Chapter 6

Arithmetic Properties for Certain Restricted ℓ-Regular Partitions

6.1 Introduction

We recall from Section 1.8 that the generating functions for the restricted ℓ-regular partition functions $\mathrm{pod}_\ell(n)$ and $\mathrm{ped}_\ell(n)$ are given by

$$\sum_{n=0}^{\infty} \mathrm{pod}_\ell(n) q^n = \frac{f_2 f_\ell f_{4\ell}}{f_1 f_4 f_{2\ell}} \tag{6.1}$$

and

$$\sum_{n=0}^{\infty} \mathrm{ped}_\ell(n) q^n = \frac{f_4 f_\ell}{f_1 f_{4\ell}}. \tag{6.2}$$

Recently, Veena and Fathima [122] proved that for a fixed positive integer k, $\mathrm{pod}_3(n)$ is almost always divisible by 3^k. Ray [102] studied the divisibility of $\mathrm{pod}_p(n)$ modulo p^j for any prime p. Singh [112] showed that the series $\sum_{n=0}^{\infty} \mathrm{ped}_t(2n+1)q^n$ is lacunary modulo arbitrary powers of 2 for $t = 3$, 5, and 9, and $\sum_{n=0}^{\infty} \mathrm{ped}_7(2n+1)q^n$ is lacunary modulo 2.

In this chapter, we study the arithmetic densities of $\mathrm{pod}_3(2n)$, $\mathrm{pod}_5(2n)$, and $\mathrm{pod}_9(2n+1)$ modulo arbitray powers of 2 and $\mathrm{pod}_7(2n+1)$, $\mathrm{pod}_{13}(4n+1)$,

The contents of this chapter have been submitted for publication [119].

$\mathrm{pod}_{17}(4n+3)$, $\mathrm{ped}_{13}(4n+2)$, and $\mathrm{ped}_{17}(4n+3)$ modulo 2. To be specific, we prove the following theorems.

Theorem 6.1. *Let k be a positive integer. Then the series $\sum_{n=0}^{\infty} \mathrm{pod}_t(2n)q^n$ is lacunary modulo 2^k where $t = 3$ and 5.*

Theorem 6.2. *Let k be a positive integer. Then the series $\sum_{n=0}^{\infty} \mathrm{pod}_9(2n+1)q^n$ is lacunary modulo 2^k.*

Theorem 6.3. *The series $\sum_{n=0}^{\infty} \mathrm{pod}_7(2n+1)q^n$ is lacunary modulo 2.*

Theorem 6.4. *The series $\sum_{n=0}^{\infty} \mathrm{pod}_{13}(4n+1)q^n$ and $\sum_{n=0}^{\infty} \mathrm{ped}_{13}(4n+2)q^n$ are lacunary modulo 2.*

Theorem 6.5. *The series $\sum_{n=0}^{\infty} \mathrm{pod}_{17}(4n+3)q^n$ and $\sum_{n=0}^{\infty} \mathrm{ped}_{17}(4n+3)q^n$ are lacunary modulo 2.*

The eta-quotients associated with the generating functions mentioned in Theorems 6.1–6.5 do not satisfy the hypotheses of Theorem 4.2 of Cotron et al. [48]. We employ Theorem 4.17 for proving Theorems 6.1 and 6.2. We establish Theorems 6.3–6.5 using a result of Landau [81].

In 2017, Gireesh, Hirschhorn, and Naika [56] proved some internal congruences for the function $\mathrm{pod}_3(n)$. Hemanthkumar, Bharadwaj, and Naika [67] established a number of infinite families of congruences for $\mathrm{pod}_9(n)$ modulo 16 and 32, and some internal congruences modulo small powers of 3 using theta functions and q-series manipulations. They also found a relation between $\mathrm{pod}_9(n)$ and $\mathrm{ped}_9(n)$. Using the theory of Hecke eigenforms, Veena and Fathima [122] proved some infinite families of congruences for $\mathrm{pod}_3(n)$ modulo 3. With the aid of Ramanujan's theta functions, Saikia [107] established some infinite families of congruences for $\mathrm{pod}_3(n)$ modulo 2 and 3. Recently, Yu [134] proved some congruences for $\mathrm{pod}_5(n)$ and $\mathrm{pod}_{25}(n)$ modulo 5, $\mathrm{pod}_7(n)$ modulo 7, and $\mathrm{pod}_9(n)$ modulo 3, respectively using q-series techniques. Ray [102] also proved an infinite family of internal congruence for $\mathrm{pod}_3(n)$ modulo 3 and two multiplicative relations for $\mathrm{pod}_5(n)$ and $\mathrm{pod}_7(n)$ modulo 5 and 7, respectively.

