Chapter 5

Two-Grid DG Method for the Navier-Stokes Equations

In this chapter, we apply a two-grid scheme to the DG formulation of the transient NSEs. The two-grid algorithm consists of the following steps: **Step 1** involves solving the nonlinear system on a coarse mesh with mesh size H, and **Step 2** involves linearizing the nonlinear system by using the coarse grid solution on a fine mesh of mesh size h and solving the resulting system to produce an approximate solution with desired accuracy. We establish optimal error estimates of the two-grid DG approximations for the velocity and pressure in energy and L^2-norms, respectively, for an appropriate choice of coarse and fine mesh parameters. We further discretize the two-grid DG model in time, using the backward Euler method, and derive the fully discrete error estimates. Finally, numerical results are presented to confirm the efficiency of the proposed scheme. This work has been published in [142].

5.1 Introduction

The use of DG techniques can result in algebraic equations with extremely high degrees of freedom, which can be computationally expensive and present a significant challenge to solve them, especially when we solve a nonlinear system. This chapter presents a cost-effective two-grid technique in conjunction with the DG method to deal with this issue for the time-dependent incompressible NSEs, which is the first attempt in this direction to the best of authors knowledge. The two-grid techniques are currently recognised as an efficient discretization approach for solving nonlinear problems. They could solve the system relatively inexpensively while maintaining a certain degree of accuracy. Construction of two shape-regular triangulations of $\bar{\Omega}$ is the key mechanism in this algorithm; a coarse mesh $\mathcal{E}_H$ and a fine mesh $\mathcal{E}_h$ with different mesh sizes H and h ($h \ll H$). Based on triangulations $\mathcal{E}_H$ and $\mathcal{E}_h$, we define discontinuous finite element spaces $(\mathbf{X}_H, M_H)$ and $(\mathbf{X}_h, M_h)$, which will be referred to as coarse and fine space, respectively. We further define the weakly divergence free subspaces $\mathbf{V}_\lambda$ of $\mathbf{X}_\lambda$, as follows

$$\mathbf{V}_\lambda \;=\; \{\mathbf{v}_\lambda \in \mathbf{X}_\lambda :\; b\,(\mathbf{v}_\lambda,\, q_\lambda) = 0,\; \forall q_\lambda \in M_\lambda\},$$

where $\lambda = H, h$.

The main algorithm for two-grid method for any nonlinear problem is stated as follow:

- **Step 1:** Solve the nonlinear problem over the coarse mesh $\mathcal{E}_H$.
- **Step 2:** Solve a time-dependent/independent linearized system over the fine mesh $\mathcal{E}_h$.

As mentioned earlier, two-grid DG methods have never been applied for the transient NSEs. Therefore, we take a brief literature survey, for results available in CG-two-grid-NSEs as well in DG-two-grid in related problems. Note that efficiency of such a method is measured in terms of the scaling between the coarse mesh size and fine mesh size, that is, between H and h. We will look into these scaling in the related problems to take a cue for our own problem.

There is plenty of literature available for NSEs using two-grid methods in the context of CG setting. For example, in the context of steady state NSEs, we refer to [52, 67, 108, 109]. The work has been extended to the transient NSEs in [68] by Girault *et al.*. They have established optimal $L^2(\mathbf{H}^1)$ and $L^2(L^2)$-norm error estimates for velocity and pressure, respectively, for $h = \mathcal{O}(H^2)$ and for Mini-element. A fully discrete two-grid technique which is second order in space and time has been analyzed for unsteady NSEs, and optimal velocity and pressure errors in $L^\infty(\mathbf{H}^1)$ and $L^2(L^2)$-norms, respectively, are established by Abboud *et al.* [1] with a choice $h^2 = H^3 = \Delta t^2$. This work has been extended to less regular solution at $t = 0$ by Frutos *et al.* [54] and Goswami *et al.* [74] with $\mathbf{u}_0 \in \mathbf{H}^2(\Omega)$ and $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$, respectively. In [54] and [74], the authors have shown optimal $L^\infty(\mathbf{H}^1)$-norm velocity error and $L^\infty(L^2)$-norm pressure error with the choice $h = \mathcal{O}(H^2)$ and $h = \mathcal{O}(t^{-1/2}H^2)$, respectively.

Literature related to two-grid DG methods has already been discussed in Section 1.5.4. In the context of nonlinear elliptic and parabolic problems, optimal energy norm error estimates have been shown with the choice of $h^r = \mathcal{O}(H^{r+1})$, when r^{th} order polynomial approximation has been carried out, see [21, 22, 51, 176, 177, 180, 189].

Similar scaling can be observed in other attempted problems as well [179, 190]. In this chapter, we thrive to obtain similar results for our problem.

We revisit the DG variational formulation of NSEs from Chapter 2 before presenting the two-grid DG algorithm of this chapter: Find $(\mathbf{u}(t), p(t)) \in \mathbf{X} \times M,\ t > 0$, such that

$$(\mathbf{u}_t(t), \mathbf{v}) + \nu\, a(\mathbf{u}(t),\, \mathbf{v}) + c^{\mathbf{u}(t)}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) + b(\mathbf{v},\, p(t)) = (\mathbf{f}(t),\, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{X}, \tag{5.1}$$

$$b(\mathbf{u}(t),\, q) = 0 \quad \forall\, q \in M, \tag{5.2}$$

$$(\mathbf{u}(0),\, \mathbf{v}) = (\mathbf{u}_0,\, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{X}. \tag{5.3}$$

The algorithm employed in this chapter involves the following steps:

- **Step 1:** Solve the nonlinear problem over a coarse mesh $\mathcal{E}_H$ to provide an approximate solution, say $\mathbf{u}_H$.
- **Step 2:** Linearize the nonlinear system with one Newton iteration around the coarse grid solution $\mathbf{u}_H$ and solve it over a fine mesh $\mathcal{E}_h$ to obtain the solution, say $\mathbf{u}_h$.

We now introduce a DG two-grid semi-discrete algorithm applied to (5.1)-(5.3) which is described as follows:

Step 1 (Nonlinear system on a coarse grid): Find $(\mathbf{u}_H, p_H) \in \mathbf{X}_H \times M_H$ such that for all $(\boldsymbol{\phi}_H, q_H) \in \mathbf{X}_H \times M_h$ for $\mathbf{u}_H(0) = \mathbf{P}_H\mathbf{u}_0$ and $t > 0$

$$\left.\begin{aligned}(\mathbf{u}_{Ht}(t),\, \boldsymbol{\phi}_H) + \nu\, a\,(\mathbf{u}_H(t),\, \boldsymbol{\phi}_H) + c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t), \mathbf{u}_H(t),\, \boldsymbol{\phi}_H)&\\ +b\,(\boldsymbol{\phi}_H,\, p_H(t)) = (\mathbf{f}(t),\, \boldsymbol{\phi}_H),&\\ b(\mathbf{u}_H(t),\, q_H) = 0.&\end{aligned}\right\} \tag{5.4}$$

Step 2 (Update on a finer mesh with one Newton iteration): Find $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times M_h$ such that for all $(\boldsymbol{\phi}_h, q_h) \in \mathbf{X}_h \times M_h$ for $\mathbf{u}_h(0) = \mathbf{P}_h\mathbf{u}_0$ and $t > 0$

$$\left.\begin{aligned}(\mathbf{u}_{ht}(t),\, \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{u}_h(t),\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t),\, \mathbf{u}_H(t),\, \boldsymbol{\phi}_h)&\\ +c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t),\, \mathbf{u}_h(t),\, \boldsymbol{\phi}_h) + b\,(\boldsymbol{\phi}_h,\, p_h(t)) = (\mathbf{f}(t),\, \boldsymbol{\phi}_h)&\\ + c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t),\, \mathbf{u}_H(t),\, \boldsymbol{\phi}_h),&\\ b(\mathbf{u}_h(t),\, q_h) = 0.&\end{aligned}\right\} \tag{5.5}$$

An equivalent DG two-grid semi-discrete algorithm corresponding to the scheme (5.4)–(5.5) on the space $\mathbf{V}_\lambda$ is the following:

Step 1 (Nonlinear system on a coarse grid): Find $\mathbf{u}_H \in \mathbf{V}_H$ such that for all $\boldsymbol{\phi}_H \in \mathbf{V}_H$ for $\mathbf{u}_H(0) = \mathbf{P}_H\mathbf{u}_0$ and $t > 0$

$$(\mathbf{u}_{Ht}(t),\, \boldsymbol{\phi}_H) + \nu\, a\,(\mathbf{u}_H(t),\, \boldsymbol{\phi}_H) + c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t),\, \mathbf{u}_H(t),\, \boldsymbol{\phi}_H) = (\mathbf{f}(t),\, \boldsymbol{\phi}_H). \qquad (5.6)$$

Step 2 (Update on a finer mesh with one Newton iteration): Find $\mathbf{u}_h \in \mathbf{V}_h$ such that for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ for $\mathbf{u}_h(0) = \mathbf{P}_h\mathbf{u}_0$ and $t > 0$

$$\begin{aligned}(\mathbf{u}_{ht}(t), \boldsymbol{\phi}_h) &+ \nu\, a\,(\mathbf{u}_h(t),\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_h(t)}(\mathbf{u}_h(t),\, \mathbf{u}_H(t),\, \boldsymbol{\phi}_h)\\ &+ c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t),\, \mathbf{u}_h(t),\, \boldsymbol{\phi}_h) = (\mathbf{f}(t),\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H(t)}(\mathbf{u}_H(t),\, \mathbf{u}_H(t),\, \boldsymbol{\phi}_h). \qquad (5.7)\end{aligned}$$

Below, we present a summary of the main findings from this chapter:

- Optimal semi-discrete error estimates for the two-grid DG velocity approximation in energy norm when $h = \mathcal{O}(H^{\frac{r+1}{r}})$ and pressure approximation in $L^\infty(L^2)$-norm when $h = \mathcal{O}(H^{\frac{r+1}{r}})$ are derived.

- Under the smallness assumption on the given data, uniform in time velocity and pressure error estimates in energy and $L^\infty(L^2)$-norms, respectively, are established.

- Error estimates for the fully-discrete backward Euler velocity and pressure approximations are derived. Numerical experiments are carried out to show the performance of the scheme.

This chapter is organized as follows: *A priori* bounds of semi-discrete solutions and some relevant estimates are discussed Section 5.2. In Section 5.3, the semi-discrete velocity and pressure error estimates are shown to be optimal. Fully discrete scheme is presented in Section 5.4. We have employed the backward Euler method, and error estimates for the velocity and pressure are derived. We carry out numerical experiments in Section 5.5, and the results are analyzed. Finally, Section 5.6 concludes this chapter by summarizing the results briefly.
Throughout this chapter, we will use C, $K(>0)$ as generic constants that depend on the given data, ν, α, K_1, K_2, C_2 but do not depend on h and Δt. Note that, K and C may grow algebraically with ν^{-1}. Further, the notations $K(t)$ and K_T will be used when they grow exponentially in time.

5.2 Regularity Bounds and Some Useful Estimates

In this section, we present *a priori* and regularity bounds for the **Step 1** and **Step 2** semi-discrete velocity approximations, and an estimate related to upwinding term $l(\cdot,\cdot,\cdot)$ which will be useful for **Step 2** error analysis.
In Lemmas 5.1 and 5.2, we recall from Lemmas 2.8, 2.13, 2.16, 2.14 and Theorems 2.1, 2.2 of Chapter 2, the **Step 1** *a priori* and regularity estimates, and error estimates, respectively, which play crucial role in the derivation of **Step 2** error estimates.

Lemma 5.1. *Suppose the assumption (***A1***) holds true and* $0<\alpha<\frac{\nu K_1}{2C_2}$*. Then, for the semi-discrete DG velocity* $\boldsymbol{u}_H(t)$*,* $t>0$ *for* **step 1***, the following holds true:*

$$\|\boldsymbol{u}_H(t)\| + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{u}_H(s)\|_\varepsilon^2\,ds \le C, \tag{5.8}$$

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{u}_{Hs}(s)\|_\varepsilon^2\,ds + e^{-2\alpha t}\int_0^t e^{2\alpha s}\,\|\boldsymbol{u}_{Hss}(s)\|_{-1,H}^2\,ds \le C, \tag{5.9}$$

$$\limsup_{t\to\infty}\|\boldsymbol{u}_H(t)\|_\varepsilon \le \frac{C_2}{K_1\nu}\|\mathbf{f}\|_{L^\infty(L^2(\Omega))}, \tag{5.10}$$

where

$$\|\boldsymbol{u}_{Htt}\|_{-1,H} = \sup\left\{\frac{\langle \boldsymbol{u}_{Htt},\boldsymbol{\phi}_H\rangle}{\|\boldsymbol{\phi}_H\|_\varepsilon},\ \boldsymbol{\phi}_H\in\boldsymbol{X}_H,\boldsymbol{\phi}_H\neq 0\right\}.$$

Lemma 5.2. *Suppose the assumption (***A1***) holds true and let* $0<\alpha<\frac{\nu K_1}{2C_2}$*. In addition, let the semi-discrete initial velocity* $\boldsymbol{u}_H(0)\in\boldsymbol{V}_H$ *with* $\boldsymbol{u}_H(0)=\boldsymbol{P}_H\boldsymbol{u}_0$*. Then, there exists a constant* $K>0$*, such that for* $0<t\le T$*,*

$$\|(\boldsymbol{u}-\boldsymbol{u}_H)(t)\| + H\|(\boldsymbol{u}-\boldsymbol{u}_H)(t)\|_\varepsilon \le K(t)H^{r+1},$$

$$\|(p-p_H)(t)\| \le K(t)H^r,$$

$$e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\|(\boldsymbol{u}-\boldsymbol{u}_H)_\tau(\tau)\|_\varepsilon^2\,d\tau \le K(t)H^{2r}.$$

Next in Lemma 5.3, we derive *a priori* estimates of **Step 2** solution $\mathbf{u}_h$.

Lemma 5.3. *Let* $0<\alpha<\frac{\nu K_1}{2C_2}$*. Then, for the semi-discrete DG velocity* $\boldsymbol{u}_h(t)$*,* $t>0$ *for* **step 2***, the following holds true*

$$\|\boldsymbol{u}_h(t)\| + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{u}_h(s)\|_\varepsilon^2\,ds \le K(t).$$

Proof. Choose $\boldsymbol{\phi}_h=\mathbf{u}_h$ in (5.7), and apply Lemma 1.6, positivity property (1.19), the Cauchy-Schwarz inequality and Young's inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_h\|^2 + \nu K_1\|\mathbf{u}_h\|_\varepsilon^2 \le \frac{\nu K_1}{4}\|\mathbf{u}_h\|_\varepsilon^2 + C\|\mathbf{f}\|^2 + |c^{\mathbf{u}_h}(\mathbf{u}_h,\,\mathbf{u}_H,\,\mathbf{u}_h)|$$

$$+\,|c^{\mathbf{u}_H}(\mathbf{u}_H,\,\mathbf{u}_H,\,\mathbf{u}_h)|. \tag{5.11}$$

The estimates (2.55), (2.57) and Young's inequality lead to the following bound:

$$\begin{aligned}|c^{\mathbf{u}_h}(\mathbf{u}_h,\,\mathbf{u}_H,\,\mathbf{u}_h)| + |c^{\mathbf{u}_H}(\mathbf{u}_H,\,\mathbf{u}_H,\,\mathbf{u}_h)| &\leq C\big(\|\mathbf{u}_h\|^{1/2}\|\mathbf{u}_h\|_\varepsilon^{3/2}\|\mathbf{u}_H\|_\varepsilon + \|\mathbf{u}_H\|_\varepsilon^2\|\mathbf{u}_h\|_\varepsilon\big)\\ &\leq \frac{\nu K_1}{4}\|\mathbf{u}_h\|_\varepsilon^2 + C\big(\|\mathbf{u}_h\|^2\|\mathbf{u}_H\|_\varepsilon^4 + \|\mathbf{u}_H\|_\varepsilon^4\big).\end{aligned}$$

Multiplying (5.11) by $e^{2\alpha t}$, integrating from 0 to t and applying the the above inequality, we obtain

$$\begin{aligned}e^{2\alpha t}\|\mathbf{u}_h(t)\|^2 + (\nu K_1 - 2\alpha C_2)\int_0^t e^{2\alpha s}\|\mathbf{u}_h(s)\|_\varepsilon^2\,ds &\leq \|\mathbf{u}_h(0)\|^2 + \int_0^t e^{2\alpha s}\|\mathbf{f}(s)\|^2\,ds\\ &\quad + C\int_0^t e^{2\alpha s}\big(\|\mathbf{u}_h(s)\|^2\|\mathbf{u}_H(s)\|_\varepsilon^4 + \|\mathbf{u}_H(s)\|_\varepsilon^4\big)\,ds. \end{aligned}\tag{5.12}$$

Note that, using triangle inequality and Lemma 5.2, we find

$$\|\mathbf{u}_H\|_\varepsilon \leq \|\mathbf{u}-\mathbf{u}_H\|_\varepsilon + \|\mathbf{u}\|_1 \leq C. \tag{5.13}$$

Now, multiplying (5.12) by $e^{-2\alpha t}$ and using (5.13), the fact

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\,ds = \frac{1}{2\alpha}(1-e^{-2\alpha t}),$$

Gronwall's inequality and choosing $\alpha < \frac{\nu K_1}{2C_2}$, we finally arrive at the desired estimate of this lemma. □

Now from the coercivity result in Lemma 1.6, the positivity (1.19) and the inf-sup condition in Lemma 1.8, the existence and uniqueness of the discrete solutions of (5.5) (or (5.7)) in **Step 2** will follow easily, see [98] for details.
The next lemma is an auxiliary result for the upwinding term $l(\cdot,\cdot,\cdot)$ which will be useful for deriving the error estimates.

