int_E\nabla\mathbf{u}(\tau):\nabla\boldsymbol{\phi} - \sum_{e\in\Gamma_h}\int_e[\nabla\mathbf{u}(\tau)\mathbf{n}_e\cdot\boldsymbol{\phi}]\Big)d\tau\\ =&\int_0^t\beta(t-\tau)a(\mathbf{u}(\tau),\boldsymbol{\phi})\,d\tau.\end{aligned}$$

With the incompresibility condition (4.2), and regularity of $\mathbf{u}$ and p, one can find

$$\sum_{E\in\mathcal{E}_h}\int_E\mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi} = c^{\mathbf{u}}(\mathbf{u},\mathbf{u},\boldsymbol{\phi})$$

and

$$-\sum_{E\in\mathcal{E}_h}\int_E p\nabla\cdot\boldsymbol{\phi} + \sum_{e\in\Gamma_h}\int_e[p\boldsymbol{\phi}\cdot\mathbf{n}_e] = b(\mathbf{u},p).$$

This completes the proof of this lemma. □

Below we present an important result, the positive property of the kernel β connected with the integral term in (4.1), which only holds for the SIPG case.

Lemma 4.2. *Suppose σ_e is large enough. Then, for the symmetric form of $a(\cdot,\cdot)$ and for arbitrary $\alpha>0,\ t^*>0$, the following positive definite property holds true:*

$$\int_0^{t^*}\int_0^t exp(-\alpha(t-s))a(\boldsymbol{\phi}_h(s),\boldsymbol{\phi}_h(t))\,ds\,dt \ge 0 \quad \forall\boldsymbol{\phi}_h\in\boldsymbol{X}_h. \qquad (4.11)$$

Proof. First of all we expand the left hand side of (4.11) as

$$\begin{aligned}
&\int_0^{t^*}\int_0^t exp(-\alpha(t-s))a(\boldsymbol{\phi}_h(s),\boldsymbol{\phi}_h(t))\,ds\,dt\\
=&\int_0^{t^*}\int_0^t exp(-\alpha(t-s))\Big(\sum_{E\in\mathcal{E}_h}\int_E \nabla\boldsymbol{\phi}_h(s):\nabla\boldsymbol{\phi}_h(t)-\sum_{e\in\Gamma_h}\int_e\{\nabla\boldsymbol{\phi}_h(s)\}\mathbf{n}_e\cdot[\boldsymbol{\phi}_h(t)]\\
&-\sum_{e\in\Gamma_h}\int_e\{\nabla\boldsymbol{\phi}_h(t)\}\mathbf{n}_e\cdot[\boldsymbol{\phi}_h(s)]+\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\int_e[\boldsymbol{\phi}_h(s)]\cdot[\boldsymbol{\phi}_h(t)]\Big)\,ds\,dt\\
=&B_1+B_2+B_3+B_4. \qquad (4.12)
\end{aligned}$$

B_1 can be rewritten as follows

$$\begin{aligned}
B_1=&\sum_{E\in\mathcal{E}_h}\int_E\Big(\int_0^{t^*}e^{-2\alpha t}\int_0^t e^{\alpha s}\nabla\boldsymbol{\phi}_h(s):e^{\alpha t}\nabla\boldsymbol{\phi}_h(t)\,ds\,dt\Big)\\
=&\frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E\Big(\int_0^{t^*}e^{-2\alpha t}\frac{d}{dt}\Big(\int_0^t e^{\alpha s}\nabla\boldsymbol{\phi}_h(s)\,ds:\int_0^t e^{\alpha s}\nabla\boldsymbol{\phi}_h(s)\,ds\Big)\,dt\Big)
\end{aligned}$$

Let us denote, $\tilde{\boldsymbol{\phi}}_h(t)=\int_0^t e^{\alpha s}\boldsymbol{\phi}_h(s)\,ds$. From the integration by parts and the definition of $\|\cdot\|_{L^2(E)}$-norm, we have

$$B_1=\frac{1}{2}\sum_{E\in\mathcal{E}_h}\Big(e^{-2\alpha t^*}\|\nabla\tilde{\boldsymbol{\phi}}_h(t^*)\|^2_{L^2(E)}+2\alpha\int_0^{t^*}e^{-2\alpha t}\|\nabla\tilde{\boldsymbol{\phi}}_h(t)\|^2_{L^2(E)}\,dt\Big). \qquad (4.13)$$

Again, by applying integration by parts, one can obtain

$$\begin{aligned}
B_2+B_3=&-\sum_{e\in\Gamma_h}\int_e\Big(e^{-2\alpha t^*}\{\nabla\tilde{\boldsymbol{\phi}}_h(t^*)\}\mathbf{n}_e\cdot[\tilde{\boldsymbol{\phi}}_h(t^*)]\Big)\\
&-2\alpha\sum_{e\in\Gamma_h}\int_e\Big(\int_0^{t^*}e^{-2\alpha t}\{\nabla\tilde{\boldsymbol{\phi}}_h(t)\}\mathbf{n}_e\cdot[\tilde{\boldsymbol{\phi}}_h(t)]\,dt\Big).
\end{aligned}$$

Now, a use of the Cauchy-Schwarz inequality yield a lower bound of B_2+B_3:

$$\begin{aligned}
B_2+B_3\ge&-\sum_{e\in\Gamma_h}\Big(e^{-2\alpha t^*}\|\{\nabla\tilde{\boldsymbol{\phi}}_h(t^*)\}\mathbf{n}_e\|_{L^2(e)}|e|^{1/2-1/2}\|[\tilde{\boldsymbol{\phi}}_h(t^*)]\|_{L^2(e)}\Big)\\
&-2\alpha\sum_{e\in\Gamma_h}\Big(\int_0^{t^*}e^{-2\alpha t}\|\{\nabla\tilde{\boldsymbol{\phi}}_h(t)\}\mathbf{n}_e\|_{L^2(e)}|e|^{1/2-1/2}\|[\tilde{\boldsymbol{\phi}}_h(t)]\|_{L^2(e)}\,dt\Big).
\end{aligned}$$

Let m_0 be the maximum number of neighbours of an element ($m_0=3$ for a triangle and $m_0=4$ for a quadrilateral). Thus, using the trace inequality (1.36) and the Cauchy-Schwarz inequality, one can find

$$B_2+B_3\ge-Cm_0^{1/2}\Big(e^{-2\alpha t^*}\sum_{e\in\Gamma_h}\frac{1}{|e|}\|[\tilde{\boldsymbol{\phi}}_h(t^*)]\|^2_{L^2(e)}\Big)^{1/2}\Big(e^{-2\alpha t^*}\sum_{E\in\mathcal{E}_h}\|\nabla\tilde{\boldsymbol{\phi}}_h(t^*)\|^2_{L^2(E)}\Big)^{1/2}$$

$$- 2C\alpha m_0^{1/2}\Big(\int_0^{t^*} e^{-2\alpha t}\sum_{e\in\Gamma_h}\frac{1}{|e|}\|[\tilde{\boldsymbol{\phi}}_h(t)]\|^2_{L^2(e)}\,dt\Big)^{1/2} \times\Big(\int_0^{t^*} e^{-2\alpha t}\sum_{E\in\mathcal{E}_h}\|\nabla\tilde{\boldsymbol{\phi}}_h(t)\|^2_{L^2(E)}\,dt\Big)^{1/2}$$

Using Young's inequality, for $l > 0$, we have

$$\begin{aligned} B_2 + B_3 \geq & -\frac{l}{2}e^{-2\alpha t^*}\sum_{E\in\mathcal{E}_h}\|\nabla\tilde{\boldsymbol{\phi}}_h(t^*)\|^2_{L^2(E)} - \frac{2C^2m_0}{l}e^{-2\alpha t^*}\sum_{e\in\Gamma_h}\frac{1}{|e|}\|[\tilde{\boldsymbol{\phi}}_h(t^*)]\|^2_{L^2(e)} \\ & - l\alpha\int_0^{t^*} e^{-2\alpha t}\sum_{E\in\mathcal{E}_h}\|\nabla\tilde{\boldsymbol{\phi}}_h(t)\|_{L^2(E)}\,dt \\ & - \frac{4C^2\alpha m_0}{l}\int_0^{t^*} e^{-2\alpha t}\sum_{e\in\Gamma_h}\frac{1}{|e|}\|[\tilde{\boldsymbol{\phi}}_h(t)]\|^2_{L^2(e)}\,dt. \end{aligned} \tag{4.14}$$

Similar to B_1, we arrive at the following form of B_4:

$$B_4 = \frac{1}{2}\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\Big(e^{-2\alpha t^*}\|[\tilde{\boldsymbol{\phi}}_h(t^*)]\|^2_{L^2(e)} + 2\alpha\int_0^{t^*} e^{-2\alpha t}\|[\tilde{\boldsymbol{\phi}}_h(t)]\|^2_{L^2(e)}\,dt\Big). \tag{4.15}$$

Combining (4.13)-(4.15) in (4.12), we arrive at

$$\begin{aligned} &\int_0^{t^*}\int_0^t exp(-\alpha(t-s))a(\boldsymbol{\phi}_h(s),\boldsymbol{\phi}_h(t))\,ds\,dt \geq \frac{1}{2}(1-l)\sum_{E\in\mathcal{E}_h} e^{-2\alpha t^*}\|\nabla\tilde{\boldsymbol{\phi}}_h(t^*)\|^2_{L^2(E)} \\ &+ \alpha(1-l)\int_0^{t^*} e^{-2\alpha t}\sum_{E\in\mathcal{E}_h}\|\nabla\tilde{\boldsymbol{\phi}}_h(t)\|^2_{L^2(E)}\,dt + \frac{1}{2}e^{-2\alpha t^*}\sum_{e\in\Gamma_h}\frac{\sigma_e - \frac{4C^2m_0}{l}}{|e|}\|[\tilde{\boldsymbol{\phi}}_h(t^*)]\|^2_{L^2(e)} \\ &+ \alpha\int_0^{t^*} e^{-2\alpha t}\sum_{e\in\Gamma_h}\frac{\sigma_e - \frac{4C^2m_0}{l}}{|e|}\|[\tilde{\boldsymbol{\phi}}_h(t)]\|^2_{L^2(e)}\,dt. \end{aligned}$$

Choosing l appropriately (for example $l = 1/2$) and choosing $\sigma_e \geq \frac{4C^2m_0}{l}$, we finally obatin (4.11). This completes the proof of this lemma. □

The next lemma states the *a priori* bound for the discrete solution $\mathbf{u}_h$.

Lemma 4.3. *Let $0 < \alpha < \min(\delta, \frac{\mu K_1}{2C_2})$. Then, for $t > 0$, the semi-discrete DG approximate solution $\boldsymbol{u}_h$ of the velocity $\boldsymbol{u}$ satisfies*

$$\|\boldsymbol{u}_h(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{u}_h(s)\|^2_{\varepsilon}ds \leq C, \tag{4.16}$$

where C is a positive constant depend only on the given data.

Proof. Choose $\boldsymbol{\phi}_h = \mathbf{u}_h$ in (4.6), $q_h = p_h$ in (4.7), and apply (1.14), (1.19), Lemma 1.6 and the Cauchy-Schwarz inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_h\|^2 + \mu K_1\|\mathbf{u}_h\|^2_{\varepsilon} + \int_0^t \beta(t-s)a(\mathbf{u}_h(s),\mathbf{u}_h(t))\,ds \leq C\|\mathbf{u}_h\|_{\varepsilon}\|\mathbf{f}\|. \tag{4.17}$$

First, we consider $a(\cdot,\cdot)$ is symmetric. A use of Young's inequality, multiply (4.17) by $e^{2\alpha t}$, integrate from 0 to t with respect to time and noting that the resulting double integral term becomes non-negative, due to (4.11), provided $\delta > \alpha > 0$, we arrive at

$$
\begin{aligned}
&e^{2\alpha t^*}\|\mathbf{u}_h(t^*)\|^2 - 2\alpha\int_0^{t^*} e^{2\alpha t}\|\mathbf{u}_h(t)\|^2\,dt + \mu K_1\int_0^{t^*} e^{2\alpha t}\|\mathbf{u}_h(t)\|_\varepsilon^2\,dt\\
&\le \|\mathbf{u}_h(0)\|^2 + C\int_0^{t^*} e^{2\alpha t}\|\mathbf{f}(t)\|^2 dt.
\end{aligned}
$$

By using L^p estimate (1.14) and multiplying by $e^{-2\alpha t^*}$, we obtain

$$
\begin{aligned}
&\|\mathbf{u}_h(t^*)\|^2 + (\mu K_1 - 2C_2\alpha)e^{-2\alpha t^*}\int_0^{t^*} e^{2\alpha t}\|\mathbf{u}_h(t)\|_\varepsilon^2\,dt\\
&\le e^{-2\alpha t^*}\|\mathbf{u}_h(0)\|^2 + Ce^{-2\alpha t^*}\int_0^{t^*} e^{2\alpha t}\|\mathbf{f}(t)\|^2 dt. \qquad (4.18)
\end{aligned}
$$

Note that, the second term on the left-hand side of (4.18) is non-negative provided $\alpha < \frac{\mu K_1}{2C_2}$. With $0 < \alpha < \min(\delta, \frac{\mu K_1}{2C_2})$, we arrive at (4.16) for the SIPG case.
If $a(\cdot,\cdot)$ is non-symmetric, then multiply (4.17) by $e^{2\alpha t}$, integrate with respect to time and use Young's inequality to obtain

$$
\begin{aligned}
&e^{2\alpha t^*}\|\mathbf{u}_h(t^*)\|^2 - 2\alpha\int_0^{t^*} e^{2\alpha t}\|\mathbf{u}_h(t)\|^2\,dt + 2\mu K_1\int_0^{t^*} e^{2\alpha t}\|\mathbf{u}_h(t)\|_\varepsilon^2\,dt\\
&+2\int_0^{t^*} e^{2\alpha t}\int_0^t \beta(t-s)a(\mathbf{u}_h(s),\mathbf{u}_h(t))\,ds\,dt\\
&\le \|\mathbf{u}_h(0)\|^2 + C\int_0^{t^*} e^{2\alpha t}\|\mathbf{f}(t)\|\|\mathbf{u}_h(t)\|_\varepsilon\,dt. \qquad (4.19)
\end{aligned}
$$

Using Lemma 1.7, Hölder's and Young's inequalities, we find that

$$
\begin{aligned}
&2\left|\int_0^{t^*} e^{2\alpha t}\int_0^t \beta(t-s)a(\mathbf{u}_h(s),\mathbf{u}_h(t))\,ds\,dt\right|\\
&\le C\int_0^{t^*} e^{2\alpha t}\int_0^t \beta(t-s)\|\mathbf{u}_h(s)\|_\varepsilon\|\mathbf{u}_h(t)\|_\varepsilon\,ds\,dt\\
&\le \frac{\mu K_1}{2}\int_0^{t^*} e^{2\alpha t}\|\mathbf{u}_h(t)\|_\varepsilon^2\,dt + C\gamma^2\int_0^{t^*}\left(\int_0^t e^{-(\delta-\alpha)(t-s)}e^{\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon\,ds\right)^2 dt\\
&\le \frac{\mu K_1}{2}\int_0^{t^*} e^{2\alpha t}\|\mathbf{u}_h(t)\|_\varepsilon^2\,dt + \frac{C\gamma^2}{2(\delta-\alpha)}\int_0^{t^*}\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\,ds\,dt. \qquad (4.20)
\end{aligned}
$$

We incorporate (4.20) in (4.19), and apply (1.14) and Young's inequality to arrive at

$$
\begin{aligned}
&e^{2\alpha t^*}\|\mathbf{u}_h(t^*)\|^2 + (\mu K_1 - 2C_2\alpha)\int_0^{t^*} e^{2\alpha t}\|\mathbf{u}_h(t)\|_\varepsilon^2\,dt \le \|\mathbf{u}_h(0)\|^2\\
&+C\int_0^{t^*} e^{2\alpha t}\|\mathbf{f}(t)\|^2 dt + C\int_0^{t^*}\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\,ds\,dt. \qquad (4.21)
\end{aligned}
$$

With $0 < \alpha < \min(\delta, \frac{\mu K_1}{2C_2})$ and using Gronwall's lemma, and multiplying the resulting inequality by $e^{-2\alpha t^*}$, we arrive at

$$\|\mathbf{u}_h(t^*)\|^2 + e^{-2\alpha t^*} \int_0^{t^*} e^{2\alpha t} \|\mathbf{u}_h(t)\|_\varepsilon^2 \, dt \le C e^{Ct^*}. \tag{4.22}$$

The above inequality is valid only for finite time t^*.

For large time, we consider the case $t^* \to \infty$. We first multiply (4.19) by $e^{-2\alpha t^*}$ to find

$$\begin{aligned} \|\mathbf{u}_h(t^*)\|^2 + 2\mu K_1 e^{-2\alpha t^*} \int_0^{t^*} e^{2\alpha t} \|\mathbf{u}_h(t)\|_\varepsilon^2 \, dt + 2e^{-2\alpha t^*} \int_0^{t^*} e^{2\alpha t} \dots \\ \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{u}_h(t)) \, ds \, dt \\ \le e^{-2\alpha t^*} \|\mathbf{u}_h(0)\|^2 + 2\alpha e^{-2\alpha t^*} \int_0^{t^*} e^{2\alpha t} \|\mathbf{u}_h(t)\|^2 \, dt + C e^{-2\alpha t^*} \int_0^{t^*} e^{2\alpha t} \|\mathbf{f}(t)\| \|\mathbf{u}_h(t)\|_\varepsilon \, dt. \end{aligned} \tag{4.23}$$

Note that, from L'Hôpital's rule and Lemma 1.6, we can find

$$\begin{aligned} &\limsup_{t^* \to \infty} 2e^{-2\alpha t^*} \int_0^{t^*} e^{2\alpha t} \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \mathbf{u}_h(t)) \, ds \, dt \\ &= \frac{\gamma}{\alpha\delta} \limsup_{t^* \to \infty} a(\mathbf{u}_h(t^*), \mathbf{u}_h(t^*)) \ge \frac{K_1 \gamma}{\alpha\delta} \limsup_{t^* \to \infty} \|\mathbf{u}_h(t^*)\|_\varepsilon^2, \end{aligned} \tag{4.24}$$

and

$$2\mu K_1 \limsup_{t^* \to \infty} e^{-2\alpha t^*} \int_0^{t^*} e^{2\alpha t} \|\mathbf{u}_h(t)\|_\varepsilon^2 \, dt = \frac{\mu K_1}{\alpha} \limsup_{t^* \to \infty} \|\mathbf{u}_h(t^*)\|_\varepsilon^2. \tag{4.25}$$

Take $t^* \to \infty$ in (4.23), and employ L'Hôpital's rule, (4.24) and (4.25) to arrive at

$$\left(\frac{\mu K_1}{\alpha} + \frac{K_1 \gamma}{\alpha\delta} \right) \limsup_{t^* \to \infty} \|\mathbf{u}_h(t^*)\|_\varepsilon^2 \le \frac{C}{\alpha} \limsup_{t^* \to \infty} \|\mathbf{f}(t^*)\| \|\mathbf{u}_h(t^*)\|_\varepsilon,$$

which implies

$$\limsup_{t^* \to \infty} \|\mathbf{u}_h(t^*)\|_\varepsilon \le C. \tag{4.26}$$

Applying (4.26) in (4.21) and multiplying by $e^{-2\alpha t^*}$, we obtain

$$\|\mathbf{u}_h(t^*)\|^2 + e^{-2\alpha t^*} \int_0^{t^*} e^{2\alpha t} \|\mathbf{u}_h(t)\|_\varepsilon^2 \, dt \le C. \tag{4.27}$$

Combining (4.17) and (4.22) with finite time t^*, we obtain (4.16) for NIPG and IIPG cases. This completes the rest of the proof. □

The existence and uniqueness of the discrete velocity $\mathbf{u}_h$ is an immediate consequence of the coercivity property from Lemma 1.6, positivity property of $c(\cdot,\cdot,\cdot)$ (1.19) and Lemma 4.3. Existence and uniqueness of the discrete pressure p_h follows from the discrete inf-sup condition in Lemma 1.8.

Before proceeding to the next section, we state a lemma for the error involving $a(\cdot,\cdot)$. The proof is similar to that of Lemma 2.3 and hence we give a miss.

Lemma 4.4. *There is a constant $C > 0$, independent of h, such that for all $\boldsymbol{u} \in \boldsymbol{H}^{r+1}(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$, we have*

$$|a(\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}, \mathbf{v}_h)| \leq Ch^r |\boldsymbol{u}|_{r+1} \|\mathbf{v}_h\|_\varepsilon, \quad \forall \mathbf{v}_h \in \boldsymbol{X}_h.$$

4.3 Fully Discrete Scheme

Since the fully discrete scheme has been presented in the introduction, we begin this section by working on the well-posedness of the scheme. For this, it is crucial to present the positivity property of the quadrature rule, i.e., the right rectangle rule in terms of the bilinear form $a(\cdot,\cdot)$, which is valid only if $a(\cdot,\cdot)$ is symmetric.

Lemma 4.5. *Suppose σ_e is large enough. Then for the symmetric form of $a(\cdot,\cdot)$ and for arbitrary $\alpha_0 > 0$, the following positivity property holds*

$$\Delta t \sum_{n=1}^{m} \Delta t \sum_{i=1}^{n} e^{-\alpha_0(t_n - t_i)} a(\boldsymbol{\phi}_h^i, \boldsymbol{\phi}_h^n) \geq 0 \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{X}_h.$$

Proof. From the definition of $a(\cdot,\cdot)$, we find

$$\begin{aligned}
\Delta t \sum_{n=1}^{m} \Delta t \sum_{i=1}^{n} e^{-\alpha_0(t_n - t_i)} a(\boldsymbol{\phi}_h^i, \boldsymbol{\phi}_h^n) =& \Delta t^2 \sum_{n=1}^{m} e^{-2\alpha_0 t_n} \sum_{i=1}^{n} \sum_{E \in \mathcal{E}_h} \int_E e^{\alpha_0 t_i} \nabla \boldsymbol{\phi}_h^i : e^{\alpha_0 t_n} \nabla \boldsymbol{\phi}_h^n \\
&- \Delta t^2 \sum_{n=1}^{m} e^{-2\alpha_0 t_n} \sum_{i=1}^{n} \sum_{e \in \Gamma_h} \int_e e^{\alpha_0 t_i} \{\nabla \boldsymbol{\phi}_h^i\} \mathbf{n}_e \cdot e^{\alpha_0 t_n} [\boldsymbol{\phi}_h^n] \\
&- \Delta t^2 \sum_{n=1}^{m} e^{-2\alpha_0 t_n} \sum_{i=1}^{n} \sum_{e \in \Gamma_h} \int_e e^{\alpha_0 t_n} \{\nabla \boldsymbol{\phi}_h^n\} \mathbf{n}_e \cdot e^{\alpha_0 t_i} [\boldsymbol{\phi}_h^i] \\
&+ \Delta t^2 \sum_{n=1}^{m} e^{-2\alpha_0 t_n} \sum_{i=1}^{n} \sum_{e \in \Gamma_h} \frac{\sigma_e}{|e|} \int_e e^{\alpha_0 t_i} [\boldsymbol{\phi}_h^i] \cdot e^{\alpha_0 t_n} [\boldsymbol{\phi}_h^n] \\
=& F_1 + F_2 + F_3 + F_4.
\end{aligned}$$

Let us define, $\boldsymbol{\psi}^n = \Delta t \sum_{i=1}^{n} e^{\alpha t_i} \boldsymbol{\phi}_h^i$. After a two times use of summation by parts on F_1 and simplifying the terms, we arrive at

$$\begin{aligned} F_1 =& \frac{1}{2} e^{-2\alpha_0 t_m} \sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{\psi}^m\|^2_{L^2(E)} + \frac{1}{2} \sum_{n=2}^{m-1} (e^{-2\alpha_0 t_n} - e^{-2\alpha_0 t_{n+1}}) \sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{\psi}^n\|^2_{L^2(E)} \\ &+ \frac{\Delta t^2}{2} \sum_{n=2}^{m} \sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{\phi}_h^n\|^2_{L^2(E)} + \frac{\Delta t^2}{2} e^{-2\alpha_0 t_{m-1}} \sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{\phi}_h^1\|^2_{L^2(E)} \\ &+ \frac{\Delta t^2}{2} (2 - e^{-2\alpha_0 t_2}) \sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{\phi}_h^1\|^2_{L^2(E)}. \end{aligned}$$