Congruences for $\mathrm{ped}_\ell(n)$ have also been studied in the literature. Drema and Saikia [52] proved congruences modulo 2 and 4 for $\mathrm{ped}_t(n)$ when $t = 3$, 5, 7, and 11. They also proved infinite families of congruences modulo 9, 12, 18, and 24 for $\mathrm{ped}_9(n)$. Very recently, Singh [112] established certain infinite families of congruences for $\mathrm{ped}_5(n)$ and $\mathrm{ped}_9(n)$ modulo 2, 8, 12, and 18 using the theory of Hecke eigenforms.

We prove new multiplicative relations for $\mathrm{pod}_5(n)$, $\mathrm{pod}_9(n)$, $\mathrm{ped}_5(n)$, and $\mathrm{ped}_9(n)$ modulo small powers of 2. We state our results in the following theorems.

Theorem 6.6. *Let k be a positive integer and p be a prime number such that $p \equiv 3 \pmod 4$. Let r be a non-negative integer such that p divides $4r+3$. Then*

$$\mathrm{pod}_5\left(2p^{k+1}n + 2pr + \frac{3p-1}{2}\right) \equiv v(p)\ \mathrm{pod}_5\left(2p^{k-1}n + \frac{4r+3-p}{2p}\right) \pmod 2,$$

where $v(p)$ is defined by

$$v(p) = \begin{cases} -1, & \text{if } p \equiv 3, 7 \pmod{20}, \\ 1, & \text{if } p \equiv 11, 19 \pmod{20}. \end{cases}$$

Theorem 6.7. *Let k be a positive integer and p be a prime such that $p \equiv i \pmod{12}$, where $i \in \{5, 7, 11\}$. Let r be a non-negative integer such that p divides $12r+i$. Then*

$$\mathrm{pod}_9\left(72p^{k+1}n + 72pr + 6pi - 1\right) \equiv w(p)\ \mathrm{pod}_9\left(72p^{k-1}n + \frac{72r+6i-p}{p}\right) \pmod{16},$$

where $w(p)$ is defined by

$$w(p) = \begin{cases} -1, & \text{if } p \equiv 5 \pmod{12}, \\ 1, & \text{if } p \equiv 7, 11 \pmod{12}. \end{cases}$$

Theorem 6.8. *Let k be a positive integer and p be a prime such that $p \equiv 3 \pmod 4$. Let r be a non-negative integer such that p divides $4r+3$. Then*

$$\mathrm{ped}_5\left(2p^{k+1}n + 2pr + \frac{3p+1}{2}\right) \equiv v(p)\ \mathrm{ped}_5\left(2p^{k-1}n + \frac{4r+3+p}{2p}\right) \pmod 2,$$

where $v(p)$ is as defined in Theorem 6.6.

Theorem 6.9. *Let k be a positive integer and p be a prime such that $p \equiv i \pmod{12}$, where $i \in \{5, 7, 11\}$. Let r be a non-negative integer such that p divides $12r+i$. Then*

$$\mathrm{ped}_9\left(12p^{k+1}n + 12pr + pi + 1\right) \equiv w(p)\ \mathrm{ped}_9\left(12p^{k-1}n + \frac{12r+i+p}{p}\right) \pmod 8,$$

where $w(p)$ is as defined in Theorem 6.7.

Theorem 6.10. *Let k be a positive integer and p be a prime such that $p \equiv i \pmod{12}$, where $i \in \{5, 7, 11\}$. Let r be a non-negative integer such that p divides $12r + i$. Then*

$$\mathrm{ped}_9\left(24p^{k+1}n + 24pr + 2pi + 1\right) \equiv w(p)\ \mathrm{ped}_9\left(24p^{k-1}n + \frac{24r + 2i - p}{p}\right) \pmod{12},$$

where $w(p)$ is as defined in Theorem 6.7.