Lemma 5.4. *For all $\boldsymbol{u}, \mathbf{v} \in \boldsymbol{X}$ and $\boldsymbol{w}_h, \boldsymbol{\phi}_h, \boldsymbol{\psi}_h \in \boldsymbol{X}_h$, there exists a positive constant C independent of h, such that, the following estimate holds true:*

$$|l^{\boldsymbol{w}_h}(\boldsymbol{u},\mathbf{v},\boldsymbol{\phi}_h) - l^{\boldsymbol{\psi}_h}(\boldsymbol{u},\mathbf{v},\boldsymbol{\phi}_h)| \leq Ch^{r/(r+1)}\|\boldsymbol{u}\|_{L^\infty(\Omega)}\|\mathbf{v}\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon.$$

Proof. The derivation of this lemma closely follows the proof of [70, Proposition 4.10]. Let $e \in \Gamma_h \setminus \partial\Omega$ be an edge adjacent to E_1 and E_2 with $\mathbf{n}_e = \mathbf{n}_{E_1}$. Then, for any $\boldsymbol{\theta}_h \in \mathbf{X}_h$, the contribution of e to the term $l^{\boldsymbol{\theta}_h}(\mathbf{u},\mathbf{v},\boldsymbol{\phi}_h)$ reduces to

$$\int_e (\{\mathbf{u}\}\cdot\mathbf{n}_e)[\mathbf{v}]\cdot\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h},$$

where $\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}|_e = \boldsymbol{\phi}_h|_{E_1}$ if $\{\boldsymbol{\theta}_h\} \cdot \mathbf{n}_e < 0$, $\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}|_e = \boldsymbol{\phi}_h|_{E_2}$ if $\{\boldsymbol{\theta}_h\} \cdot \mathbf{n}_e > 0$, and $\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}|_e = \mathbf{0}$ if $\{\boldsymbol{\theta}_h\} \cdot \mathbf{n}_e = 0$. In a similar way, if $e \in \partial\Omega \cap E$, then we have $\mathbf{n}_e = \mathbf{n}_{\partial\Omega}$. Then, the contribution corresponding to e is

$$\int_e (\mathbf{u} \cdot \mathbf{n}_e)\mathbf{v} \cdot \boldsymbol{\phi}_h^{\boldsymbol{\theta}_h},$$

where $\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}|_e = \boldsymbol{\phi}_h|_E$ if $\boldsymbol{\theta}_h \cdot \mathbf{n}_e < 0$ and $\boldsymbol{\phi}_h^{\boldsymbol{\theta}_h}|_e = \mathbf{0}$ otherwise. Set $B = l^{\mathbf{w}_h}(\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}_h) - l^{\boldsymbol{\psi}_h}(\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}_h)$. Then, following the above notation, B can be rewritten as

$$B = \sum_{e \in \Gamma_h} \int_e (\{\mathbf{u}\} \cdot \mathbf{n}_e)[\mathbf{v}](\boldsymbol{\phi}_h^{\mathbf{w}_h} - \boldsymbol{\phi}_h^{\boldsymbol{\psi}_h}).$$

The domain of integration can be partitioned as follows:

$$\Gamma_h = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4,$$

where

$$\begin{aligned}
\mathcal{G}_1 &= \{e : \{\mathbf{w}_h\} \cdot \mathbf{n}_e \neq 0 \text{ and } \{\boldsymbol{\psi}_h\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\}, \\
\mathcal{G}_2 &= \{e : \{\mathbf{w}_h\} \cdot \mathbf{n}_e = 0 \text{ and } \{\boldsymbol{\psi}_h\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\}, \\
\mathcal{G}_3 &= \{e : \{\boldsymbol{\psi}_h\} \cdot \mathbf{n}_e = 0 \text{ and } \{\mathbf{w}_h\} \cdot \mathbf{n}_e \neq 0 \text{ a.e on } e\}, \\
\mathcal{G}_4 &= \Gamma_h \setminus (\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3).
\end{aligned}$$

Now, applying Hölder's inequality and Lemma 1.10, we can deduce

$$\begin{aligned}
|B| =& \left| \sum_{e \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4} \int_e (\{\mathbf{u}\} \cdot \mathbf{n}_e)[\mathbf{v}](\boldsymbol{\phi}_h^{\mathbf{w}_h} - \boldsymbol{\phi}_h^{\boldsymbol{\psi}_h}) \right| \\
\leq& C\|\mathbf{u}\|_{L^\infty(\Omega)} \sum_{e \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4} |e|^{r/2(r+1)} \|[\mathbf{v}]\|_{L^2(e)} \|\boldsymbol{\phi}_h\|_{L^{2(r+1)}(e)} \\
\leq& C\|\mathbf{u}\|_{L^\infty(\Omega)} \sum_{e \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4} \sum_{i=1}^{2} \sigma_e^{1/2} |e|^{r/(r+1)} |e|^{-1/2} \|[\mathbf{v}]\|_{L^2(e)} \|\boldsymbol{\phi}_h\|_{L^{2(r+1)}(E_i)}
\end{aligned}$$

Furthermore, using Hölder's inequality, Jensen's inequality and estimate (1.14), one can obtain

$$\begin{aligned}
|B| \leq& Ch^{r/(r+1)} \|\mathbf{u}\|_{L^\infty(\Omega)} \left(\sum_{e \in \Gamma_h} \frac{\sigma_e}{|e|} \|[\mathbf{v}]\|_{L^2(e)}^2 \right)^{1/2} \|\boldsymbol{\phi}_h\|_{L^{2(r+1)}(\Omega)} \\
\leq& Ch^{r/(r+1)} \|\mathbf{u}\|_{L^\infty(\Omega)} \|\mathbf{v}\|_\varepsilon \|\boldsymbol{\phi}_h\|_\varepsilon.
\end{aligned}$$

This completes the proof of this lemma. □

5.3 Semi-discrete Error Estimates in Step 2

This section deals with semi-discrete velocity and pressure error estimates in **Step 2**.

5.3.1 Velocity Error Estimates

In this subsection, we derive the bounds of semi-discrete velocity error in **Step 2** for two-grid algorithm. Define $\mathbf{e}_H = \mathbf{u} - \mathbf{u}_H$ and $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$. Using the equations (5.1) and (5.7), we obtain

$$(\mathbf{e}_t, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{e}, \boldsymbol{\phi}_h) + c^{\mathbf{u}_h}(\mathbf{e}, \mathbf{u}_H, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{e}, \boldsymbol{\phi}_h) + l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h)$$
$$-l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, p) = -c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h. \tag{5.14}$$

Lemma 5.5. *Suppose the assumption (**A1**) holds true and let $0 < \alpha < \frac{\nu K_1}{2C_2}$. Then, there exists a constant $K > 0$, such that for $0 < t \le T$,*

$$\|\boldsymbol{e}(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\boldsymbol{e}(s)\|_\varepsilon^2\, ds \le K(t)(h^{2r} + h^{2r/(r+1)} H^{2r} + H^{2r+2}),$$

where $K(t)$ grows exponentially in time.

Proof. Set $\boldsymbol{\phi}_h = \mathbf{P}_h \mathbf{e} = \mathbf{e} + (\mathbf{P}_h \mathbf{u} - \mathbf{u})$ in (5.14). An application of Lemma 1.6 and definition of L^2-projection, one can find

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{P}_h\mathbf{e}\|^2 + \nu K_1 \|\mathbf{P}_h \mathbf{e}\|_\varepsilon^2 + c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{P}_h\mathbf{e}, \mathbf{P}_h\mathbf{e}) \le \nu\, a(\mathbf{P}_h\mathbf{u} - \mathbf{u}, \mathbf{P}_h\mathbf{e})$$
$$-c^{\mathbf{u}_h}(\mathbf{P}_h\mathbf{e}, \mathbf{u}_H, \mathbf{P}_h\mathbf{e}) - c^{\mathbf{u}_h}(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{u}_H, \mathbf{P}_h\mathbf{e}) - c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{P}_h\mathbf{e})$$
$$-c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}) + l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}) - l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}) - b(\mathbf{P}_h\mathbf{e}, p). \tag{5.15}$$

The third term on the left hand side of (5.15) is non-negative due to (1.19). A use of Lemma 2.3 and Young's inequality yields a bound for the first term on the right hand side of (5.15) as

$$\nu |a(\mathbf{P}_h\mathbf{u} - \mathbf{u}, \mathbf{P}_h\mathbf{e})| \le C\nu h^r |\mathbf{u}|_{r+1} \|\mathbf{P}_h\mathbf{e}\|_\varepsilon \le \frac{\nu K_1}{64}\|\mathbf{P}_h\mathbf{e}\|_\varepsilon^2 + Ch^{2r}|\mathbf{u}|_{r+1}^2. \tag{5.16}$$

Using estimate (2.57) and Young's inequality, we arrive at

$$|c^{\mathbf{u}_h}(\mathbf{P}_h\mathbf{e}, \mathbf{u}_H, \mathbf{P}_h\mathbf{e})| \le C\|\mathbf{P}_h\mathbf{e}\|^{1/2}\|\mathbf{u}_H\|_\varepsilon \|\mathbf{P}_h\mathbf{e}\|_\varepsilon^{3/2}$$
$$\le C\|\mathbf{P}_h\mathbf{e}\|^2\|\mathbf{u}_H\|_\varepsilon^4 + \frac{\nu K_1}{64}\|\mathbf{P}_h\mathbf{e}\|_\varepsilon^2. \tag{5.17}$$

Apply estimate (2.56), Lemma 2.2 and Young's inequality, we obtain

$$|c^{\mathbf{u}_h}(\mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{u}_H, \mathbf{P}_h\mathbf{e}) + c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{u} - \mathbf{P}_h\mathbf{u}, \mathbf{P}_h\mathbf{e}) + c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_H, \mathbf{P}_h\mathbf{e})|$$

$$\leq C\|\mathbf{u}-\mathbf{P}_h\mathbf{u}\|_\varepsilon\|\mathbf{u}_H\|_\varepsilon\|\mathbf{P}_h\mathbf{e}\|_\varepsilon + C\|\mathbf{e}_H\|_\varepsilon^2\|\mathbf{P}_h\mathbf{e}\|_\varepsilon$$
$$\leq Ch^{2r}|\mathbf{u}|_{r+1}^2\|\mathbf{u}_H\|_\varepsilon^2 + C\|\mathbf{e}_H\|_\varepsilon^4 + \frac{\nu K_1}{64}\|\mathbf{P}_h\mathbf{e}\|_\varepsilon^2. \tag{5.18}$$

Following Lemma 5.4, and using Lemma 1.3 and Young's inequality, we bound the sixth and seventh term on the right hand side of (5.15) as

$$|l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}) - l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e})| \leq Ch^{r/(r+1)}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\mathbf{e}_H\|_\varepsilon\|\mathbf{P}_h\mathbf{e}\|_\varepsilon$$
$$\leq Ch^{2r/(r+1)}\|\mathbf{u}\|_2^2\|\mathbf{e}_H\|_\varepsilon^2 + \frac{\nu K_1}{64}\|\mathbf{P}_h\mathbf{e}\|_\varepsilon^2. \tag{5.19}$$

Due to the space $\mathbf{V}_h$, the pressure term becomes

$$b(\mathbf{P}_h\mathbf{e},p) = b(\mathbf{P}_h\mathbf{e},p-r_hp).$$

We use Young's inequality and Lemma 2.4 to arrive at

$$|b(\mathbf{P}_h\mathbf{e},p-r_hp)| \leq \frac{K_1\nu}{10}\|\mathbf{P}_h\mathbf{e}\|_\varepsilon^2 + Ch^{2r}|p|_r^2. \tag{5.20}$$

Collecting the bounds (5.16)-(5.20) in (5.15) and multiplying the resulting inequality by $e^{2\alpha t}$, we arrive at

$$\frac{d}{dt}\big(e^{2\alpha t}\|\mathbf{P}_h\mathbf{e}\|^2\big) + (\nu K_1 - 2C_2\alpha)e^{2\alpha t}\|\mathbf{P}_h\mathbf{e}\|_\varepsilon^2 \leq Ce^{2\alpha t}\|\mathbf{P}_h\mathbf{e}\|^2\|\mathbf{u}_H\|_\varepsilon^4$$
$$+Ch^{2r/(r+1)}e^{2\alpha t}\|\mathbf{u}\|_2^2\|\mathbf{e}_H\|_\varepsilon^2 + Ce^{2\alpha t}\|\mathbf{e}_H\|_\varepsilon^4$$
$$+Ce^{2\alpha t}|\mathbf{u}|_{r+1}^2(h^{2r} + h^{2r}\|\mathbf{u}_H\|_\varepsilon^2) + Ch^{2r}e^{2\alpha t}|p|_r^2. \tag{5.21}$$

Integrate (5.21) with respect to time, use $\|\mathbf{P}_h\mathbf{e}(0)\| = 0$, (5.13), Lemma 5.2, Gronwall's lemma and assumption (**A1**) to obtain

$$\|\mathbf{P}_h\mathbf{e}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{P}_h\mathbf{e}(s)\|_\varepsilon^2\,ds \leq C(t)(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}).$$

Finally, a use of triangle inequality $\|\mathbf{e}\|_\varepsilon \leq \|\mathbf{P}_h\mathbf{e}\|_\varepsilon + \|\mathbf{u}-\mathbf{P}_h\mathbf{u}\|_\varepsilon$ and Lemma 2.2 leads to the desired result. □

Theorem 5.1. *Under the assumptions of Lemma 5.5, the error $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$ in* **Step 2** *for approximating the velocity satisfies*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{e}_s(s)\|^2\,ds + \|\boldsymbol{e}(t)\|_\varepsilon^2 \leq K(t)(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2})$$
$$+ K(t)h^{-1}H^{2r}(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}).$$

Proof. Choose $\boldsymbol{\phi}_h = \mathbf{P}_h\mathbf{e}_t$ in (5.14) to obtain

$$\|\mathbf{P}_h\mathbf{e}_t\|^2 + \frac{\nu}{2}\frac{d}{dt}\big(a(\mathbf{P}_h\mathbf{e},\mathbf{P}_h\mathbf{e})\big) + c^{\mathbf{u}_h}(\mathbf{e},\mathbf{u}_H,\mathbf{P}_h\mathbf{e}_t) + c^{\mathbf{u}_H}(\mathbf{u}_H,\mathbf{e},\mathbf{P}_h\mathbf{e}_t)$$

$$+l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t) - l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t) + b(\mathbf{P}_h\mathbf{e}_t,p) = (\mathbf{P}_h\mathbf{u}_t - \mathbf{u}_t,\mathbf{P}_h\mathbf{e}_t)$$
$$+\nu\, a(\mathbf{P}_h\mathbf{u}-\mathbf{u},\mathbf{P}_h\mathbf{e}_t) - c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t).$$

Multiply the above equation by $e^{2\alpha t}$ to find

$$\begin{aligned}
e^{2\alpha t}\|\mathbf{P}_h\mathbf{e}_t\|^2 + \frac{\nu}{2}\frac{d}{dt}\left(e^{2\alpha t}a(\mathbf{P}_h\mathbf{e},\mathbf{P}_h\mathbf{e})\right) &= e^{2\alpha t}(\mathbf{P}_h\mathbf{u}_t - \mathbf{u}_t,\mathbf{P}_h\mathbf{e}_t)\\
+\nu\, e^{2\alpha t}a(\mathbf{P}_h\mathbf{u}-\mathbf{u},\mathbf{P}_h\mathbf{e}_t) + \nu\alpha e^{2\alpha t}\, a(\mathbf{P}_h\mathbf{e},\mathbf{P}_h\mathbf{e}) &- e^{2\alpha t}c^{\mathbf{u}_h}(\mathbf{e},\mathbf{u}_H,\mathbf{P}_h\mathbf{e}_t)\\
-e^{2\alpha t}c^{\mathbf{u}_H}(\mathbf{u}_H,\mathbf{e},\mathbf{P}_h\mathbf{e}_t) - e^{2\alpha t}c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t) &- e^{2\alpha t}l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t)\\
+e^{2\alpha t}l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t) &- e^{2\alpha t}b(\mathbf{P}_h\mathbf{e}_t,p).
\end{aligned} \tag{5.22}$$