In a similar way, F_4 can be presented as follows

$$\begin{aligned} F_4 =& \frac{1}{2} e^{-2\alpha_0 t_m} \sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\boldsymbol{\psi}^m]\|^2_{L^2(e)} + \frac{1}{2} \sum_{n=2}^{m-1} (e^{-2\alpha_0 t_n} - e^{-2\alpha_0 t_{n+1}}) \sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\boldsymbol{\psi}^n]\|^2_{L^2(e)} \\ &+ \frac{\Delta t^2}{2} \sum_{n=2}^{m} \sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\boldsymbol{\phi}_h^n]\|^2_{L^2(e)} + \frac{\Delta t^2}{2} e^{-2\alpha_0 t_{m-1}} \sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\boldsymbol{\phi}_h^1]\|^2_{L^2(e)} \\ &+ \frac{\Delta t^2}{2} (2 - e^{-2\alpha_0 t_2}) \sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \|[\boldsymbol{\phi}_h^1]\|^2_{L^2(e)}. \end{aligned}$$

Again, a repeated application of summation by parts, we find a combined form of F_2 and F_3 as

$$\begin{aligned} F_2 + F_3 =& - \sum_{e\in\Gamma_h} \int_e e^{-2\alpha_0 t_m} \{\nabla \boldsymbol{\psi}^m\} \mathbf{n}_e \cdot [\boldsymbol{\psi}^m] \\ &- \sum_{n=2}^{m-1} \sum_{e\in\Gamma_h} \int_e (e^{-2\alpha_0 t_n} - e^{-2\alpha_0 t_{n+1}}) \{\nabla \boldsymbol{\psi}^n\} \mathbf{n}_e \cdot [\boldsymbol{\psi}^n] \\ &- \Delta t^2 \sum_{n=2}^{m} \sum_{e\in\Gamma_h} \int_e \{\nabla \boldsymbol{\phi}_h^n\} \mathbf{n}_e \cdot [\boldsymbol{\phi}_h^n] - \Delta t^2 (2 - e^{-2\alpha_0 t_2}) \sum_{e\in\Gamma_h} \int_e \{\nabla \boldsymbol{\phi}_h^1\} \mathbf{n}_e \cdot [\boldsymbol{\phi}_h^1]. \end{aligned}$$

Recall that, m_0 is the maximum number of neighbours of an element. Now, we use the Cauchy-Schwarz inequality to the terms on the right hand side of the above equality. Then use (1.36) in the resulting inequality to the terms containing the function $\{\cdot\}$. Finally, by applying Young's inequality, we write

$$\begin{aligned} F_1 + F_2 + F_3 + F_4 \geq& \frac{1}{2} (1 - l) e^{-2\alpha_0 t_m} \sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{\psi}^m\|^2_{L^2(E)} \\ &+ \frac{1}{2} (1 - l) \sum_{n=2}^{m-1} (e^{-2\alpha_0 t_n} - e^{-2\alpha_0 t_{n+1}}) \sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{\psi}^n\|^2_{L^2(E)} \\ &+ \frac{\Delta t^2}{2} (1 - l) \sum_{n=2}^{m} \sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{\phi}_h^n\|^2_{L^2(E)} + \frac{\Delta t^2}{2} (1 - l)(2 - e^{-2\alpha_0 t_2}) \sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{\phi}_h^1\|^2_{L^2(E)} \\ &+ \frac{\Delta t^2}{2} e^{-2\alpha_0 t_{m-1}} \sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{\phi}_h^1\|^2_{L^2(E)} + \frac{1}{2} e^{-2\alpha_0 t_m} \sum_{e\in\Gamma_h} \frac{\sigma_e - \frac{4C^2 m_0}{l}}{|e|} \|[\boldsymbol{\psi}^m]\|^2_{L^2(e)} \end{aligned}$$

$$+\frac{1}{2}\sum_{n=2}^{m-1}(e^{-2\alpha_0 t_n}-e^{-2\alpha_0 t_{n+1}})\sum_{e\in\Gamma_h}\frac{\sigma_e-\frac{4C^2m_0}{l}}{|e|}\|[\boldsymbol{\psi}^n]\|^2_{L^2(e)}$$
$$+\frac{\Delta t^2}{2}\sum_{n=2}^{m}\sum_{e\in\Gamma_h}\frac{\sigma_e-\frac{4C^2m_0}{l}}{|e|}\|[\boldsymbol{\phi}_h^n]\|^2_{L^2(e)}+\frac{\Delta t^2}{2}e^{-2\alpha_0 t_{m-1}}\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\boldsymbol{\phi}_h^1]\|^2_{L^2(e)}$$
$$+\frac{\Delta t^2}{2}(2-e^{-2\alpha_0 t_2})\sum_{e\in\Gamma_h}\frac{\sigma_e-\frac{4C^2m_0}{l}}{|e|}\|[\boldsymbol{\phi}_h^1]\|^2_{L^2(e)}.$$

In the above inequality C and l are the positive constants corresponding to trace inequality (1.36) and Young's inequality, respectively. A suitable choice of l and σ_e so that $(1-l)\geq 0$ and $\sigma_e-\frac{4C^2m_0}{l}\geq 0$, completes the rest of the proof. □

Next, we will establish the stability of the scheme in the following lemma:

Lemma 4.6. *Let $0<\alpha<\min\left(\delta,\frac{\mu K_1}{2C_2}\right)$. If the bilinear form $a(\cdot,\cdot)$ is symmetric, the discrete solution $\boldsymbol{U}_h^n,\ n\geq 1$ of (4.9)-(4.10) satisfies the following estimate:*

$$\|\boldsymbol{U}_h^m\|^2+e^{-2\alpha t_m}\Delta t\sum_{n=1}^{m}e^{2\alpha t_n}\|\boldsymbol{U}_h^n\|_\varepsilon^2\leq C.$$

Furthermore, for the non-symmetric form of $a(\cdot,\cdot)$, choose k_0 small so that $0<\Delta t\leq k_0$ and the following estimate hold true

$$\|\boldsymbol{U}_h^m\|^2+e^{-2\alpha t_m}\Delta t\sum_{n=1}^{m}e^{2\alpha t_n}\|\boldsymbol{U}_h^n\|_\varepsilon^2\leq K_T.$$

Here, $K_T>0$ depends on T.

Proof. Put $\mathbf{v}_h=\mathbf{U}_h^n$ in (4.9) and $q_h=P_h^n$ in (4.10). Observe that

$$(\partial_t\mathbf{U}_h^n,\mathbf{U}_h^n)=\frac{1}{\Delta t}(\mathbf{U}_h^n-\mathbf{U}_h^{n-1},\mathbf{U}_h^n)\geq\frac{1}{2\Delta t}(\|\mathbf{U}_h^n\|^2-\|\mathbf{U}_h^{n-1}\|^2)=\frac{1}{2}\partial_t\|\mathbf{U}_h^n\|^2,$$

and from the positivity of $c(\cdot,\cdot,\cdot)$ in (1.19) and the coercivity property in Lemma 1.6, we obtain

$$\partial_t\|\mathbf{U}_h^n\|^2+2\mu K_1\|\mathbf{U}_h^n\|_\varepsilon^2+2a(q_r^n(\mathbf{U}_h),\mathbf{U}_h^n)\leq 2\|\mathbf{f}^n\|\|\mathbf{U}_h^n\|. \tag{4.28}$$

First of all, we prove this lemma for the SIPG discetization. Noting that

$$\sum_{n=1}^{m}\Delta t e^{2\alpha t_n}\partial_t\|\mathbf{U}_h^n\|^2=\sum_{n=1}^{m}e^{2\alpha t_n}(\|\mathbf{U}_h^n\|^2-\|\mathbf{U}_h^{n-1}\|^2)$$
$$=e^{2\alpha t_m}\|\mathbf{U}_h^m\|^2-\sum_{n=1}^{m-1}e^{2\alpha t_n}(e^{2\alpha\Delta t}-1)\|\mathbf{U}_h^n\|^2-e^{2\alpha\Delta t}\|\mathbf{U}_h^0\|^2,$$

and due to the Lemma 4.5, we have

$$\sum_{n=1}^{m} \Delta t e^{2\alpha t_n} a(q_r^n(\mathbf{U}_h), \mathbf{U}_h^n) = \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \Delta t \sum_{i=1}^{n} \beta(t_n - t_i) a(\mathbf{U}_h^i, \mathbf{U}_h^n) \geq 0.$$

Multiply (4.28) by $\Delta t e^{2\alpha t_n}$, sum over $n = 1$ to m, and using (1.14) and Young's inequality, we have

$$\begin{aligned} e^{2\alpha t_m} \|\mathbf{U}_h^m\|^2 &+ \left(\mu K_1 - \frac{C_2(e^{2\alpha\Delta t} - 1)}{\Delta t}\right) \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{U}_h^n\|_\varepsilon^2 \\ &\leq e^{2\alpha\Delta t} \|\mathbf{U}_h^0\|^2 + \frac{C_2^2}{\mu K_1} \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{f}\|^2. \end{aligned} \tag{4.29}$$

Choose α in such a way that

$$1 + \frac{\mu K_1 \Delta t}{C_2} \geq e^{2\alpha \Delta t}.$$

On multiplying (4.29) through out by $e^{-2\alpha t_m}$, we establish our desired estimate for the SIPG case.

For the NIPG or IIPG case, the third term on the left hand side of (4.28) can be bounded using Lemma 1.7 and Young's inequality as follows:

$$\begin{aligned} |a(q_r^n(\mathbf{U}_h), \mathbf{U}_h^n)| &\leq C \Delta t \sum_{i=1}^{n} \beta(t_n - t_i) \|\mathbf{U}_h^i\|_\varepsilon \|\mathbf{U}_h^n\|_\varepsilon \\ &\leq \frac{K_1 \mu}{64} \|\mathbf{U}_h^n\|_\varepsilon^2 + C \left(\Delta t \sum_{i=1}^{n} \beta(t_n - t_i) \|\mathbf{U}_h^i\|_\varepsilon\right)^2. \end{aligned} \tag{4.30}$$

Insert (4.30) in (4.28), multiply the resulting inequality by $\Delta t e^{2\alpha t_n}$, sum over $n = 1$ to m and using (1.14), we find

$$\begin{aligned} e^{2\alpha t_m} \|\mathbf{U}_h^m\|^2 + \left(\mu K_1 - \frac{C_2(e^{2\alpha\Delta t} - 1)}{\Delta t}\right) \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{U}_h^n\|_\varepsilon^2 \leq e^{2\alpha\Delta t} \|\mathbf{U}_h^0\|^2 \\ + \frac{C_2^2}{\mu K_1} \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{f}\|^2 + C \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \left(\Delta t \sum_{i=1}^{n} \beta(t_n - t_i) \|\mathbf{U}_h^i\|_\varepsilon\right)^2. \end{aligned} \tag{4.31}$$

The last term in the right hand side of (4.31) is bounded using Hölder's inequality as follows:

$$\begin{aligned} \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \left(\Delta t \sum_{i=1}^{n} \beta(t_n - t_i) \|\mathbf{U}_h^i\|_\varepsilon\right)^2 \leq \gamma^2 \Delta t \sum_{n=1}^{m} \left(\Delta t \sum_{i=1}^{n} e^{-2(\delta-\alpha)(t_n - t_i)}\right) \\ \times \left(\Delta t \sum_{i=1}^{n} e^{2\alpha t_i} \|\mathbf{U}_h^i\|_\varepsilon^2\right) \end{aligned}$$

$$\leq \frac{\gamma^2 e^{2(\delta-\alpha)\Delta t}}{2(\delta-\alpha)} \Delta t^2 \sum_{n=1}^{m} \sum_{i=1}^{n} e^{2\alpha t_i} \|\mathbf{U}_h^i\|_\varepsilon^2. \quad (4.32)$$

Combining (4.32) with (4.31), choosing α similar to the SIPG case, applying discrete Gronwall's lemma and after a multiplication of the resulting inequality by $e^{-2\alpha t_m}$, we conclude our desired estimate for the NIPG or IIPG case. □

The existence and uniqueness of the fully discrete solutions to the discrete problem (4.9)-(4.10) can be achieved using (1.19), Lemmas 1.6, 1.8, 4.5, 4.6, and following similar steps as in [72].

4.3.1 Fully Discrete Error Analysis

We are now going to discuss about the error bounds of the fully discrete scheme. We first observe that due to conformity, the exact solution pair $(\mathbf{u}(t), p(t))$ of (4.1)-(4.3) satisfy the system (4.4)-(4.5), that is, for $(\mathbf{u}(t), p(t)) \in (\mathbf{X}_h, M_h)$,

$$(\mathbf{u}_t(t), \mathbf{v}_h) + \mu\, a(\mathbf{u}(t), \mathbf{v}_h) + c^{\mathbf{u}(t)}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}_h)$$
$$+ \int_0^t \beta(t-s) a(\mathbf{u}(s), \mathbf{v}_h)\, ds + b(\mathbf{v}_h, p(t)) = (\mathbf{f}(t), \mathbf{v}_h), \quad (4.33)$$
$$b(\mathbf{u}(t), q_h) = 0, \quad \text{and} \quad (\mathbf{u}(0), \mathbf{v}_h) = (\mathbf{u}_0, \mathbf{v}_h), \quad (4.34)$$

for $(\mathbf{v}_h, q_h) \in (\mathbf{X}_h, M_h)$.

Denote $\mathbf{E}^n = \mathbf{u}^n - \mathbf{U}_h^n$ with $n = 1, 2, \ldots, M$. At $t = t_n$, the equations (4.33) and (4.34) become

$$(\mathbf{u}_t^n, \mathbf{v}_h) + \mu a(\mathbf{u}^n, \mathbf{v}_h) + c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}_h) + \int_0^{t_n} \beta(t_n - s) a(\mathbf{u}(s), \mathbf{v}_h)\, ds$$
$$+ b(\mathbf{v}_h, p^n) = (\mathbf{f}^n, \mathbf{v}_h), \quad (4.35)$$
$$b(\mathbf{u}^n, q_h) = 0, \quad (4.36)$$

for $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times M_h$. We now obtain the error equations from (4.9), (4.10), (4.35) and (4.36) in the following form:

$$(\partial_t \mathbf{E}^n, \mathbf{v}_h) + \mu\, a(\mathbf{E}^n, \mathbf{v}_h) + a(q_r^n(\mathbf{E}), \mathbf{v}_h) + c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}_h) - c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \mathbf{v}_h)$$
$$+ b(\mathbf{v}_h, p^n - P_h^n) = -(\mathbf{u}_t^n, \mathbf{v}_h) + (\partial_t \mathbf{u}^n, \mathbf{v}_h) + a(q_r^n(\mathbf{u}), \mathbf{v}_h)$$
$$- \int_0^{t_n} \beta(t_n - s) a(\mathbf{u}(s), \mathbf{v}_h)\, ds, \quad \forall \mathbf{v}_h \in \mathbf{X}_h, \quad (4.37)$$
$$b(\mathbf{E}^n, q_h) = 0, \quad \forall q_h \in M_h. \quad (4.38)$$

Before looking into the error analysis, we will need the error associated the quadrature rule. For $\boldsymbol{\psi} \in C^1[0, t_n]$, the error associated with the rule (4.8) is given by

$$\left| \int_0^{t_n} \beta(t_n - s)\boldsymbol{\psi}(s)\, ds - q_r^n(\boldsymbol{\psi}) \right| \leq C\Delta t \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left| \frac{\partial}{\partial s}\big(\beta(t_n - s)\boldsymbol{\psi}(s)\big) \right| ds. \tag{4.39}$$

The above estimate can be derived with the help of the following estimate: For $\boldsymbol{\psi} \in C^1[t_{n-1}, t_n]$

$$\boldsymbol{\psi}(t_n) - \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \boldsymbol{\psi}(s)\, ds = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} (s - t_{n-1})\boldsymbol{\psi}_s(s)\, ds. \tag{4.40}$$

The next theorem establishes a velocity error estimate for SIPG method in the fully discrete discretization.

Theorem 4.1. *Let the assumption (**A3**) be hold true and $0 < \alpha < \min\left(\delta, \frac{\mu K_1}{2C_2}\right)$. Then, if $a(\cdot,\cdot)$ is symmetric, there is a constant K_T independent of h and Δt, such that*

$$\|\mathbf{E}^m\|^2 + \Delta t e^{-2\alpha t_m} \sum_{n=1}^{m} e^{2\alpha t_n} \|\mathbf{E}^n\|_\varepsilon^2 \leq K_T(h^{2r} + \Delta t^2).$$

Proof. First, we define $\boldsymbol{\zeta}_n = \mathbf{U}_h^n - (\boldsymbol{\Pi}_h(\mathbf{u}))^n$ and $\boldsymbol{\eta}_n = \mathbf{u}^n - (\boldsymbol{\Pi}_h(\mathbf{u}))^n$. We choose $\mathbf{v}_h = \boldsymbol{\zeta}_n$ in (4.37) and observing the fact

$$(\partial_t \boldsymbol{\zeta}_n, \boldsymbol{\zeta}_n) \geq \frac{1}{2}\partial_t \|\boldsymbol{\zeta}_n\|^2,$$

and using Lemma 1.6, we find that

$$\begin{aligned}
\partial_t \|\boldsymbol{\zeta}_n\|^2 &+ 2K_1\mu\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + 2a(q_r^n(\boldsymbol{\zeta}), \boldsymbol{\zeta}_n) - 2c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\zeta}_n) + 2c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\zeta}_n) \\
&\leq |2(\mathbf{u}_t^n, \boldsymbol{\zeta}_n) - 2(\partial_t \mathbf{u}^n, \boldsymbol{\zeta}_n)| + |2(\partial_t \boldsymbol{\eta}_n, \boldsymbol{\zeta}_n)| + 2\mu|a(\boldsymbol{\eta}_n, \boldsymbol{\zeta}_n)| + |2b(\boldsymbol{\zeta}_n, p^n - P_h^n)| \\
&\quad + 2\left| a(q_r^n(\mathbf{u}), \boldsymbol{\zeta}_n) - \int_0^{t_n} \beta(t_n - s) a(\mathbf{u}(s), \boldsymbol{\zeta}_n)\, ds \right| + \left| 2a(q_r^n(\boldsymbol{\eta}), \boldsymbol{\zeta}_n) \right| \\
&= H_1 + H_2 + H_3 + H_4 + H_5 + H_6.
\end{aligned} \tag{4.41}$$

Using the continuity of $\mathbf{u}$, we eliminate the superscripts from the trilinear forms and then rewrite them as

$$\begin{aligned}
c(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\zeta}_n) - c(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\zeta}_n) &= c(\mathbf{U}_h^{n-1}, \boldsymbol{\zeta}_n, \boldsymbol{\zeta}_n) - c(\boldsymbol{\zeta}_{n-1}, \boldsymbol{\eta}_n, \boldsymbol{\zeta}_n) \\
+ c(\boldsymbol{\zeta}_{n-1}, \mathbf{u}^n, \boldsymbol{\zeta}_n) - c(\boldsymbol{\eta}_{n-1}, (\boldsymbol{\Pi}_h(\mathbf{u}))^n, \boldsymbol{\zeta}_n) &- c(\mathbf{u}^{n-1}, \boldsymbol{\eta}_n, \boldsymbol{\zeta}_n) - c(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^n, \boldsymbol{\zeta}_n).
\end{aligned}$$

From the positivity property (1.19), the term $c(\mathbf{U}_h^{n-1}, \boldsymbol{\zeta}_n, \boldsymbol{\zeta}_n)$ is non-negative. The other nonlinear terms can be bounded similarly as in the proof of Theorem 5.2 in [98].

$$|c(\boldsymbol{\zeta}_{n-1}, \boldsymbol{\eta}_n, \boldsymbol{\zeta}_n)| \leq \frac{K_1\mu}{64}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + C\|\boldsymbol{\zeta}_{n-1}\|^2,$$

$$
\begin{aligned}
|c(\boldsymbol{\zeta}_{n-1}, \mathbf{u}^n, \boldsymbol{\zeta}_n)| &\le \frac{K_1\mu}{64}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + C\|\boldsymbol{\zeta}_{n-1}\|^2,\\
|c(\boldsymbol{\eta}_{n-1}, (\mathbf{R}_h(\mathbf{u}))^n, \boldsymbol{\zeta}_n)| &\le \frac{K_1\mu}{64}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + Ch^{2r}|\mathbf{u}^n|_{r+1}^2,\\
|c(\mathbf{u}^{n-1}, \boldsymbol{\eta}_n, \boldsymbol{\zeta}_n)| &\le \frac{K_1\mu}{64}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + Ch^{2r}|\mathbf{u}^n|_{r+1}^2,\\
|c(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^n, \boldsymbol{\zeta}_n)| &\le \frac{K_1\mu}{64}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + C\Delta t\|\mathbf{u}_t\|_{L^2(t_{n-1},t_n;H^1(\Omega))}^2.
\end{aligned}
$$

Regarding the term H_1, (2.138), and the Cauchy-Schwarz and Young's inequalities yield

$$
H_1 \le C\Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{tt}(t)\|^2\,dt\right)^{1/2}\|\boldsymbol{\zeta}_n\|_\varepsilon \le \frac{K_1\mu}{64}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + C\Delta t\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{tt}(t)\|^2\,dt.
$$

For the term H_2, containing ∂_t, we can write

$$
\partial_t\boldsymbol{\eta}_n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\frac{\partial}{\partial t}(\boldsymbol{\eta}(s))\,ds.
$$

Hence, H_2 can be bounded by using (1.14), (1.28), and the Cauchy-Schwarz and Young's inequalities as

$$
H_2 \le C\frac{h^r}{\Delta t^{1/2}}\|\mathbf{u}_t\|_{L^2(t_{n-1},t_n;H^r(\Omega))}\|\boldsymbol{\zeta}_n\|_\varepsilon \le \frac{K_1\mu}{64}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + C\frac{h^{2r}}{\Delta t}\|\mathbf{u}_t\|_{L^2(t_{n-1},t_n;H^r(\Omega))}^2.
$$

Using (1.26), (4.38), Lemmas 4.4 and 2.4 and Young's inequality, we can bound the bounds for the linear terms H_3 and H_4 as follows

$$
\begin{aligned}
&H_3 \le \frac{K_1\mu}{64}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + Ch^{2r}|\mathbf{u}^n|_{r+1}^2,\\
&H_4 \le 2|b(\boldsymbol{\zeta}_n, p^n - (r_h(p))^n)| + 2|b(\boldsymbol{\zeta}_n, (r_h(p))^n - P_h^n)| \le \frac{K_1\mu}{64}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + Ch^{2r}|p^n|_r^2.
\end{aligned}
$$

Note that, since $\mathbf{u}$ is continuous, the two jump terms containing $\mathbf{u}$ in H_5 will vanish. Using the trace inequality (1.24), definition of the bilinear form $a(\cdot,\cdot)$, estimate (1.14) and (4.39), we obtain

$$
\begin{aligned}
H_5 \le \Bigg(C\Delta t\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\big(&\beta_s(t_n-s)(\|\mathbf{u}(s)\|_1 + h\|\mathbf{u}(s)\|_2)\\
&+ \beta(t_n-s)(\|\mathbf{u}_s(s)\|_1 + h\|\mathbf{u}_s(s)\|_2)\big)\,ds\Bigg)\|\boldsymbol{\zeta}_n\|_\varepsilon.
\end{aligned}
$$

Similar to H_3, the term H_6 can be bounded using Lemma 4.4 as

$$
H_6 \le Ch^r\Delta t\sum_{i=1}^{n}\beta(t_n-t_i)|\mathbf{u}^i|_{r+1}\|\boldsymbol{\zeta}_n\|_\varepsilon.
$$

Collecting the bounds for the nonlinear terms and H_1, H_2, H_3, H_4, H_5, H_6 in (4.41) and using Young's inequality, we obtain

$$\begin{aligned}\partial_t\|\boldsymbol{\zeta}_n\|^2 + K_1\mu\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + a(q_r^n(\boldsymbol{\zeta}),\boldsymbol{\zeta}_n) &\le C\|\boldsymbol{\zeta}_{n-1}\|^2 + Ch^{2r}(|\mathbf{u}^n|_{r+1}^2 + |p^n|_r^2)\\ &+ C\Delta t\int_{t_{n-1}}^{t_n}(\|\mathbf{u}_t(t)\|_1^2 + \|\mathbf{u}_{tt}(t)\|^2)\,dt + C\frac{h^{2r}}{\Delta t}\int_{t_{n-1}}^{t_n}|\mathbf{u}_t(t)|_r^2\,dt\\ &+ C\Delta t^2\int_0^{t_n} e^{-2\delta(t_n-t)}(\|\mathbf{u}(t)\|_1^2 + h^2\|\mathbf{u}(t)\|_2^2 + \|\mathbf{u}_t(t)\|_1^2 + h^2\|\mathbf{u}_t(t)\|_2^2)\,dt\\ &+ Ch^{2r}\Delta t^2\Big(\sum_{i=1}^n\beta(t_n-t_i)|\mathbf{u}^i|_{r+1}\Big)^2. \end{aligned} \tag{4.42}$$