Theorem 6.11. *Let k be a positive integer and p be a prime such that $p \equiv 2 \pmod{3}$. Let r be a non-negative integer such that p divides $3r + 2$. Then*

$$\mathrm{ped}_9\left(12p^{k+1}n + 12pr + 8p + 1\right) \equiv (-p)\ \mathrm{ped}_9\left(12p^{k-1}n + \frac{12r + 8 + p}{p}\right) \pmod{18}.$$

The results of this chapter are proved with the aid of the theory of modular forms and Hecke eigenforms. In Section 6.2, we establish Theorems 6.1–6.5 whereas Theorems 6.6–6.11 are deduced in Section 6.3.

6.2 Proofs of Theorems 6.1–6.5

Proof of Theorem 6.1. Putting $\ell = 3$ in (6.1) and then using (2.18) of [130], we have

$$\sum_{n=0}^{\infty} \mathrm{pod}_3(n)q^n = \frac{f_2 f_3 f_{12}}{f_1 f_4 f_6} = \frac{f_2 f_{12}}{f_4 f_6}\left(\frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_{48} f_8^2}{f_2^2 f_{16} f_{24}}\right).$$

Extracting the even powers of q from the above, we find that

$$\sum_{n=0}^{\infty} \mathrm{pod}_3(2n)q^n = \frac{f_8 f_{12}^2}{f_1 f_4 f_{24}}. \tag{6.3}$$

Let

$$A(z) := \prod_{n=1}^{\infty} \frac{(1 - q^{96n})^2}{(1 - q^{192n})} = \frac{\eta^2(96z)}{\eta(192z)}.$$

Using the binomial theorem, we have

$$A^{2^k}(z) = \frac{\eta^{2^{k+1}}(96z)}{\eta^{2^k}(192z)} \equiv 1 \pmod{2^{k+1}}.$$

Define $B_k(z)$ by

$$B_k(z) := \left(\frac{\eta(64z)\eta^2(96z)}{\eta(8z)\eta(32z)\eta(192z)}\right) A^{2^k}(z) = \frac{\eta(64z)\eta^{2^{k+1}+2}(96z)}{\eta(8z)\eta(32z)\eta^{2^k+1}(192z)},$$

which under modulo 2^{k+1} reduces to

$$B_k(z) \equiv \frac{\eta(64z)\eta^2(96z)}{\eta(8z)\eta(32z)\eta(192z)} \equiv q\frac{f_{64}f_{96}^2}{f_8 f_{32} f_{192}} \pmod{2^{k+1}}. \tag{6.4}$$

Combining (6.3) and (6.4), we find that

$$B_k(z) \equiv \sum_{n=0}^{\infty} \mathrm{pod}_3(2n) q^{8n+1} \pmod{2^{k+1}}. \tag{6.5}$$

Now, $B_k(z)$ is an eta-quotient with $N = 192$. Our goal is to prove that $B_k(z)$ is a modular form for all $k \geq 3$. We know that the cusps of $\Gamma_0(192)$ are represented by fractions c/d , where $d \mid 192$ and $\gcd(c, d) = 1$. By Theorem 4.16, we find that $B_k(z)$ is holomorphic at a cusp c/d if and only if

$$\begin{aligned}\mathcal{L} := \left(2^{k+2}+4\right)\frac{\gcd(d,96)^2}{\gcd(d,192)^2} - 24\frac{\gcd(d,8)^2}{\gcd(d,192)^2} + 3\frac{\gcd(d,64)^2}{\gcd(d,192)^2} \\ - 6\frac{\gcd(d,32)^2}{\gcd(d,192)^2} - 2^k - 1 \geq 0.\end{aligned}$$

From the following table, we conclude that $\mathcal{L} \geq 0$ for all $d \mid 192$ for $k \geq 3$.