First we rewrite

$$c^{\mathbf{u}_h}(\mathbf{e},\mathbf{u}_H,\mathbf{P}_h\mathbf{e}_t) = -c^{\mathbf{u}_h}(\mathbf{e},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t) + c^{\mathbf{u}_h}(\mathbf{e},\mathbf{u},\mathbf{P}_h\mathbf{e}_t).$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
&|c^{\mathbf{u}_h}(\mathbf{e},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t)|\\
&= \Big|\sum_{E\in\mathcal{E}_h}\int_E(\mathbf{e}\cdot\nabla\mathbf{e}_H)\cdot\mathbf{P}_h\mathbf{e}_t + \sum_{E\in\mathcal{E}_h}\int_{\partial E_-}|\{\mathbf{e}\}\cdot\mathbf{n}_E|(\mathbf{e}_H^{int}-\mathbf{e}_H^{ext})\cdot\mathbf{P}_h\mathbf{e}_t^{int}\\
&\quad + \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E(\nabla\cdot\mathbf{e})\mathbf{e}_H\cdot\mathbf{P}_h\mathbf{e}_t - \frac{1}{2}\sum_{e\in\Gamma_h}\int_e[\mathbf{e}]\cdot\mathbf{n}_e\{\mathbf{e}_H\cdot\mathbf{P}_h\mathbf{e}_t\}\Big|\\
&\le \sum_{E\in\mathcal{E}_h}\|\mathbf{e}\|_{L^4(E)}\|\nabla\mathbf{e}_H\|_{L^2(E)}\|\mathbf{P}_h\mathbf{e}_t\|_{L^4(E)} + \sum_{e\in\Gamma_h}\|\{\mathbf{e}\}\cdot\mathbf{n}_e\|_{L^4(e)}\|[\mathbf{e}_H]\|_{L^2(e)}\|\mathbf{P}_h\mathbf{e}_t\|_{L^4(e)}\\
&\quad + \frac{1}{2}\sum_{E\in\mathcal{E}_h}\|\nabla\mathbf{e}\|_{L^2(E)}\|\mathbf{e}_H\|_{L^4(E)}\|\mathbf{P}_h\mathbf{e}_t\|_{L^4(E)} + \frac{1}{2}\sum_{e\in\Gamma_h}\|[\mathbf{e}]\cdot\mathbf{n}_e\|_{L^2(e)}\|\{\mathbf{e}_H\cdot\mathbf{P}_h\mathbf{e}_t\}\|_{L^2(e)}.
\end{aligned}$$

We now estimate the terms on the edges. Let us consider the elements E_1 and E_2 adjacent to e. Now, Lemmas 1.5 and 1.10 lead to

$$\begin{aligned}
&\|\{\mathbf{e}\}\cdot\mathbf{n}_e\|_{L^4(e)}\|[\mathbf{e}_H]\|_{L^2(e)}\|\mathbf{P}_h\mathbf{e}_t\|_{L^4(e)} \le \frac{1}{2}\sum_{i,j=1}^{2}\|\mathbf{e}\cdot\mathbf{n}_e|_{E_i}\|_{L^4(e)}\|[\mathbf{e}_H]\|_{L^2(e)}\|\mathbf{P}_h\mathbf{e}_t|_{E_j}\|_{L^4(e)}\\
&\le Ch_{E_j}^{-1/2}\sum_{i,j=1}^{2}(\|\mathbf{e}\|_{L^4(E_i)} + h_{E_i}^{1/2}\|\nabla\mathbf{e}\|_{L^2(E_i)})\frac{1}{|e|^{1/2}}\|[\mathbf{e}_H]\|_{L^2(e)}\|\mathbf{P}_h\mathbf{e}_t\|_{L^2(E_j)}.
\end{aligned} \tag{5.23}$$

With an identical approach as above, we can derive the following

$$\begin{aligned}
&\|[\mathbf{e}]\cdot\mathbf{n}_e\|_{L^2(e)}\|\{\mathbf{e}_H\cdot\mathbf{P}_h\mathbf{e}_t\}\|_{L^2(e)}\\
&\quad\le Ch_{E_j}^{-1/2}\sum_{i,j=1}^{2}\frac{1}{|e|^{1/2}}\|[\mathbf{e}]\|_{L^2(e)}(\|\mathbf{e}_H\|_{L^4(E_i)} + h_{E_i}^{1/2}\|\nabla\mathbf{e}_H\|_{L^2(E_i)})\|\mathbf{P}_h\mathbf{e}_t\|_{L^2(E_j)}.
\end{aligned} \tag{5.24}$$

Using Hölder's and Young's inequalities, (5.23), (5.24), (1.14) and Lemma 1.10, we arrive at

$$|c^{\mathbf{u}_h}(\mathbf{e},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t)| \le Ch^{-1/2}\|\mathbf{e}\|_\varepsilon\|\mathbf{e}_H\|_\varepsilon\|\mathbf{P}_h\mathbf{e}_t\| \le \frac{1}{6}\|\mathbf{P}_h\mathbf{e}_t\|^2 + Ch^{-1}\|\mathbf{e}\|_\varepsilon^2\|\mathbf{e}_H\|_\varepsilon^2. \quad (5.25)$$

Since $\mathbf{u}$ is continuous, Hölder's and Young's inequalities, Lemmas 1.3 and 1.10 yield

$$\begin{aligned}
&|c^{\mathbf{u}_h}(\mathbf{e},\mathbf{u},\mathbf{P}_h\mathbf{e}_t)| \\
&=\Big|\sum_{E\in\mathcal{E}_h}\int_E(\mathbf{e}\cdot\nabla\mathbf{u})\cdot\mathbf{P}_h\mathbf{e}_t + \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E(\nabla\cdot\mathbf{e})\mathbf{u}\cdot\mathbf{P}_h\mathbf{e}_t \\
&\quad -\frac{1}{2}\sum_{e\in\Gamma_h}\int_e[\mathbf{e}]\cdot n_e\,\{\mathbf{u}\cdot\mathbf{P}_h\mathbf{e}_t\}\Big| \\
&\le \sum_{E\in\mathcal{E}_h}\|\mathbf{e}\|_{L^4(E)}\|\nabla\mathbf{u}\|_{L^4(E)}\|\,\mathbf{P}_h\mathbf{e}_t\|_{L^2(E)} + C\|\mathbf{u}\|_{L^\infty(\Omega)}\sum_{E\in\mathcal{E}_h}\|\nabla\mathbf{e}\|_{L^2(E)}\,\|\mathbf{P}_h\mathbf{e}_t\|_{L^2(E)} \\
&\quad + C\,\|\mathbf{u}\|_{L^\infty(\Omega)}\Big(\sum_{e\in\Gamma_h}\frac{1}{|e|}\,\|[\mathbf{e}]\|_{L^2(e)}^2\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h}\|\mathbf{P}_h\mathbf{e}_t\|_{L^2(E)}^2\Big)^{1/2} \\
&\le \frac{1}{6}\|\mathbf{P}_h\mathbf{e}_t\|^2 + C\|\mathbf{u}\|_2^2\|\mathbf{e}\|_\varepsilon^2. \qquad (5.26)
\end{aligned}$$

In a similar manner, we can bound the fifth terms on the right hand side of (5.22) as follows

$$\begin{aligned}
|c^{\mathbf{u}_H}(\mathbf{u}_H,\mathbf{e},\mathbf{P}_h\mathbf{e}_t)| &= |c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e},\mathbf{P}_h\mathbf{e}_t)| + |c^{\mathbf{u}_H}(\mathbf{u},\mathbf{e},\mathbf{P}_h\mathbf{e}_t)| \\
&\le Ch^{-1/2}\|\mathbf{e}_H\|_\varepsilon\|\mathbf{e}\|_\varepsilon\|\mathbf{P}_h\mathbf{e}_t\| + C\|\mathbf{u}\|_2\|\mathbf{e}\|_\varepsilon\|\mathbf{P}_h\mathbf{e}_t\| \\
&\le \frac{1}{6}\|\mathbf{P}_h\mathbf{e}_t\|^2 + Ch^{-1}\|\mathbf{e}\|_\varepsilon^2\|\mathbf{e}_H\|_\varepsilon^2 + C\|\mathbf{u}\|_2^2\|\mathbf{e}\|_\varepsilon^2. \qquad (5.27)
\end{aligned}$$

Again, we rewrite the sixth term on the right hand side of the equality (5.22) as

$$\begin{aligned}
e^{2\alpha t}c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t) &= \frac{d}{dt}\big(e^{2\alpha t}c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_H,\mathbf{P}_h\mathbf{e})\big) - 2\alpha e^{2\alpha t}c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_H,\mathbf{P}_h\mathbf{e}) \\
&\quad - e^{2\alpha t}c^{\mathbf{u}_H}(\mathbf{e}_{Ht},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}) - e^{2\alpha t}c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_{Ht},\mathbf{P}_h\mathbf{e}).
\end{aligned}$$

Apply estimate (2.56) to arrive at

$$\begin{aligned}
&|2\alpha e^{2\alpha t}c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_H,\mathbf{P}_h\mathbf{e})| + |e^{2\alpha t}c^{\mathbf{u}_H}(\mathbf{e}_{Ht},\mathbf{e}_H,\mathbf{P}_h\mathbf{e})| + |e^{2\alpha t}c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_{Ht},\mathbf{P}_h\mathbf{e})| \\
&\qquad \le Ce^{2\alpha t}\big(\|\mathbf{e}_H\|_\varepsilon^2 + \|\mathbf{e}_{Ht}\|_\varepsilon\|\mathbf{e}_H\|_\varepsilon\big)\|\mathbf{P}_h\mathbf{e}\|_\varepsilon. \qquad (5.28)
\end{aligned}$$

Rewrite seventh and eighth terms on the right hand side of (5.22) as

$$\begin{aligned}
&e^{2\alpha t}\big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t) - l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t)\big) = \frac{d}{dt}\big(e^{2\alpha t}\big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}) - l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e})\big)\big) \\
&- 2\alpha e^{2\alpha t}\big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}) - l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e})\big) - e^{2\alpha t}\big(l^{\mathbf{u}_H}(\mathbf{u}_t,\mathbf{e}_H,\mathbf{P}_h\mathbf{e}) - l^{\mathbf{u}_h}(\mathbf{u}_t,\mathbf{e}_H,\mathbf{P}_h\mathbf{e})\big)
\end{aligned}$$

$$-\, e^{2\alpha t}\big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_{Ht},\mathbf{P}_h\mathbf{e}) - l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_{Ht},\mathbf{P}_h\mathbf{e})\big). \tag{5.29}$$

Similar to Lemma 5.5, using Lemmas 1.3 and 5.4, we can bound:

$$\begin{aligned}
&|2\alpha e^{2\alpha t}\big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e}) - l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\mathbf{P}_h\mathbf{e})\big)| + |e^{2\alpha t}\big(l^{\mathbf{u}_H}(\mathbf{u}_t,\mathbf{e}_H,\mathbf{P}_h\mathbf{e}) - l^{\mathbf{u}_h}(\mathbf{u}_t,\mathbf{e}_H,\mathbf{P}_h\mathbf{e})\big)| \\
&\qquad \le Ch^{\frac{r}{r+1}} e^{2\alpha t}(\|\mathbf{u}\|_2 + \|\mathbf{u}_t\|_2)\|\mathbf{e}_H\|_\varepsilon \|\mathbf{P}_h\mathbf{e}\|_\varepsilon,
\end{aligned} \tag{5.30}$$

and

$$|e^{2\alpha t}\big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_{Ht},\mathbf{P}_h\mathbf{e}) - l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_{Ht},\mathbf{P}_h\mathbf{e})\big)| \le Ch^{\frac{r}{r+1}} e^{2\alpha t}\|\mathbf{u}\|_2 \|\mathbf{e}_{Ht}\|_\varepsilon \|\mathbf{P}_h\mathbf{e}\|_\varepsilon. \tag{5.31}$$

Finally, the second and ninth term on the right hand side of (5.22) can be rewritten as follows

$$\begin{aligned}
\nu\, e^{2\alpha t} a(\mathbf{P}_h\mathbf{u} - \mathbf{u}, \mathbf{P}_h\mathbf{e}_t) =& \nu \frac{d}{dt}\big(e^{2\alpha t} a(\mathbf{P}_h\mathbf{u} - \mathbf{u}, \mathbf{P}_h\mathbf{e})\big) - 2\alpha\nu\, e^{2\alpha t} a(\mathbf{P}_h\mathbf{u} - \mathbf{u}, \mathbf{P}_h\mathbf{e}) \\
&- \nu\, e^{2\alpha t} a(\mathbf{P}_h\mathbf{u}_t - \mathbf{u}_t, \mathbf{P}_h\mathbf{e})
\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}
e^{2\alpha t} b(\mathbf{P}_h\mathbf{e}_t, p) =& \frac{d}{dt}\big(e^{2\alpha t} b(\mathbf{P}_h\mathbf{e}, p - r_h p)\big) - 2\alpha e^{2\alpha t} b(\mathbf{P}_h\mathbf{e}, p - r_h p) \\
&- e^{2\alpha t} b(\mathbf{P}_h\mathbf{e}, p_t - r_h p_t).
\end{aligned} \tag{5.33}$$

Collecting the bounds (5.25)-(5.33) in (5.22), integrating the resulting inequality from 0 to t, applying Lemmas 1.6, 1.7, 2.2 along with the Cauchy-Schwarz and Young's inequalities one can find

$$\begin{aligned}
&\int_0^t e^{2\alpha\tau}\|\mathbf{P}_h\mathbf{e}_\tau(\tau)\|^2\,d\tau + \nu K_1\, e^{2\alpha t}\|\mathbf{P}_h\mathbf{e}(t)\|_\varepsilon^2 \\
&\le Ch^{2r}\int_0^t e^{2\alpha\tau}\big(|\mathbf{u}(\tau)|_{r+1}^2 + |\mathbf{u}_\tau(\tau)|_{r+1}^2 + |p(\tau)|_r^2 + |p_\tau(\tau)|_r^2\big)\,d\tau \\
&\quad + Ch^{2r}e^{2\alpha t}\big(|\mathbf{u}(t)|_{r+1}^2 + |p(t)|_r^2\big) + C\int_0^t e^{2\alpha\tau}\|\mathbf{P}_h\mathbf{e}(\tau)\|_\varepsilon^2\,d\tau \\
&\quad + Ch^{-1}\int_0^t e^{2\alpha\tau}\|\mathbf{e}(\tau)\|_\varepsilon^2\|\mathbf{e}_H(\tau)\|_\varepsilon^2\,d\tau + C\int_0^t e^{2\alpha\tau}\|\mathbf{u}(\tau)\|_2^2\|\mathbf{e}(\tau)\|_\varepsilon^2\,d\tau + Ce^{2\alpha t}\|\mathbf{e}_H(t)\|_\varepsilon^4 \\
&\quad + C\int_0^t e^{2\alpha\tau}\big(\|\mathbf{e}_H(\tau)\|_\varepsilon^4 + \|\mathbf{e}_{H\tau}(\tau)\|_\varepsilon^2\|\mathbf{e}_H(\tau)\|_\varepsilon^2\big)\,d\tau + Ch^{\frac{2r}{r+1}}e^{2\alpha t}\|\mathbf{u}(t)\|_2^2\|\mathbf{e}_H(t)\|_\varepsilon^2 \\
&\quad + Ch^{\frac{2r}{r+1}}\int_0^t e^{2\alpha\tau}\big(\|\mathbf{u}(\tau)\|_2^2 + \|\mathbf{u}_\tau(\tau)\|_2^2\big)\|\mathbf{e}_H(\tau)\|_\varepsilon^2\,d\tau + Ch^{\frac{2r}{r+1}}\int_0^t e^{2\alpha\tau}\|\mathbf{u}\|_2^2\|\mathbf{e}_{H\tau}(\tau)\|_\varepsilon^2\,d\tau.
\end{aligned} \tag{5.34}$$

Using triangle inequality $\|\mathbf{e}\|_\varepsilon \le \|\mathbf{u} - \mathbf{P}_h\mathbf{u}\|_\varepsilon + \|\mathbf{P}_h\mathbf{e}\|_\varepsilon$, Lemmas 2.2, 5.2 and 5.5, and assumption (**A1**) lead us to the desired estimate of Theorem 5.1. □

Remark 5.1. *Under the smallness condition on the data, that is,*

$$N = \sup_{\boldsymbol{\phi}_h, \mathbf{v}_h, \boldsymbol{w}_h \in \boldsymbol{V}_h} \frac{c^{\boldsymbol{\phi}_h}(\boldsymbol{w}_h, \mathbf{v}_h, \boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_\varepsilon^2 \|\mathbf{v}_h\|_\varepsilon} \quad \text{and} \quad \frac{NC_2}{K_1^2\nu^2}\|\mathbf{f}\| < 1. \tag{5.35}$$

the bounds of Lemma 5.5 are uniform in time, that is,

$$\|\boldsymbol{e}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{e}(s)\|_\varepsilon^2\, ds \le C(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}).$$

where the constant $C > 0$ is independent of time t.