Multiplying (4.42) by $\Delta t e^{2\alpha t_n}$, summing the resulting inequality from $n=1$ to $n=m$ ($m\le M$), noting

$$\sum_{n=1}^m \Delta t e^{2\alpha t_n}\partial_t\|\boldsymbol{\zeta}_n\|^2 = e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|^2 - \sum_{n=1}^{m-1} e^{2\alpha t_n}(e^{2\alpha\Delta t}-1)\|\boldsymbol{\zeta}_n\|^2,$$

and due to Lemma 4.5

$$\sum_{n=1}^m \Delta t e^{2\alpha t_n} a(q_r^n(\boldsymbol{\zeta}),\boldsymbol{\zeta}_n) = \Delta t\sum_{n=1}^m e^{2\alpha t_n}\Delta t\sum_{i=1}^n \beta(t_n-t_i)a(\boldsymbol{\zeta}_i,\boldsymbol{\zeta}_n) \ge 0,$$

we arrive at

$$\begin{aligned} e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|^2 + K_1\mu\Delta t\sum_{n=1}^m e^{2\alpha t_n}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 &\le \sum_{n=1}^{m-1} e^{2\alpha t_n}(e^{2\alpha\Delta t}-1)\|\boldsymbol{\zeta}_n\|^2 + C\Delta t\sum_{n=1}^m e^{2\alpha t_n}\|\boldsymbol{\zeta}_{n-1}\|^2\\ &+Ch^{2r}\Delta t\sum_{n=1}^m e^{2\alpha t_n}(|\mathbf{u}^n|_{r+1}^2 + |p^n|_r^2) + C\Delta t^2\sum_{n=1}^m e^{2\alpha t_n}\int_{t_{n-1}}^{t_n}(\|\mathbf{u}_t(t)\|_1^2 + \|\mathbf{u}_{tt}(t)\|^2)\,dt\\ &+C\Delta t^3\sum_{n=1}^m e^{2\alpha t_n}\int_0^{t_n} e^{-2\delta(t_n-t)}(\|\mathbf{u}(t)\|_1^2 + h^2\|\mathbf{u}(t)\|_2^2 + \|\mathbf{u}_t(t)\|_1^2 + h^2\|\mathbf{u}_t(t)\|_2^2)\,dt\\ &+Ch^{2r}\sum_{n=1}^m e^{2\alpha t_n}\int_{t_{n-1}}^{t_n}|\mathbf{u}_t(t)|_r^2\,dt + Ch^{2r}\Delta t\sum_{n=1}^m e^{2\alpha t_n}\Delta t^2\Big(\sum_{i=1}^n\beta(t_n-t_i)|\mathbf{u}^i|_{r+1}\Big)^2. \end{aligned} \tag{4.43}$$

Since $e^{2\alpha\Delta t}-1\le C(\alpha)\Delta t$, the first and second terms on right hand side of (4.43) can be combined. Using assumption (**A3**), we now observe that

$$\begin{aligned} &C\Delta t^2\sum_{n=1}^m e^{2\alpha t_n}\int_{t_{n-1}}^{t_n}(\|\mathbf{u}_t(t)\|_1^2 + \|\mathbf{u}_{tt}(t)\|^2)\,dt\\ &= C\Delta t^2\sum_{n=1}^m\int_{t_{n-1}}^{t_n} e^{2\alpha(t_n-t)}e^{2\alpha t}(\|\mathbf{u}_t(t)\|_1^2 + \|\mathbf{u}_{tt}(t)\|^2)\,dt\\ &\le C\Delta t^2 e^{2\alpha\Delta t}\int_0^{t_m} e^{2\alpha t}(\|\mathbf{u}_t(t)\|_1^2 + \|\mathbf{u}_{tt}(t)\|^2)\,dt \le C\Delta t^2 e^{2\alpha t_{m+1}}. \end{aligned} \tag{4.44}$$

Similarly, one can obtain

$$C\Delta t^3 \sum_{n=1}^{m} e^{2\alpha t_n} \int_0^{t_n} e^{-2\delta(t_n-t)} (\|\mathbf{u}(t)\|_1^2 + h^2\|\mathbf{u}(t)\|_2^2 + \|\mathbf{u}_t(t)\|_1^2 + h^2\|\mathbf{u}_t(t)\|_2^2)\, dt$$
$$\leq C\Delta t^2 e^{2\alpha t_m}. \tag{4.45}$$

Now, by using the Cauchy-Schwarz inequality and change of order of summation, we obtain

$$Ch^{2r}\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \left(\Delta t \sum_{i=1}^{n} \beta(t_n - t_i)|\mathbf{u}^i|_{r+1} \right)^2 \leq Ch^{2r} \frac{\gamma^2 e^{2(\delta-\alpha)\Delta t}}{(\delta-\alpha)^2} \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} |\mathbf{u}^n|_{r+1}^2. \tag{4.46}$$

Insert (4.44)–(4.46) in (4.43) and apply assumption (**A3**) to arrive at

$$e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|^2 + K_1\mu\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 \leq C\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\boldsymbol{\zeta}_{n-1}\|^2 + Ch^{2r}e^{2\alpha t_m}$$
$$+ C\Delta t^2 e^{2\alpha t_{m+1}}.$$

Using discrete Gronwall's lemma and multiplying by $e^{-2\alpha t_m}$, we obatin

$$\|\boldsymbol{\zeta}_m\|^2 + K_1\mu\Delta t \sum_{n=1}^{m} e^{2\alpha(t_n - t_m)}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 \leq Ce^{CT}(h^{2r} + \Delta t^2). \tag{4.47}$$

Noting that, $\mathbf{E}^n = \boldsymbol{\eta}_n - \boldsymbol{\zeta}_n$, and an application of triangle inequality, approximation properties (1.28) and (1.29), and assumption (**A3**) give us the desired result. □

In the case of NIPG or IIPG discretizaton, the analysis will slightly differ since the Lemma 4.5 will not hold. In fact, the analysis for NIPG or IIPG case will be identical as Theorem 4.1, except for the quadrature term, $a(q_r^n(\boldsymbol{\zeta}), \boldsymbol{\zeta}_n)$, which needs to be bounded. The following theorem establishes an error estimate of velocity for these two cases.

Theorem 4.2. *Under the assumptions of Theorem 4.1, and if* $a(\cdot,\cdot)$ *is non-symmetric, then there exists a constant* $K_T > 0$*, such that, the following estimates hold true:*

$$\|\mathbf{E}^m\|^2 + \Delta t e^{-2\alpha t_m} \sum_{n=1}^{m} e^{2\alpha t_n}\|\mathbf{E}^n\|_\varepsilon^2 \leq K_T(h^{2r} + \Delta t^2).$$

Proof. Following the proof of Lemma 4.6 for the NIPG or IIPG case, the term $a(q_r^n(\boldsymbol{\zeta}), \boldsymbol{\zeta}_n)$ can be bounded. The only difference is that, we have to incorporate $\boldsymbol{\zeta}_n$ in place of $\mathbf{U}_h^n$ in (4.30). Multiply (4.42) by $\Delta t e^{2\alpha t_n}$, sum over $1 \leq n \leq m \leq M$ and proceed similarly as Theorem 4.1, we obtain

$$e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|^2 + K_1\mu\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 \leq C\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\boldsymbol{\zeta}_{n-1}\|^2 + Ch^{2r}e^{2\alpha t_m}$$

$$+ C\Delta t^2 e^{2\alpha t_{m+1}} + C\Delta t^2 \sum_{n=1}^{m}\sum_{i=1}^{n} e^{2\alpha t_i}\|\boldsymbol{\zeta}_i\|_\varepsilon^2.$$

Using discrete Gronwall's lemma and triangle inequality, we complete the proof of this theorem. □

For deriving fully discrete pressure error estimate, we next present an auxiliary lemma. We only take into consideration the scenario when $a(\cdot,\cdot)$ is symmetric since the non-symmetric form of $a(\cdot,\cdot)$ generates sub-optimal rates of convergence.

Lemma 4.7. *Suppose the assumptions of Theorem 4.1 are satisfied. Then, for SIPG discretization, the following estimate holds true:*

$$e^{-2\alpha t_m}\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\partial_t \mathbf{E}^n\|^2 + \|\mathbf{E}^m\|_\varepsilon^2 \le K_T(h^{2r} + \Delta t^2). \tag{4.48}$$

Proof. We put $\mathbf{v}_h = \partial_t\boldsymbol{\zeta}_n$ and $q_h = P_h^n$ in the error equations (4.37) and (4.38), respectively, to find that

$$\begin{aligned}
&\|\partial_t\boldsymbol{\zeta}_n\|^2 + \mu a(\boldsymbol{\zeta}_n, \partial_t\boldsymbol{\zeta}_n) + a(q_r^n(\boldsymbol{\zeta}), \partial_t\boldsymbol{\zeta}_n) + c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \partial_t\boldsymbol{\zeta}_n) - c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \partial_t\boldsymbol{\zeta}_n)\\
&\le |(\mathbf{u}_t^n, \partial_t\boldsymbol{\zeta}_n) - (\partial_t\mathbf{u}^n, \partial_t\boldsymbol{\zeta}_n)| + |(\partial_t\boldsymbol{\eta}_n, \partial_t\boldsymbol{\zeta}_n)| + \mu|a(\boldsymbol{\eta}_n, \partial_t\boldsymbol{\zeta}_n)| + \left|a(q_r^n(\boldsymbol{\eta}), \partial_t\boldsymbol{\zeta}_n)\right|\\
&\quad + |b(\partial_t\boldsymbol{\zeta}_n, p^n - P_h^n)| + \left|\int_0^{t_n}\beta(t_n - s)a(\mathbf{u}(s), \partial_t\boldsymbol{\zeta}_n)\,ds - a(q_r^n(\mathbf{u}), \partial_t\boldsymbol{\zeta}_n)\right|\\
&= H_7 + H_8 + \cdots + H_{12}.
\end{aligned} \tag{4.49}$$

The superscripts are removed and the trilinear forms are rewritten as follows

$$\begin{aligned}
c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \partial_t\boldsymbol{\zeta}_n) - c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^n, \mathbf{u}^n, \partial_t\boldsymbol{\zeta}_n) &= -c(\mathbf{u}^{n-1}, \mathbf{E}^n, \partial_t\boldsymbol{\zeta}_n)\\
-c(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^n, \partial_t\boldsymbol{\zeta}_n) + c(\mathbf{E}^{n-1}, \mathbf{u}^n, \partial_t\boldsymbol{\zeta}_n) &+ c(\boldsymbol{\eta}_{n-1}, \mathbf{E}^n, \partial_t\boldsymbol{\zeta}_n)\\
-c(\boldsymbol{\zeta}_{n-1}, \boldsymbol{\eta}_n, \partial_t\boldsymbol{\zeta}_n) &+ c(\boldsymbol{\zeta}_{n-1}, \boldsymbol{\zeta}_n, \partial_t\boldsymbol{\zeta}_n).
\end{aligned} \tag{4.50}$$

Similar to Theorem 4.1, the first five terms on the right hand side of (4.50) can be bounded as follows:

$$\begin{aligned}
&|c(\mathbf{u}^{n-1}, \mathbf{E}^n, \partial_t\boldsymbol{\zeta}_n) + c(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^n, \partial_t\boldsymbol{\zeta}_n) + c(\mathbf{E}^{n-1}, \mathbf{u}^n, \partial_t\boldsymbol{\zeta}_n)\\
&+ c(\boldsymbol{\eta}_{n-1}, \mathbf{E}^n, \partial_t\boldsymbol{\zeta}_n) + c(\boldsymbol{\zeta}_{n-1}, \boldsymbol{\eta}_n, \partial_t\boldsymbol{\zeta}_n)| \le \frac{1}{64}\|\partial_t\boldsymbol{\zeta}_n\|^2\\
&+ C\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}\Big(\|\mathbf{E}^n\|_\varepsilon^2 + \|\mathbf{E}^{n-1}\|_\varepsilon^2 + \|\boldsymbol{\zeta}_{n-1}\|_\varepsilon^2 + \Delta t\|\mathbf{u}_t\|^2_{L^2(t_{n-1},t_n;H^1(\Omega))}\Big).
\end{aligned} \tag{4.51}$$

The term $c(\boldsymbol{\zeta}_{n-1}, \boldsymbol{\zeta}_n, \partial_t\boldsymbol{\zeta}_n)$ in (4.50) is estimated by applying Lemma 1.10 as

$$|c(\boldsymbol{\zeta}_{n-1}, \boldsymbol{\zeta}_n, \partial_t\boldsymbol{\zeta}_n)| \le \frac{1}{64}\|\partial_t\boldsymbol{\zeta}_n\|^2 + C\frac{1}{\min\limits_{E\in\mathcal{E}_h} h_E}\|\boldsymbol{\zeta}_{n-1}\|_\varepsilon^2\|\boldsymbol{\zeta}_n\|_\varepsilon^2. \tag{4.52}$$

The terms H_7 and H_8 can be bounded similar to the terms H_1 and H_2 in Theorem 4.1 with an exception that, here we have to bound $\partial_t\boldsymbol{\zeta}_n$ in $\mathbf{L}^2$-norm.
We now multiply (4.49) by $\Delta t e^{2\alpha t_n}$ and then sum from $n=1$ to $n=m$ $(m \le M)$. In the terms corresponding to H_9, H_{10} and H_{11}, we shift the discrete time derivative to the other arguments in the bilinear forms by using summation by parts. For instance, the term corresponding to H_9 is bounded using Lemma 4.4 and Young's inequality as follows

$$\begin{aligned}
\mu\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}|a(\boldsymbol{\eta}_n,\partial_t\boldsymbol{\zeta}_n)| \le{}& \mu e^{2\alpha t_m}|a(\boldsymbol{\eta}_m,\boldsymbol{\zeta}_m)| + \mu|a(\boldsymbol{\eta}_0,\boldsymbol{\zeta}_0)| \\
&+ \mu\sum_{n=1}^{m}\Big(1-e^{2\alpha\Delta t}\Big)e^{2\alpha t_{n-1}}|a(\boldsymbol{\eta}_{n-1},\boldsymbol{\zeta}_{n-1})| + \mu\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}|a(\partial_t\boldsymbol{\eta}_n,\boldsymbol{\zeta}_{n-1})| \\
\le{}& \frac{\mu K_1}{64}e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|_\varepsilon^2 + C\Delta t\sum_{n=1}^{m}(e^{2\alpha t_{n-1}}+e^{2\alpha t_n})\|\boldsymbol{\zeta}_{n-1}\|_\varepsilon^2 + Ch^{2r}e^{2\alpha t_m}|\mathbf{u}^m|_{r+1}^2 \\
&+ Ch^{2r}|\mathbf{u}_0|_{r+1}^2 + Ch^{2r}\Delta t\sum_{n=1}^{m} e^{2\alpha t_{n-1}}|\mathbf{u}^{n-1}|_{r+1}^2 + Ch^{2r}\sum_{n=1}^{m} e^{2\alpha t_n}\int_{t_{n-1}}^{t_n}|\mathbf{u}_t(t)|_{r+1}^2\,dt.
\end{aligned} \tag{4.53}$$

Similarly, the terms corresponding to H_{10} and H_{11} can be bounded as

$$\begin{aligned}
\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\Big|a(q_r^n(\boldsymbol{\eta}),\partial_t\boldsymbol{\zeta}_n)\Big| \le{}& \frac{\mu K_1}{64}e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|_\varepsilon^2 + C\Delta t\sum_{n=1}^{m}(e^{2\alpha t_n}+e^{2\alpha t_{n-1}})\|\boldsymbol{\zeta}_{n-1}\|_\varepsilon^2 \\
&+ Ch^{2r}\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}|\mathbf{u}^n|_{r+1}^2,
\end{aligned} \tag{4.54}$$

and

$$\begin{aligned}
\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}|b(\partial_t\boldsymbol{\zeta}_n, p^n - P_h^n)| \le{}& \frac{\mu K_1}{64}e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|_\varepsilon^2 + C\Delta t\sum_{n=1}^{m}(e^{2\alpha t_{n-1}}+e^{2\alpha t_n})\|\boldsymbol{\zeta}_{n-1}\|_\varepsilon^2 \\
+ Ch^{2r}e^{2\alpha t_m}|p^m|_r^2 + Ch^{2r}\Delta t&\sum_{n=1}^{m} e^{2\alpha t_{n-1}}|p^{n-1}|_r^2 + Ch^{2r}\sum_{n=1}^{m} e^{2\alpha t_n}\int_{t_{n-1}}^{t_n}|p_t(t)|_r^2\,dt.
\end{aligned} \tag{4.55}$$

Again, by using summation by parts, the Cauchy-Schwarz, Young's inequalities and (4.39), and observing $e^{2\alpha\Delta t}-1 \le C(\alpha)\Delta t$ and $e^{\delta\Delta t}-1 \le C(\delta)\Delta t$, one can obtain

$$\begin{aligned}
&\Big|\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\int_0^{t_n}\beta(t_n-s)a(\mathbf{u}(s),\partial_t\boldsymbol{\zeta}_n)\,ds - a(q_r^n(\mathbf{u}),\partial_t\boldsymbol{\zeta}_n)\Big| \\
\le{}& e^{2\alpha t_m}\Big|\int_0^{t_m}\beta(t_m-s)a(\mathbf{u}(s),\boldsymbol{\zeta}_m)\,ds - a(q_r^m(\mathbf{u}),\boldsymbol{\zeta}_m)\Big| \\
&+ (e^{2\alpha\Delta t}-1)\Big|\sum_{n=1}^{m-1} e^{2\alpha t_n}\Big(\int_0^{t_n}\beta(t_n-s)a(\mathbf{u}(s),\boldsymbol{\zeta}_n)\,ds - a(q_r^n(\mathbf{u}),\boldsymbol{\zeta}_n)\Big)\Big| \\
&+ \Big|\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\Big(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\beta(t_n-s)a(\mathbf{u}(s),\boldsymbol{\zeta}_{n-1})\,ds - \beta(0)a(\mathbf{u}^n,\boldsymbol{\zeta}_{n-1})\Big)\Big|
\end{aligned}$$

$$
\begin{aligned}
&+ e^{-\delta\Delta t}(e^{\delta\Delta t} - 1)\left|\sum_{n=1}^{m-1} e^{2\alpha t_{n+1}}\left(\int_0^{t_n} \beta(t_n - s)a(\mathbf{u}(s), \boldsymbol{\zeta}_n)\, ds - a(q_r^n(\mathbf{u}), \boldsymbol{\zeta}_n)\right)\right| \\
\leq& \frac{\mu K_1}{64} e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|_\varepsilon^2 + C\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}(\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + \|\boldsymbol{\zeta}_{n-1}\|_\varepsilon^2) \\
&+ C\Delta t^3 \sum_{n=1}^{m} e^{2\alpha t_n} \int_0^{t_n} e^{-2\delta(t_n - t)}(\|\mathbf{u}(t)\|_1^2 + h^2\|\mathbf{u}(t)\|_2^2 + \|\mathbf{u}_t(t)\|_1^2 + h^2\|\mathbf{u}_t(t)\|_2^2)\, dt \\
&+ C\Delta t^2 e^{2\alpha t_m} \int_0^{t_m} (\|\mathbf{u}(t)\|_1^2 + h^2\|\mathbf{u}(t)\|_2^2 + \|\mathbf{u}_t(t)\|_1^2 + h^2\|\mathbf{u}_t(t)\|_2^2)\, dt. \qquad (4.56)
\end{aligned}
$$

Similar to the above estimate and with an application of Lemma 1.7, we find

$$
\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\left|a(q_r^n(\boldsymbol{\zeta}), \partial_t \boldsymbol{\zeta}_n)\right| \leq \frac{\mu K_1}{64} e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|_\varepsilon^2 + C\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}(\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + \|\boldsymbol{\zeta}_{n-1}\|_\varepsilon^2). \qquad (4.57)
$$

Since $a(\cdot, \cdot)$ is symmetric, one can obtain

$$
\begin{aligned}
\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} a(\boldsymbol{\zeta}_n, \partial_t \boldsymbol{\zeta}_n) =& \frac{1}{2} e^{2\alpha t_m} a(\boldsymbol{\zeta}_m, \boldsymbol{\zeta}_m) - \frac{1}{2} a(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_0) \\
&- \frac{1}{2}\sum_{n=0}^{m-1} e^{2\alpha t_n}(e^{2\alpha\Delta t} - 1)a(\boldsymbol{\zeta}_n, \boldsymbol{\zeta}_n) + \frac{1}{2}\Delta t^2 \sum_{n=1}^{m} e^{2\alpha t_n} a(\partial_t\boldsymbol{\zeta}_n, \partial_t\boldsymbol{\zeta}_n). \qquad (4.58)
\end{aligned}
$$

Combining (4.50)–(4.58) in the resulting inequality, and using Lemmas 1.6 and 1.7, we obtain

$$
\begin{aligned}
&\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\partial_t\boldsymbol{\zeta}_n\|^2 + \mu K_1 e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|_\varepsilon^2 + K_1\Delta t^2 \sum_{n=1}^{m} e^{2\alpha t_n}\|\partial_t\boldsymbol{\zeta}_n\|_\varepsilon^2 \leq C\Delta t \sum_{n=1}^{m-1} e^{2\alpha t_n}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 \\
&+ \Delta t \sum_{n=1}^{m-1} e^{2\alpha t_{n+1}}\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + C\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}(\|\mathbf{E}^n\|_\varepsilon^2 + \|\mathbf{E}^{n-1}\|_\varepsilon^2 + \|\boldsymbol{\zeta}_{n-1}\|_\varepsilon^2 + \|\boldsymbol{\zeta}_n\|_\varepsilon^2) \\
&+ C\frac{\Delta t}{\min\limits_{E\in\mathcal{E}_h} h_E} \sum_{n=1}^{m} e^{2\alpha t_n}\|\boldsymbol{\zeta}_{n-1}\|_\varepsilon^2\|\boldsymbol{\zeta}_n\|_\varepsilon^2 + C\Delta t^2 \int_0^{t_m} e^{2\alpha t}\|\mathbf{u}_t(t)\|_1^2\, dt \\
&+ C\Delta t^2 \int_0^{t_m} e^{2\alpha t}\|\mathbf{u}_{tt}(t)\|^2\, dt + Ch^{2r}e^{2\alpha t_m}(|\mathbf{u}^m|_{r+1}^2 + |p^m|_r^2) + Ch^{2r}\Delta t \sum_{n=0}^{m} e^{2\alpha t_n}|\mathbf{u}^n|_{r+1}^2 \\
&+ Ch^{2r}\Delta t \sum_{n=1}^{m} e^{2\alpha t_{n-1}}|p^{n-1}|_r^2 + Ch^{2r} \int_0^{t_m} e^{2\alpha t}(|\mathbf{u}_t(t)|_{r+1}^2 + |p_t(t)|_r^2)\, dt \\
&+ C\Delta t^2 \int_0^{t_m} e^{2\alpha t}(\|\mathbf{u}(t)\|_1^2 + h^2\|\mathbf{u}(t)\|_2^2 + h^2\|\mathbf{u}_t(t)\|_2^2)\, dt + Ch^{2r}|\mathbf{u}_0|_{r+1}^2. \qquad (4.59)
\end{aligned}
$$

An application of the discrete Gronwall's lemma, estimate (4.47), Theorem 4.1 and assumption (**A3**) yield

$$
\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\partial_t\boldsymbol{\zeta}_n\|^2 + \mu K_1 e^{2\alpha t_m}\|\boldsymbol{\zeta}_m\|_\varepsilon^2 \leq Ce^L(h^{2r} + \Delta t^2),
$$

where $L = \frac{h^{2r}+\Delta t^2}{\min\limits_{E\in\mathcal{E}_h} h_E}$. In this case, we require that the mesh satisfies for some constant $B > 0$, independent of h and Δt, $L \leq B$. Finally, an application of triangle inequality, estimates (1.28) and (1.29), and assumption (**A3**) complete the rest of the proof. □

The next theorem is for the fully discrete error estimate on pressure which is an immediate consequence of the discrete inf-sup condition in Lemma 1.8, Lemma 4.7 and Theorem 4.1.

