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DECLARATION BY THE CANDIDATE

I, Dharmaraj Deka, hereby declare that the subject matter in this thesis entitled “Nonuniform Padé based compact schemes for fluid and heat flow problems: development and application” is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

Date: 29-11-2024

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This is to certify that the thesis entitled, “**Nonuniform Padé based compact schemes for fluid and heat flow problems: development and application**” submitted to the School of Sciences of Tezpur University in partial fulfilment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by **Dharmaraj Deka** under my supervision and guidance. All the help received by him from various sources have been duly acknowledged. No part of this thesis has been submitted elsewhere for award of any other degree.

Date: 29-11-2024

Place: Tezpur

Shuvam Sen

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