Proof. In order to derive the estimates, which are valid uniformly for all $t > 0$, let us first bound the nonlinear term $c^{\mathbf{u}_h}(\mathbf{P}_h\mathbf{e}, \mathbf{u}_H, \mathbf{P}_h\mathbf{e})$ by using (5.35) as follows:

$$|c^{\mathbf{u}_h}(\mathbf{P}_h\mathbf{e}, \mathbf{u}_H, \mathbf{P}_h\mathbf{e})| \le N\|\mathbf{u}_H\|_\varepsilon\|\mathbf{P}_h\mathbf{e}\|_\varepsilon^2.$$

Now, the proof of Lemma 5.5 is modified in the following manner: Rewrite (5.15) to obtain

$$\frac{d}{dt}\|\mathbf{P}_h\mathbf{e}\|^2 + 2(\nu K_1 - N\|\mathbf{u}_H\|_\varepsilon)\|\mathbf{P}_h\mathbf{e}\|_\varepsilon^2 \le C\Big(h^{r/(r+1)}\|\mathbf{u}\|_2\|\mathbf{e}_H\|_\varepsilon + C\|\mathbf{e}_H\|_\varepsilon^2 + h^r|\mathbf{u}|_{r+1} + h^r\|\mathbf{u}_H\|_\varepsilon|\mathbf{u}|_{r+1} + h^r|p|_r\Big)\|\mathbf{P}_h\mathbf{e}\|_\varepsilon. \tag{5.36}$$

Multiply (5.36) by $e^{2\alpha t}$, integrate from 0 to t, and use Lemma 5.1, Theorem 5.2 and assumption (**A1**). After a final multiplication of the resulting equation by $e^{-2\alpha t}$, we arrive at

$$\begin{aligned}
\|\mathbf{P}_h\mathbf{e}(t)\|^2 &+ 2e^{-2\alpha t}\int_0^t e^{2\alpha s}(\nu K_1 - N\|\mathbf{u}_H\|_\varepsilon)\|\mathbf{P}_h\mathbf{e}(s)\|_\varepsilon^2\, ds \\
&\le e^{-2\alpha t}\|\mathbf{P}_h\mathbf{e}(0)\| + 2\alpha e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{P}_h\mathbf{e}(s)\|^2\, ds \\
&\quad + C(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2})e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{P}_h\mathbf{e}(s)\|_\varepsilon ds.
\end{aligned}$$

Take $t \to \infty$, employ L'Hôpital's rule and (5.10) to obtain

$$\begin{aligned}
&\frac{1}{\alpha}\big(\nu K_1 - \frac{NC_2}{K_1\nu}\|\mathbf{f}\|_{L^\infty(0,\infty;L^2(\Omega))}\big)\limsup_{t\to\infty}\|\mathbf{P}_h\mathbf{e}(t)\|_\varepsilon^2 \\
&\le \frac{1}{\alpha}C(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2})\limsup_{t\to\infty}\|\mathbf{P}_h\mathbf{e}(t)\|_\varepsilon.
\end{aligned}$$

Due to the condition (5.35), there holds

$$\limsup_{t\to\infty}\|\mathbf{P}_h\mathbf{e}(t)\|_\varepsilon \le C(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}).$$

From (1.14), we now obtain

$$\limsup_{t\to\infty}\|\mathbf{P}_h\mathbf{e}(t)\| \le C(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}). \tag{5.37}$$

Substitute (5.37) in (5.21), integrate the resulting inequality with respect to time, use $\|\mathbf{P}_h\mathbf{e}(0)\| = 0$, (5.13), Lemmas 2.2 and 5.2, triangle inequality and assumption (**A1**) to obtain

$$\|\mathbf{e}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{e}(s)\|_\varepsilon^2\,ds \le C(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}).$$

Here, $C > 0$ is valid uniformly for all $t > 0$. $\square$

5.3.2 Pressure Error Estimates

This subsection is devoted to the derivation of two-grid pressure error estimates. Before establishing the main result, we obtain the bounds for $\mathbf{e}_t$, which will play a significant role for achieving pressure error estimates.

Lemma 5.6. *Under the assumptions of Lemma 5.5, the error* $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$ *in* **Step 2** *for approximating the velocity satisfies*

$$\|\boldsymbol{e}_t(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{e}_s(s)\|_\varepsilon^2\,ds \le K(t)(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}) + K(t)h^{-1}H^{2r}(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}).$$

Proof. Differentiate (5.14) with respect to time and substitute $\boldsymbol{\phi}_h = \mathbf{P}_h\mathbf{e}_t = (\mathbf{P}_h\mathbf{u}_t - \mathbf{u}_t) + \mathbf{e}_t$ in the resulting equation to find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{P}_h\mathbf{e}_t\|^2 + \nu K_1\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{P}_h\mathbf{e}_t, \mathbf{P}_h\mathbf{e}_t) = (\mathbf{P}_h\mathbf{u}_{tt} - \mathbf{u}_{tt}, \mathbf{P}_h\mathbf{e}_t)\\
&+ \nu\, a(\mathbf{P}_h\mathbf{u}_t - \mathbf{u}_t, \mathbf{P}_h\mathbf{e}_t) - b(\mathbf{P}_h\mathbf{e}_t, p_t) - c^{\mathbf{u}_h}(\mathbf{P}_h\mathbf{e}_t, \mathbf{u}_H, \mathbf{P}_h\mathbf{e}_t)\\
&- c^{\mathbf{u}_h}(\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \mathbf{u}_H, \mathbf{P}_h\mathbf{e}_t) - c^{\mathbf{u}_h}(\mathbf{e}, \mathbf{u}_{Ht}, \mathbf{P}_h\mathbf{e}_t) - c^{\mathbf{u}_H}(\mathbf{u}_{Ht}, \mathbf{e}, \mathbf{P}_h\mathbf{e}_t)\\
&- c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \mathbf{P}_h\mathbf{e}_t) - c^{\mathbf{u}_H}(\mathbf{e}_{Ht}, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_t) - c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_{Ht}, \mathbf{P}_h\mathbf{e}_t)\\
&+ \big(l^{\mathbf{u}_H}(\mathbf{u}_t, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_t) - l^{\mathbf{u}_h}(\mathbf{u}_t, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_t)\big) + \big(l^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_{Ht}, \mathbf{P}_h\mathbf{e}_t) - l^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_{Ht}, \mathbf{P}_h\mathbf{e}_t)\big).
\end{aligned} \tag{5.38}$$

An application of (2.57), (2.56), Lemma 2.2 and Young's inequality implies

$$\begin{aligned}
&|c^{\mathbf{u}_h}(\mathbf{P}_h\mathbf{e}_t, \mathbf{u}_H, \mathbf{P}_h\mathbf{e}_t) + c^{\mathbf{u}_h}(\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \mathbf{u}_H, \mathbf{P}_h\mathbf{e}_t) + c^{\mathbf{u}_h}(\mathbf{e}, \mathbf{u}_{Ht}, \mathbf{P}_h\mathbf{e}_t)\\
&+ c^{\mathbf{u}_H}(\mathbf{u}_{Ht}, \mathbf{e}, \mathbf{P}_h\mathbf{e}_t) + c^{\mathbf{u}_H}(\mathbf{u}_H, \mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t, \mathbf{P}_h\mathbf{e}_t) + c^{\mathbf{u}_H}(\mathbf{e}_{Ht}, \mathbf{e}_H, \mathbf{P}_h\mathbf{e}_t)\\
&+ c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_{Ht}, \mathbf{P}_h\mathbf{e}_t)| \le C\|\mathbf{P}_h\mathbf{e}_t\|^{1/2}\|\mathbf{u}_H\|_\varepsilon\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^{3/2}\\
&+ C\big(\|\mathbf{u}_t - \mathbf{P}_h\mathbf{u}_t\|_\varepsilon\|\mathbf{u}_H\|_\varepsilon + \|\mathbf{e}\|_\varepsilon\|\mathbf{u}_{Ht}\|_\varepsilon + \|\mathbf{e}_H\|_\varepsilon\|\mathbf{e}_{Ht}\|_\varepsilon\big)\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon\\
\le\ & \frac{\nu K_1}{64}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2 + C\big(\|\mathbf{u}_H\|_\varepsilon^4\|\mathbf{P}_h\mathbf{e}_t\|^2 + h^{2r}|\mathbf{u}_t|_{r+1}^2\|\mathbf{u}_H\|_\varepsilon^2\\
&+ \|\mathbf{e}\|_\varepsilon^2\|\mathbf{u}_{Ht}\|_\varepsilon^2 + \|\mathbf{e}_H\|_\varepsilon^2\|\mathbf{e}_{Ht}\|_\varepsilon^2\big).
\end{aligned} \tag{5.39}$$

Again, a use of Lemmas 1.3 and 5.4, and Young's inequality, one finds

$$\begin{aligned}\big|\big(l^{\mathbf{u}_H}(\mathbf{u}_t,\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t)-l^{\mathbf{u}_h}(\mathbf{u}_t,\mathbf{e}_H,\mathbf{P}_h\mathbf{e}_t)\big)+\big(l^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_{Ht},\mathbf{P}_h\mathbf{e}_t)-l^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_{Ht},\mathbf{P}_h\mathbf{e}_t)\big)\big|\\ \le Ch^{\frac{r}{r+1}}(\|\mathbf{u}_t\|_{L^\infty(\Omega)}\|\mathbf{e}_H\|_\varepsilon+\|\mathbf{u}\|_{L^\infty(\Omega)}\|\mathbf{e}_{Ht}\|_\varepsilon)\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon\\ \le Ch^{\frac{2r}{r+1}}(\|\mathbf{u}_t\|_2^2\|\mathbf{e}_H\|_\varepsilon^2+\|\mathbf{u}\|_2^2\|\mathbf{e}_{Ht}\|_\varepsilon^2)+\frac{\nu K_1}{64}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2. \qquad (5.40)\end{aligned}$$

Additionally, similar to the bounds (5.16) and (5.20) in the proof of Lemma 5.5, we obtain

$$\nu|a(\mathbf{P}_h\mathbf{u}_t-\mathbf{u}_t,\mathbf{P}_h\mathbf{e}_t)|+|b(\mathbf{P}_h\mathbf{e}_t,p_t)|\le\frac{\nu K_1}{64}\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2+Ch^{2r}(|\mathbf{u}_t|_{r+1}^2+|p_t|_r^2). \qquad (5.41)$$

Apply (5.39)-(5.41) in (5.38), using (1.19) and the definition of L^2-projection, we arrive at

$$\begin{aligned}\frac{d}{dt}\|\mathbf{P}_h\mathbf{e}_t\|^2+\nu K_1\|\mathbf{P}_h\mathbf{e}_t\|_\varepsilon^2\le C\big(\|\mathbf{u}_H\|_\varepsilon^4\|\mathbf{P}_h\mathbf{e}_t\|^2+h^{2r}|\mathbf{u}_t|_{r+1}^2\|\mathbf{u}_H\|_\varepsilon^2+\|\mathbf{e}\|_\varepsilon^2\|\mathbf{u}_{Ht}\|_\varepsilon^2\\ +\|\mathbf{e}_H\|_\varepsilon^2\|\mathbf{e}_{Ht}\|_\varepsilon^2+h^{\frac{2r}{r+1}}\|\mathbf{u}_t\|_2^2\|\mathbf{e}_H\|_\varepsilon^2+h^{\frac{2r}{r+1}}\|\mathbf{u}\|_2^2\|\mathbf{e}_{Ht}\|_\varepsilon^2\big)+Ch^{2r}(|\mathbf{u}_t|_{r+1}^2+|p_t|_r^2).\end{aligned}$$

Multiply the above inequality by $e^{2\alpha t}$ and integrate with respect to time. Then a use of Gronwall's lemma, triangle inequality, (5.13), Lemmas 2.2, 5.1, 5.2, and Theorem 5.1, and assumption (**A1**) complete the rest of the proof. □

Theorem 5.2. *Under the assumptions of Lemma 5.5, there exists a constant $K>0$, such that, the following error estimates hold true:*

$$\begin{aligned}\|p-p_h\|^2\le K(t)(h^{2r}+h^{2r/(r+1)}H^{2r}+H^{2r+2})\\ +K(t)h^{-1}H^{2r}(h^{2r}+h^{2r/(r+1)}H^{2r}+H^{2r+2}).\end{aligned}$$

Proof. We can write the error equation (5.14) as follows:

$$\begin{aligned}(\mathbf{e}_t,\boldsymbol{\phi}_h)+\nu\,a(\mathbf{e},\boldsymbol{\phi}_h)+c^{\mathbf{u}_h}(\mathbf{e},\mathbf{u}_H,\boldsymbol{\phi}_h)+c^{\mathbf{u}_H}(\mathbf{u}_H,\mathbf{e},\boldsymbol{\phi}_h)+c^{\mathbf{u}_h}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\phi}_h)\\ -c^{\mathbf{u}_H}(\mathbf{u},\mathbf{e}_H,\boldsymbol{\phi}_h)+c^{\mathbf{u}_H}(\mathbf{e}_H,\mathbf{e}_H,\boldsymbol{\phi}_h)+b(\boldsymbol{\phi}_h,p-r_h(p))=b(\boldsymbol{\phi}_h,p_h-r_h(p)), \qquad (5.42)\end{aligned}$$

for $\boldsymbol{\phi}_h\in\mathbf{X}_h$. By virtue of the inf-sup condition presented in Lemma 1.8, there is $\boldsymbol{\phi}_h\in\mathbf{X}_h$ such that

$$b(\boldsymbol{\phi}_h,p_h-r_h(p))=-\|p_h-r_h(p)\|^2,\quad \|\boldsymbol{\phi}_h\|_\varepsilon\le\frac{1}{\beta^*}\|p_h-r_h(p)\|. \qquad (5.43)$$

Therefore, from (5.42), we obtain

$$\|p_h-r_h(p)\|^2=(\mathbf{e}_t,\boldsymbol{\phi}_h)+\nu\,a(\mathbf{e},\boldsymbol{\phi}_h)+c^{\mathbf{u}_h}(\mathbf{e},\mathbf{u}_H,\boldsymbol{\phi}_h)+c^{\mathbf{u}_H}(\mathbf{u}_H,\mathbf{e},\boldsymbol{\phi}_h)$$

$$+c^{\mathbf{u}_h}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h) - c^{\mathbf{u}_H}(\mathbf{u}, \mathbf{e}_H, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, p - r_h(p)). \qquad (5.44)$$

The terms on the right hand side of (5.44) can be bounded as in Lemma 5.5 and Theorem 5.1. Then, applying (5.43), the equality (5.44) becomes

$$\|p_h - r_h(p)\|^2 \leq C\big(\|\mathbf{e}_t\|^2 + \|\mathbf{e}\|_\varepsilon^2 + \|\mathbf{u}_H\|_\varepsilon^2\|\mathbf{e}\|_\varepsilon^2 + \|\mathbf{u}\|_2^2\|\mathbf{e}_H\|^2 + h^2\|\mathbf{u}\|_2^2\|\mathbf{e}_H\|_\varepsilon^2 + \|\mathbf{e}_H\|_\varepsilon^4 + h^{2r}|\mathbf{u}|_{r+1}^2 + h^{2r}|p|_r^2\big).$$

Finally, using triangle inequality, estimate (1.31), (5.13), Lemmas 5.2, 5.6 and Theorem 5.1, and assumption (**A1**), we complete the rest of the proof. □

Remark 5.2. *Under the condition (5.35) the estimate of Theorem 5.2 is uniform in time.*

5.4 Fully Discrete DG Two-Grid Method

For discretization in time variable, we employ the backward Euler scheme in this section. We describe below the backward Euler scheme for the semi-discrete DG Two-grid algorithm (5.4)-(5.5) as follows:

Step 1 (Nonlinear system on a coarse grid): Find $(\mathbf{U}_H^n, P_H^n)_{n\geq 1} \in \mathbf{X}_H \times M_H$ such that for all $(\boldsymbol{\phi}_H, q_H) \in \mathbf{X}_H \times M_H$ and for $\mathbf{U}_H^0 = \mathbf{P}_H\mathbf{u}_0$

$$\left.\begin{aligned} (\partial_t\mathbf{U}_H^n, \boldsymbol{\phi}_H) + \nu\, a(\mathbf{U}_H^n, \boldsymbol{\phi}_H) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_H) & \\ +b(\boldsymbol{\phi}_H, P_H^n) = (\mathbf{f}^n, \boldsymbol{\phi}_H), & \\ b(\mathbf{U}_H^n, q_H) = 0. & \end{aligned}\right\} \qquad (5.45)$$

Step 2 (Update on a finer mesh with one Newton iteration): Find $(\mathbf{U}_h^n, P_h^n)_{n\geq 1} \in \mathbf{X}_h \times M_h$ such that for all $(\boldsymbol{\phi}_h, q_h) \in \mathbf{X}_h \times M_h$ and for $\mathbf{U}_h^0 = \mathbf{P}_h\mathbf{u}_0$

$$\left.\begin{aligned} (\partial_t\mathbf{U}_h^n, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{U}_h^n, \boldsymbol{\phi}_h) &+ c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_h^n, \boldsymbol{\phi}_h) \\ +b(\boldsymbol{\phi}_h, P_h^n) &= (\mathbf{f}^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_h), \\ b(\mathbf{U}_h^n, q_h) &= 0. \end{aligned}\right\} \qquad (5.46)$$

The DG backward Euler scheme applied to (5.6)-(5.7) is described below in the form of the following algorithm:

Step 1 (Nonlinear system on a coarse grid): Find $\mathbf{U}_H^n \in \mathbf{V}_H$ such that for all $\boldsymbol{\phi}_H \in \mathbf{V}_H$ and for $\mathbf{U}_H^0 = \mathbf{P}_H\mathbf{u}_0$

$$(\partial_t\mathbf{U}_H^n, \boldsymbol{\phi}_H) + \nu\, a(\mathbf{U}_H^n, \boldsymbol{\phi}_H) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \boldsymbol{\phi}_H) = (\mathbf{f}^n, \boldsymbol{\phi}_H). \qquad (5.47)$$

Step 2 (Update on a finer mesh with one Newton iteration): Find $\mathbf{U}_h^n \in \mathbf{V}_h$ such that for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$ and for $\mathbf{U}_h^0 = \mathbf{P}_h \mathbf{u}_0$

$$(\partial_t \mathbf{U}_h^n,\, \boldsymbol{\phi}_h) + \nu\, a\,(\mathbf{U}_h^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_h^n}(\mathbf{U}_h^n,\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_h^n,\, \boldsymbol{\phi}_h) = (\mathbf{f}^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_h). \tag{5.48}$$

5.4.1 *A priori* Bounds

Lemma 5.7. *Let the assumption (**A1**) be satisfied and $0 < \alpha < \frac{\nu K_1}{2C_2}$. Then, for the semi-discrete DG velocity $\boldsymbol{u}_h(t)$, $t > 0$ for* **step 2**, *the following holds true*

$$\|\boldsymbol{u}_h\|_\varepsilon^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s}\, \|\boldsymbol{u}_{hss}(s)\|_{-1,h}^2\, ds \le C,$$

where

$$\|\boldsymbol{u}_{htt}\|_{-1,h} = \sup \left\{ \frac{\langle \boldsymbol{u}_{htt}, \boldsymbol{\phi}_h \rangle}{\|\boldsymbol{\phi}_h\|_\varepsilon},\ \boldsymbol{\phi}_h \in \boldsymbol{X}_h, \boldsymbol{\phi}_h \neq 0 \right\}.$$

Proof. First of all, we use triangle inequality and Theorem 5.1 to bound

$$\|\mathbf{u}_h\|_\varepsilon \le \|\mathbf{u} - \mathbf{u}_h\|_\varepsilon + \|\mathbf{u}\|_1 \le C. \tag{5.49}$$

Now, differentiating (5.7) with respect to t, we obtain

$$\begin{aligned}(\mathbf{u}_{htt},\, \boldsymbol{\phi}_h) &+ \nu\, a\,(\mathbf{u}_{ht},\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_h}(\mathbf{u}_{ht},\, \mathbf{u}_H,\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_h}(\mathbf{u}_h,\, \mathbf{u}_{Ht},\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_{Ht},\, \mathbf{u}_h,\, \boldsymbol{\phi}_h)\\ &+ c^{\mathbf{u}_H}(\mathbf{u}_H,\, \mathbf{u}_{ht},\, \boldsymbol{\phi}_h) = (\mathbf{f}_t,\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_{Ht},\, \mathbf{u}_H,\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_H,\, \mathbf{u}_{Ht},\, \boldsymbol{\phi}_h),\end{aligned} \tag{5.50}$$

for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$. Take $\boldsymbol{\phi}_h = \mathbf{u}_{ht}$ in (5.50), apply Lemma 1.6 and positivity result (1.19) to find

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}_{ht}\|^2 + \nu K_1 \|\mathbf{u}_{ht}\|_\varepsilon^2 &\le C\|\mathbf{f}_t\|^2 + |c^{\mathbf{u}_h}(\mathbf{u}_{ht},\, \mathbf{u}_H,\, \mathbf{u}_{ht})| + |c^{\mathbf{u}_h}(\mathbf{u}_h,\, \mathbf{u}_{Ht},\, \mathbf{u}_{ht})|\\ &+ |c^{\mathbf{u}_H}(\mathbf{u}_{Ht},\, \mathbf{u}_h,\, \mathbf{u}_{ht})| + |c^{\mathbf{u}_H}(\mathbf{u}_{Ht},\, \mathbf{u}_H,\, \mathbf{u}_{ht})| + |c^{\mathbf{u}_H}(\mathbf{u}_H,\, \mathbf{u}_{Ht},\, \mathbf{u}_{ht})|.\end{aligned} \tag{5.51}$$

A use of estimates (2.55) and (2.57) implies

$$\begin{aligned}&|c^{\mathbf{u}_h}(\mathbf{u}_{ht},\, \mathbf{u}_H,\, \mathbf{u}_{ht})| + |c^{\mathbf{u}_h}(\mathbf{u}_h,\, \mathbf{u}_{Ht},\, \mathbf{u}_{ht})| + |c^{\mathbf{u}_H}(\mathbf{u}_{Ht},\, \mathbf{u}_h,\, \mathbf{u}_{ht})|\\ &+ |c^{\mathbf{u}_H}(\mathbf{u}_{Ht},\, \mathbf{u}_H,\, \mathbf{u}_{ht})| + |c^{\mathbf{u}_H}(\mathbf{u}_H,\, \mathbf{u}_{Ht},\, \mathbf{u}_{ht})| \le C\|\mathbf{u}_{ht}\|^{1/2} \|\mathbf{u}_H\|_\varepsilon \|\mathbf{u}_{ht}\|_\varepsilon^{3/2}\\ &+ C\big(\|\mathbf{u}_h\|_\varepsilon \|\mathbf{u}_{Ht}\|_\varepsilon + \|\mathbf{u}_{Ht}\|_\varepsilon \|\mathbf{u}_H\|_\varepsilon\big) \|\mathbf{u}_{ht}\|_\varepsilon.\end{aligned} \tag{5.52}$$

Apply (5.52) and Young's inequality in (5.51), multiply the resulting inequality by $e^{2\alpha t}$ and integrating with respect time from 0 to t, we find that

$$e^{2\alpha t}\|\mathbf{u}_{ht}(t)\|^2 + (\nu K_1 - 2\alpha C_2) \int_0^t e^{2\alpha s} \|\mathbf{u}_{ht}(s)\|_\varepsilon^2\, ds$$

$$\le \|\mathbf{u}_{ht}(0)\|^2 + C\int_0^t e^{2\alpha s}\|\mathbf{u}_{ht}(s)\|^2\|\mathbf{u}_H(s)\|_\varepsilon^4\,ds$$
$$+ C\int_0^t e^{2\alpha s}\big(\|\mathbf{u}_h(s)\|_\varepsilon^2\|\mathbf{u}_{Hs}(s)\|_\varepsilon^2 + \|\mathbf{u}_{Hs}(s)\|_\varepsilon^2\|\mathbf{u}_H(s)\|_\varepsilon^2\big)\,ds + C\int_0^t e^{2\alpha s}\|\mathbf{f}_t(s)\|^2\,ds.$$

Choosing $\alpha < \frac{\nu K_1}{2C_2}$, applying (5.49), (5.13), Lemma 5.1, Gronwall's lemma and after a final multiplication by $e^{-2\alpha t}$, we obtain

$$\|\mathbf{u}_{ht}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_{ht}(s)\|_\varepsilon^2\,ds \le C. \tag{5.53}$$

Again, we use (5.50) as follows:

$$\begin{aligned}(\mathbf{u}_{htt},\, \boldsymbol{\phi}_h) =\; & -\nu\, a\,(\mathbf{u}_{ht},\, \boldsymbol{\phi}_h) - c^{\mathbf{u}_h}(\mathbf{u}_{ht},\, \mathbf{u}_H,\, \boldsymbol{\phi}_h) - c^{\mathbf{u}_h}(\mathbf{u}_h,\, \mathbf{u}_{Ht},\, \boldsymbol{\phi}_h)\\ & - c^{\mathbf{u}_H}(\mathbf{u}_{Ht},\, \mathbf{u}_h,\, \boldsymbol{\phi}_h) - c^{\mathbf{u}_H}(\mathbf{u}_H,\, \mathbf{u}_{ht},\, \boldsymbol{\phi}_h) + c^{\mathbf{u}_H}(\mathbf{u}_{Ht},\, \mathbf{u}_H,\, \boldsymbol{\phi}_h)\\ & + c^{\mathbf{u}_H}(\mathbf{u}_H,\, \mathbf{u}_{Ht},\, \boldsymbol{\phi}_h) + (\mathbf{f}_t,\, \boldsymbol{\phi}_h), \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h.\end{aligned}$$

Using estimate (1.14), (2.55) and Lemma 1.7, we obtain

$$\|\mathbf{u}_{htt}\|_{-1,h}^2 \le C\big(\|\mathbf{u}_{ht}\|_\varepsilon^2 + \|\mathbf{u}_{ht}\|_\varepsilon^2\|\mathbf{u}_H\|_\varepsilon^2 + \|\mathbf{u}_h\|_\varepsilon^2\|\mathbf{u}_{Ht}\|_\varepsilon^2 + \|\mathbf{u}_{Ht}\|_\varepsilon^2\|\mathbf{u}_H\|_\varepsilon^2 + \|\mathbf{f}_t\|^2\big).$$

Multiply the above inequality by $e^{2\alpha t}$ and integrate from 0 to t. Then again multiply by $e^{-2\alpha t}$, use (5.13), (5.49), (5.53) and Lemma 5.1 to complete the rest of the proof. □

Below in Lemma 5.8, we state *a priori* bounds of the **Step 1** fully discrete solution $\mathbf{U}_H^n$. For a proof, one may refer to Lemma 2.15 of Chapter 2.

Lemma 5.8. *Suppose the assumption (*$\mathbf{A1}$*) is satisfied and* $0 < \alpha < \frac{\nu K_1}{2C_2}$*. Further, let* $\boldsymbol{U}_H^0 = \boldsymbol{P}_H\boldsymbol{u}_0$*. Then, there exists a constant* $C > 0$*, such that, the solution* $\{\boldsymbol{U}_H^n\}_{n\ge1}$ *of (5.47) satisfies the following a priori bounds:*

$$\|\boldsymbol{U}_H^n\|^2 \,+\, e^{-2\alpha t_M}\,\Delta t\sum_{n=1}^{M} e^{2\alpha t_n}\|\boldsymbol{U}_H^n\|_\varepsilon^2 \,\le\, C, \quad n = 1, \cdots, M,$$

Now, we provide a proof of *a priori estimates* of the solution $\mathbf{U}_h^n$ of (5.48).

Lemma 5.9. *Let the assumption (*$\mathbf{A1}$*) be satisfied , choose* k_0 *small so that* $0 < \Delta t \le k_0$ *and* $0 < \alpha < \frac{\nu K_1}{2C_2}$*. Further, let* $\boldsymbol{U}_h^0 = \boldsymbol{P}_h\boldsymbol{u}_0$*. Then, there exists a constant* $K_T > 0$*, that depends on* T*, such that, the solution* $\{\boldsymbol{U}_h^n\}_{n\ge1}$ *of (5.48) satisfies the following a priori bounds:*

$$\|\boldsymbol{U}_h^n\|^2 \,+\, e^{-2\alpha t_M}\,\Delta t\sum_{n=1}^{M} e^{2\alpha t_n}\|\boldsymbol{U}_h^n\|_\varepsilon^2 \,\le\, K_T, \quad n = 1, \cdots, M.$$

Proof. First of all, we choose $\boldsymbol{\phi}_h = \mathbf{U}_h^n$ in (5.48). Note that

$$(\partial_t \mathbf{U}_h^n, \mathbf{U}_h^n) = \frac{1}{\Delta t}(\mathbf{U}_h^n - \mathbf{U}_h^{n-1}, \mathbf{U}_h^n) \geq \frac{1}{2\Delta t}(\|\mathbf{U}_h^n\|^2 - \|\mathbf{U}_h^{n-1}\|^2) = \frac{1}{2}\partial_t \|\mathbf{U}_h^n\|^2,$$

and from (1.19) and the coercivity property in Lemma 1.6, we obtain

$$\partial_t \|\mathbf{U}_h^n\|^2 + 2\nu K_1 \|\mathbf{U}_h^n\|_\varepsilon^2 \leq 2\|\mathbf{f}^n\|\|\mathbf{U}_h^n\| + 2|c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{U}_H^n, \mathbf{U}_h^n)| + 2|c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{U}_h^n)|. \tag{5.54}$$

A use of (2.55) and (2.57) yield

$$\begin{aligned} & 2|c^{\mathbf{U}_h^n}(\mathbf{U}_h^n, \mathbf{U}_H^n, \mathbf{U}_h^n)| + 2|c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{U}_H^n, \mathbf{U}_h^n)| \\ & \qquad \leq C\big(\|\mathbf{U}_h^n\|^{1/2}\|\mathbf{U}_H^n\|_\varepsilon \|\mathbf{U}_h^n\|_\varepsilon^{3/2} + \|\mathbf{U}_H^n\|_\varepsilon^2 \|\mathbf{U}_h^n\|_\varepsilon\big) \end{aligned} \tag{5.55}$$

Observe that

$$\begin{aligned} \sum_{n=1}^{m} \Delta t e^{2\alpha t_n} \partial_t \|\mathbf{U}_h^n\|^2 &= \sum_{n=1}^{m} e^{2\alpha t_n}(\|\mathbf{U}_h^n\|^2 - \|\mathbf{U}_h^{n-1}\|^2) \\ &= e^{2\alpha t_m}\|\mathbf{U}_h^m\|^2 - \sum_{n=1}^{m-1} e^{2\alpha t_n}(e^{2\alpha\Delta t} - 1)\|\mathbf{U}_h^n\|^2 - e^{2\alpha\Delta t}\|\mathbf{U}_h^0\|^2. \end{aligned}$$

Multiply (5.54) by $\Delta t e^{2\alpha t_n}$, sum over $n = 1$ to m, and using (1.14) and Young's inequality, we have

$$\begin{aligned} e^{2\alpha t_m}\|\mathbf{U}_h^m\|^2 + (\nu K_1 - C_2(e^{2\alpha\Delta t} - 1))\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\|\mathbf{U}_h^n\|_\varepsilon^2 &\leq e^{2\alpha\Delta t}\|\mathbf{U}_h^0\|^2 \\ + C\Delta t \sum_{n=1}^{m} e^{2\alpha t_n}\big(\|\mathbf{U}_h^n\|^2\|\mathbf{U}_H^n\|_\varepsilon^4 &+ \|\mathbf{U}_H^n\|_\varepsilon^4 + \|\mathbf{f}\|^2\big). \end{aligned} \tag{5.56}$$

Choose α in such a way that

$$1 + \frac{\nu K_1}{C_2} \geq e^{2\alpha\Delta t}.$$

Also, with an application of (5.13) and Lemma 5.10, we have $\|\mathbf{U}_H^n\|_\varepsilon \leq C$. Using discrete Gronwall's inequality and multiplying the resulting inequality through out by $e^{-2\alpha t_m}$, we establish our desired estimate. □

Using (1.19) and Lemmas 1.6, 1.8, 5.9, the existence and uniqueness of the discrete solutions to the discrete problem (5.46) (or (5.48)) in **Step 2** can be achieved following similar steps as in [72].