Theorem 4.3. *Under the assumptions of Theorem 4.1, the following hold true for SIPG case:*

$$e^{-2\alpha t_m}\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\|p^n - P_h^n\|^2 \leq K_T(h^{2r} + \Delta t^2).$$

4.4 Optimal Error Estimates for Velocity in $L^\infty(\mathbf{L}^2)$-norm

In this section, we focus on deriving optimal estimates for SIPG discretization of the velocity error $\mathbf{E}^n = \mathbf{u}^n - \mathbf{U}_h^n$ in $L^\infty(\mathbf{L}^2)$-norm, where $0 \leq n \leq M$. For this reason, we consider the scheme (4.9)-(4.10) on the space $\mathbf{V}_h$.
Now, the fully discrete scheme on $\mathbf{V}_h$ is described as: For $\mathbf{v}_h \in \mathbf{V}_h$, we seek $\{\mathbf{U}_h^n\}_{n\geq 1} \in \mathbf{V}_h$ such that

$$\begin{aligned}(\partial_t \mathbf{U}_h^n, \mathbf{v}_h) + \mu a(\mathbf{U}_h^n, \mathbf{v}_h) + a(q_r^n(\mathbf{U}_h), \mathbf{v}_h) + c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \mathbf{v}_h)&\\ = (\mathbf{f}^n, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h.&\end{aligned} \tag{4.60}$$

We initiate by examining the linearized error and consequently introduce the solution $\mathbf{v}_h(t) \in \mathbf{V}_h$, where $0 \leq t \leq T$, derived from a DG approximation of a linearized problem. In other words, $\mathbf{v}_h(t)$ stands as the solution for:

$$(\mathbf{v}_{ht}, \boldsymbol{\phi}_h) + \mu a(\mathbf{v}_h, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s)a(\mathbf{v}_h(s), \boldsymbol{\phi}_h)\,ds = (\mathbf{f}, \boldsymbol{\phi}_h) - c^{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h), \tag{4.61}$$

for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$. With the help of $\mathbf{v}_h(t)$ at $t = t_n$, we split $\mathbf{E}^n$ into two parts as $\mathbf{E}^n = (\mathbf{u}^n - \mathbf{v}_h^n) + (\mathbf{v}_h^n - \mathbf{U}_h^n) = \boldsymbol{\xi}^n + \boldsymbol{\eta}^n$. By considering (4.61) and (4.33), we can derive the equation involving $\boldsymbol{\xi}(t)$ for $t > 0$:

$$(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \mu a(\boldsymbol{\xi}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s)a(\boldsymbol{\xi}(s), \boldsymbol{\phi}_h)\,ds = -b(\boldsymbol{\phi}_h, p), \quad \boldsymbol{\phi}_h \in \mathbf{V}_h. \tag{4.62}$$

In order to derive optimal $L^\infty(\mathbf{L}^2)$-norm estimates of $\boldsymbol{\xi}$, we define the following auxiliary projection $\mathbf{S}_h^{vol}\mathbf{u} : [0,\infty) \to \mathbf{V}_h$ satisfying,

$$\mu\, a(\mathbf{u}-\mathbf{S}_h^{vol}\mathbf{u},\boldsymbol{\phi}_h)+\int_0^t \beta(t-s)\, a((\mathbf{u}-\mathbf{S}_h^{vol}\mathbf{u})(s),\boldsymbol{\phi}_h)\, ds + b(\boldsymbol{\phi}_h,p)=0, \quad \forall \boldsymbol{\phi}_h\in\mathbf{V}_h \tag{4.63}$$

and call it modified Stokes-Volterra projection. We now split the error ξ into two parts as follows:

$$\boldsymbol{\xi}=(\mathbf{u}-\mathbf{S}_h^{vol}\mathbf{u})+(\mathbf{S}_h^{vol}\mathbf{u}-\mathbf{v}_h)=\boldsymbol{\zeta}+\boldsymbol{\theta}.$$

First of all, we will derive optimal error bounds for $\boldsymbol{\zeta}$.

Lemma 4.8. *The term $\boldsymbol{\zeta}$ satisfies the following estimates:*

$$\begin{aligned}\|\boldsymbol{\zeta}(t)\|^2+h^2\|\boldsymbol{\zeta}(t)\|_\varepsilon^2 \le{}& Ch^{2r+2}\Big(|\boldsymbol{u}(t)|_{r+1}^2+|p(t)|_r^2\\ &+e^{-2\alpha t}\int_0^t e^{2\alpha s}(|\boldsymbol{u}(s)|_{r+1}^2+|p(s)|_r^2)\, ds\Big), \end{aligned}\tag{4.64}$$

$$\begin{aligned}\|\boldsymbol{\zeta}_t(t)\|^2+h^2\|\boldsymbol{\zeta}_t(t)\|_\varepsilon^2 \le{}& Ch^{2r+2}\Big(|\boldsymbol{u}(t)|_{r+1}^2+|p(t)|_r^2+|\boldsymbol{u}_t(t)|_{r+1}^2+|p_t(t)|_r^2\\ &+e^{-2\alpha t}\int_0^t e^{2\alpha s}(|\boldsymbol{u}(s)|_{r+1}^2+|p(s)|_r^2)\, ds\Big). \end{aligned}\tag{4.65}$$

Proof. With $\boldsymbol{\zeta}=\mathbf{u}-\mathbf{S}_h^{vol}\mathbf{u}$, we rewrite the equation (4.63) as

$$\mu\, a(\boldsymbol{\zeta},\boldsymbol{\phi}_h)+\int_0^t \beta(t-s)\, a(\boldsymbol{\zeta}(s),\boldsymbol{\phi}_h)\, ds=-b(\boldsymbol{\phi}_h,p), \quad \forall\boldsymbol{\phi}_h\in\mathbf{V}_h. \tag{4.66}$$

Set $\boldsymbol{\phi}_h=\mathbf{P}_h\boldsymbol{\zeta}$ in (4.66) and use the definition of the space $\mathbf{V}_h$ to arrive at

$$\begin{aligned}\mu\, a(\mathbf{P}_h\boldsymbol{\zeta},\mathbf{P}_h\boldsymbol{\zeta})+\int_0^t \beta(t-s)\, a(\mathbf{P}_h\boldsymbol{\zeta}(s),\mathbf{P}_h\boldsymbol{\zeta})\, ds ={}& -\mu\, a(\mathbf{u}-\mathbf{P}_h\mathbf{u},\mathbf{P}_h\boldsymbol{\zeta})\\ &-b(\mathbf{P}_h\boldsymbol{\zeta},p-r_h(p))-\int_0^t \beta(t-s)\, a((\mathbf{u}-\mathbf{P}_h\mathbf{u})(s),\mathbf{P}_h\boldsymbol{\zeta})\, ds.\end{aligned}\tag{4.67}$$

With the help of Lemma 2.3, we can bound the last integral term in (4.67) by using Young's inequality as

$$\Big|\int_0^t \beta(t-s)\, a((\mathbf{u}-\mathbf{P}_h\mathbf{u})(s),\mathbf{P}_h\boldsymbol{\zeta})\, ds\Big| \le \frac{K_1\mu}{4}\|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon^2+Ch^{2r}\left(\int_0^t \beta(t-s)|\mathbf{u}(s)|_{r+1}\, ds\right)^2. \tag{4.68}$$

Applying Lemma 1.6, (4.68) in (4.67), and estimating the first two terms on the right hand side of (4.67) similar to Lemma 2.5, we find

$$K_1\mu\|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon^2+2\int_0^t \beta(t-s)\, a(\mathbf{P}_h\boldsymbol{\zeta}(s),\mathbf{P}_h\boldsymbol{\zeta})\, ds\le Ch^{2r}(|\mathbf{u}|_{r+1}^2+|p|_r^2)$$

$$+Ch^{2r}\left(\int_0^t \beta(t-s)|\mathbf{u}(s)|_{r+1}\,ds\right)^2. \tag{4.69}$$

We now use triangle inequality $\|\boldsymbol{\zeta}\|_\varepsilon \le \|\mathbf{P}_h\boldsymbol{\zeta}\|_\varepsilon + \|\mathbf{u}-\mathbf{P}_h\mathbf{u}\|_\varepsilon$ with Lemma 2.2 in inequality (4.69). Then, multiply the resulting inequality by $e^{2\alpha t}$, integrate with respect to time and note that the resulting double integral on the left hand side is non-negative due to Lemma 4.2 to finally obtain

$$\begin{aligned}\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|_\varepsilon^2\,ds \le& Ch^{2r}\int_0^t e^{2\alpha s}\big(|\mathbf{u}(s)|_{r+1}^2+|p(s)|_r^2\big)\,ds\\ &+Ch^{2r}\int_0^t e^{2\alpha s}\bigg(\int_0^s \beta(s-\tau)|\mathbf{u}(\tau)|_{r+1}\,d\tau\bigg)^2\,ds.\end{aligned} \tag{4.70}$$

For the second term, i.e. the double integral term on the right-hand side of (4.70), we bound it similar to the I term in the proof of [134, Theorem 2.1] by

$$Ch^{2r}\int_0^t e^{2\alpha s}|\mathbf{u}(s)|_{r+1}^2\,ds.$$

Thus, from (4.70), we arrive at

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|_\varepsilon^2\,ds \le Ch^{2r}\int_0^t e^{2\alpha s}\big(|\mathbf{u}(s)|_{r+1}^2+|p(s)|_r^2\big)\,ds. \tag{4.71}$$

Now multiply (4.69) by $e^{2\alpha t}$, and use Lemma 1.7, Hölder's and Young's inequalities. Again, multiply by $e^{-2\alpha t}$, and employ triangle inequality and Lemma 2.2 to find

$$\begin{aligned}\|\boldsymbol{\zeta}\|_\varepsilon^2 \le& Ch^{2r}\big(|\mathbf{u}|_{r+1}^2+|p|_r^2\big)+Ch^{2r}e^{-2\alpha t}\int_0^t e^{2\alpha s}|\mathbf{u}(s)|_{r+1}^2\,ds\\ &+Ce^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{P}_h\boldsymbol{\zeta}(s)\|_\varepsilon^2\,ds.\end{aligned} \tag{4.72}$$

Again, using triangle inequality and Lemma 2.2, we can easily find that

$$\begin{aligned}\|\boldsymbol{\zeta}\|_\varepsilon^2 \le& Ch^{2r}\big(|\mathbf{u}|_{r+1}^2+|p|_r^2\big)+Ch^{2r}e^{-2\alpha t}\int_0^t e^{2\alpha s}|\mathbf{u}(s)|_{r+1}^2\,ds\\ &+Ce^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|_\varepsilon^2\,ds.\end{aligned} \tag{4.73}$$

Using (4.71), from (4.73), we finally able to manage

$$\|\boldsymbol{\zeta}\|_\varepsilon^2 \le Ch^{2r}\big(|\mathbf{u}|_{r+1}^2+|p|_r^2\big)+Ch^{2r}e^{-2\alpha t}\int_0^t e^{2\alpha s}\big(|\mathbf{u}(s)|_{r+1}^2+|p(s)|_r^2\big)\,ds. \tag{4.74}$$

In order to estimate $\boldsymbol{\zeta}$ in $\mathbf{L}^2$-norm, we utilize Aubin-Nitsche duality argument. Let $(\mathbf{w},q)\in\mathbf{H}_0^1(\Omega)\in L^2(\Omega)/\mathbb{R}$ be the solution of the following dual problem:

$$-\mu\Delta\mathbf{w}+\nabla q=\boldsymbol{\zeta}\quad\text{in }\Omega,\quad \nabla\cdot\mathbf{w}=0\quad\text{in }\Omega,\quad \mathbf{w}|_{\partial\Omega}=0. \tag{4.75}$$

Further, $(\mathbf{w}, q)$ satisfies the following bound

$$\|\mathbf{w}\|_2 + \|q\|_1 \le C\|\boldsymbol{\zeta}\|. \tag{4.76}$$

Take L^2 inner product between (4.75) and $\boldsymbol{\zeta}$. Now utilize (4.66) with $\boldsymbol{\phi}_h = \mathbf{P}_h\mathbf{w}$, and incompressiblity condition satisfied by $\mathbf{w}$, regularity of $\mathbf{w}$ and q to obtain

$$\begin{aligned}
\|\boldsymbol{\zeta}(t)\|^2 &= \mu \sum_{T\in\mathcal{T}_h} \int_T \nabla(\mathbf{w} - \mathbf{P}_h\mathbf{w}) : \nabla\boldsymbol{\zeta} + \mu \sum_{e\in\Gamma_h} \int_e \{\nabla\boldsymbol{\zeta}\}\mathbf{n}_e \cdot [\mathbf{P}_h\mathbf{w} - \mathbf{w}] \\
&- \mu \sum_{e\in\Gamma_h} \int_e \{\nabla(\mathbf{w} - \mathbf{P}_h\mathbf{w})\}\mathbf{n}_e \cdot [\boldsymbol{\zeta}] - \mu \sum_{e\in\Gamma_h} \frac{\sigma_e}{|e|} \int_e [\mathbf{w} - \mathbf{P}_h\mathbf{w}] \cdot [\boldsymbol{\zeta}] \\
&- b(\mathbf{P}_h\mathbf{w} - \mathbf{w}, p - r_h(p)) + b(\boldsymbol{\zeta}, q) + \int_0^t \beta(t-s) a(\boldsymbol{\zeta}(s), \mathbf{w} - \mathbf{P}_h\mathbf{w})\, ds \\
&- \int_0^t \beta(t-s) a(\boldsymbol{\zeta}(s), \mathbf{w})\, ds.
\end{aligned} \tag{4.77}$$

Following similar procedure used in deriving the bounds of the right hand terms of (2.34) (see Lemma 2.5), and by using (4.76) in place of (2.33) and Young's inequality, one can bound all the right hand side terms of (4.77), except the last two terms, by

$$\frac{1}{6}\|\boldsymbol{\zeta}\|^2 + Ch^2\|\boldsymbol{\zeta}\|_\varepsilon^2 + Ch^{2r+2}(|\mathbf{u}|_{r+1}^2 + |p|_r^2). \tag{4.78}$$

Again, Lemma 2.2, the regularity result (4.76), Young's and Hölder's inequalities yield

$$\begin{aligned}
&\left| \int_0^t \beta(t-s) a(\boldsymbol{\zeta}(s), \mathbf{w} - \mathbf{P}_h\mathbf{w})\, ds \right| \\
&\le \int_0^t \beta(t-s)(Ch\|\boldsymbol{\zeta}(s)\|_\varepsilon \|\mathbf{w}\|_2 + Ch^{r+1}|\mathbf{u}(s)|_{r+1}\|\mathbf{w}\|_2)\, ds \\
&\le \frac{1}{6}\|\boldsymbol{\zeta}\|^2 + Ch^2\Big(\int_0^t \beta(t-s)\|\boldsymbol{\zeta}(s)\|_\varepsilon\, ds\Big)^2 \\
&\quad + Ch^{2r+2}\Big(\int_0^t \beta(t-s)|\mathbf{u}(s)|_{r+1}\, ds\Big)^2.
\end{aligned} \tag{4.79}$$

From the consistency of diffusion and pressure term, and using (4.75), the eighth term on the right hand side of (4.77) is written as

$$\begin{aligned}
-\int_0^t \beta(t-s) a(\boldsymbol{\zeta}(s), \mathbf{w})\, ds = &-\frac{1}{\mu}\int_0^t \beta(t-s)(\boldsymbol{\zeta}(s), \boldsymbol{\zeta}(t))\, ds \\
&+ \frac{1}{\mu}\int_0^t \beta(t-s) b(\boldsymbol{\zeta}(s), q)\, ds.
\end{aligned} \tag{4.80}$$

Furthermore, similar to the bound of $b(\boldsymbol{\zeta}, q)$, using (4.76), Young's and Hölder's inequalities, we bound the last term on the right hand side of (4.80) as

$$\left| \frac{1}{\mu}\int_0^t \beta(t-s) b(\boldsymbol{\zeta}(s), q)\, ds \right| \le \frac{1}{6}\|\boldsymbol{\zeta}\|^2 + Ch^2\Big(\int_0^t \beta(t-s)\|\boldsymbol{\zeta}(s)\|_\varepsilon\, ds\Big)^2$$

$$+ Ch^{2r+2}\bigg(\int_0^t \beta(t-s)|\mathbf{u}(s)|_{r+1}\, ds \bigg)^2. \qquad (4.81)$$

Substitute (4.78)–(4.81) in (4.77) and rewrite the resulting equation, we find that

$$\begin{aligned} \|\boldsymbol{\zeta}(t)\|^2 + \frac{2}{\mu}\int_0^t \beta(t-s)(\boldsymbol{\zeta}(s),\boldsymbol{\zeta}(t))\, ds &\le Ch^2\|\boldsymbol{\zeta}(t)\|_\varepsilon^2 + Ch^{2r+2}(|\mathbf{u}(t)|_{r+1}^2 + |p(t)|_r^2) \\ + Ch^2\bigg(\int_0^t \beta(t-s)\|\boldsymbol{\zeta}(s)\|_\varepsilon\, ds \bigg)^2 &+ Ch^{2r+2}\bigg(\int_0^t \beta(t-s)|\mathbf{u}(s)|_{r+1}\, ds \bigg)^2. \qquad (4.82) \end{aligned}$$

Multiply (4.82) by $e^{2\alpha t}$ and integrate from 0 to t. We now drop the double integral term on the left-hand side of the resulting inequality, being non-negative. The double integral terms on the right-hand side is handled exactly similar to the double integral term of (4.70). Finally use (4.71) to conclude

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2\, ds \le Ch^{2r+2}\int_0^t e^{2\alpha s}\left(|\mathbf{u}(s)|_{r+1}^2 + |p(s)|_r^2\right)\, ds. \qquad (4.83)$$

From (4.82), it easily follows that

$$\begin{aligned} \|\boldsymbol{\zeta}(t)\|^2 \le & Ch^2\|\boldsymbol{\zeta}(t)\|_\varepsilon^2 + C\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2\, ds + Ch^2 e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|_\varepsilon^2\, ds \\ & + Ch^{2r+2}(|\mathbf{u}(t)|_{r+1}^2 + |p(t)|_r^2) + Ch^{2r+2}e^{-2\alpha t}\int_0^t e^{2\alpha s}|\mathbf{u}(s)|_{r+1}^2\, ds. \qquad (4.84) \end{aligned}$$

Incorporate (4.71), (4.74) and (4.83) in (4.84), we establish (4.64).

For the estimates involving $\boldsymbol{\zeta}_t$, we differentiate (4.66) with respect to the temporal variable t and use Leibniz integral rule to find that

$$\mu\, a(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) + \beta(0)\, a(\boldsymbol{\zeta}(t), \boldsymbol{\phi}_h) + \int_0^t \beta_t(t-s)\, a(\boldsymbol{\zeta}(s), \boldsymbol{\phi}_h)\, ds = -b(\boldsymbol{\phi}_h, p_t), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h. \qquad (4.85)$$

Choose $\boldsymbol{\phi}_h = \mathbf{P}_h\boldsymbol{\zeta}_t = \boldsymbol{\zeta}_t - (\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t)$ in (4.85), and notice that $\beta(0) = \gamma$ and $\beta_t(t-s) = -\delta\beta(t-s)$ to observe that

$$\begin{aligned} \mu\, a(\mathbf{P}_h\boldsymbol{\zeta}_t, \mathbf{P}_h\boldsymbol{\zeta}_t) = & -\mu\, a(\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \mathbf{P}_h\boldsymbol{\zeta}_t) - \gamma\, a(\boldsymbol{\zeta}(t), \mathbf{P}_h\boldsymbol{\zeta}_t) \\ & + \delta\int_0^t \beta(t-s)a(\boldsymbol{\zeta}(s), \mathbf{P}_h\boldsymbol{\zeta}_t)\, ds - b(\mathbf{P}_h\boldsymbol{\zeta}_t, p_t). \end{aligned}$$

By using the similar set of arguments as used for deriving the estimates of $\boldsymbol{\zeta}(t)$ and with the help of estimate of $\|\boldsymbol{\zeta}(t)\|_\varepsilon$, we obtain

$$\begin{aligned} \|\boldsymbol{\zeta}_t\|_\varepsilon^2 \le & Ch^{2r}\bigg(|\mathbf{u}(t)|_{r+1}^2 + |p(t)|_r^2 + |\mathbf{u}_t(t)|_{r+1}^2 + |p_t(t)|_r^2 \\ & + e^{-2\alpha t}\int_0^t e^{2\alpha s}(|\mathbf{u}(s)|_{r+1}^2 + |p(s)|_r^2)\, ds \bigg). \end{aligned}$$

Finally, for the estimation of $\boldsymbol{\zeta}_t$ in L^2-norm, we again appeal the dual problem (4.75), replacing $\boldsymbol{\zeta}$ by $\boldsymbol{\zeta}_t$. Using similar steps required for deriving $\|\boldsymbol{\zeta}\|$ estimate, we can establish the estimate for $\|\boldsymbol{\zeta}_t\|$. This completes the proof of Lemma 4.8. $\square$

We can now estimate $\boldsymbol{\xi}$ in the $\mathbf{L}^2$ and $\|\cdot\|_\varepsilon$-norms, for $t > 0$. Since $\boldsymbol{\xi} = \boldsymbol{\zeta} + \boldsymbol{\theta}$ and the estimates of $\boldsymbol{\zeta}$ are known from Lemma 4.8, it is sufficient to estimate $\boldsymbol{\theta}$. In order to derive estimates of $\boldsymbol{\theta}$, we consider the folowing equation in $\boldsymbol{\theta}$:

$$(\boldsymbol{\theta}_t, \boldsymbol{\phi}_h) + \mu\, a(\boldsymbol{\theta}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\boldsymbol{\theta}(s), \boldsymbol{\phi}_h)\, ds = -(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h. \tag{4.86}$$

Lemma 4.9. *Under the assumption (*$\mathbf{A3}$*) and* $0 < \alpha < \min\left(\delta, \frac{\mu K_1}{2C_2}\right)$*, there is a positive constant* C *such that for* $t > 0$*,* $\boldsymbol{\xi}$ *satisfies the following estimates*

$$\|\boldsymbol{\xi}(t)\| + h\|\boldsymbol{\xi}(t)\|_\varepsilon \leq C h^{2r+2}, \quad 0 \leq t \leq T. \tag{4.87}$$

Proof. We choose $\boldsymbol{\phi}_h = \boldsymbol{\theta}$ in (4.86), multiply the resulting equation by $e^{2\alpha t}$, and use the coercivity result of Lemma 1.6 and the Cauchy-Schwarz inequality to arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(e^{2\alpha t}\|\boldsymbol{\theta}(t)\|^2\Big) - \alpha e^{2\alpha t}\|\boldsymbol{\theta}(t)\|^2 + \mu K_1 e^{2\alpha t}\|\boldsymbol{\theta}(t)\|_\varepsilon^2 \\ + e^{2\alpha t}\int_0^t \beta(t-\tau) a(\boldsymbol{\theta}(\tau), \boldsymbol{\theta}(t))\, d\tau \leq e^{2\alpha t}\|\boldsymbol{\zeta}_t(t)\|\|\boldsymbol{\theta}(t)\|.\end{aligned}$$

Now, integrate the above inequality with respect to time, use (1.14) and Young's inequality to find

$$\begin{aligned}e^{2\alpha t}\|\boldsymbol{\theta}(t)\|^2 + (\mu K_1 - 2\alpha C_2)\int_0^t e^{2\alpha s}\|\boldsymbol{\theta}(s)\|_\varepsilon^2\, ds \\ + \gamma \int_0^t \int_0^s e^{-(\delta-\alpha)(s-\tau)} a(e^{\alpha\tau}\boldsymbol{\theta}(\tau), e^{\alpha s}\boldsymbol{\theta}(s))\, d\tau\, ds \leq C \int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}_s(s)\|^2\, ds.\end{aligned}$$

Consequently, multiplying the above inequality by $e^{-2\alpha t}$, using the estimate (4.65), Lemma 4.2 and assumption ($\mathbf{A3}$), and choosing $0 < \alpha < \min\left(\delta, \frac{\mu K_1}{2C_2}\right)$ in the above inequality, we arrive at

$$\|\boldsymbol{\theta}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\theta}(s)\|_\varepsilon^2\, ds \leq C h^{2r+2}.$$

By using the inverse relation (1.37), one can obtain

$$\|\boldsymbol{\theta}(t)\|^2 + h^2\|\boldsymbol{\theta}(t)\|_\varepsilon^2 \leq C h^{2r+2}. \tag{4.88}$$

Finally, combining (4.88) with (4.64) and using assumption ($\mathbf{A3}$), we obtain the desired result. $\square$

Below, we derive some regularity bounds for $\mathbf{v}_h$.

Lemma 4.10. *Let $0 < \alpha < \min\left(\delta, \frac{\mu K_1}{2C_2}\right)$. Then, the solution $\mathbf{v}_h(t)$, $t > 0$, for the problem (4.61) satisfies*

$$e^{-2\alpha t} \int_0^t e^{2\alpha s}(\|\mathbf{v}_h(s)\|_\varepsilon^2 + \|\mathbf{v}_{hs}(s)\|_\varepsilon^2)\, ds \le C, \tag{4.89}$$

$$e^{-2\alpha t} \int_0^t e^{2\alpha s}\, \|\mathbf{v}_{hss}(s)\|_{-1,h}^2\, ds \le C, \tag{4.90}$$

where C is a positive constant.