d such that $d\mid 192$	$\frac{\gcd(d,8)^2}{\gcd(d,192)^2}$	$\frac{\gcd(d,32)^2}{\gcd(d,192)^2}$	$\frac{\gcd(d,64)^2}{\gcd(d,192)^2}$	$\frac{\gcd(d,96)^2}{\gcd(d,192)^2}$	Value of $\mathcal{L}$
1,2,4,8	1	1	1	1	$3\left(2^k-8\right)$
16	1/4	1	1	1	$3\left(2^k-2\right)$
32	1/16	1	1	1	$3 \times 2^k - 3/2$
64	1/64	1/4	1	1/4	9/8
3,6,12,24	1/9	1/9	1/9,	1	3×2^k
48	1/36	1/9	1/9	1	$3 \times 2^k + 2$
96	1/144	1/9	1/9	1	$3 \times 2^k + 5/2$
192	1/576	1/36	1/9	1/4	1/8

Hence, $B_k(z)$ is holomorphic at every cusp c/d for all $k \geq 3$. The weight of $B_k(z)$ is 2^{k-1} and the associated character is given by $\chi_8(\bullet) = \left(\frac{2^{2^{k+2}+2}3^{2^k+1}}{\bullet}\right)$. Thus, $B_k(z) \in M_{2^{k-1}}\left(\Gamma_0(192), \chi_8\right)$. Again, the Fourier coefficients of $B_k(z)$ are all integers. Therefore by Theorem 4.17, the Fourier coefficients of $B_k(z)$ are almost divisible by $m = 2^k$. Due to (6.5), this holds for $\mathrm{pod}_3(2n)$. Hence, $\sum_{n=0}^{\infty} \mathrm{pod}_3(2n)q^n$ is lacunary modulo 2^k.

Now, we prove the lacunarity of $\sum_{n=0}^{\infty} \text{pod}_5(2n)q^n$ modulo 2^k.

Putting $\ell = 5$ in (6.1) and then employing Theorem 2.1 of [69], we have

$$\sum_{n=0}^{\infty} \text{pod}_5(n)q^n = \frac{f_2 f_5 f_{20}}{f_1 f_4 f_{10}} = \frac{f_2 f_{20}}{f_4 f_{10}} \left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_{10} f_{40} f_4^3}{f_2^3 f_8 f_{20}} \right).$$

Extracting the even powers of q from both sides of the above, we find that

$$\sum_{n=0}^{\infty} \text{pod}_5(2n)q^n = \frac{f_4 f_{10}^3}{f_1 f_2 f_5 f_{20}}. \tag{6.6}$$

Let

$$C(z) := \prod_{n=1}^{\infty} \frac{(1-q^{40n})^2}{(1-q^{80n})} = \frac{\eta^2(40z)}{\eta(80z)}.$$

In view of the binomial theorem, we have

$$C^{2^k}(z) = \frac{\eta^{2^{k+1}}(40z)}{\eta^{2^k}(80z)} \equiv 1 \pmod{2^{k+1}}.$$

Define $D_k(z)$ by

$$D_k(z) := \left(\frac{\eta(16z)\eta^3(40z)}{\eta(4z)\eta(8z)\eta(20z)\eta(80z)} \right) C^{2^k}(z) = \frac{\eta(16z)\eta^{2^{k+1}+3}(40z)}{\eta(4z)\eta(8z)\eta(20z)\eta^{2^k+1}(80z)},$$

which under modulo 2^{k+1} reduces to

$$D_k(z) \equiv \frac{\eta(16z)\eta^3(40z)}{\eta(4z)\eta(8z)\eta(20z)\eta(80z)} \equiv q \frac{f_{16} f_{40}^3}{f_4 f_8 f_{20} f_{80}} \pmod{2^{k+1}}. \tag{6.7}$$

Combining (6.6) and (6.7), we obtain

$$D_k(z) \equiv \sum_{n=0}^{\infty} \text{pod}_5(2n)q^{4n+1} \pmod{2^{k+1}}. \tag{6.8}$$

Now, $D_k(z)$ is an eta-quotient with $N = 80$. We next prove that $D_k(z)$ is a modular form for all $k \geq 3$. We know that the cusps of $\Gamma_0(80)$ are represented by fractions c/d, where $d \mid 80$ and $\gcd(c, d) = 1$. By Theorem 4.16, we find that $D_k(z)$ is holomorphic at a cusp c/d if and only if

$$\begin{aligned}\mathcal{L} := \left(2^{k+2} + 6\right) \frac{\gcd(d, 40)^2}{\gcd(d, 80)^2} - 20 \frac{\gcd(d, 4)^2}{\gcd(d, 80)^2} - 10 \frac{\gcd(d, 8)^2}{\gcd(d, 80)^2} - 4 \frac{\gcd(d, 20)^2}{\gcd(d, 80)^2} \\ + 5 \frac{\gcd(d, 16)^2}{\gcd(d, 80)^2} - 2^k - 1 \geq 0.\end{aligned}$$