5.4.2 Fully Discrete Step 2 Error Estimates

Next, the error estimates of backward Euler method are discussed. Considering the semi-discrete scheme (5.6)-(5.7) at $t = t_n$ and subtracting from (5.47)-(5.48), we arrive at

Step 1. Set $\mathbf{e}_H^n = \mathbf{U}_H^n - \mathbf{u}_H(t_n) = \mathbf{U}_H^n - \mathbf{u}_H^n$, for fixed $n \in \mathbb{N}$, $1 \leq n \leq M$. Then, for all $\boldsymbol{\phi}_H \in \mathbf{V}_H$

$$\begin{aligned}(\partial_t \mathbf{e}_H^n, \boldsymbol{\phi}_H) + \nu\, a(\mathbf{e}_H^n,\, \boldsymbol{\phi}_H) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{U}_H^n,\, \boldsymbol{\phi}_H) - c^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_H)& \\ = (\mathbf{u}_{Ht}^n,\, \boldsymbol{\phi}_H) - (\partial_t \mathbf{u}_H^n,\, \boldsymbol{\phi}_H).& \end{aligned} \tag{5.57}$$

Step 2. Set $\mathbf{e}_h^n = \mathbf{U}_h^n - \mathbf{u}_h(t_n) = \mathbf{U}_h^n - \mathbf{u}_h^n$, for fixed $n \in \mathbb{N}$, $1 \leq n \leq M$. Then, for all $\boldsymbol{\phi}_h \in \mathbf{V}_h$

$$\begin{aligned}(\partial_t \mathbf{e}_h^n, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{e}_h^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{e}_h^n,\, \boldsymbol{\phi}_h) = (\mathbf{u}_{ht}^n,\, \boldsymbol{\phi}_h) - (\partial_t \mathbf{u}_h^n,\, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\, \mathbf{e}_H^n,\, \boldsymbol{\phi}_h)& \\ -c^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\, \mathbf{e}_H^n,\, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h) - c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\, \mathbf{u}_h^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\, \mathbf{e}_H^n,\, \boldsymbol{\phi}_h)& \\ +c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\, \mathbf{e}_H^n,\, \boldsymbol{\phi}_h) + \big(c^{\mathbf{u}_h^n}(\mathbf{u}_h^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h)\big)& \\ +\big(c^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_h^n,\, \boldsymbol{\phi}_h) - c^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_h^n,\, \boldsymbol{\phi}_h)\big) + \big(c^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h) - c^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h)\big).&\end{aligned} \tag{5.58}$$

The following lemma provides us bounds for the **Step 1** fully discrete error $\mathbf{e}_H^n$.

Lemma 5.10. *Suppose the assumptions of Lemma 5.8 hold true. Then, there exists a constant $K_T > 0$, such that, the following estimates hold true:*

$$\|\boldsymbol{e}_H^n\|^2 + e^{-2\alpha t_M} \Delta t \sum_{n=1}^{M} e^{2\alpha t_n} \|\boldsymbol{e}_H^n\|_\varepsilon^2 \leq K_T \Delta t^2, \tag{5.59}$$

$$e^{-2\alpha t_m}\, \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\partial_t \boldsymbol{e}_H^n\|^2 \;+\; \|\boldsymbol{e}_H^m\|_\varepsilon^2 \;\leq\; K_T \Delta t. \tag{5.60}$$

Proof. The first estimate is similar to Lemma 2.17 of Chapter 2. The only difference in the estimates of the nonlinear terms. For that, we set $\boldsymbol{\phi}_H = \mathbf{e}_H^n$ in (5.57) and rewrite the nonlinear terms to find

$$\begin{aligned}(\partial_t \mathbf{e}_H^n, \mathbf{e}_H^n) + \nu a(\mathbf{e}_H^n, \mathbf{e}_H^n) + c^{\mathbf{U}_H^n}(\mathbf{U}_H^n, \mathbf{e}_H^n, \mathbf{e}_H^n) = (\mathbf{u}_{Ht}^n, \mathbf{e}_H^n) - (\partial_t \mathbf{u}_H^n, \mathbf{e}_H^n)& \\ +c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}^n - \mathbf{u}_H^n, \mathbf{e}_H^n) - c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}^n, \mathbf{e}_H^n) + l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \mathbf{e}_H^n) - l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \mathbf{e}_H^n).&\end{aligned} \tag{5.61}$$

With a similar technique as in the proof of Lemma 2.17 of Chapter 2 and using Lemma 1.3, we can obtain

$$|c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}^n - \mathbf{u}_H^n, \mathbf{e}_H^n)| + |c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}^n, \mathbf{e}_H^n)| \leq C\|\mathbf{e}_H^n\|\|\mathbf{e}_H^n\|_\varepsilon.$$

Since $\mathbf{u}^n$ is continuous, apply (2.60), Theorem 5.2, inequalities (1.35), (1.38) and (1.14) to arrive at

$$\begin{aligned}
&|l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \mathbf{e}_H^n) - l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \mathbf{e}_H^n)| \\
&\quad = |l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}^n - \mathbf{u}_H^n, \mathbf{e}_H^n) - l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}^n - \mathbf{u}_H^n, \mathbf{e}_H^n)| \\
&\quad \le C\|\mathbf{e}_H^n\|_{L^4(\Omega)}\|\mathbf{u} - \mathbf{u}_H^n\|_\varepsilon \|\mathbf{e}_H^n\|_{L^4(\Omega)} \le C\|\mathbf{e}_H^n\|_\varepsilon \|\mathbf{u} - \mathbf{u}_H^n\|_\varepsilon \|\mathbf{e}_H^n\|_\varepsilon \le C\|\mathbf{e}_H^n\|\|\mathbf{e}_H^n\|_\varepsilon.
\end{aligned}$$

Now, proceed similar to Lemma 2.17 of Chapter 2 will follow the estimate (5.59).
To derive the estimate (5.60), we again rewrite the nonlinear terms of (5.57) and set $\boldsymbol{\phi}_H = \partial_t \mathbf{e}_H^n$ to find

$$\begin{aligned}
(\partial_t \mathbf{e}_H^n, \partial_t \mathbf{e}_H^n) + \nu a(\mathbf{e}_H^n, \partial_t \mathbf{e}_H^n) &= (\mathbf{u}_{Ht}^n, \boldsymbol{\phi}_H) - (\partial_t \mathbf{u}_H^n, \boldsymbol{\phi}_H) + c^{\mathbf{U}_H^n}(\mathbf{u}^n - \mathbf{u}_H^n, \mathbf{e}_H^n, \partial_t \mathbf{e}_H^n) \\
&\quad - c^{\mathbf{U}_H^n}(\mathbf{u}^n, \mathbf{e}_H^n, \partial_t \mathbf{e}_H^n) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}^n - \mathbf{u}_H^n, \partial_t \mathbf{e}_H^n) - c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}^n, \partial_t \mathbf{e}_H^n) \\
&\quad - c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{e}_H^n, \partial_t \mathbf{e}_H^n) + l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \partial_t \mathbf{e}_H^n) - l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \partial_t \mathbf{e}_H^n). \qquad (5.62)
\end{aligned}$$

Following similar steps as in Lemma 2.18 of Chapter 2, we obtain

$$\begin{aligned}
&|c^{\mathbf{U}_H^n}(\mathbf{u}^n - \mathbf{u}_H^n, \mathbf{e}_H^n, \partial_t \mathbf{e}_H^n)| + |c^{\mathbf{U}_H^n}(\mathbf{u}^n, \mathbf{e}_H^n, \partial_t \mathbf{e}_H^n)| + |c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}^n - \mathbf{u}_H^n, \partial_t \mathbf{e}_H^n)| \\
&\quad + |c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}^n, \partial_t \mathbf{e}_H^n)| \le C\|\mathbf{e}_H^n\|_\varepsilon \|\partial_t \mathbf{e}_H^n\|. \qquad (5.63)
\end{aligned}$$

A use of estimate (2.55) and Young's inequality implies

$$|c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{e}_H^n, \partial_t \mathbf{e}_H^n)| \le C\|\mathbf{e}_H^n\|_\varepsilon^2 \|\partial_t \mathbf{e}_H^n\|_\varepsilon \le \frac{C}{\Delta t}\|\mathbf{e}_H^n\|_\varepsilon^2 + \frac{C}{\Delta t}\|\mathbf{e}_H^n\|_\varepsilon^2 (\|\mathbf{e}_H^n\|_\varepsilon^2 + \|\mathbf{e}_H^{n-1}\|_\varepsilon^2). \qquad (5.64)$$

Similar to the bound of $l(\cdot;\cdot,\cdot,\cdot)$, one can find

$$\begin{aligned}
&|l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \partial_t \mathbf{e}_H^n) - l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \partial_t \mathbf{e}_H^n)| \\
&= |l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}^n - \mathbf{u}_H^n, \partial_t \mathbf{e}_H^n) - l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}^n - \mathbf{u}_H^n, \partial_t \mathbf{e}_H^n)| \\
&\le C\|\mathbf{e}_H^n\|_\varepsilon \|\mathbf{u} - \mathbf{u}_H^n\|_\varepsilon \|\partial_t \mathbf{e}_H^n\|_\varepsilon \le C\|\mathbf{e}_H^n\|_\varepsilon \|\partial_t \mathbf{e}_H^n\|. \qquad (5.65)
\end{aligned}$$

Substitute (5.63)-(5.65) in (5.62) and following Lemma 2.18 of Chapter 2, we arrive at the estimate (5.60). This completes the rest of the proof. □

The next lemma establishes the bounds for the **Step 2** fully discrete error $\mathbf{e}_h^n$.

Lemma 5.11. *Suppose the assumptions of Theorem 5.2 and Lemma 5.8 hold true and choose k_0 small so that $0 < \Delta t \le k_0$. Then, there exists a constant $K_T > 0$, such that, the following estimates hold true:*

$$\|\boldsymbol{e}_h^n\|^2 + e^{-2\alpha t_M}\Delta t \sum_{n=1}^{M} e^{2\alpha t_n}\|\boldsymbol{e}_h^n\|_\varepsilon^2 \le K_T \Delta t^2.$$

Proof. We put $\boldsymbol{\phi}_h = \mathbf{e}_h^n$ in error equation (5.58). With the observation

$$(\partial_t \mathbf{e}_h^n, \mathbf{e}_h^n) \geq \frac{1}{2}\partial_t \|\mathbf{e}_h^n\|^2,$$

and a use of Lemma 1.6 and (1.19) yields

$$\begin{aligned}
\partial_t\|\mathbf{e}_h^n\|^2 + 2K_1\,\nu\,\|\mathbf{e}_h^n\|_\varepsilon^2 \;&\leq\; 2\big((\mathbf{u}_{ht}^n, \mathbf{e}_h^n) - (\partial_t \mathbf{u}_h^n, \mathbf{e}_h^n)\big) - c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\, \mathbf{e}_H^n,\, \mathbf{e}_h^n)\\
&- c^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\, \mathbf{e}_H^n,\, \mathbf{e}_h^n) - c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\, \mathbf{u}_H^n,\, \mathbf{e}_h^n) - c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\, \mathbf{u}_h^n,\, \mathbf{e}_h^n) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\, \mathbf{e}_H^n,\, \mathbf{e}_h^n)\\
&+ c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\, \mathbf{u}_H^n,\, \mathbf{e}_h^n) + c^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\, \mathbf{e}_H^n,\, \mathbf{e}_h^n) + \big(l^{\mathbf{u}_h^n}(\mathbf{u}_h^n,\, \mathbf{u}_H^n,\, \mathbf{e}_h^n) - l^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\, \mathbf{u}_H^n,\, \mathbf{e}_h^n)\big)\\
&+ \big(l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_h^n,\, \mathbf{e}_h^n) - l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_h^n,\, \mathbf{e}_h^n)\big) + \big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_H^n,\, \mathbf{e}_h^n) - l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_H^n,\, \mathbf{e}_h^n)\big)\\
=& I_1 + I_2 + I_3 + \cdots + I_{11}.
\end{aligned} \tag{5.66}$$

A use of (2.55), the Cauchy-Schwarz and Young's inequalities leads to the following bound:

$$\begin{aligned}
|I_3| &+ |I_5| + |I_6| + |I_7| + |I_8|\\
&\leq C\big(\|\mathbf{u}_h^n\|_\varepsilon \|\mathbf{e}_H^n\|_\varepsilon \|\mathbf{e}_h^n\|_\varepsilon + \|\mathbf{e}_H^n\|_\varepsilon^2 \|\mathbf{e}_h^n\|_\varepsilon + \|\mathbf{e}_H^n\|_\varepsilon \|\mathbf{u}_H^n\|_\varepsilon \|\mathbf{e}_h^n\|_\varepsilon\big)\\
&\leq \frac{K_1\nu}{64}\|\mathbf{e}_h^n\|_\varepsilon^2 + C\big(\|\mathbf{u}_h^n\|_\varepsilon^2 + \|\mathbf{e}_H^n\|_\varepsilon^2 + \|\mathbf{u}_H^n\|_\varepsilon^2\big)\|\mathbf{e}_H^n\|_\varepsilon^2.
\end{aligned} \tag{5.67}$$

The terms I_2 and I_4 are bounded using the same technique as in Lemma 5.5 for estimating $c^{\mathbf{u}_h}(\mathbf{P}_h\mathbf{e}, \mathbf{u}_H, \mathbf{P}_h\mathbf{e})$. An application of the Cauchy-Schwarz inequality, Young inequality and 2.57 leads to

$$\begin{aligned}
|I_2| + |I_4| &\leq C\,\|\mathbf{e}_h^n\|^{1/2}\big(\|\mathbf{e}_H^n\|_\varepsilon + \|\mathbf{u}_H^n\|_\varepsilon\big)\|\mathbf{e}_h^n\|_\varepsilon^{3/2}\\
&\leq \frac{K_1\nu}{64}\|\mathbf{e}_h^n\|_\varepsilon^2 + C\,\|\mathbf{e}_h^n\|^2\big(\|\mathbf{e}_H^n\|_\varepsilon^4 + \|\mathbf{u}_H^n\|_\varepsilon^4\big).
\end{aligned} \tag{5.68}$$

Use a result in (2.60), Lemma 2.6 and estimate (1.14), we find that

$$|I_9| \leq C\|\mathbf{e}_h^n\|_{L^4(\Omega)}\|\mathbf{u}_H^n\|_\varepsilon\|\mathbf{e}_h^n\|_\varepsilon \leq \frac{K_1\nu}{64}\|\mathbf{e}_h^n\|_\varepsilon^2 + C\,\|\mathbf{e}_h^n\|^2\|\mathbf{u}_H^n\|_\varepsilon^4 \tag{5.69}$$

$$|I_{10}| \leq C\|\mathbf{e}_H^n\|_{L^4(\Omega)}\|\mathbf{u}_h^n\|_\varepsilon\|\mathbf{e}_h^n\|_\varepsilon \leq \frac{K_1\nu}{64}\|\mathbf{e}_h^n\|_\varepsilon^2 + C\,\|\mathbf{e}_H^n\|_\varepsilon^2\|\mathbf{u}_h^n\|_\varepsilon^2 \tag{5.70}$$

$$|I_{11}| \leq C\|\mathbf{e}_H^n\|_{L^4(\Omega)}\|\mathbf{u}_H^n\|_\varepsilon\|\mathbf{e}_h^n\|_\varepsilon \leq \frac{K_1\nu}{64}\|\mathbf{e}_h^n\|_\varepsilon^2 + C\,\|\mathbf{e}_H^n\|_\varepsilon^2\|\mathbf{u}_H^n\|_\varepsilon^2. \tag{5.71}$$

From (2.138), the Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned}
|I_1| &\leq C\,\Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}(s)\|_{-1,h}^2\,ds\right)^{1/2}\|\mathbf{e}_h^n\|_\varepsilon\\
&\leq \frac{K_1\nu}{64}\|\mathbf{e}_h^n\|_\varepsilon^2 \;+\; C\Delta t\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hss}(s)\|_{-1,h}^2\,ds.
\end{aligned} \tag{5.72}$$