Proof. Choose $\boldsymbol{\phi}_h = \mathbf{v}_h$ in (4.61), and apply Lemma 1.6, estimates (1.14) and (2.56), the Cauchy-Schwarz inequality and Young's inequality to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}_h(t)\|^2 + \mu K_1\|\mathbf{v}_h(t)\|_\varepsilon^2 + \int_0^t \beta(t-s)a(\mathbf{v}_h(s), \mathbf{v}_h(t))\, ds \\ \le \frac{\mu K_1}{2}\|\mathbf{v}_h(t)\|_\varepsilon^2 + C\|\mathbf{f}(t)\|^2 + C\|\mathbf{u}(t)\|_1^4.\end{aligned} \tag{4.91}$$

Multiplying (4.91) by $e^{2\alpha t}$, integrating from 0 to t and using (1.14), observing that the resulting double integral on the left hand side is non-negative due to Lemma 4.2, we find that

$$\begin{aligned}e^{2\alpha t}\|\mathbf{v}_h(t)\|^2 + (\mu K_1 - 2\alpha C_2)\int_0^t e^{2\alpha s}\|\mathbf{v}_h(s)\|_\varepsilon^2\, ds \le \|\mathbf{v}_h(0)\|^2 + C\int_0^t e^{2\alpha s}\|\mathbf{f}(s)\|^2\, ds \\ + C\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega))}^2 \int_0^t e^{2\alpha s}\|\mathbf{u}(s)\|_1^2\, ds.\end{aligned}$$

Multiplying the above inequality by $e^{-2\alpha t}$, and using assumption (**A3**), and choosing $\alpha < \frac{\mu K_1}{2C_2}$, we obtain

$$\|\mathbf{v}_h(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{v}_h(s)\|_\varepsilon^2\, ds \le C. \tag{4.92}$$

Differentiate (4.61) with respect to t to find

$$\begin{aligned}(\mathbf{v}_{htt}(t), \boldsymbol{\phi}_h) + \mu\, a(\mathbf{v}_{ht}(t), \boldsymbol{\phi}_h) + \beta(0)a(\mathbf{v}_h(t), \boldsymbol{\phi}_h) + \int_0^t \beta_t(t-s)a(\mathbf{v}_h(s), \boldsymbol{\phi}_h)\, ds \\ = (\mathbf{f}_t(t), \boldsymbol{\phi}_h) - c^{\mathbf{u}(t)}(\mathbf{u}_t(t), \mathbf{u}(t), \boldsymbol{\phi}_h) - c^{\mathbf{u}(t)}(\mathbf{u}(t), \mathbf{u}_t(t), \boldsymbol{\phi}_h),\ \forall\, \boldsymbol{\phi}_h \in \mathbf{V}_h.\end{aligned} \tag{4.93}$$

Take $\boldsymbol{\phi}_h = \mathbf{v}_{ht}(t)$ in (4.93), and apply (1.14), (2.56), Lemmas 1.6 and 1.7, the Cauchy-Schwarz inequality and Young's inequality, and noting that $\beta(0) = \gamma$ and $\beta_t(t-s) = -\delta\beta(t-s)$, we arrive at

$$\frac{d}{dt}\|\mathbf{v}_{ht}(t)\|^2 + \mu K_1\|\mathbf{v}_{ht}(t)\|_\varepsilon^2 \le C\|\mathbf{f}_t(t)\|^2 + C\|\mathbf{v}_h(t)\|_\varepsilon^2 + C\left(\int_0^t \beta(t-s)\|\mathbf{v}_h(s)\|_\varepsilon\, ds\right)^2$$

$$+ C\|\mathbf{u}(t)\|_1^2\|\mathbf{u}_t(t)\|_1^2.$$

Now, multiply the above inequality by $e^{2\alpha t}$, integrate from 0 to t and handle the resulting double integral term similar to the double integral term of (4.70) as

$$C\int_0^t e^{2\alpha s}\Big(\int_0^s \beta(s-\tau)\|\mathbf{v}_h(\tau)\|_\varepsilon\, d\tau\Big)^2 ds \le C\int_0^t e^{2\alpha s}\|\mathbf{v}_h(s)\|_\varepsilon^2\, ds$$

to obtain

$$e^{2\alpha t}\|\mathbf{v}_{ht}(t)\|^2 + (\mu K_1 - 2\alpha C_2)\int_0^t e^{2\alpha s}\|\mathbf{v}_{hs}(s)\|_\varepsilon^2\, ds \le \|\mathbf{v}_{ht}(0)\|^2 + C\int_0^t e^{2\alpha s}\|\mathbf{v}_h(s)\|_\varepsilon^2\, ds$$
$$+C\|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega))}\int_0^t e^{2\alpha s}\|\mathbf{u}_s(s)\|_1^2\, ds + C\int_0^t e^{2\alpha s}\|\mathbf{f}_s(s)\|^2\, ds.$$

Choosing $\alpha < \frac{\mu K_1}{2C_2}$, after a multiplication by $e^{-2\alpha t}$, and utilizing (4.92) and assumption (**A3**), we obtain

$$\|\mathbf{v}_{ht}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{v}_{hs}(s)\|_\varepsilon^2\, ds \le C. \tag{4.94}$$

Combining estimates (4.92) and (4.94), we arrive at (4.89). Finally, consider (4.93) to derive the estimate (4.90). Thus an application of (1.14), (2.56) and Lemma 1.7 to find

$$(\mathbf{v}_{htt}(t), \boldsymbol{\phi}_h) \le C\Big(\|\mathbf{v}_{ht}(t)\|_\varepsilon + \|\mathbf{v}_h(t)\|_\varepsilon + \int_0^t \beta_t(t-s)\|\mathbf{v}_h(s)\|_\varepsilon\, ds + \|\mathbf{f}_t(t)\|$$
$$+ \|\mathbf{u}_t(t)\|_1\|\mathbf{u}(t)\|_1\Big)\|\boldsymbol{\phi}_h\|_\varepsilon.$$

Now, apply the definition of $\|\cdot\|_{-1,h}$, square both sides of the above inequality. Further, multiply the resulting inequality by $e^{2\alpha t}$, integrate from 0 to t, and then multiply by $e^{-2\alpha t}$, and use (4.92), (4.94) and assumption (**A3**) to find estimate (4.90). This completes the rest of the proof. □

The next lemma establishes the estimates for $\boldsymbol{\eta}^n = \mathbf{v}_h^n - \mathbf{U}_h^n$.

Lemma 4.11. *Suppose the assumption (**A3**) is satisfied and* $0 < \alpha < \min(\delta, \frac{\mu K_1}{2C_2})$. *Let* $\mathbf{v}_h(t) \in \boldsymbol{V}_h$ *be a solution of (4.61) corresponding to the initial value* $\mathbf{v}_h^0 = \boldsymbol{P}_h\boldsymbol{u}_0$. *Then, the error* $\boldsymbol{\eta}^n = \mathbf{v}_h^n - \boldsymbol{U}_h^n$, *satisfies*

$$\|\boldsymbol{\eta}^m\|^2 + \Delta t e^{-2\alpha t_m}\sum_{n=1}^m e^{2\alpha t_n}\|\boldsymbol{\eta}^n\|_\varepsilon^2 \le K_T(h^{2r+2} + \Delta t^2).$$

Proof. We consider the equations (4.61) at $t = t_n$ and (4.60), satisfied by $\mathbf{v}_h$ and $\mathbf{U}_h^n$, respectively, to obtain

$$(\partial_t \boldsymbol{\eta}^n, \boldsymbol{\phi}_h) + \mu a(\boldsymbol{\eta}^n, \boldsymbol{\phi}_h) + a(q_r^n(\boldsymbol{\eta}), \boldsymbol{\phi}_h) = (\partial_t \mathbf{v}_h^n - \mathbf{v}_{ht}^n, \boldsymbol{\phi}_h) + a(q_r^n(\mathbf{v}_h), \boldsymbol{\phi}_h)$$
$$- \int_0^{t_n} \beta(t_n - s) a(\mathbf{v}_h(s), \boldsymbol{\phi}_h)\, ds + c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\phi}_h) - c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\phi}_h),$$

for $\boldsymbol{\phi}_h \in \mathbf{V}_h$. Then, setting $\boldsymbol{\phi}_h = \boldsymbol{\eta}^n$ and using Lemma 1.6, we find that

$$\frac{1}{2}\partial_t \|\boldsymbol{\eta}^n\|^2 + \mu K_1 \|\boldsymbol{\eta}^n\|_\varepsilon + a(q_r^n(\boldsymbol{\eta}), \boldsymbol{\eta}^n) \leq (\partial_t \mathbf{v}_h^n - \mathbf{v}_{ht}^n, \boldsymbol{\eta}^n) + a(q_r^n(\mathbf{v}_h), \boldsymbol{\eta}^n)$$
$$- \int_0^{t_n} \beta(t_n - s) a(\mathbf{v}_h(s), \boldsymbol{\eta}^n)\, ds + c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\eta}^n) - c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\eta}^n). \tag{4.95}$$

We now estimate the terms on the right hand side of the above inequality. An application of (2.138), Hölder's and Young's inequalities lead to

$$|(\partial_t \mathbf{v}_h^n - \mathbf{v}_{ht}^n, \boldsymbol{\eta}^n)| \leq C\Delta t^{1/2} \left(\int_{t_{n-1}}^{t_n} \|\mathbf{v}_{htt}(t)\|_{-1,h}^2\, dt \right)^{1/2} \|\boldsymbol{\eta}^n\|_\varepsilon$$
$$\leq \frac{K_1 \mu}{64} \|\boldsymbol{\eta}^n\|_\varepsilon^2 + C\Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{v}_{htt}(t)\|_{-1,h}^2\, dt. \tag{4.96}$$

The estimate (4.39) and Lemma 1.7, Hölder's and Young's inequalities yield

$$\left| a(q_r^n(\mathbf{v}_h), \boldsymbol{\eta}^n) - \int_0^{t_n} \beta(t_n - s) a(\mathbf{v}_h(s), \boldsymbol{\eta}^n)\, ds \right|$$
$$\leq C \bigg(\Delta t \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \big(\beta_s(t_n - s)\|\mathbf{v}_h(s)\|_\varepsilon + \beta(t_n - s)\|\mathbf{v}_{hs}(s)\|_\varepsilon\big)\, ds \bigg) \|\boldsymbol{\eta}^n\|_\varepsilon$$
$$\leq \frac{K_1 \mu}{64} \|\boldsymbol{\eta}^n\|_\varepsilon^2 + C \bigg(\Delta t \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \big(\beta_s(t_n - s)\|\mathbf{v}_h(s)\|_\varepsilon + \beta(t_n - s)\|\mathbf{v}_{hs}(s)\|_\varepsilon\big)\, ds \bigg)^2. \tag{4.97}$$

Let us consider the nonlinear terms from (4.95). Due to the regularity of $\mathbf{u}^n$, we have zero jump of of $\mathbf{u}^n$ and hence can rewrite the nonlinear terms as follows

$$c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\eta}^n) - c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\eta}^n) = c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\eta}^n) - c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\eta}^n)$$
$$= - c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \boldsymbol{\eta}^n, \boldsymbol{\eta}^n) - c^{\mathbf{U}_h^{n-1}}(\boldsymbol{\xi}^{n-1}, \mathbf{u}^n, \boldsymbol{\eta}^n) - c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^n, \boldsymbol{\eta}^n)$$
$$+ c^{\mathbf{U}_h^{n-1}}(\boldsymbol{\xi}^{n-1}, \boldsymbol{\xi}^n, \boldsymbol{\eta}^n) + c^{\mathbf{U}_h^{n-1}}(\boldsymbol{\eta}^{n-1}, \boldsymbol{\xi}^n, \boldsymbol{\eta}^n) - c^{\mathbf{U}_h^{n-1}}(\boldsymbol{\eta}^{n-1}, \mathbf{u}^n, \boldsymbol{\eta}^n)$$
$$- c^{\mathbf{u}^{n-1}}(\mathbf{u}^{n-1}, \boldsymbol{\xi}^n, \boldsymbol{\eta}^n) + l^{\mathbf{u}^{n-1}}(\mathbf{u}^{n-1}, \boldsymbol{\xi}^n, \boldsymbol{\eta}^n) - l^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1}, \boldsymbol{\xi}^n, \boldsymbol{\eta}^n). \tag{4.98}$$

From inequality (1.19), the term $c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \boldsymbol{\eta}^n, \boldsymbol{\eta}^n)$ is non-negative. The remaining nonlinear terms on the right hand side of (4.98) can be estimated following the estimation techniques of (2.94)-(2.99) and (2.144) as follows

$$|c^{\mathbf{U}_h^{n-1}}(\boldsymbol{\xi}^{n-1}, \mathbf{u}^n, \boldsymbol{\eta}^n)| \leq \frac{K_1 \mu}{64} \|\boldsymbol{\eta}^n\|_\varepsilon^2 + C\|\mathbf{u}^n\|_2^2 (\|\boldsymbol{\xi}^{n-1}\|^2 + h^2 \|\boldsymbol{\xi}^{n-1}\|_\varepsilon^2), \tag{4.99}$$

$$|c^{\mathbf{u}^{n-1}}(\mathbf{u}^{n-1},\boldsymbol{\xi}^n,\boldsymbol{\eta}^n)| \leq \frac{K_1\mu}{64}\|\boldsymbol{\eta}^n\|_\varepsilon^2 + C\|\mathbf{u}^{n-1}\|_2^2(\|\boldsymbol{\xi}^n\|^2 + h^2\|\boldsymbol{\xi}^n\|_\varepsilon^2), \tag{4.100}$$

$$|c^{\mathbf{U}_h^{n-1}}(\boldsymbol{\xi}^{n-1},\boldsymbol{\xi}^n,\boldsymbol{\eta}^n)| \leq \frac{K_1\mu}{64}\|\boldsymbol{\eta}^n\|_\varepsilon^2 + C\|\boldsymbol{\xi}^{n-1}\|_\varepsilon^2\|\boldsymbol{\xi}^n\|_\varepsilon^2, \tag{4.101}$$

$$|c^{\mathbf{U}_h^{n-1}}(\boldsymbol{\eta}^{n-1},\boldsymbol{\xi}^n,\boldsymbol{\eta}^n)| \leq \frac{K_1\mu}{64}\|\boldsymbol{\eta}^n\|_\varepsilon^2 + Ch^{-2}\|\boldsymbol{\xi}^n\|_\varepsilon^2\|\boldsymbol{\eta}^{n-1}\|^2, \tag{4.102}$$

$$|c^{\mathbf{U}_h^{n-1}}(\boldsymbol{\eta}^{n-1},\mathbf{u}^n,\boldsymbol{\eta}^n)| \leq \frac{K_1\mu}{64}\|\boldsymbol{\eta}^n\|_\varepsilon^2 + C\|\mathbf{u}^n\|_2^2\|\boldsymbol{\eta}^{n-1}\|^2, \tag{4.103}$$

$$|c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^n-\mathbf{u}^{n-1},\mathbf{u}^n,\boldsymbol{\eta}^n)| \leq \frac{K_1\mu}{64}\|\boldsymbol{\eta}^n\|_\varepsilon^2 + C\Delta t\|\mathbf{u}^n\|_2\int_{t_{n-1}}^{t_n}\|\mathbf{u}_t(t)\|^2\,dt, \tag{4.104}$$

$$\begin{aligned}|l^{\mathbf{u}^{n-1}}(\mathbf{u}^{n-1},\boldsymbol{\xi}^n,\boldsymbol{\eta}^n) - l^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1},\boldsymbol{\xi}^n,\boldsymbol{\eta}^n)| \leq{}& \frac{K_1\mu}{64}\|\boldsymbol{\eta}^n\|_\varepsilon^2 + Ch^{-2}\|\boldsymbol{\xi}^n\|_\varepsilon^2\|\boldsymbol{\eta}^{n-1}\|^2\\ &+ C\|\boldsymbol{\xi}^{n-1}\|_\varepsilon^2\|\boldsymbol{\xi}^n\|_\varepsilon^2.\end{aligned} \tag{4.105}$$

It is easy to check that

$$\Delta t\sum_{n=1}^m e^{2\alpha t_n}\partial_t\|\boldsymbol{\eta}^n\|^2 = e^{2\alpha t_m}\|\boldsymbol{\eta}^m\|^2 - e^{2\alpha t_0}\|\boldsymbol{\eta}^0\|^2 - (e^{2\alpha\Delta t}-1)\sum_{n=0}^{m-1}e^{2\alpha t_n}\|\boldsymbol{\eta}^n\|^2.$$

Substitute (4.96)-(4.105) in (4.95), and multiply the resulting equation by $\Delta t e^{2\alpha t_n}$ and sum over $1 \leq n \leq m \leq M$, where $T = M\Delta t$ and apply Lemma 4.5 to observe that

$$\Delta t\sum_{n=1}^m e^{2\alpha t_n}a(q_r^n(\boldsymbol{\eta}),\boldsymbol{\eta}^n) = \gamma\Delta t\sum_{n=1}^m\Delta t\sum_{i=1}^n e^{-(\delta-\alpha)(t_n-t_i)}a(e^{\alpha t_i}\boldsymbol{\eta}^i, e^{\alpha t_n}\boldsymbol{\eta}^n) \geq 0$$

and find that

$$\begin{aligned}&e^{2\alpha t_m}\|\boldsymbol{\eta}^m\|^2 + K_1\mu\Delta t\sum_{n=1}^m e^{2\alpha t_n}\|\boldsymbol{\eta}^n\|_\varepsilon^2 \leq (e^{2\alpha\Delta t}-1)\sum_{n=0}^{m-1}e^{2\alpha t_n}\|\boldsymbol{\eta}^n\|^2 \\ &+ C\Delta t\sum_{n=1}^m e^{2\alpha t_n}(\|\mathbf{u}^n\|_2^2 + h^{-2}\|\boldsymbol{\xi}^n\|_\varepsilon^2)\|\boldsymbol{\eta}^{n-1}\|^2 + C\Delta t\sum_{n=1}^m e^{2\alpha t_n}\|\mathbf{u}^{n-1}\|_2^2(\|\boldsymbol{\xi}^n\|^2 + h^2\|\boldsymbol{\xi}^n\|_\varepsilon^2)\\ &+ C\Delta t\sum_{n=1}^m e^{2\alpha t_n}\|\mathbf{u}^n\|_2^2(\|\boldsymbol{\xi}^{n-1}\|^2 + h^2\|\boldsymbol{\xi}^{n-1}\|_\varepsilon^2) + C\Delta t\sum_{n=1}^m e^{2\alpha t_n}\|\boldsymbol{\xi}^{n-1}\|_\varepsilon^2\|\boldsymbol{\xi}^n\|_\varepsilon^2\\ &+ C\Delta t^3\sum_{n=1}^m e^{2\alpha t_n}\int_0^{t_n}e^{-2\delta(t_n-t)}\big(\|\mathbf{v}_h(t)\|_\varepsilon^2 + \|\mathbf{v}_{ht}(t)\|_\varepsilon^2\big)\,dt\\ &+ C\Delta t^2\sum_{n=1}^m e^{2\alpha t_n}\int_{t_{n-1}}^{t_n}\|\mathbf{v}_{htt}(t)\|_{-1,h}^2\,dt + C\|\mathbf{u}\|_{L^\infty(0,T;H^2(\Omega))}\Delta t^2\sum_{n=1}^m e^{2\alpha t_n}\int_{t_{n-1}}^{t_n}\|\mathbf{u}_t(t)\|^2\,dt.\end{aligned} \tag{4.106}$$

We now bound the terms involving $\mathbf{v}_h$, $\mathbf{v}_{ht}$ and $\mathbf{v}_{htt}$ using estimates (4.89) and (4.90). For instance, we only show the following bounding technique

$$\begin{aligned}&C\Delta t^3\sum_{n=1}^m e^{2\alpha t_n}\int_0^{t_n}e^{-2\delta(t_n-t)}\big(\|\mathbf{v}_h(t)\|_\varepsilon^2 + \|\mathbf{v}_{ht}(t)\|_\varepsilon^2\big)\,dt\\ &= C\Delta t^3\sum_{n=1}^m\int_0^{t_n}e^{-(2\delta-2\alpha)(t_n-t)}e^{2\alpha t}\big(\|\mathbf{v}_h(t)\|_\varepsilon^2 + \|\mathbf{v}_{ht}(t)\|_\varepsilon^2\big)\,dt\end{aligned}$$

$$\leq C\Delta t^3 \sum_{n=1}^{m} \int_0^{t_n} e^{2\alpha t}\big(\|\mathbf{v}_h(t)\|_\varepsilon^2 + \|\mathbf{v}_{ht}(t)\|_\varepsilon^2\big)\, dt \leq Ce^{2\alpha t_m}\Delta t^2.$$

In a similar manner, the last two terms on the right hand side of (4.106) are estimated by $C\Delta t^2 e^{2\alpha t_{m+1}}$. Thus, with the above estimates, and from (4.106), Lemma 4.9, assumption (**A3**) and $\|\boldsymbol{\eta}^0\| = 0$, one can find

$$e^{2\alpha t_m}\|\boldsymbol{\eta}^m\|^2 + K_1\mu\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\boldsymbol{\eta}^n\|_\varepsilon^2 \leq C\Delta t \sum_{n=1}^{m-1} e^{2\alpha t_n}\|\boldsymbol{\eta}^n\|^2 + Ch^{2r+2}e^{2\alpha t_m} + C\Delta t^2 e^{2\alpha t_{m+1}}.$$

A use of the discrete Gronwall's lemma and after a final multiplication of the resulting inequality by $e^{-2\alpha t_m}$ completes the rest of the proof. □

The next theorem states optimal $L^\infty(\mathbf{L}^2)$-norm fully discrete error estimate of the velocity.

Theorem 4.4. *Suppose the assumption (**A3**) is satisfied. Further, let the discrete initial velocity $\boldsymbol{U}_h^0 \in \boldsymbol{V}_h$ with $\boldsymbol{U}_h^0 = \boldsymbol{P}_h\boldsymbol{u}_0$, and $0 < \alpha < \min\left(\delta, \frac{\mu K_1}{2C_2}\right)$. Then, there exists a positive constant K_T, such that*

$$\|\boldsymbol{u}^n - \boldsymbol{U}_h^n\| \leq K_T(h^{r+1} + \Delta t).$$

Proof. A combination of Lemmas 4.9 and 4.11, completes the proof of this theorem. □

4.5 Error Estimates for Pressure in $L^\infty(L^2)$-norm

In this section, we derive error estimates for the DG approximation in $L^\infty(L^2)$-norm of the pressure. Before proving our main theorem we need an auxiliary lemma. First, we define the following discretization error $\boldsymbol{\chi}^n = \mathbf{P}_h\mathbf{u}^n - \mathbf{U}_h^n$. From the equations (4.33) at time level $t = t_n$ and (4.60), and applying the definition of L^2-projection $\mathbf{P}_h$, we find the equation for $\boldsymbol{\chi}^n$ as follows

$$\begin{aligned}(\partial_t\boldsymbol{\chi}^n, \boldsymbol{\phi}_h) + \mu\, a(\boldsymbol{\chi}^n, \boldsymbol{\phi}_h) + a(q_r^n(\boldsymbol{\chi}), \boldsymbol{\phi}_h) &= -(\mathbf{u}_t^n - \partial_t\mathbf{u}^n, \boldsymbol{\phi}_h) - \mu\, a(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n, \boldsymbol{\phi}_h) \\ &+ a(q_r^n(\mathbf{P}_h\mathbf{u}), \boldsymbol{\phi}_h) - \int_0^{t_n} \beta(t_n - s)a(\mathbf{P}_h\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \\ &- \int_0^{t_n} \beta(t_n - s)a(\mathbf{u}(s) - \mathbf{P}_h\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds - b(\boldsymbol{\phi}_h, p^n) \\ &+ c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\phi}_h) - c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h. \end{aligned} \tag{4.107}$$

Before proceeding, we mention an important property of $\mathbf{P}_h$ which can be easily derived employing triangle inequality, the standard Lagrange interpolant, (1.36), (1.37) and Lemma 2.2:

$$\|\mathbf{P}_h\mathbf{v}\|_\varepsilon \leq C|\mathbf{v}|_1, \quad \forall\ \mathbf{v}\in\mathbf{J}_1. \tag{4.108}$$

The following lemma establishes an estimate for $\|\partial_t\mathbf{E}^n\|$.