From the following table, we conclude that $\mathcal{L} \geq 0$ for all $d \mid 80$ for $k \geq 3$.

d such that $d \mid 80$	$\frac{\gcd(d,4)^2}{\gcd(d,80)^2}$	$\frac{\gcd(d,8)^2}{\gcd(d,80)^2}$	$\frac{\gcd(d,16)^2}{\gcd(d,80)^2}$	$\frac{\gcd(d,20)^2}{\gcd(d,80)^2}$	$\frac{\gcd(d,40)^2}{\gcd(d,80)^2}$	Value of $\mathcal{L}$
1,2,4	1	1	1	1	1	$3\left(2^k-8\right)$
8	1/4	1	1	1/4	1	$3\left(2^k-2\right)$
16	1/16	1/4	1	1/16	1/4	3/2
5,10,20	1/25	1/25	1/25	1	1	3×2^k
40	1/100	1/25	1/25	1/4	1	$3\times 2^k+18/5$
80	1/400	1/100	1/25	1/16	1/4	3/10

Hence, $D_k(z)$ is holomorphic at every cusp c/d for all $k \geq 3$. Thus, $D_k(z) \in M_{2^{k-1}}\left(\Gamma_0(80)\right)$. Now, using Theorem 4.17 and (6.8) and arguing similarly as in the earlier proof, we establish the lacunarity of $\sum\limits_{n=0}^{\infty} \mathrm{pod}_5(2n)q^n$ modulo 2^k. This completes the proof of Theorem 6.1. $\square$

Proof of Theorem 6.2. Putting $\ell = 9$ in (6.1) and then using (3.14) of [127], we have

$$\sum_{n=0}^{\infty} \mathrm{pod}_9(n)q^n = \frac{f_2 f_9 f_{36}}{f_1 f_4 f_{18}} = \frac{f_2 f_{36}}{f_4 f_{18}}\left(\frac{f_{18} f_{12}^3}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}\right).$$

We extract the odd powers of q from above to arrive at

$$\sum_{n=0}^{\infty} \mathrm{pod}_9(2n+1)q^n = \frac{f_2 f_3 f_{18}^2}{f_1^2 f_6 f_9}. \tag{6.9}$$

Let

$$E(z) := \prod_{n=1}^{\infty} \frac{(1-q^{9n})^2}{(1-q^{18n})} = \frac{\eta^2(9z)}{\eta(18z)}.$$

Using the binomial theorem, we have

$$E^{2^k}(z) = \frac{\eta^{2^{k+1}}(9z)}{\eta^{2^k}(18z)} \equiv 1 \pmod{2^{k+1}}.$$

Define $F_k(z)$ by

$$F_k(z) := \left(\frac{\eta(2z)\eta(3z)\eta^2(18z)}{\eta^2(z)\eta(6z)\eta(9z)}\right) E^{2^k}(z) = \frac{\eta(2z)\eta(3z)\eta^{2^{k+1}-1}(9z)}{\eta^2(z)\eta(6z)\eta^{2^k-2}(18z)},$$

which under modulo 2^{k+1} reduces to

$$F_k(z) \equiv \frac{\eta(2z)\eta(3z)\eta^2(18z)}{\eta^2(z)\eta(6z)\eta(9z)} \equiv q\frac{f_2 f_3 f_{18}^2}{f_1^2 f_6 f_9} \pmod{2^{k+1}}. \tag{6.10}$$

Combining (6.9) and (6.10), we arrive at

$$F_k(z) \equiv \sum_{n=0}^{\infty} \mathrm{pod}_9(2n+1)q^{n+1} \pmod{2^{k+1}}. \tag{6.11}$$