Combine (5.67)-(5.72), multiply (5.66) by $\Delta t e^{2\alpha n\Delta t}$, sum the resulting inequality from $n=1$ to $m\,(\leq M)$ and obtain

$$\begin{aligned} e^{2\alpha m\Delta t}\|\mathbf{e}_h^m\|^2 + K_1\nu\Delta t\sum_{n=1}^{m} e^{2\alpha n\Delta t}\|\mathbf{e}_h^n\|_\varepsilon^2 \leq& \sum_{n=1}^{m-1} e^{2\alpha n\Delta t}(e^{2\alpha\Delta t}-1)\|\mathbf{e}_h^n\|^2 \\ &+ C\Delta t\sum_{n=1}^{m} e^{2\alpha n\Delta t}\|\mathbf{e}_h^n\|^2\big(\|\mathbf{e}_H^n\|_\varepsilon^4 + \|\mathbf{u}_H^n\|_\varepsilon^4\big) \\ &+C\Delta t\sum_{n=1}^{m} e^{2\alpha n\Delta t}\big(\|\mathbf{u}_h^n\|_\varepsilon^2 + \|\mathbf{e}_H^n\|_\varepsilon^2 + \|\mathbf{u}_H^n\|_\varepsilon^2\big)\|\mathbf{e}_H^n\|_\varepsilon^2 \\ &+ C\Delta t^2\sum_{n=1}^{m} e^{2\alpha n\Delta t}\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hss}(s)\|_{-1,h}^2\,ds. \end{aligned} \tag{5.73}$$

We bound the terms involving $\mathbf{u}_h$ using Lemma 5.7. Observe that

$$\begin{aligned} C\Delta t^2\sum_{n=1}^{m} e^{2\alpha n\Delta t}\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hss}(s)\|_{-1,h}^2\,ds =& \; C\Delta t^2\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} e^{2\alpha(t_n-s)}e^{2\alpha s}\|\mathbf{u}_{hss}(s)\|_{-1,h}^2\,ds \\ \leq C\Delta t^2 e^{2\alpha\Delta t}\int_0^{t_m} e^{2\alpha s}\|\mathbf{u}_{hss}(s)\|_{-1,h}^2\,ds \leq& \; C\Delta t^2 e^{2\alpha(m+1)\Delta t}. \end{aligned} \tag{5.74}$$

Applying (5.74), (5.13), and Lemmas 5.7 and 5.10 in (5.73) and using the fact $e^{2\alpha\Delta t}-1\leq C(\alpha)\Delta t$, we obtain

$$\begin{aligned} e^{2\alpha m\Delta t}\|\mathbf{e}_h^m\|^2 + K_1\nu\Delta t\sum_{n=1}^{m} e^{2\alpha n\Delta t}\|\mathbf{e}_h^n\|_\varepsilon^2 \leq& C\Delta t\sum_{n=1}^{m} e^{2\alpha n\Delta t}\|\mathbf{e}_h^n\|^2 + C\Delta t^2 e^{2\alpha m\Delta t} \\ &+ C\Delta t^2 e^{2\alpha(m+1)\Delta t}. \end{aligned}$$

Now the desired result is achieved by employing discrete Gronwall's lemma. □

Lemma 5.12. *Suppose the assumptions of Lemma 5.11 hold true. Then, the fully discrete velocity error* $\boldsymbol{e}_h^n = \boldsymbol{U}_h^n - \boldsymbol{u}_h^n$ *satisfies*

$$e^{-2\alpha t_m}\,\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\|\partial_t\boldsymbol{e}_h^n\|^2 \;+\; \nu\,K_1\,\|\boldsymbol{e}_h^m\|_\varepsilon^2 \;\leq\; K_T\,\Delta t.$$

Proof. In (5.58), we choose $\boldsymbol{\phi}_h = \partial_t\mathbf{e}_h^n$ and rewrite the nonlinear terms to obtain

$$\begin{aligned} &\|\partial_t\mathbf{e}_h^n\|^2 + \nu\,a(\mathbf{e}_h^n,\partial_t\mathbf{e}_h^n) = (\mathbf{u}_{ht}^n,\partial_t\mathbf{e}_h^n) - (\partial_t\mathbf{u}_h^n,\partial_t\mathbf{e}_h^n) - c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\,\mathbf{e}_h^n,\,\partial_t\mathbf{e}_h^n) \\ &- c^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\,\mathbf{e}_h^n,\,\partial_t\mathbf{e}_h^n) - c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\,\mathbf{e}_H^n,\,\partial_t\mathbf{e}_h^n) - c^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\,\mathbf{e}_H^n,\,\partial_t\mathbf{e}_h^n) - c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\,\mathbf{u}_H^n,\,\partial_t\mathbf{e}_h^n) \\ &- c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\,\mathbf{u}_h^n,\,\partial_t\mathbf{e}_h^n) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\,\mathbf{e}_H^n,\,\partial_t\mathbf{e}_h^n) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\,\mathbf{u}_H^n,\,\partial_t\mathbf{e}_h^n) + c^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\,\mathbf{e}_H^n,\,\partial_t\mathbf{e}_h^n) \\ &+ \big(l^{\mathbf{u}_h^n}(\mathbf{u}_h^n,\,\mathbf{u}_H^n,\,\partial_t\mathbf{e}_h^n) - l^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\,\mathbf{u}_H^n,\,\partial_t\mathbf{e}_h^n)\big) + \big(l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_h^n,\,\partial_t\mathbf{e}_h^n) - l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_h^n,\,\partial_t\mathbf{e}_h^n)\big) \\ &+ \big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_H^n,\,\partial_t\mathbf{e}_h^n) - l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\,\mathbf{u}_H^n,\,\partial_t\mathbf{e}_h^n)\big). \end{aligned} \tag{5.75}$$

From (2.138), we have

$$(\mathbf{u}_{ht}^n, \partial_t \mathbf{e}_h^n) - (\partial_t \mathbf{u}_h^n, \partial_t \mathbf{e}_h^n) \leq C\Delta t^{1/2} \left(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}(s)\|_{-1,h}^2 \, ds \right)^{1/2} \|\partial_t \mathbf{e}_h^n\|_\varepsilon$$
$$\leq C \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}(s)\|_{-1,h}^2 \, ds + \frac{C}{\Delta t} \|\mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_\varepsilon^2. \tag{5.76}$$

The remaining terms on the right hand side of (5.75) can be bounded above similar to Lemma 5.11 by

$$\frac{C}{\Delta t}\big(\|\mathbf{e}_h^n\|_\varepsilon^2 + \|\mathbf{e}_h^{n-1}\|_\varepsilon^2\big) + \frac{C}{\Delta t}\big(\|\mathbf{e}_h^n\|_\varepsilon^2 \|\mathbf{e}_H^n\|_\varepsilon^2 + \|\mathbf{u}_h^n\|_\varepsilon^2 \|\mathbf{e}_H^n\|_\varepsilon^2 + \|\mathbf{e}_h^n\|_\varepsilon^2 \|\mathbf{u}_H^n\|_\varepsilon^2$$
$$+ \|\mathbf{e}_H^n\|_\varepsilon^4 + \|\mathbf{e}_H^n\|_\varepsilon^2 \|\mathbf{u}_H^n\|_\varepsilon^2\big) \tag{5.77}$$

Since $a(\cdot, \cdot)$ is symmetric, one can obtain

$$a(\mathbf{e}_h^n, \partial_t \mathbf{e}_h^n) = \frac{1}{2}\bigg(\frac{1}{\Delta t} a(\mathbf{e}_h^n, \mathbf{e}_h^n) - \frac{1}{\Delta t} a(\mathbf{e}_h^{n-1}, \mathbf{e}_h^{n-1}) + \Delta t a(\partial_t \mathbf{e}_h^n, \partial_t \mathbf{e}_h^n) \bigg). \tag{5.78}$$

Again, we have

$$\sum_{n=1}^{m} e^{2\alpha t_n} \Big(a(\mathbf{e}_h^n, \mathbf{e}_h^n) - a(\mathbf{e}_h^{n-1}, \mathbf{e}_h^{n-1}) \Big) = e^{2\alpha t_m} a(\mathbf{e}_h^m, \mathbf{e}_h^m) - \sum_{n=1}^{m-1} e^{2\alpha t_n} (e^{2\alpha \Delta t} - 1) a(\mathbf{e}_h^n, \mathbf{e}_h^n). \tag{5.79}$$

Combining (5.76)–(5.79), multiply (5.75) by $\Delta t e^{2\alpha t_n}$, sum over $n = 1$ to $m\,(\leq M)$ and using Lemmas 1.6, 1.7, we obtain

$$\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \|\partial_t \mathbf{e}_h^n\|^2 + \nu K_1 e^{2\alpha t_m} \|\mathbf{e}_h^m\|_\varepsilon^2$$
$$\leq C\,\Delta t \sum_{n=1}^{m-1} e^{2\alpha t_n} \|\mathbf{e}_h^n\|_\varepsilon^2 + \frac{C}{\Delta t} \Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \big(\|\mathbf{e}_h^n\|_\varepsilon^2 + \|\mathbf{e}_h^{n-1}\|_\varepsilon^2\big)$$
$$+ \frac{C}{\Delta t}\,\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \big(\|\mathbf{e}_h^n\|_\varepsilon^2 \|\mathbf{e}_H^n\|_\varepsilon^2 + \|\mathbf{u}_h^n\|_\varepsilon^2 \|\mathbf{e}_H^n\|_\varepsilon^2 + \|\mathbf{e}_h^n\|_\varepsilon^2 \|\mathbf{u}_H^n\|_\varepsilon^2$$
$$+ \|\mathbf{e}_H^n\|_\varepsilon^4 + \|\mathbf{e}_H^n\|_\varepsilon^2 \|\mathbf{u}_H^n\|_\varepsilon^2\big) + C\,\Delta t \sum_{n=1}^{m} e^{2\alpha t_n} \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{hss}(s)\|_{-1,h}^2 \, ds. \tag{5.80}$$

Finally, a use of (5.13), Lemmas 5.7, 5.10 and 5.11 give us the desired estimate. This completes the proof. □

Now, from Theorem 5.1, Lemmas 5.11 and 5.12, we conclude the **Step 2** fully discrete energy norm estimate for velocity:

Theorem 5.3. *Suppose the assumptions of Theorem 5.1 and Lemma 5.11 are satisfied. Then, for $1 \leq m \leq M$, the following estimates hold true:*

$$
\begin{aligned}
e^{-2\alpha t_m}\Delta t\sum_{n=1}^{m} e^{2\alpha t_n}\|\boldsymbol{u}^n - \boldsymbol{U}_h^n\|_\varepsilon^2 \leq K_T\big(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2} \\
+ h^{-1}H^{2r}(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}) + \Delta t^2\big),\\
\|\boldsymbol{u}^m - \boldsymbol{U}_h^m\|_\varepsilon^2 \leq K_T\big(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2} \\
+ h^{-1}H^{2r}(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}) + \Delta t\big).
\end{aligned}
$$

Lemma 5.13. *Suppose the assumptions of Lemma 5.11 hold true. Then, the fully discrete velocity error $\boldsymbol{e}_h^n = \boldsymbol{U}_h^n - \boldsymbol{u}_h^n$ satisfies*

$$\|\partial_t \boldsymbol{e}_h^n\|_{-1,h} \leq K_T\,\Delta t^{1/2}.$$

Proof. Rewrite the nonlinear terms of (5.58) to obtain

$$
\begin{aligned}
&(\partial_t\mathbf{e}_h^n, \boldsymbol{\phi}_h) = (\mathbf{u}_{ht}^n, \boldsymbol{\phi}_h) - (\partial_t\mathbf{u}_h^n, \boldsymbol{\phi}_h) - \nu\, a(\mathbf{e}_h^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{e}_h^n, \boldsymbol{\phi}_h)\\
&- c^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{e}_h^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{e}_h^n, \mathbf{e}_H^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{u}_h^n, \mathbf{e}_H^n, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{e}_h^n, \mathbf{u}_H^n, \boldsymbol{\phi}_h)\\
&- c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{e}_H^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n, \mathbf{u}_H^n, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{e}_H^n, \boldsymbol{\phi}_h)\\
&+ \big(l^{\mathbf{u}_h^n}(\mathbf{u}_h^n, \mathbf{u}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{U}_h^n}(\mathbf{u}_h^n, \mathbf{u}_H^n, \boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h) - l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h)\big)\\
&+ \big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \boldsymbol{\phi}_h) - l^{\mathbf{u}_H^n}(\mathbf{u}_H^n, \mathbf{u}_H^n, \boldsymbol{\phi}_h)\big). \qquad (5.81)
\end{aligned}
$$

Similar to Lemma 5.11 and apply Lemma 5.7 to arrive at

$$(\mathbf{u}_{ht}^n, \boldsymbol{\phi}_h) - (\partial_t\mathbf{u}_h^n, \boldsymbol{\phi}_h) \leq C\Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{hss}(s)\|_{-1,h}^2\,ds\right)^{1/2}\|\boldsymbol{\phi}_h\|_\varepsilon \leq C\Delta t^{1/2}\|\boldsymbol{\phi}_h\|_\varepsilon.$$

An application of Lemma 1.7 implies

$$\nu|a(\mathbf{e}_h^n, \boldsymbol{\phi}_h)| \leq \|\mathbf{e}_h^n\|_\varepsilon\|\boldsymbol{\phi}_h\|_\varepsilon.$$

Other terms on the right hand side of (5.81) is bounded similar to Lemma 5.11 by

$$C\big(\|\mathbf{e}_h^n\|_\varepsilon\|\mathbf{e}_H^n\|_\varepsilon + \|\mathbf{u}_h^n\|_\varepsilon\|\mathbf{e}_H^n\|_\varepsilon + \|\mathbf{e}_h^n\|_\varepsilon\|\mathbf{u}_H^n\|_\varepsilon + \|\mathbf{e}_H^n\|_\varepsilon^2 + \|\mathbf{e}_H^n\|_\varepsilon\|\mathbf{u}_H^n\|_\varepsilon\big)\|\boldsymbol{\phi}_h\|_\varepsilon.$$

Combining the above bounds in (5.81), and applying the definition of $\|\cdot\|_{-1,h}$, (5.13), Lemmas 5.7, 5.10 and 5.12, we arrive at the desired estimate. □

Lemma 5.14. *Suppose the hypotheses of Lemma 5.11 be satsified. Then, there exists a constant $C = C(\nu, \alpha, K_1, C_2,\ \lambda_1, M_0) > 0$, such that, the following estimates hold true:*

$$\|P_h^n - p_h^n\| \leq K_T\,\Delta t^{1/2},\ 1 \leq n \leq M.$$

Proof. Subtract (5.5) from (5.46) to find

$$\begin{aligned}
b(\boldsymbol{\phi}_h, P_h^n - p_h^n) &= -(\partial_t \mathbf{e}_h^n, \boldsymbol{\phi}_h) - \nu\, a(\mathbf{e}_h^n,\, \boldsymbol{\phi}_h) + (\mathbf{u}_{ht}^n,\, \boldsymbol{\phi}_h) - (\partial_t \mathbf{u}_h^n,\, \boldsymbol{\phi}_h) \\
&- c^{\mathbf{U}_H^n}(\mathbf{U}_H^n,\, \mathbf{e}_h^n,\, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\, \mathbf{e}_H^n,\, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\, \mathbf{e}_H^n,\, \boldsymbol{\phi}_h) - c^{\mathbf{U}_h^n}(\mathbf{e}_h^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h) \\
&- c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\, \mathbf{u}_h^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\, \mathbf{e}_H^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{e}_H^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h) + c^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\, \mathbf{e}_H^n,\, \boldsymbol{\phi}_h) \\
&+ \big(l^{\mathbf{u}_h^n}(\mathbf{u}_h^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h) - l^{\mathbf{U}_h^n}(\mathbf{u}_h^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h)\big) + \big(l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_h^n,\, \boldsymbol{\phi}_h) - l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_h^n,\, \boldsymbol{\phi}_h)\big) \\
&+ \big(l^{\mathbf{U}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h) - l^{\mathbf{u}_H^n}(\mathbf{u}_H^n,\, \mathbf{u}_H^n,\, \boldsymbol{\phi}_h)\big). \qquad (5.82)
\end{aligned}$$

Using Lemma 1.8 and bounding the terms on the right hand side of (5.82) following the steps involved in the proof of Lemma 5.11, we arrive at

$$\|P_h^n - p_h^n\| \le C\|\partial_t \mathbf{e}_h^n\|_{-1,h} + C\|\mathbf{e}_h^n\|_\varepsilon + C\Delta t^{1/2}.$$

An application of Lemmas 5.12 and 5.13 completes the proof of this lemma. □

Combining Theorem 5.2 and Lemma 5.14, we arrive at the **Step 2** fully discrete pressure error estimate:

Theorem 5.4. *Suppose the assumptions of Theorem 5.2 and Lemma 5.14 are satisfied. Then, for $1 \le n \le M$, the following estimates hold true:*

$$\begin{aligned}
\|p^n - P_h^n\|_\varepsilon^2 \le K_T\big(&h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2} \\
&+ h^{-1}H^{2r}(h^{2r} + h^{2r/(r+1)}H^{2r} + H^{2r+2}) + \Delta t\big).
\end{aligned}$$

Remark 5.3. *In this chapter, we have obtained optimal estimates in energy norm for velocity error and in L^2-norm for pressure error, by employing two-step two-grid scheme, in the DG framework. However, this is not sufficient to obtain optimal L^2-norm error estimate for velocity, and an additional correction step is needed. Such a result is available in the context of CG, see [16]. A correction step would be employed in the next two chapters for the two linear viscoelastic models, as these models have been our main focus to develop these DG schemes.*

5.5 Numerical Experiments

In this section, a few numerical experiments are performed and the theoretical findings are confirmed. For space discretization $\mathbb{P}_r - \mathbb{P}_{r-1}$, $r = 1, 2$, DG finite elements are employed and for time discretization, backward Euler method is applied. We choose the domain $\Omega = [0, 1]^2$. We have considered here three examples, where the first two are computed on the time interval $[0,1]$ with the final time $T=1$, the time step $\Delta t=\mathcal{O}(h^r)$ and $h=\mathcal{O}(H^{\frac{r+1}{r}})$. And the third example is analyzed on the time interval $[0,100]$ with final time $T=100$.