Lemma 4.12. *The error $\partial_t\mathbf{E}^n$ satisfies*

$$\|\partial_t\mathbf{E}^m\|^2 + e^{-2\alpha t_m}\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\|\partial_t\mathbf{E}^n\|_\varepsilon^2 \leq K_T(h^{2r}+\Delta t).$$

Proof. Subtract (4.107) at time t_{n-1} from (4.107) at time t_n and substitute $\boldsymbol{\phi}_h = \partial_t\boldsymbol{\chi}^n$ and use Lemma 1.6 to find

$$\begin{aligned}
&\Delta t(\partial_t(\partial_t\boldsymbol{\chi}^n),\partial_t\boldsymbol{\chi}^n) + \mu K_1\Delta t\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon^2 + \Delta t a(\partial_t q_r^n(\boldsymbol{\chi}),\partial_t\boldsymbol{\chi}^n) \leq -\Delta t(\partial_t(\mathbf{u}_t^n-\partial_t\mathbf{u}^n),\partial_t\boldsymbol{\chi}^n)\\
&-\mu\Delta t a(\partial_t(\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n),\partial_t\boldsymbol{\chi}^n) + \Delta t\, a(\partial_t q_r^n(\mathbf{P}_h\mathbf{u}),\partial_t\boldsymbol{\chi}^n)\\
&-\Big(\int_0^{t_n}\beta(t_n-s)a(\mathbf{P}_h\mathbf{u}(s),\partial_t\boldsymbol{\chi}^n)\,ds - \int_0^{t_{n-1}}\beta(t_{n-1}-s)a(\mathbf{P}_h\mathbf{u}(s),\partial_t\boldsymbol{\chi}^n)\,ds\Big)\\
&-\Big(\int_0^{t_n}\beta(t_n-s)a(\mathbf{u}(s)-\mathbf{P}_h\mathbf{u}(s),\partial_t\boldsymbol{\chi}^n)\,ds\\
&-\int_0^{t_{n-1}}\beta(t_{n-1}-s)a(\mathbf{u}(s)-\mathbf{P}_h\mathbf{u}(s),\partial_t\boldsymbol{\chi}^n)\,ds\Big)\\
&-\Delta t b(\partial_t\boldsymbol{\chi}^n,\partial_t p^n) + c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1},\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n)\\
&-c^{\mathbf{U}_h^{n-2}}(\mathbf{U}_h^{n-2},\mathbf{U}_h^{n-1},\partial_t\boldsymbol{\chi}^n) - c^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{u}^n,\partial_t\boldsymbol{\chi}^n) + c^{\mathbf{u}^{n-1}}(\mathbf{u}^{n-1},\mathbf{u}^{n-1},\partial_t\boldsymbol{\chi}^n).
\end{aligned} \tag{4.109}$$

Let us rewrite the non-linear terms as follows

$$\begin{aligned}
&c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1},\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n) - c^{\mathbf{U}_h^{n-2}}(\mathbf{U}_h^{n-2},\mathbf{U}_h^{n-1},\partial_t\boldsymbol{\chi}^n) - c^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{u}^n,\partial_t\boldsymbol{\chi}^n)\\
&+c^{\mathbf{u}^{n-1}}(\mathbf{u}^{n-1},\mathbf{u}^{n-1},\partial_t\boldsymbol{\chi}^n)\\
=&-\Delta t c^{\mathbf{U}_h^{n-1}}(\partial_t\mathbf{u}^{n-1},\mathbf{E}^n,\partial_t\boldsymbol{\chi}^n) - \Delta t c^{\mathbf{U}_h^{n-1}}(\partial_t(\mathbf{u}^n-\mathbf{u}^{n-1}),\mathbf{u}^n,\partial_t\boldsymbol{\chi}^n)\\
&-\Delta t c^{\mathbf{U}_h^{n-1}}(\partial_t\mathbf{E}^{n-1},\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n) - \Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1},\partial_t\mathbf{E}^n,\partial_t\boldsymbol{\chi}^n)\\
&-\Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1}-\mathbf{u}^{n-2},\partial_t\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n) - \Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{E}^{n-2},\partial_t\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n)\\
&+l^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-2},\mathbf{U}_h^{n-1},\partial_t\boldsymbol{\chi}^n) - l^{\mathbf{U}_h^{n-2}}(\mathbf{U}_h^{n-2},\mathbf{U}_h^{n-1},\partial_t\boldsymbol{\chi}^n).
\end{aligned} \tag{4.110}$$

An application of Hölder's and Young's inequalities, Taylor expansions, estimates (1.14) and (2.56), the fact $\mathbf{u}^{n-1}-\mathbf{u}^{n-2} = \int_{t_{n-2}}^{t_{n-1}}\mathbf{u}_t(t)\,dt$, and noting that $\mathbf{u}^i$, $i = n, n-1, n-2$ satisfies incompressibility condition and is continuous, one can find

$$\Delta t|c^{\mathbf{U}_h^{n-1}}(\partial_t\mathbf{u}^{n-1},\mathbf{E}^n,\partial_t\boldsymbol{\chi}^n)| \leq C\Delta t\|\partial_t\mathbf{u}^{n-1}\|_1\|\mathbf{E}^n\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon$$

$$\leq C\|\mathbf{E}^n\|_\varepsilon^2 \int_{t_{n-2}}^{t_{n-1}} \|\mathbf{u}_t(t)\|_1^2\,dt + \frac{\mu K_1 \Delta t}{64}\|\partial_t \boldsymbol{\chi}^n\|_\varepsilon^2, \tag{4.111}$$

$$\Delta t|c^{\mathbf{U}_h^{n-1}}(\partial_t(\mathbf{u}^n-\mathbf{u}^{n-1}),\mathbf{u}^n,\partial_t\boldsymbol{\chi}^n)| \leq C\Delta t\int_{t_{n-2}}^{t_n} \|\mathbf{u}_{tt}(t)\|\|\nabla \mathbf{u}^n\|_{L^4(\Omega)}\|\partial_t\boldsymbol{\chi}^n\|_{L^4(\Omega)}\,dt$$

$$\leq C\|\mathbf{u}^n\|_2^2\Delta t^2 \int_{t_{n-2}}^{t_n} \|\mathbf{u}_{tt}(t)\|^2\,dt + \frac{\mu K_1 \Delta t}{64}\|\partial_t \boldsymbol{\chi}^n\|_\varepsilon^2. \tag{4.112}$$

Since $\mathbf{E}^{n-1} = \mathbf{u}^{n-1} - \mathbf{P}_h\mathbf{u}^{n-1} + \boldsymbol{\chi}^{n-1}$, we can rewrite $c^{\mathbf{U}_h^{n-1}}(\partial_t\mathbf{E}^{n-1},\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n)$ as follows

$$\Delta t\, c^{\mathbf{U}_h^{n-1}}(\partial_t\mathbf{E}^{n-1},\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n) = \Delta t\, c^{\mathbf{U}_h^{n-1}}(\partial_t(\mathbf{u}^{n-1}-\mathbf{P}_h\mathbf{u}^{n-1}),\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n) + \Delta t\, c^{\mathbf{U}_h^{n-1}}(\partial_t\boldsymbol{\chi}^{n-1},\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n). \tag{4.113}$$

Using estimate (2.56), approximation property of Lemma 2.2 and Young's inequality, one can derive

$$\Delta t|c^{\mathbf{U}_h^{n-1}}(\partial_t(\mathbf{u}^{n-1}-\mathbf{P}_h\mathbf{u}^{n-1}),\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n)| \leq C\Delta t\|\partial_t(\mathbf{u}^{n-1}-\mathbf{P}_h\mathbf{u}^{n-1})\|_\varepsilon\|\mathbf{U}_h^n\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon$$

$$\leq Ch^{2r}\|\mathbf{U}_h^n\|_\varepsilon^2 \int_{t_{n-2}}^{t_{n-1}} |\mathbf{u}_t(t)|_{r+1}^2\,dt + \frac{\mu K_1\Delta t}{64}\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon^2. \tag{4.114}$$

An application of Hölder's and Young's inequalities, (1.14), trace inequality (1.37), Lemma 2.6 and definition of $\|\cdot\|_\varepsilon$-norm imply

$$\begin{aligned}
&\Delta t\,|c^{\mathbf{U}_h^{n-1}}(\partial_t\boldsymbol{\chi}^{n-1},\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n)| \\
&\leq \Delta t\Big(\sum_{T\in\mathcal{T}_h}\|\partial_t\boldsymbol{\chi}^{n-1}\|_{L^4(T)}^4\Big)^{1/4}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla\mathbf{U}_h^n\|_{L^2(T)}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\partial_t\boldsymbol{\chi}^n\|_{L^4(T)}^4\Big)^{1/4} \\
&+ C\Delta t\Big(\sum_{T\in\mathcal{T}_h}\|\partial_t\boldsymbol{\chi}^{n-1}\|_{L^4(T)}^4\Big)^{1/4}\Big(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\mathbf{U}_h^n]\|_{L^2(e)}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\partial_t\boldsymbol{\chi}^n\|_{L^4(T)}^4\Big)^{1/4} \\
&+ \Delta t\Big(\sum_{T\in\mathcal{T}_h}\|\nabla\partial_t\boldsymbol{\chi}^{n-1}\|_{L^2(T)}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\mathbf{U}_h^n\|_{L^4(T)}^4\Big)^{1/4}\Big(\sum_{T\in\mathcal{T}_h}\|\partial_t\boldsymbol{\chi}^n\|_{L^4(T)}^4\Big)^{1/4} \\
&+ C\Delta t\Big(\sum_{e\in\Gamma_h}\frac{\sigma_e}{|e|}\|[\partial_t\boldsymbol{\chi}^{n-1}]\|_{L^2(e)}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\mathbf{U}_h^n\|_{L^4(T)}^4\Big)^{1/4}\Big(\sum_{T\in\mathcal{T}_h}\|\partial_t\boldsymbol{\chi}^n\|_{L^4(T)}^4\Big)^{1/4} \\
&\leq C\Delta t\|\partial_t\boldsymbol{\chi}^{n-1}\|_\varepsilon\|\mathbf{U}_h^n\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|_{L^4(\Omega)} \\
&\leq \frac{\mu K_1\Delta t}{64}e^{-2\alpha\Delta t}\|\partial_t\boldsymbol{\chi}^{n-1}\|_\varepsilon^2 + C\Delta t e^{2\alpha\Delta t}\|\mathbf{U}_h^n\|_\varepsilon^2\|\partial_t\boldsymbol{\chi}^n\|\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon \\
&\leq \frac{\mu K_1\Delta t}{64}e^{-2\alpha\Delta t}\|\partial_t\boldsymbol{\chi}^{n-1}\|_\varepsilon^2 + \frac{\mu K_1\Delta t}{64}\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon^2 + C\Delta t e^{4\alpha\Delta t}\|\mathbf{U}_h^n\|_\varepsilon^4\|\partial_t\boldsymbol{\chi}^n\|^2. \qquad (4.115)
\end{aligned}$$

Furthermore, Hölder's and Young's inequalities, (1.14), (1.25), (1.37), (2.56), and Lemmas 2.2 and 1.3 yield

$$\Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1},\partial_t\mathbf{E}^n,\partial_t\boldsymbol{\chi}^n) = \Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1},\partial_t(\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n),\partial_t\boldsymbol{\chi}^n)$$

$$+\Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1},\partial_t\boldsymbol{\chi}^n,\partial_t\boldsymbol{\chi}^n)$$
$$\le C\|\mathbf{u}^{n-1}\|_1\int_{t_{n-1}}^{t_n}\|\mathbf{u}_t(t)-\mathbf{P}_h\mathbf{u}_t(t)\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon\,dt+C\Delta t\|\mathbf{u}^{n-1}\|_2\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|$$
$$\le\frac{\mu K_1\Delta t}{64}\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon^2+Ch^{2r}\|\mathbf{u}^{n-1}\|_1^2\int_{t_{n-1}}^{t_n}|\mathbf{u}_t(t)|_{r+1}^2\,dt+C\Delta t\|\mathbf{u}^{n-1}\|_2^2\|\partial_t\boldsymbol{\chi}^n\|^2. \quad (4.116)$$

Note that, $\Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1}-\mathbf{u}^{n-2},\partial_t\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n)$ can be rewritten as

$$\Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1}-\mathbf{u}^{n-2},\partial_t\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n)=-\Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1}-\mathbf{u}^{n-2},\partial_t\boldsymbol{\chi}^n,\partial_t\boldsymbol{\chi}^n)$$
$$-\Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1}-\mathbf{u}^{n-2},\partial_t(\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n),\partial_t\boldsymbol{\chi}^n)+\Delta t c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1}-\mathbf{u}^{n-2},\partial_t\mathbf{u}^n,\partial_t\boldsymbol{\chi}^n).$$

Thus, $\Delta t|c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1}-\mathbf{u}^{n-2},\partial_t\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n)|$ is bounded with the help of Hölder's and Young's inequalities, (1.14), (1.37) and Lemma 2.2 as follows

$$\Delta t|c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^{n-1}-\mathbf{u}^{n-2},\partial_t\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n)|\le C\Delta t\|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega))}\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|$$
$$+C\|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega))}\int_{t_{n-1}}^{t_n}\|\mathbf{u}_t(t)-\mathbf{P}_h\mathbf{u}_t(t)\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|\,dt$$
$$+C\int_{t_{n-2}}^{t_{n-1}}\|\mathbf{u}_t(t)\|_1\,dt\int_{t_{n-1}}^{t_n}\|\mathbf{u}_t(t)\|_1\,dt\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon$$
$$\le\frac{\mu K_1\Delta t}{64}\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon^2+C\Delta t\|\mathbf{u}\|^2_{L^\infty(0,T;L^\infty(\Omega))}\|\partial_t\boldsymbol{\chi}^n\|^2+Ch^{2r}\int_{t_{n-1}}^{t_n}|\mathbf{u}_t(t)|_{r+1}^2\,dt$$
$$+C\Delta t^2\|\mathbf{u}_t\|^2_{L^\infty(0,T;H^1(\Omega))}\int_{t_{n-1}}^{t_n}\|\mathbf{u}_t(t)\|_1^2\,dt. \quad (4.117)$$

In a similar manner, using triangle inequality and Lemma 2.6 one can show that

$$\Delta t|c^{\mathbf{U}_h^{n-1}}(\mathbf{E}^{n-2},\partial_t\mathbf{U}_h^n,\partial_t\boldsymbol{\chi}^n)|\le\Delta t|c^{\mathbf{U}_h^{n-1}}(\mathbf{E}^{n-2},\partial_t\boldsymbol{\chi}^n,\partial_t\boldsymbol{\chi}^n)|$$
$$+\Delta t|c^{\mathbf{U}_h^{n-1}}(\mathbf{E}^{n-2},\partial_t(\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n),\partial_t\boldsymbol{\chi}^n)|+\Delta t|c^{\mathbf{U}_h^{n-1}}(\mathbf{E}^{n-2},\partial_t\mathbf{u}^n,\partial_t\boldsymbol{\chi}^n)|$$
$$\le C\Delta t(\|\mathbf{E}^{n-2}\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon^{3/2}\|\partial_t\boldsymbol{\chi}^n\|^{1/2}+\|\mathbf{E}^{n-2}\|_\varepsilon\|\partial_t(\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n)\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon$$
$$+\|\mathbf{E}^{n-2}\|_\varepsilon\|\partial_t\mathbf{u}^n\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon)$$
$$\le\frac{\mu K_1\Delta t}{64}\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon^2+C\Delta t\|\mathbf{E}^{n-2}\|_\varepsilon^4\|\partial_t\boldsymbol{\chi}^n\|^2+C\|\mathbf{E}^{n-2}\|_\varepsilon^2\int_{t_{n-1}}^{t_n}|\mathbf{u}_t(t)|_1^2\,dt. \quad (4.118)$$

Using estimate (2.60) and regularity of $\mathbf{u}^{n-1}$, we arrive at

$$|l^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-2},\mathbf{U}_h^{n-1},\partial_t\boldsymbol{\chi}^n)-l^{\mathbf{U}_h^{n-2}}(\mathbf{U}_h^{n-2},\mathbf{U}_h^{n-1},\partial_t\boldsymbol{\chi}^n)|$$
$$\le C\Delta t\|\partial_t\mathbf{U}_h^{n-1}\|_\varepsilon\|\mathbf{E}^{n-1}\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|_{L^4(\Omega)}$$
$$\le C\Delta t(\|\partial_t\boldsymbol{\chi}^{n-1}\|_\varepsilon+\|\partial_t\mathbf{P}_h\mathbf{u}^{n-1}\|_\varepsilon)\|\mathbf{E}^{n-1}\|_\varepsilon\|\partial_t\boldsymbol{\chi}^n\|_{L^4(\Omega)}.$$

A use of Young's inequality and Lemma 2.6 imply

$$|l^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-2},\mathbf{U}_h^{n-1},\partial_t\boldsymbol{\chi}^n)-l^{\mathbf{U}_h^{n-2}}(\mathbf{U}_h^{n-2},\mathbf{U}_h^{n-1},\partial_t\boldsymbol{\chi}^n)|$$

$$\leq \frac{\mu K_1 \Delta t}{64} e^{-2\alpha\Delta t} \|\partial_t \boldsymbol{\chi}^{n-1}\|_\varepsilon^2 + \frac{\mu K_1 \Delta t}{64} \|\partial_t \boldsymbol{\chi}^n\|_\varepsilon^2 + C\Delta t e^{4\alpha\Delta t} \|\mathbf{E}^{n-1}\|_\varepsilon^4 \|\partial_t \boldsymbol{\chi}^n\|^2$$
$$+ C\|\mathbf{E}^{n-1}\|_\varepsilon^2 \int_{t_{n-2}}^{t_{n-1}} \|\mathbf{P}_h \mathbf{u}_t(t)\|_\varepsilon^2 \, dt. \tag{4.119}$$

Now, $\Delta t \, a(\partial_t q_r^n(\boldsymbol{\chi}), \partial_t \boldsymbol{\chi}^n)$ can be rewritten as

$$\Delta t a(\partial_t q_r^n(\boldsymbol{\chi}), \partial_t \boldsymbol{\chi}^n) = \Delta t \beta(0) a(\boldsymbol{\chi}^n, \partial_t \boldsymbol{\chi}^n) + (e^{-\delta\Delta t} - 1) a(q_r^{n-1}(\boldsymbol{\chi}), \partial_t \boldsymbol{\chi}^n).$$

Therefore, Lemma 1.7, Hölder's inequality and Young's inequality lead to

$$\Delta t \sum_{n=2}^m e^{2\alpha t_n} |a(\partial_t q_r^n(\boldsymbol{\chi}), \partial_t \boldsymbol{\chi}^n)| \leq \frac{\mu K_1 \Delta t}{64} \sum_{n=2}^m e^{2\alpha t_n} \|\partial_t \boldsymbol{\chi}^n\|_\varepsilon^2$$
$$+ C\Delta t \sum_{n=2}^m e^{2\alpha t_n} \|\boldsymbol{\chi}^n\|_\varepsilon^2. \tag{4.120}$$

An application of Hölder's inequality, Young's inequality and Lemma 2.3 imply

$$\mu \Delta t \sum_{n=2}^m e^{2\alpha t_n} |a(\partial_t(\mathbf{u}^n - \mathbf{P}_h \mathbf{u}^n), \partial_t \boldsymbol{\chi}^n)| \leq C h^{2r} e^{2\alpha\Delta t} \int_{t_1}^{t_m} e^{2\alpha t} |\mathbf{u}_t(t)|_{r+1}^2 \, dt$$
$$+ \frac{\mu K_1 \Delta t}{64} \sum_{n=2}^m e^{2\alpha t_n} \|\partial_t \boldsymbol{\chi}^n\|_\varepsilon^2. \tag{4.121}$$

To handle the first term on the right hand side of (4.109), we rewrite it as

$$\Delta t(\partial_t(\mathbf{u}_t^n - \partial_t \mathbf{u}^n), \partial_t \boldsymbol{\chi}^n) = \Delta t \partial_t(\mathbf{u}_t^n - \partial_t \mathbf{u}^n, \partial_t \boldsymbol{\chi}^n) - \Delta t(\mathbf{u}_t^{n-1} - \partial_t \mathbf{u}^{n-1}, \partial_t(\partial_t \boldsymbol{\chi}^n)).$$

Therefore, we use summation by parts, (2.138), the Cauchy-Schwarz inequality, Hölder's inequality and Young's inequality to derive

$$\Delta t \sum_{n=2}^m e^{2\alpha t_n} (\partial_t(\mathbf{u}_t^n - \partial_t \mathbf{u}^n), \partial_t \boldsymbol{\chi}^n) = e^{2\alpha t_m} (\mathbf{u}_t^m - \partial_t \mathbf{u}^m, \partial_t \boldsymbol{\chi}^m) - e^{2\alpha t_1} (\mathbf{u}_t^1 - \partial_t \mathbf{u}^1, \partial_t \boldsymbol{\chi}^1)$$
$$- (e^{2\alpha\Delta t} - 1) \sum_{n=2}^m e^{2\alpha t_{n-1}} (\mathbf{u}_t^{n-1} - \partial_t \mathbf{u}^{n-1}, \partial_t \boldsymbol{\chi}^{n-1})$$
$$- \Delta t \sum_{n=2}^m e^{2\alpha t_n} (\mathbf{u}_t^{n-1} - \partial_t \mathbf{u}^{n-1}, \partial_t(\partial_t \boldsymbol{\chi}^n))$$
$$\leq \frac{1}{8} e^{2\alpha t_m} \|\partial_t \boldsymbol{\chi}^m\|^2 + C\Delta t e^{2\alpha t_m} \int_{t_{m-1}}^{t_m} \|\mathbf{u}_{tt}(t)\|^2 \, dt + C e^{2\alpha t_1} \|\partial_t \boldsymbol{\chi}^1\|^2$$
$$+ C\Delta t e^{2\alpha t_1} \int_0^{t_1} \|\mathbf{u}_{tt}(t)\|^2 \, dt + C\Delta t \sum_{n=2}^m e^{2\alpha t_{n-1}} \|\partial_t \boldsymbol{\chi}^{n-1}\|^2$$
$$+ C\Delta t (e^{4\alpha\Delta t} + e^{2\alpha\Delta t}) \int_0^{t_{m-1}} e^{2\alpha t} \|\mathbf{u}_{tt}(t)\|^2 \, dt + \frac{\Delta t^2}{4} \sum_{n=2}^m e^{2\alpha t_n} \|\partial_t(\partial_t \boldsymbol{\chi}^n)\|^2. \tag{4.122}$$

Also, we can rewrite

$$\Delta t\, a(\partial_t q_r^n(\mathbf{P}_h\mathbf{u}), \partial_t\boldsymbol{\chi}^n) = \Delta t\beta(0)a(\mathbf{P}_h\mathbf{u}^n, \partial_t\boldsymbol{\chi}^n) + (e^{-\delta\Delta t} - 1)a(q_r^{n-1}(\mathbf{P}_h\mathbf{u}), \partial_t\boldsymbol{\chi}^n),$$

and

$$\begin{aligned} &\int_0^{t_n} \beta(t_n - s)a(\mathbf{P}_h\mathbf{u}(s), \partial_t\boldsymbol{\chi}^n)\, ds - \int_0^{t_{n-1}} \beta(t_{n-1} - s)a(\mathbf{P}_h\mathbf{u}(s), \partial_t\boldsymbol{\chi}^n)\, ds \\ &= \int_{t_{n-1}}^{t_n} \beta(t_n - s)a(\mathbf{P}_h\mathbf{u}(s), \partial_t\boldsymbol{\chi}^n)\, ds \\ &\quad + (e^{-\delta\Delta t} - 1)\int_0^{t_{n-1}} \beta(t_{n-1} - s)a(\mathbf{P}_h\mathbf{u}(s), \partial_t\boldsymbol{\chi}^n)\, ds. \end{aligned} \tag{4.123}$$

With the help of above two equalities, (4.39), (4.40), Lemma 1.7, Hölder's inequality and Young's inequality, one can find

$$\begin{aligned} &\sum_{n=2}^m e^{2\alpha t_n}\Big(\Delta t a(\partial_t q_r^n(\mathbf{P}_h\mathbf{u}), \partial_t\boldsymbol{\chi}^n) - \int_0^{t_n} \beta(t_n - s)a(\mathbf{P}_h\mathbf{u}(s), \partial_t\boldsymbol{\chi}^n)\, ds \\ &\qquad + \int_0^{t_{n-1}} \beta(t_{n-1} - s)a(\mathbf{P}_h\mathbf{u}(s), \partial_t\boldsymbol{\chi}^n)\, ds\Big) \\ &\qquad \le \frac{\mu K_1\Delta t}{64}\sum_{n=2}^m e^{2\alpha t_n}\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon^2 + C\Delta t^2 e^{2\alpha\Delta t}\int_{t_1}^{t_m} e^{2\alpha t}\|\mathbf{P}_h\mathbf{u}_t(t)\|_\varepsilon^2\, dt \\ &\qquad + C\Delta t^3 e^{2\alpha\Delta t}\sum_{n=2}^m\int_0^{t_{n-1}} e^{2\alpha t}(\|\mathbf{P}_h\mathbf{u}(t)\|_\varepsilon^2 + \|\mathbf{P}_h\mathbf{u}_t(t)\|_\varepsilon^2)\, dt. \end{aligned} \tag{4.124}$$