Now, $F_k(z)$ is an eta-quotient with $N = 18$. We next prove that $F_k(z)$ is a modular form for all $k \geq 3$. We know that the cusps of $\Gamma_0(18)$ are represented by fractions c/d, where $d \mid 18$ and $\gcd(c,d) = 1$. By Theorem 4.16, we find that $F_k(z)$ is holomorphic at a cusp c/d if and only if

$$\begin{aligned}\mathcal{L} := \left(2^{k+2}-2\right)\frac{\gcd(d,9)^2}{\gcd(d,18)^2} &+ 9\frac{\gcd(d,2)^2}{\gcd(d,18)^2} - 3\frac{\gcd(d,6)^2}{\gcd(d,18)^2} \\ &+ 6\frac{\gcd(d,3)^2}{\gcd(d,18)^2} - \frac{36}{\gcd(d,18)^2} - 2^k + 2 \geq 0.\end{aligned}$$

From the following table, we conclude that $\mathcal{L} \geq 0$ for all $d \mid 18$ for $k \geq 3$.

d such that $d \mid 18$	$\frac{\gcd(d,2)^2}{\gcd(d,18)^2}$	$\frac{\gcd(d,3)^2}{\gcd(d,18)^2}$	$\frac{\gcd(d,6)^2}{\gcd(d,18)^2}$	$\frac{\gcd(d,9)^2}{\gcd(d,18)^2}$	Value of $\mathcal{L}$
1	1	1	1	1	$3(2^k-8)$
2	1	1/4	1	1/4	0
3	1/9	1	1	1	3×2^k
6	1/9	1/4	1	1/4	0
9	1/81	1/9	1/9	1	3×2^k
18	1/81	1/36	1/9	1/4	4/3

Hence, $F_k(z)$ is holomorphic at every cusp c/d for all $k \geq 3$. Thus, $F_k(z) \in M_{2^{k-1}}\left(\Gamma_0(18)\right)$. Again, using Theorem 4.17 and proceeding in a similar manner as in the proof of Theorem 6.1, we prove the lacunarity of $\sum_{n=0}^{\infty} \mathrm{pod}_9(2n+1)q^n$ due to (6.11). Thus, we complete the proof. □

Proofs of Theorems 6.3–6.5 are based on Lemma 4.19 due to Landau [81].

Since the proofs are similar in nature, we only proof of Theorems 6.3 and 6.5 here and then present the outlines for the proof of Theorem 6.4.

Proof of Theorem 6.3. Putting $\ell = 7$ in (6.1), we get

$$\sum_{n=0}^{\infty} \mathrm{pod}_7(n)q^n = \frac{f_2 f_7 f_{28}}{f_1 f_4 f_{14}} \equiv \frac{f_{14}}{f_2} \cdot \frac{f_7}{f_1} \pmod 2. \tag{6.12}$$

Employing the relation $\frac{f_7}{f_1} \equiv f_1^6 + q f_1^2 f_7^4 + q^2 \frac{f_7^8}{f_1^2} \pmod{2}$ from [78], and then extracting the terms involving odd powers of q from the both sides of (6.12), we arrive at

$$\sum_{n=0}^{\infty} \mathrm{pod}_7(2n+1)q^n \equiv f_7 f_{14} \pmod{2}. \tag{6.13}$$

Again, from (4.24), we have

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \equiv \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} \pmod{2}, \quad \text{where } |\, q \,| < 1. \tag{6.14}$$

Magnifying (6.14) by $q \to q^7$ and $q \to q^{14}$, we find that

$$f_7 \equiv \sum_{n=-\infty}^{\infty} q^{7n(3n+1)/2} \pmod{2} \quad \text{and} \quad f_{14} \equiv \sum_{n=-\infty}^{\infty} q^{7n(3n+1)} \pmod{2}. \tag{6.15}$$

Combining (6.13) and (6.15) and then applying Lemma 4.19, we complete the proof. $\square$

Proof of Theorem 6.5. Putting $\ell = 17$ in (6.1) and (6.2), we get

$$\sum_{n=0}^{\infty} \mathrm{pod}_{17}(n)q^n = \frac{f_2 f_{17} f_{68}}{f_1 f_4 f_{34}} \equiv \frac{f_{68}}{f_2} \cdot \frac{1}{f_1 f_{17}} \pmod{2}. \tag{6.16}$$

Again, Zhao, Jin, and Yao [135] proved the following congruence relation:

$$\frac{1}{f_1 f_{17}} \equiv \sum_{n=0}^{\infty} \Delta_8(2n) q^{2n} + q f_2^3 + q^5 f_{34}^3 \pmod{2}, \tag{6.17}$$

where $\sum_{n=0}^{\infty} \Delta_8(n) q^n = \frac{f_2 f_{17}}{f_1^3 f_{34}}$.