Example 5.1. *Consider the transient NSEs with exact solution* $(\boldsymbol{u},p)=\big((u_1(x,y,t),$ $u_2(x,y,t)),p(x,y,t)\big)$ *as*

$$\begin{aligned} u_1(x,y,t) =& 10x^2(x-1)^2y(y-1)(2y-1)\cos(t),\\ u_2(x,y,t) =& -10y^2(y-1)^2x(x-1)(2x-1)\cos(t)\\ p(x,y,t) =& 10\cos(t)(3y^2-1). \end{aligned}$$

In Tables 5.1 and 5.2, we represent the computational errors and orders of convergence for the two-grid DG solution of (5.45)-(5.46) for $r=1$ and 2 with viscositiy $\nu=1$, respectively. Further, Tables 5.3 and 5.4 represent numerical errors and orders of convergence for $r=1$ and 2, respectively, with $\nu=1/100$. We set the penalty parameter $\sigma_e=20$ and 40 for $r=1$ and 2, respectively. We notice that the numerical outcomes of Tables 5.1-5.4 confirm the theoretically derived convergence orders, which is of $\mathcal{O}(h^r)$ in energy and L^2-norms for velocity and pressure, respectively.

Table 5.1: Errors for two-grid DG approximations and order of convergence for Example 5.1 with $r=1$ and $\nu=1$.

h	$\|\mathbf{u}(t_M)-\mathbf{U}_h^M\|_\varepsilon$	Rate	$\|p(t_M)-P_h^M\|$	Rate
1/4	0.253901		0.192905	
1/8	0.110618	1.1986	0.092012	1.0679
1/16	0.049209	1.1685	0.048684	0.9183
1/32	0.022671	1.1180	0.025302	0.9441
1/64	0.010805	1.0691	0.012949	0.9664

Table 5.2: Errors for two-grid DG approximations and order of convergence for Example 5.1 with $r=2$ and $\nu=1$.

h	$\|\mathbf{u}(t_M)-\mathbf{U}_h^M\|_\varepsilon$	Rate	$\|p(t_M)-P_h^M\|$	Rate
1/4	0.020730		0.172635	
1/8	0.004967	2.0612	0.043092	2.0022
1/16	0.001197	2.0524	0.010762	2.0014
1/32	0.000295	2.0203	0.002689	2.0007

Table 5.3: Errors for two-grid DG approximations and order of convergence for Example 5.1 with $r = 1$ and $\nu = 1/100$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}_h^M\|_\varepsilon$	Rate	$\|p(t_M) - P_h^M\|$	Rate
1/4	4.074005		0.991889	
1/8	1.329392	1.6156	0.110527	3.1657
1/16	0.565533	1.2330	0.046946	1.2353
1/32	0.259262	1.1251	0.023482	0.9994
1/64	0.122572	1.0807	0.011924	0.9776

Table 5.4: Errors for two-grid DG approximations and order of convergence for Example 5.1 with $r = 2$ and $\nu = 1/100$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}_h^M\|_\varepsilon$	Rate	$\|p(t_M) - P_h^M\|$	Rate
1/4	1.400594		0.181643	
1/8	0.300850	2.2189	0.042907	2.0818
1/16	0.047228	2.6713	0.010640	2.0117
1/32	0.011684	2.0150	0.002657	2.0013

Example 5.2. *In this example, the choice of right-hand side source function* $\mathbf{f}$ *is made in such a manner that the exact solution* $(\boldsymbol{u},\, p) = \big((u_1(x,y,t),\, u_2(x,y,t)),\, p(x,y,t)\big)$ *takes the following form:*

$$\begin{aligned} u_1(x,y,t) =& t\, e^{-t^2}\, \sin(2\pi y)(1-\cos(2\pi x)),\\ u_2(x,y,t) =& t\, e^{-t^2}\, \sin(2\pi x)(\cos(2\pi y)-1),\\ p(x,y,t) =& 2\,\pi\, t\, e^{-t}(\cos(2\pi y)-\cos(2\pi x)). \end{aligned}$$

Tables 5.5 and 5.8 depict the error and convergence orders of the two-grid DG scheme for $r = 1$ and 2, respectively, with $\nu = 1/10$. The penalty parameter σ_e is chosen same as in Example 5.1. These results verify the derived theoretical results. Additionally, we compute the approximate solutions by the standard direct DG scheme to better assess the performance of our two-grid DG scheme with the same fine mesh, σ_e and Δt. Tables 5.6 and 5.9 represent the numerical error and convergence orders for $r =$ 1 and 2, respectively, employed in the direct DG scheme. By comparing Table 5.5 with Table 5.6 and Table 5.8 with Table 5.9, we observe that the accuracy of the numerical solutions by the proposed two-grid DG method is quite close to that of the direct DG method. In Tables 5.7 and 5.10, we compare the computational times taken to compute the two-grid DG solution and the direct DG solution corresponding to $r = 1$ and 2, respectively. The tables demonstrate that the proposed two-grid DG method requires significantly less computational time than the direct DG method. Additionally, as we refine the mesh more and more, the computational time gap increases between both solutions, namely the two-grid DG solution and the direct DG solution.

Table 5.5: Errors for two-grid DG approximations and order of convergence for Example 5.2 with $r = 1$ and $\nu = 1/10$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}_h^M\|_\varepsilon$	Rate	$\|p(t_M) - P_h^M\|$	Rate
1/4	1.380453		0.375059	
1/8	0.815307	0.7597	0.262572	0.5144
1/16	0.380265	1.1003	0.187638	0.4847
1/32	0.159587	1.2526	0.108098	0.7956
1/64	0.072591	1.1364	0.056563	0.9343

Table 5.6: Errors for direct DG approximations and order of convergence for Example 5.2 with $r = 1$ and $\nu = 1/10$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}_{DGh}^M\|_\varepsilon$	Rate	$\|p(t_M) - P_h^M\|$	Rate
1/4	1.391405		0.377845	
1/8	0.815558	0.7706	0.257370	0.5539
1/16	0.381141	1.0974	0.179631	0.5188
1/32	0.160671	1.2462	0.105951	0.7616
1/64	0.072459	1.1488	0.056166	0.9156

Table 5.7: Comparison of computational time between “direct DG solution” and the solution obtained by the ”two-grid DG method” for Example 5.2 with $r = 1$.

h	Two-grid DG solution (in Seconds)	Direct DG solution (in Seconds)
1/4	0.38	0.56
1/8	2.82	6.89
1/16	28.53	104.45
1/32	386.42	1705.05
1/64	5989.01	28981.81

Table 5.8: Errors for two-grid DG approximations and order of convergence for Example 5.2 with $r = 2$ and $\nu = 1/10$.

h	$\|\mathbf{u}(t_M) - \mathbf{U}_h^M\|_\varepsilon$	Rate	$\|p(t_M) - P_h^M\|$	Rate
1/4	0.434541		0.462235	
1/8	0.104623	2.0542	0.121759	1.9245
1/16	0.023376	2.1620	0.030648	1.9901
1/32	0.005668	2.0439	0.007680	1.9965

Table 5.9: Errors for direct DG approximations and order of convergence for Example 5.2 with $r = 2$ and $\nu = 1/10$.

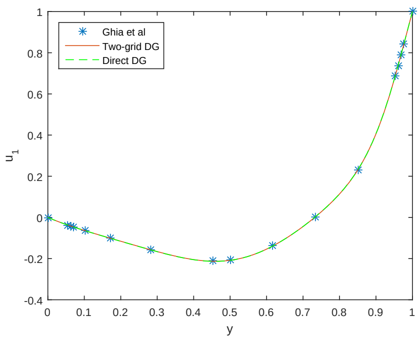
h	$\|\mathbf{u}(t_M) - \mathbf{U}_{DGh}^M\|_\varepsilon$	Rate	$\|p(t_M) - P_{DGh}^M\|$	Rate
1/4	0.429612		0.459927	
1/8	0.102943	2.0611	0.121039	1.9259
1/16	0.023159	2.1521	0.030522	1.9875
1/32	0.005511	2.0710	0.007642	1.9977

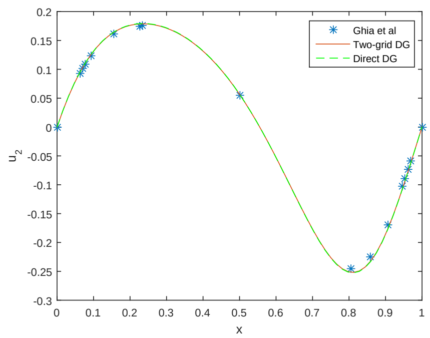
Table 5.10: Comparison of computational time between "direct DG solution" and the solution obtained by the "two-grid DG method" for Example 5.2 with $r = 2$.

h	Two-grid DG solution (in Seconds)	Direct DG solution (in Seconds)
1/4	8.00	9.25
1/8	51.46	94.88
1/16	564.98	1358.13
1/32	7560.98	22140.62

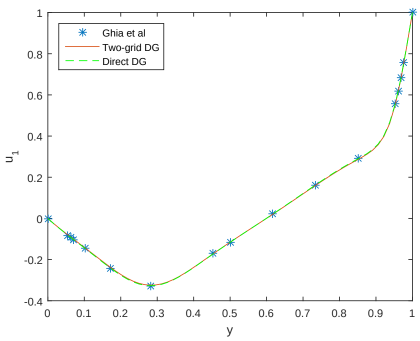
Example 5.3 (Benchmark Problem)**.** *In this case, the lid-driven cavity flow on the computational domain* $[0, 1]^2$ *is examined. The velocity at the top of the boundary* $\boldsymbol{u} = (1, 0)$*, is what majorly drives the flow of fluid. Other portions of the cavity boundaries are subject to the no-slip boundary conditions. On the body, no forces are acting i.e.,* $\mathbf{f} = (0, 0)$*.*

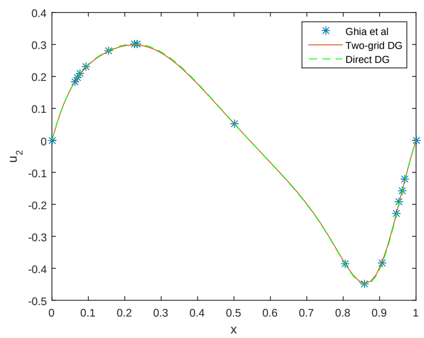
We perform a comparison along $(x, 0.5)$ and $(0.5, y)$ lines for velocity components for both two-grid DG and direct DG schemes against the data presented by Ghia et al. in [66] with $r = 1$. For the time discretization backward Euler method is employed with $\Delta t = 0.01$ and final time $T = 100$. For the sake of simplicity, numerical simulations of the direct DG and two-grid DG methods are conducted with the uniform mesh sizes $h = 1/64$ and $H = 1/32$ (only for two-grid DG) to present the stability and accuracy of our method. For this test case, we choose different $\nu = \{1/100, 1/400, 1/1000\}$, and the penalty parameter is $\sigma_e = 40$.
The comparisons of the horizontal velocity component at $x = 0.5$ and the vertical velocity component at $y = 0.5$ are shown in Figure 5.1 to indicate that the direct DG and two-grid DG methods produce similar numerical solutions that can be compared with those presented in [66].
At last, the velocity streamlines of NSEs utilising the two-grid DG scheme are shown in Figure 5.2, and they coincide well with the experimental results in [66]. As ν reduces, we can observe that the primary vortex shifts toward the cavity's center. A second vortex may also develop in the cavity's right bottom corner, and a third vortex may form in its left bottom corner. A good agreement with the results in [66] is obtained in all the cases.



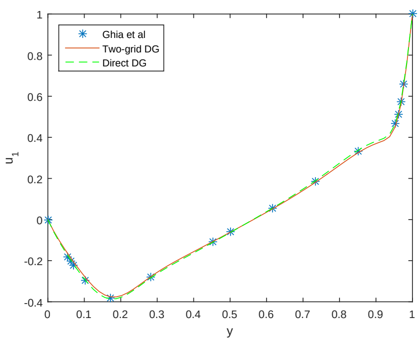


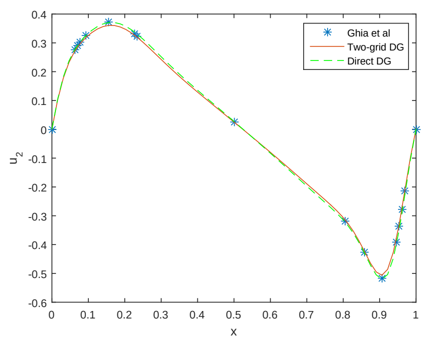
(a) First velocity component along $x = 0.5$ (b) Second velocity component along $y = 0.5$





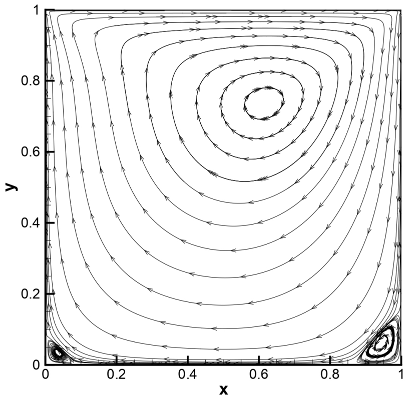
(c) First velocity component along $x = 0.5$ (d) Second velocity component along $y = 0.5$



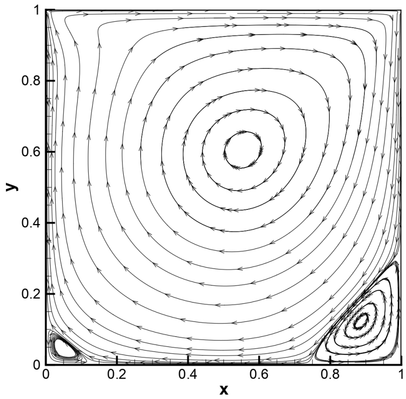


(e) First velocity component along $x = 0.5$ (f) Second velocity component along $y = 0.5$

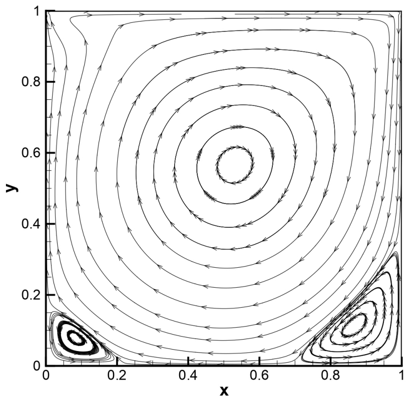
Figure 5.1: Comparison of velocity components for lid driven cavity flow with $\nu = 1/100$ (first row), $\nu = 1/400$ (second row) and $\nu = 1/1000$ (third row).



(a) Streamlines for $\nu = 1/100$.



(b) Streamlines for $\nu = 1/400$.



(c) Streamlines for $\nu = 1/1000$.

Figure 5.2: Streamlines for NSEs of the two-grid DG scheme with $\nu = 1/100, 1/400$, $1/1000$.

5.6 Conclusion

This chapter applies a two-grid scheme to the DG model of time-dependent NSEs. Optimal semi-discrete two-grid DG error estimates for velocity and pressure approximations in energy and $L^\infty(L^2)$-norms, respectively, are established for an appropriate choice of coarse and fine mesh parameters. And under smallness condition on data, these estimates are shown uniformly with time. A full discretization of the semi-discrete two-grid model is achieved by applying a backward Euler method in the time direction. Fully discrete error estimates are derived. Finally, numerical results are depicted to show the effectiveness of the scheme.