With the same technique as (4.123), and using Lemma 2.3, Hölder's inequality and Young's inequality, we arrive at

$$\begin{aligned} &\sum_{n=2}^m e^{2\alpha t_n}\Big(\int_0^{t_n} \beta(t_n - s)a(\mathbf{u}(s) - \mathbf{P}_h\mathbf{u}(s), \partial_t\boldsymbol{\chi}^n)\, ds \\ &\qquad - \int_0^{t_{n-1}} \beta(t_{n-1} - s)a(\mathbf{u}(s) - \mathbf{P}_h\mathbf{u}(s), \partial_t\boldsymbol{\chi}^n)\, ds\Big) \\ &\le \frac{\mu K_1\Delta t}{64}\sum_{n=2}^m e^{2\alpha t_n}\|\partial_t\boldsymbol{\chi}^n\|_\varepsilon^2 + Ch^{2r}e^{2\alpha\Delta t}\int_{t_1}^{t_m} e^{2\alpha t}|\mathbf{u}(t)|_{r+1}^2\, dt \\ &\qquad + Ch^{2r}\Delta t^2 e^{2\alpha\Delta t}\sum_{n=2}^m\int_0^{t_{n-1}} e^{2\alpha t}|\mathbf{u}(t)|_{r+1}^2\, dt. \end{aligned} \tag{4.125}$$

Using the definition of the space $\mathbf{V}_h$, Lemma 2.4, Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned} \Delta t\sum_{n=2}^m e^{2\alpha t_n}b(\partial_t\boldsymbol{\chi}^n, \partial_t p^n) &= \Delta t\sum_{n=2}^m e^{2\alpha t_n}b(\partial_t\boldsymbol{\chi}^n, \partial_t(p^n - r_h p^n)) \\ &= \sum_{n=2}^m e^{2\alpha t_n}\int_{t_{n-1}}^{t_n} b(\partial_t\boldsymbol{\chi}^n, p_t(t) - r_h p_t(t))\, dt \end{aligned}$$

$$\leq \frac{\mu K_1 \Delta t}{64} \sum_{n=2}^{m} e^{2\alpha t_n} \|\partial_t \boldsymbol{\chi}^n\|_\varepsilon^2 + Ch^{2r} e^{2\alpha \Delta t} \int_{t_1}^{t_m} e^{2\alpha t} |p_t(t)|_r^2 \, dt. \tag{4.126}$$

Substitute (4.111)-(4.119) in (4.110) and then in (4.109). Now, multiply the resulting inequality by $e^{2\alpha t_n}$, sum over $2 \leq n \leq m \leq M$. Note that, $\|\partial_t \boldsymbol{\chi}^n\| \leq \|\partial_t \mathbf{E}^n\| + \|\partial_t(\mathbf{u}^n - \mathbf{P}_h \mathbf{u}^n)\|$ and $\|\boldsymbol{\chi}^n\|_\varepsilon \leq \|\mathbf{E}^n\|_\varepsilon + \|\mathbf{u}^n - \mathbf{P}_h \mathbf{u}^n\|_\varepsilon$. Thus, an application of (4.108), Lemmas 2.2, 4.7 and assumption (**A3**) leads to

$$e^{2\alpha t_m} \|\partial_t \boldsymbol{\chi}^m\|^2 + \Delta t \sum_{n=2}^{m} e^{2\alpha t_n} \|\partial_t \boldsymbol{\chi}^n\|_\varepsilon^2 \leq C e^{2\alpha t_1} \|\partial_t \boldsymbol{\chi}^1\|^2 + C e^{2\alpha t_m} (h^{2r} + \Delta t). \tag{4.127}$$

It remains to bound $\|\partial_t \boldsymbol{\chi}^1\|$. To handle this term, we denote $\mathcal{G}^n = \mathbf{S}_h^{vol} \mathbf{u}^n - \mathbf{U}_h^n$. Consider the equations (4.33) and (4.63) at $t = t_n$, and from (4.60) we find

$$\begin{aligned}
(\partial_t \mathcal{G}^n, \boldsymbol{\phi}_h) + \mu\, a(\mathcal{G}^n, \boldsymbol{\phi}_h) &= -(\mathbf{u}_t^n - \partial_t \mathbf{u}^n, \boldsymbol{\phi}_h) - (\partial_t \boldsymbol{\zeta}^n, \boldsymbol{\phi}_h) - a(q_r^n(\mathcal{G}), \boldsymbol{\phi}_h) \\
&+ a(q_r^n(\mathbf{S}_h^{vol} \mathbf{u}), \boldsymbol{\phi}_h) - \int_0^{t_n} \beta(t_n - s) a(\mathbf{S}_h^{vol} \mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \\
&+ c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\phi}_h) - c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h.
\end{aligned} \tag{4.128}$$

Now, we consider (4.128) with $n = 1$, choose $\boldsymbol{\phi}_h = \partial_t \mathcal{G}^1$ and rewrite the nonlinear terms to find

$$\begin{aligned}
&\|\partial_t \mathcal{G}^1\|^2 + \mu\, a(\mathcal{G}^1, \partial_t \mathcal{G}^1) = -(\mathbf{u}_t^1 - \partial_t \mathbf{u}^1, \partial_t \mathcal{G}^1) - (\partial_t \boldsymbol{\zeta}^1, \partial_t \mathcal{G}^1) - a(q_r^1(\mathcal{G}), \partial_t \mathcal{G}^1) \\
&+ a(q_r^1(\mathbf{S}_h^{vol} \mathbf{u}), \partial_t \mathcal{G}^1) - \int_0^{t_1} \beta(t_1 - s) a(\mathbf{S}_h^{vol} \mathbf{u}(s), \partial_t \mathcal{G}^1)\, ds - c^{\mathbf{U}_h^0}(\mathbf{u}^0, \mathbf{E}^1, \partial_t \mathcal{G}^1) \\
&- c^{\mathbf{U}_h^0}(\mathbf{E}^0, \mathbf{u}^1, \partial_t \mathcal{G}^1) + c^{\mathbf{U}_h^0}(\mathbf{E}^0, \mathbf{E}^1, \partial_t \mathcal{G}^1) - c^{\mathbf{U}_h^0}(\mathbf{u}^1 - \mathbf{u}^0, \mathbf{u}^1, \partial_t \mathcal{G}^1).
\end{aligned} \tag{4.129}$$

Notice that, $\mathcal{G}^0 = 0$. Apply the coercivity property from Lemma 1.6 to find

$$\mu K_1 \Delta t \|\partial_t \mathcal{G}^1\|_\varepsilon^2 \leq \mu\, a(\mathcal{G}^1, \partial_t \mathcal{G}^1) = \Delta t \mu\, a(\partial_t \mathcal{G}^1, \partial_t \mathcal{G}^1). \tag{4.130}$$

Again, with the help of the continuity property of $a(\cdot, \cdot)$, that is, Lemma 1.7, we obtain

$$|a(q_r^1(\mathcal{G}), \partial_t \mathcal{G}^1)| = \beta(0) |a(\mathcal{G}^1, \mathcal{G}^1)| \leq C \|\mathcal{G}^1\|_\varepsilon^2. \tag{4.131}$$

In addition, a use of (4.40), Lemma 1.7, triangle inequality, the Cauchy-Schwarz inequality and Young's inequality imply

$$\begin{aligned}
&\left| a(q_r^1(\mathbf{S}_h^{vol} \mathbf{u}), \partial_t \mathcal{G}^1) - \int_0^{t_1} \beta(t_1 - s) a(\mathbf{S}_h^{vol} \mathbf{u}(s), \partial_t \mathcal{G}^1)\, ds \right| \qquad (4.132) \\
=& \frac{1}{\Delta t} \left| \int_0^{t_1} s \left(\beta_s(t_1 - s) a(\mathbf{S}_h^{vol} \mathbf{u}(s), \mathcal{G}^1) + \beta(t_1 - s) a(\mathbf{S}_h^{vol} \mathbf{u}_s(s), \mathcal{G}^1) \right) ds \right| \\
\leq& C \Delta t \int_0^{t_1} (\|\mathbf{S}_h^{vol} \mathbf{u}(s)\|_\varepsilon^2 + \|\mathbf{S}_h^{vol} \mathbf{u}_s(s)\|_\varepsilon^2)\, ds + C \|\mathcal{G}^1\|_\varepsilon^2
\end{aligned}$$

$$\leq C\Delta t\int_0^{t_1}(\|\boldsymbol{\zeta}(s)\|_\varepsilon^2+\|\boldsymbol{\zeta}_s(s)\|_\varepsilon^2)\,ds+C\Delta t\int_0^{t_1}(\|\mathbf{u}(s)\|_\varepsilon^2+\|\mathbf{u}_s(s)\|_\varepsilon^2)\,ds+C\|\mathcal{G}^1\|_\varepsilon^2.$$

The rest of the terms on the right hand side of (4.129) can be estimated identically as Lemma 4.7. Note that, $\|\mathcal{G}^1\|_\varepsilon \leq \|\boldsymbol{\zeta}^1\|_\varepsilon + \|\mathbf{E}^1\|_\varepsilon$. Thus, a combination of (4.130)-(4.132), a multiplication of the resulting inequality by $e^{2\alpha t_1}$, and an application of (4.64), (4.65), Lemmas 4.7, 4.8 and assumption (**A3**) lead to

$$e^{2\alpha t_1}\|\partial_t\mathcal{G}^1\|^2+\mu K_1 e^{2\alpha t_1}\Delta t\|\partial_t\mathcal{G}^1\|_\varepsilon^2 \leq C(h^{2r}+\Delta t).$$

We now use the above estimate, triangle inequality

$$\begin{aligned}\|\partial_t\boldsymbol{\chi}^1\| &\leq \|\partial_t(\mathbf{u}^1-\mathbf{P}_h\mathbf{u}^1)\|+\|\partial_t\boldsymbol{\zeta}^1\|+\|\partial_t\mathcal{G}^1\|\\ \|\partial_t\boldsymbol{\chi}^1\|_\varepsilon &\leq \|\partial_t(\mathbf{u}^1-\mathbf{P}_h\mathbf{u}^1)\|_\varepsilon+\|\partial_t\boldsymbol{\zeta}^1\|_\varepsilon+\|\partial_t\mathcal{G}^1\|_\varepsilon,\end{aligned}$$

(4.65) and Lemma 2.2 to observe that

$$e^{2\alpha t_1}\|\partial_t\boldsymbol{\chi}^1\|^2+\mu K_1 e^{2\alpha t_1}\Delta t\|\partial_t\boldsymbol{\chi}^1\|_\varepsilon^2 \leq C(h^{2r}+\Delta t). \tag{4.133}$$

Finally, insert (4.133) in (4.127), and use the relation $\partial_t\mathbf{E}^n=\partial_t(\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n)+\partial_t\boldsymbol{\chi}^n$, Lemma 2.2 and assumption (**A3**) to conclude the rest of the proof. □

The following theorem establishes fully discrete $L^\infty(L^2)$-norm estimate for the pressure error.

Theorem 4.5. *Under the assumptions of Theorem 4.4 , there exists a positive constant K_T such that the following error estimate holds:*

$$\|p^n-P_h^n\| \leq K_T(h^r+\Delta t^{1/2}).$$

Proof. Subtract (4.9) from the equation (4.33) at $t=t_n$ to find error equation as follows:

$$\begin{aligned}-b(\boldsymbol{\phi}_h, r_h(p^n)-P_h^n) &= (\mathbf{u}_t^n-\partial_t\mathbf{u}^n,\boldsymbol{\phi}_h)+(\partial_t\mathbf{E}^n,\boldsymbol{\phi}_h)+\mu\, a(\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n,\boldsymbol{\phi}_h)\\ &+\mu\, a(\boldsymbol{\chi}^n,\boldsymbol{\phi}_h)+a(q_r^n(\boldsymbol{\chi}),\boldsymbol{\phi}_h)+\int_0^{t_n}\beta(t_n-s)a(\mathbf{P}_h\mathbf{u}(s),\boldsymbol{\phi}_h)\,ds\\ &-a(q_r^n(\mathbf{P}_h\mathbf{u}),\boldsymbol{\phi}_h)+\int_0^{t_n}\beta(t_n-s)a(\mathbf{u}(s)-\mathbf{P}_h\mathbf{u}(s),\boldsymbol{\phi}_h)\,ds\\ &+c^{\mathbf{u}^n}(\mathbf{u}^n,\mathbf{u}^n,\boldsymbol{\phi}_h)-c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1},\mathbf{U}_h^n,\boldsymbol{\phi}_h)+b(\boldsymbol{\phi}_h,p^n-r_h(p^n)),\quad \forall\boldsymbol{\phi}_h\in\mathbf{X}_h.\end{aligned} \tag{4.134}$$

Using Lemma 1.7 and triangle inequality, one can obtain

$$|a(q_r^n(\boldsymbol{\chi}),\boldsymbol{\phi}_h)| \leq C\Delta t\sum_{i=1}^n\|\boldsymbol{\chi}^i\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon \leq C\Delta t\sum_{i=1}^n(\|\mathbf{E}^i\|_\varepsilon+\|\mathbf{u}^i-\mathbf{P}_h\mathbf{u}^i\|_\varepsilon)\|\boldsymbol{\phi}_h\|_\varepsilon.$$

A use of (4.39), Lemma 1.7 and Hölder's inequality leads to

$$\begin{aligned}
&\Big| \int_0^{t_n} \beta(t_n - s) a(\mathbf{P}_h \mathbf{u}(s), \boldsymbol{\phi}_h)\, ds - a(q_r^n(\mathbf{P}_h \mathbf{u}), \boldsymbol{\phi}_h) \Big| \\
&\le C\Delta t \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \Big| \frac{\partial}{\partial s} \Big(\beta(t_n - s) a(\mathbf{P}_h \mathbf{u}(s), \boldsymbol{\phi}_h) \Big) \Big|\, ds \\
&\le C\Delta t \Big(\int_0^{t_n} e^{2\alpha s} (\|\mathbf{P}_h \mathbf{u}(s)\|_\varepsilon^2 + \|\mathbf{P}_h \mathbf{u}_s(s)\|_\varepsilon^2)\, ds \Big)^{1/2} \|\boldsymbol{\phi}_h\|_\varepsilon.
\end{aligned}$$

Applying Lemma 2.3 and Hölder's inequality, we arrive at

$$\Big| \int_0^{t_n} \beta(t_n - s) a(\mathbf{u}(s) - \mathbf{P}_h \mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \Big| \le Ch^r \Big(\int_0^{t_n} e^{2\alpha s} |\mathbf{u}(s)|_{r+1}^2\, ds \Big)^{1/2} \|\boldsymbol{\phi}_h\|_\varepsilon.$$

In this case, the nonlinear terms can be rewritten by virtue of the continuity of $\mathbf{u}^n$ and then bounded with the help of (2.56) as follows

$$\begin{aligned}
c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\phi}_h) &= c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\phi}_h) \\
&= c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^n, \mathbf{E}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^{n-1}}(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{U}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^{n-1}}(\mathbf{E}^{n-1}, \mathbf{U}_h^n, \boldsymbol{\phi}_h) \\
&\le C(\|\mathbf{u}^n\|_1 \|\mathbf{E}^n\|_\varepsilon + \Delta t \|\mathbf{u}_t\|_{L^\infty(0,T;H^1(\Omega))} \|\mathbf{U}_h^n\|_\varepsilon + \|\mathbf{E}^{n-1}\|_\varepsilon \|\mathbf{U}_h^n\|_\varepsilon) \|\boldsymbol{\phi}_h\|_\varepsilon.
\end{aligned}$$

The remaining terms on the right hand side of (4.134) can be bounded exactly like Lemmas 4.7 and 4.12. Thus, employing the inf-sup condition from Lemma 1.8, combining all the above estimates in (4.134) and applying Lemma 4.7, we obtain

$$\|r_h(p^n) - P_h^n\| \le C\|\partial_t \mathbf{E}^n\| + C(h^r + \Delta t^{1/2}).$$

Since $\|p^n - P_h^n\| \le \|p^n - r_h(p^n)\| + \|r_h(p^n) - P_h^n\|$, using (1.31) and Lemma 4.12, we arrive at our desired estimate of this theorem. □

Remark 4.1. *If we assume $\boldsymbol{u}_{tt} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, then we can find an improved estimate of pressure error:*

$$\|p^n - P_h^n\| \le C(h^r + \Delta t).$$

To derive the above estimate , we rewrite (4.134) as

$$\begin{aligned}
&(\partial_t \boldsymbol{\chi}^n, \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, r_h(p^n) - P_h^n) = -(\mathbf{u}_t^n - \partial_t \mathbf{u}^n, \boldsymbol{\phi}_h) - (\partial_t(\mathbf{u}^n - \mathbf{P}_h \mathbf{u}^n), \boldsymbol{\phi}_h) \\
&- \mu\, a(\mathbf{u}^n - \mathbf{P}_h \mathbf{u}^n, \boldsymbol{\phi}_h) - \mu\, a(\boldsymbol{\chi}^n, \boldsymbol{\phi}_h) - a(q_r^n(\boldsymbol{\chi}), \boldsymbol{\phi}_h) - \int_0^{t_n} \beta(t_n - s) a(\mathbf{P}_h \mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \\
&+ a(q_r^n(\mathbf{P}_h \mathbf{u}), \boldsymbol{\phi}_h) - \int_0^{t_n} \beta(t_n - s) a(\mathbf{u}(s) - \mathbf{P}_h \mathbf{u}(s), \boldsymbol{\phi}_h)\, ds \\
&- c^{\mathbf{u}^n}(\mathbf{u}^n, \mathbf{u}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^{n-1}}(\mathbf{U}_h^{n-1}, \mathbf{U}_h^n, \boldsymbol{\phi}_h) - b(\boldsymbol{\phi}_h, p^n - r_h(p^n)), \quad \forall \boldsymbol{\phi}_h \in \mathbf{X}_h. \qquad (4.135)
\end{aligned}$$

The term $(\mathbf{u}_t^n - \partial_t \mathbf{u}^n, \boldsymbol{\phi}_h)$ is bounded employing (1.14), (2.138), the Cauchy-Schwarz and Hölder's inequalities as follows

$$|(\mathbf{u}_t^n - \partial_t \mathbf{u}^n, \boldsymbol{\phi}_h)| \leq C\Delta t^{1/2} \left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{tt}(t)\|^2 \, dt \right)^{1/2} \|\boldsymbol{\phi}_h\|_\varepsilon \leq C\Delta t \sup_{1\leq n\leq M} \|\mathbf{u}_{tt}^n\| \|\boldsymbol{\phi}_h\|_\varepsilon.$$

For the term $(\partial_t(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n), \boldsymbol{\phi}_h)$, using (1.14), Lemma 2.2 and the Cauchy-Schwarz inequality, we find

$$|(\partial_t(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n), \boldsymbol{\phi}_h)| \leq \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \|\mathbf{u}_s(s) - \mathbf{P}_h\mathbf{u}_s(s)\| \|\boldsymbol{\phi}_h\|_\varepsilon \, ds \leq Ch^r \sup_{1\leq n\leq M} |\mathbf{u}_t^n|_r \|\boldsymbol{\phi}_h\|_\varepsilon.$$

The remaining terms on the right hand side of (4.135) can estimated similar to Theorem 4.5. Now, Lemma 1.8 leads to

$$\|\partial_t \boldsymbol{\chi}^n\| + \beta^* \|r_h(p^n) - P_h^n\| \leq \sup_{\boldsymbol{\phi}_h \in \mathbf{X}_h \setminus \{\mathbf{0}\}} \frac{(\partial_t \boldsymbol{\chi}^n, \boldsymbol{\phi}_h)}{\|\boldsymbol{\phi}_h\|_\varepsilon} + \sup_{\boldsymbol{\phi}_h \in \mathbf{X}_h \setminus \{\mathbf{0}\}} \frac{b(\boldsymbol{\phi}_h, r_h(p^n) - P_h^n)}{\|\boldsymbol{\phi}_h\|_\varepsilon}.$$

The desired estimate is obtained by applying the above bounds in (4.135).

Remark 4.2. *Similar to Remark 2.5, the optimal order convergence rates derived in this chapter can be extended to the 3D case.*

4.6 Numerical Experiments

This section provides numerical results that corroborate the established theoretical results. To discretize space, we utilize $\mathbb{P}_r$-$\mathbb{P}_{r-1}$ $(r = 1, 2)$ mixed spaces. For discretization in time direction, we employ a backward Euler method that has first-order accuracy. The spatial computational domain Ω is selected as $[0, 1] \times [0, 1]$. Examples 4.1 and 4.2 are computed on the time interval [0, 1].

Example 4.1. *In this example, the following exact solutions* $(\boldsymbol{u}, p) = \big((u_1(x, y, t), u_2(x, y, t)), p(x, y, t)\big)$ *are considered:*

$$\begin{aligned} u_1(x, y, t) &= 2x^2(x-1)^2 y(y-1)(2y-1)\cos(t), \\ u_2(x, y, t) &= -2x(x-1)(2x-1)y^2(y-1)^2\cos(t), \\ p(x, y, t) &= 2\cos(t)(x-y). \end{aligned}$$

We compute the forcing term $\mathbf{f}$ *from the above exact solution.*

Tables 4.1 and 4.2 present the numerical errors and rates of convergence for the $\mathbb{P}_1 - \mathbb{P}_0$ mixed finite element space, whereas Tables 4.3 and 4.4 are for $\mathbb{P}_2 - \mathbb{P}_1$ space. The parameters are chosen as $\mu = 1$, $\gamma = 0.1$, $\delta = 0.1$ for Tables 4.1-4.4. The penalty parameter $\sigma_e = 10$ and 20 for $r = 1$ and 2, respectively. The time step Δt is chosen as $\mathcal{O}(h^r)$ for Tables 4.1 and 4.3, and $\mathcal{O}(h^{r+1})$ for Tables 4.2 and 4.4. We have considered three DG methods namely, SIPG, NIPG and IIPG. The convergence rates are optimal for velocity in energy norm and for pressure in L^2-norm in Tables 4.1 and 4.3, and for velocity in $\mathbf{L}^2$norm in Tables 4.2 and 4.4, as predicted by the theory.

Table 4.1: Numerical results with $\mathbb{P}_1$–$\mathbb{P}_0$ finite element for Example 4.1.

Method	h	$\\|\mathbf{u}(T) - \mathbf{U}^M\\|_\varepsilon$	Rate	$\\|p(T) - P^M\\|$	Rate
SIPG	1/4	2.8328×10^{-2}		1.0486×10^{-2}	
	1/8	1.2429×10^{-2}	1.1885	7.1664×10^{-3}	0.5491
	1/16	5.4126×10^{-3}	1.1993	4.3413×10^{-3}	0.7231
	1/32	2.4326×10^{-3}	1.1538	2.4102×10^{-3}	0.8489
	1/64	1.1362×10^{-3}	1.0982	1.2698×10^{-3}	0.9245
NIPG	1/4	1.2675×10^{-2}		1.3412×10^{-2}	
	1/8	6.2917×10^{-3}	1.0104	1.0711×10^{-2}	0.3244
	1/16	3.1137×10^{-3}	1.0147	6.9746×10^{-3}	0.6189
	1/32	1.5458×10^{-3}	1.0102	3.9503×10^{-3}	0.8201
	1/64	7.7025×10^{-4}	1.0050	2.0882×10^{-3}	0.9196
IIPG	1/4	1.3262×10^{-2}		1.2781×10^{-2}	
	1/8	6.5214×10^{-3}	1.0241	1.0541×10^{-2}	0.2780
	1/16	3.1718×10^{-3}	1.0398	6.9157×10^{-3}	0.6081
	1/32	1.5569×10^{-3}	1.0266	3.9357×10^{-3}	0.8132
	1/64	7.7221×10^{-4}	1.0116	2.0849×10^{-3}	0.9165

Table 4.2: Numerical results with $\mathbb{P}_1$–$\mathbb{P}_0$ finite element for Example 4.1.

Method	h	$\\|\mathbf{u}(T) - \mathbf{U}^M\\|$	Rate
SIPG	1/4	2.1525×10^{-3}	
	1/8	4.7248×10^{-4}	2.1876
	1/16	1.0314×10^{-4}	2.1955
	1/32	2.3388×10^{-5}	2.1408
	1/64	5.5033×10^{-6}	2.0873

Table 4.3: Numerical results with $\mathbb{P}_2$–$\mathbb{P}_1$ finite element for Example 4.1.