Employing (6.17) in (6.16) and then extracting the terms involving odd powers of q from the both sides, we find that

$$\sum_{n=0}^{\infty} \mathrm{pod}_{17}(2n+1)q^n \equiv f_2 f_{34} + q^2 f_{34}^3 \cdot \frac{1}{f_1 f_{17}} \pmod{2}. \tag{6.18}$$

Again, applying (6.17) in (6.18), and extracting the terms involving odd powers of q, we have

$$\sum_{n=0}^{\infty} \mathrm{pod}_{17}(4n+3)q^n \equiv q f_1^3 f_{17}^3 + q^3 f_{17}^6 \pmod{2}. \tag{6.19}$$

From (3.66), we have

$$f_1^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} \equiv \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod 2. \tag{6.20}$$

Magnifying (6.20) by $q \to q^{17}$, we find that

$$f_{17}^3 \equiv \sum_{n=0}^{\infty} q^{17n(n+1)/2} \pmod 2. \tag{6.21}$$

From (6.20) and (6.21) and Lemma 4.19, we observe that the first term of the right hand side of (6.19) is lacunary modulo 2. Also, taking $r(n) = s(n) = 17n(n+1)/2$ in Lemma 4.19, we conclude that the second term of the right hand side of (6.19) is also lacunary modulo 2. Thus, we establish the lacunarity of $\sum_{n=0}^{\infty} \mathrm{pod}_{17}(4n+3)q^n$.

Proceeding in a similar manner, we find that

$$\sum_{n=0}^{\infty} \mathrm{ped}_{17}(4n+3)q^n \equiv f_1^6 + q^2 f_1^3 f_{17}^3 \pmod 2. \tag{6.22}$$

Now, we consider $r(n) = s(n) = n(n+1)/2$ in Lemma 4.19, and observe that the first term of the right hand side of (6.22) is lacunary modulo 2. And from (6.20) and (6.21) and Lemma 4.19, we conclude that the second term of the right hand side of (6.22) is also lacunary modulo 2. Hence, $\sum_{n=0}^{\infty} \mathrm{ped}_{17}(4n+3)q^n$ is lacunary modulo 2. This completes the proof of Theorem 6.5 $\square$

Proof of Theorem 6.4. The proof of Theorem 6.4 is similar in nature as the above proof. So, we only note the generating functions of the corresponding functions modulo 2 and the choices of $r(n)$ and $s(n)$ for each of them.

Using $\dfrac{f_{13}}{f_1} \equiv f_4^3 + q f_2^5 f_{26} + q^6 f_{52}^3 + q^7 \dfrac{f_{26}^7}{f_2} \pmod 2$ from [38], we can easily obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_{13}(4n+1)q^n \equiv f_2 f_{13} \pmod 2 \tag{6.23}$$

and

$$\sum_{n=0}^{\infty} \mathrm{ped}_{13}(4n+2)q^n \equiv q f_1 f_{26} \pmod 2. \tag{6.24}$$

We complete the proof from the following table.

	Generating Function	$r(n)$	$s(n)$
$\sum_{n=0}^{\infty} \mathrm{pod}_{13}(4n+1)q^n$	$f_2 f_{13}$	$n(3n+1)$	$13n(3n+1)/2$
$\sum_{n=0}^{\infty} \mathrm{ped}_{13}(4n+2)q^n$	$f_1 f_{26}$	$n(3n+1)/2$	$13n(3n+1)$

$\square$

6.3 Proofs of Theorems 6.6–6.11

Proof of Theorem 6.6. Putting $\ell = 5$ in (6.1), we have

$$\sum_{n=0}^{\infty} \mathrm{pod}_5(n)q^n = \frac{f_2 f_5 f_{20}}{f_1 f_4 f_{10}} \equiv \frac{f_{10}}{f_2} \cdot \frac{f_5}{f_1} \pmod{2}.$$