Method	h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|p(T) - P^M\|$	Rate
SIPG	1/2	1.0191×10^{-2}		3.7928×10^{-3}	
	1/4	3.2147×10^{-3}	1.6646	1.5561×10^{-3}	1.2852
	1/8	7.4950×10^{-4}	2.1006	4.7252×10^{-4}	1.7195
	1/16	1.7473×10^{-4}	2.1007	1.3290×10^{-4}	1.8300
	1/32	4.2214×10^{-5}	2.0494	3.5666×10^{-5}	1.8977
NIPG	1/2	9.1793×10^{-3}		2.9873×10^{-3}	
	1/4	3.1398×10^{-3}	1.5476	2.2555×10^{-3}	0.4054
	1/8	8.2579×10^{-4}	1.9268	5.1996×10^{-4}	2.1169
	1/16	2.0821×10^{-4}	1.9876	1.1558×10^{-4}	2.1695
	1/32	5.2325×10^{-5}	1.9925	2.7201×10^{-5}	2.0871
IIPG	1/2	8.8447×10^{-3}		2.9281×10^{-3}	
	1/4	2.9530×10^{-3}	1.5825	1.9636×10^{-3}	0.5764
	1/8	7.4931×10^{-4}	1.9785	4.8520×10^{-4}	2.0168
	1/16	1.8588×10^{-4}	2.0111	1.1833×10^{-4}	2.0357
	1/32	4.6497×10^{-5}	1.9991	2.9740×10^{-5}	1.9923

Table 4.4: Numerical results with $\mathbb{P}_2$–$\mathbb{P}_1$ finite element for Example 4.1.

Method	h	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate
SIPG	1/2	9.1376×10^{-4}	
	1/4	1.4803×10^{-4}	2.6258
	1/8	2.0159×10^{-5}	2.8763
	1/16	2.6797×10^{-6}	2.9113
	1/32	3.3465×10^{-7}	3.0013

Example 4.2. *In this test case, the initial and boundary conditions and* $\mathbf{f}$ *are picked such that the analytical solutions* $(\boldsymbol{u}, p) = \big((u_1(x,y,t),\, u_2(x,y,t)),\, p(x,y,t)\big)$ *are*

$$\begin{aligned} u_1(x,y,t) &= \cos(2\pi x)\sin(2\pi y)e^{-8\pi^2\mu t}, \\ u_2(x,y,t) &= -\sin(2\pi x)\cos(2\pi y)e^{-8\pi^2\mu t}, \\ p(x,y,t) &= -\frac{1}{4}(\cos(4\pi x) + \cos(4\pi y))e^{-16\pi^2\mu t}. \end{aligned}$$

We choose the penalty parameter $\sigma_e = 20$ and 40 for $r = 1$ and 2, respectively. Other parameters are taken as $\gamma = 0.001$, $\delta = 0.01$. The computational errors and convergence orders are shown in Tables 4.5-4.8 with $\mu = 1/100$. Tables 4.5 and 4.6 are for the case $r = 1$, and Tables 4.7 and 4.8 are for $r = 2$. Optimal rates of convergence are obtained for this test problem.

Table 4.5: Numerical results with $\mathbb{P}_1$–$\mathbb{P}_0$ element for Example 4.2 .

Method	h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|p(T) - P^M\|$	Rate
SIPG	1/4	1.5340×10^0		4.1795×10^{-2}	
	1/8	7.6118×10^{-1}	1.0109	2.5388×10^{-2}	0.7191
	1/16	3.5366×10^{-1}	1.1058	1.0213×10^{-2}	1.3137
	1/32	1.7315×10^{-1}	1.0302	4.8638×10^{-3}	1.0702
	1/64	8.6291×10^{-2}	1.0047	2.4399×10^{-3}	0.9952
NIPG	1/4	1.5220×10^0		4.1153×10^{-2}	
	1/8	7.3415×10^{-1}	1.0518	2.2886×10^{-2}	0.8465
	1/16	3.4782×10^{-1}	1.0777	9.8146×10^{-3}	1.2214
	1/32	1.7227×10^{-1}	1.0136	4.8303×10^{-3}	1.0228
	1/64	8.6199×10^{-2}	0.9989	2.4366×10^{-3}	0.9872
IIPG	1/4	1.5203×10^0		1.2781×10^{-2}	
	1/8	7.4449×10^{-1}	1.0300	1.0541×10^{-2}	0.7911
	1/16	3.5002×10^{-1}	1.0888	6.9157×10^{-3}	1.2616
	1/32	1.7256×10^{-1}	1.0203	3.9357×10^{-3}	1.0444
	1/64	8.6216×10^{-2}	1.0010	2.0849×10^{-3}	0.9915

Table 4.6: Numerical results with $\mathbb{P}_1$–$\mathbb{P}_0$ element for Example 4.2 .

Method	h	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate
SIPG	1/4	1.4240×10^{-1}	
	1/8	4.8458×10^{-2}	1.5551
	1/16	1.1158×10^{-2}	2.1185
	1/32	2.5256×10^{-3}	2.1434
	1/64	6.0406×10^{-4}	2.0638

Table 4.7: Numerical results with $\mathbb{P}_2$–$\mathbb{P}_1$ element for Example 4.2 .

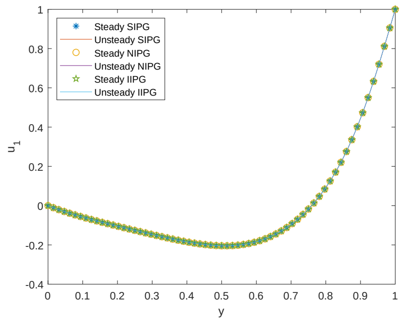
Method	h	$\|\mathbf{u}(T) - \mathbf{U}^M\|_\varepsilon$	Rate	$\|p(T) - P^M\|$	Rate
SIPG	1/2	1.6672×10^0		1.3814×10^{-1}	
	1/4	8.0392×10^{-1}	1.0523	2.9275×10^{-2}	2.2384
	1/8	1.6804×10^{-1}	2.2582	1.3203×10^{-2}	1.1487
	1/16	3.7013×10^{-2}	2.1827	3.5454×10^{-3}	1.8969
	1/32	8.4995×10^{-3}	2.1225	9.0318×10^{-4}	1.9728
NIPG	1/2	1.6849×10^0		1.3685×10^{-1}	
	1/4	7.8166×10^{-1}	1.1081	2.9688×10^{-2}	2.2046
	1/8	1.6351×10^{-1}	2.2571	1.3046×10^{-2}	1.1862
	1/16	3.6318×10^{-2}	2.1706	3.4602×10^{-3}	1.9147
	1/32	8.4214×10^{-3}	2.1085	8.7769×10^{-4}	1.9790
IIPG	1/2	1.6762×10^0		1.3746×10^{-1}	
	1/4	7.9212×10^{-1}	1.0814	2.9480×10^{-2}	2.2212
	1/8	1.6554×10^{-1}	2.2584	1.3120×10^{-2}	1.1679
	1/16	3.6423×10^{-2}	2.1843	3.5009×10^{-3}	1.9059
	1/32	8.5372×10^{-3}	2.0930	8.8994×10^{-4}	1.9759

Table 4.8: Numerical results with $\mathbb{P}_2$–$\mathbb{P}_1$ element for Example 4.2 .

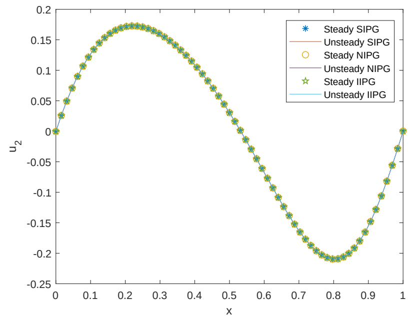
Method	h	$\|\mathbf{u}(T) - \mathbf{U}^M\|$	Rate
SIPG	1/2	1.4479×10^{-1}	
	1/4	5.3989×10^{-2}	1.4232
	1/8	6.0808×10^{-3}	3.1503
	1/16	5.5709×10^{-4}	3.4482
	1/32	6.7527×10^{-5}	3.0443

Example 4.3. *In this problem, we consider the lid-driven cavity flow on the computational domain* $[0,1]^2$ *with* $\mathbf{f} = (0,0)$*. The velocity on the upper boundary is* $\boldsymbol{u} = (1,0)$*. No slip boundary conditions are considered on the other portions of the cavity boundaries.*

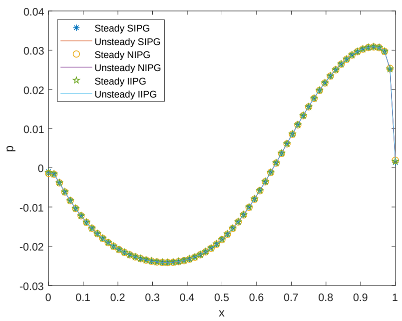
For computation, we first choose $\mathbb{P}_1 - \mathbb{P}_0$ elements with $T = 75$, $h = 1/64$ and $\Delta t = 0.01$. Here, we have considered different $\mu = \{1/100,\ 1/400,\ 1/1000\}$, $\sigma_e = 50$, and for spatial discretization SIPG, NIPG and IIPG schemes are implemented. In Figures 4.1, 4.2 and 4.3, we have shown a comparison along the lines $(0.5, y)$ and $(x, 0.5)$ between unsteady backward Euler and steady state DG velocities and pressure for $\mu = 1/100,\ 1/400,\ 1/1000$, respectively. The parameters for this experiment are $\gamma = 0.1\mu$ and $\delta = 0.1$. We repeat this experiment by employing $\mathbb{P}_2 - \mathbb{P}_1$ elements for $\mu = 1/1000$ in Figure 4.4. It can be concluded from the Figures 4.1-4.4 that for large time, the Oldroyd model DG solution coincide with the steady state DG solution which support the theoretical findings.



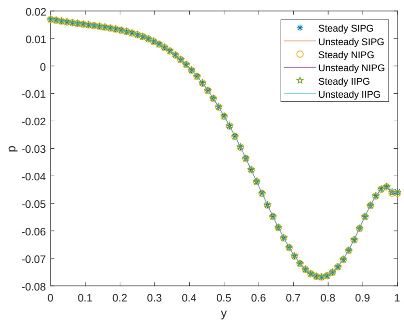
(a) First velocity component along $x = 0.5$



(b) Second velocity component along $y = 0.5$

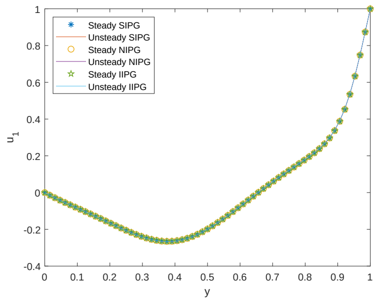


(c) Pressure along $y = 0.5$

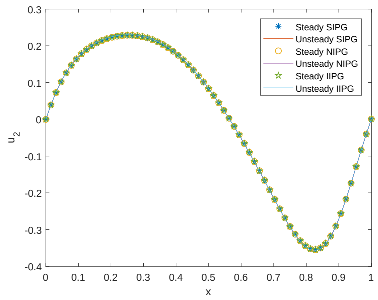


(d) Pressure along $x = 0.5$

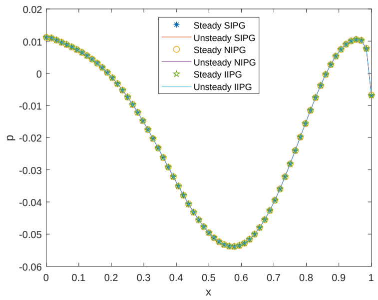
Figure 4.1: Velocity components and pressure for lid driven cavity flow with $\mu = 1/100$.



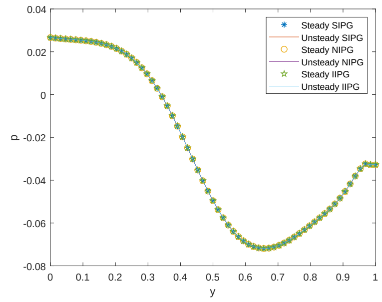
(a) First velocity component along $x = 0.5$



(b) Second velocity component along $y = 0.5$

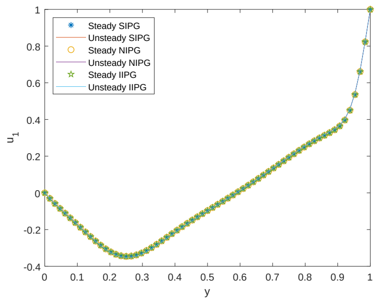


(c) Pressure along $y = 0.5$

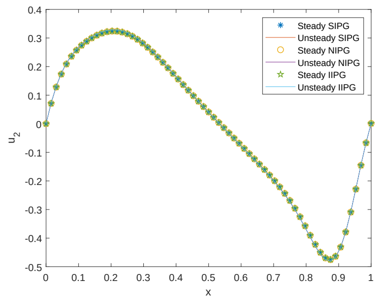


(d) Pressure along $x = 0.5$

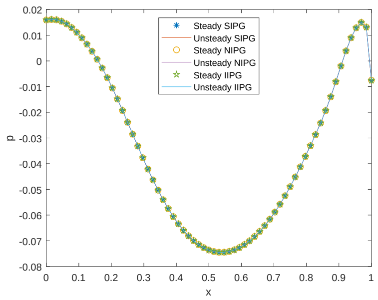
Figure 4.2: Velocity components and pressure for lid driven cavity flow with $\mu = 1/400$.



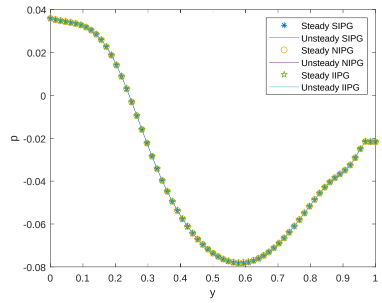
(a) First velocity component along $x = 0.5$



(b) Second velocity component along $y = 0.5$

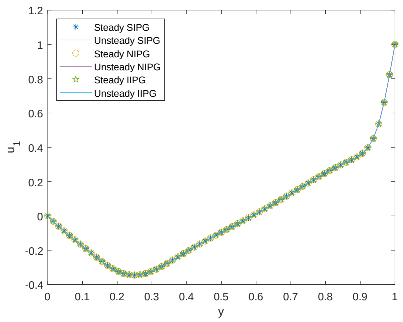


(c) Pressure along $y = 0.5$

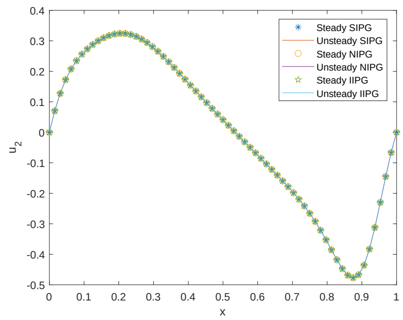


(d) Pressure along $x = 0.5$

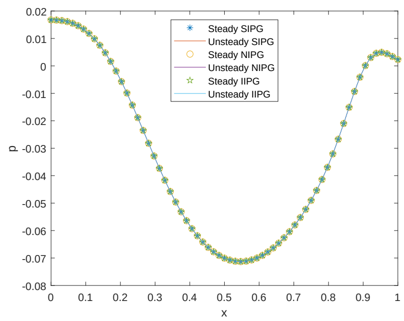
Figure 4.3: Velocity components and pressure for lid driven cavity flow with $\mu = 1/1000$.



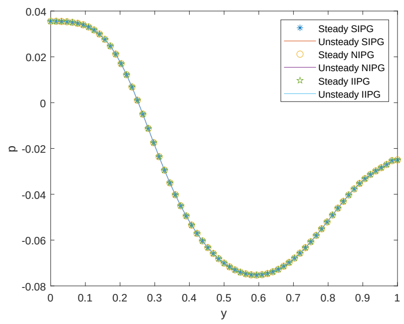
(a) First velocity component along $x = 0.5$



(b) Second velocity component along $y = 0.5$



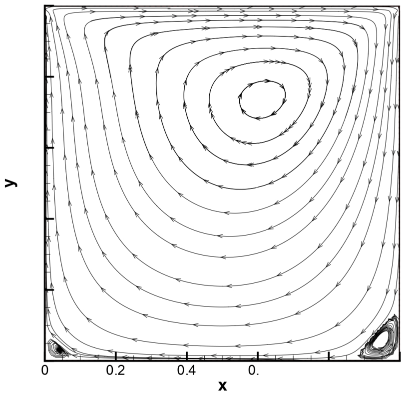
(c) Pressure along $y = 0.5$



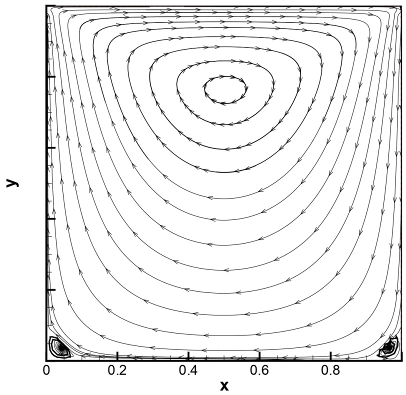
(d) Pressure along $x = 0.5$

Figure 4.4: Velocity components and pressure using $\mathbb{P}_2 - \mathbb{P}_1$ element for lid driven cavity flow with $\mu = 1/1000$.

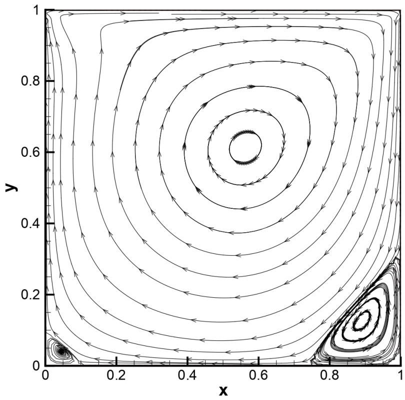
Note that, when $\gamma = 0$, the viscoelastic model under consideration (4.1) transforms into the well-known NSEs. In Figures 4.5 and 4.6, we depict for various μ the streamline plots of the NSEs and our model problem at final time $T = 10$ for SIPG discretization utilizing $\mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_2 - \mathbb{P}_1$ elements, respectively. In this case, the parameters are $\gamma = 0.01$ and $\delta = 0.01$. From these graphs, we observe that the swirls situated in the corners of the cavity from the proposed Oldroyd viscoelastic flow problem are smaller than those from the NSEs. This is due to the presence of integral term which plays a crucial role in stablizing the velocity field.



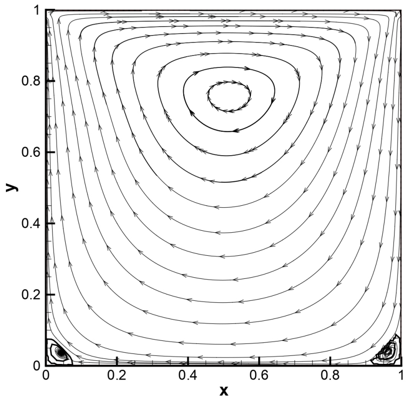
(a) NSEs with $\mu = 1/100$.



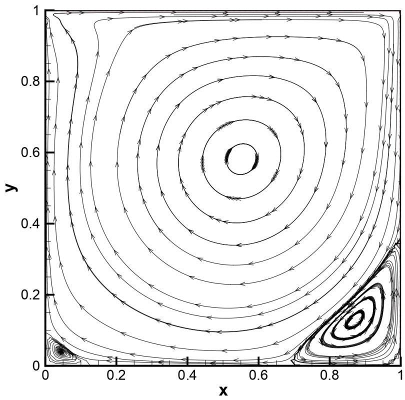
(b) Model problem with $\mu = 1/100$.



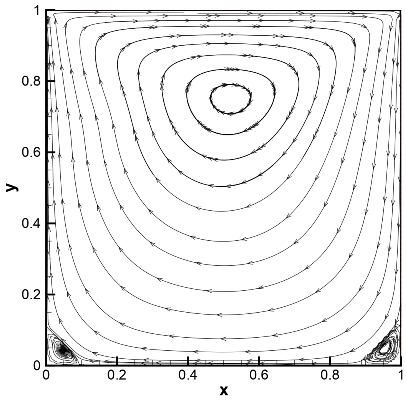
(c) NSEs with $\mu = 1/400$.



(d) Model problem with $\mu = 1/400$.

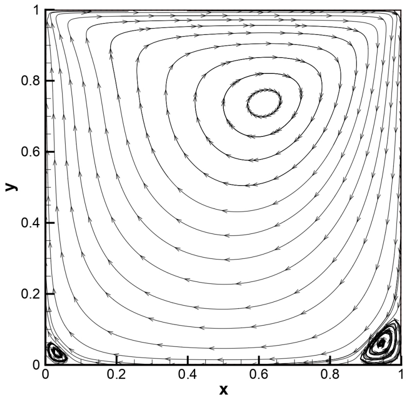


(e) NSEs with $\mu = 1/1000$.

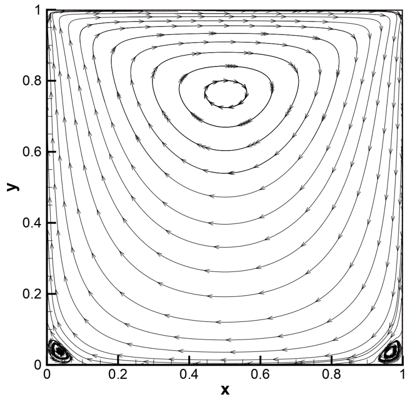


(f) Model problem with $\mu = 1/1000$.

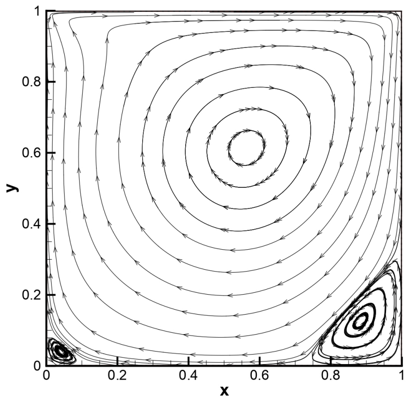
Figure 4.5: Streamlines for NSEs (first column) and viscoelatic model problem (4.1) - (4.3) (second column) using $\mathbb{P}_1 - \mathbb{P}_0$ elements with $\mu = 1/100, 1/400,\ 1/1000$ at final time T=10.



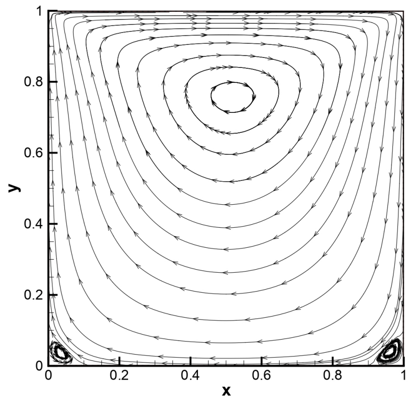
(a) NSEs with $\mu = 1/100$.



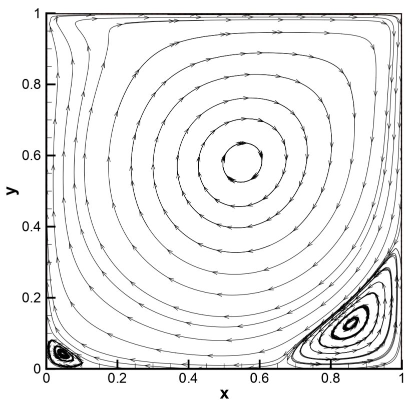
(b) Model problem with $\mu = 1/100$.



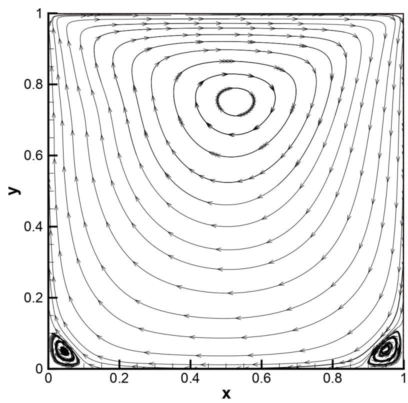
(c) NSEs with $\mu = 1/400$.



(d) Model problem with $\mu = 1/400$.



(e) NSEs with $\mu = 1/1000$.



(f) Model problem with $\mu = 1/1000$.

Figure 4.6: Streamlines for NSEs (first column) and viscoelatic model problem (4.1) - (4.3) (second column) using $\mathbb{P}_2 - \mathbb{P}_1$ elements with $\mu = 1/100, 1/400$, $1/1000$ at final time T=10.

4.7 Conclusion

In this chapter, we have applied a DG method to the Oldroyd model of order one. Regularity results of the semi-discrete DG solution, and existence, uniqueness of the discrete solutions, and consistency of the scheme have been shown. A fully discrete scheme with the backward Euler method for time discretization has been studied and *a priori* bounds of the fully discrete solution have been derived. We have established optimal error estimates of the velocity in energy norm for SIPG, NIPG and IIPG methods, and optimal error estimates for pressure in $L^2(L^2)$-norm only for SIPG method of the fully discrete approximations. Furthermore, optimal $L^\infty(\mathbf{L}^2)$ and $L^\infty(L^2)$-norms error estimates for the velocity and pressure, respectively, are derived only for the SIPG method. Our numerical results support our theoretical findings.