Now, applying the relation $\dfrac{f_5}{f_1} \equiv f_1^4 + q\dfrac{f_5^6}{f_1^2} \pmod{2}$ from [78], and then extracting the terms with even powers of q, we find that

$$\sum_{n=0}^{\infty} \mathrm{pod}_5(2n)q^n \equiv f_1 f_5 \pmod{2}, \tag{6.25}$$

which gives

$$\sum_{n=0}^{\infty} \mathrm{pod}_5(2n)q^{4n+1} \equiv \eta(4z)\eta(20z) \pmod{2}.$$

Let $\eta(4z)\eta(20z) := \sum_{n=0}^{\infty} a(n)q^n$. Then $a(n) = 0$ if $n \not\equiv 1 \pmod{4}$ and for all $n \geq 0$,

$$\mathrm{pod}_5(2n) = a(4n+1). \tag{6.26}$$

By Theorem 4.15, we have $\eta(4z)\eta(20z) \in M_1\left(\Gamma_0(80), \left(\frac{-20}{\bullet}\right)\right)$. Since $\eta(4z)\eta(20z)$ is a Hecke eigenform (see, for example [84]), hence (1.22) and (1.23) yield

$$\eta(4z)\eta(20z) \mid T_p = \sum_{n=0}^{\infty} \left(a(pn) + \left(\frac{-20}{p}\right) a\left(\frac{n}{p}\right)\right) q^n = \lambda(p) \sum_{n=0}^{\infty} a(n)q^n,$$

which gives

$$a(pn) + \left(\frac{-20}{p}\right) a\left(\frac{n}{p}\right) = \lambda(p)a(n). \tag{6.27}$$

Putting $n = 1$ and noting that $a(1) = 1$, we obtain $a(p) = \lambda(p)$. Since $a(p) = 0$

for all $p \not\equiv 1 \pmod 4$, we have $\lambda(p) = 0$. From (6.27), we obtain

$$a(pn) = (-1)\left(\frac{-20}{p}\right) a\left(\frac{n}{p}\right). \tag{6.28}$$

Next, substituting n by $4p^k n + 4r + 3$ such that p divides $4r+3$ in (6.28), we have

$$a\left(4\left(p^{k+1}n + pr + \frac{3p-1}{4}\right) + 1\right) = (-1)\left(\frac{-20}{p}\right) a\left(4\left(p^{k-1}n + \frac{4r+3-p}{4p}\right) + 1\right). \tag{6.29}$$

We note that $\frac{3p-1}{4}$ and $\frac{4r+3-p}{4p}$ are integer. Now, using (6.26) and (6.29), we arrive at

$$\mathrm{pod}_5\left(2p^{k+1}n + 2pr + \frac{3p-1}{2}\right) \equiv (-1)\left(\frac{-20}{p}\right) \mathrm{pod}_5\left(2p^{k-1}n + \frac{4r+3-p}{2p}\right) \pmod 2. \tag{6.30}$$

For a prime $p \equiv 3 \pmod 4$, we have

$$\left(\frac{-20}{p}\right) = \begin{cases} 1, & \text{if } p \equiv 3, 7 \pmod{20}, \\ -1, & \text{if } p \equiv 11, 19 \pmod{20}. \end{cases}$$

Hence, we conclude the proof of the theorem from (6.30). $\square$

Proof of Theorem 6.7. From (2.51) of [67], we have

$$\sum_{n=0}^{\infty} \mathrm{pod}_9(72n+5)q^n \equiv 4f_1^2 \pmod{16},$$

which gives

$$\sum_{n=0}^{\infty} \mathrm{pod}_9(72n+5)q^{12n+1} \equiv 4\eta^2(12z) \pmod{16}.$$

Let $\eta^2(12z) := \sum_{n=0}^{\infty} b(n)q^n$. Then $b(n) = 0$ if $n \not\equiv 1 \pmod{12}$ and for all $n \geq 0$,

$$\mathrm{pod}_9(72n+5) = b(12n+1). \tag{6.31}$$

By Theorem 4.15, we have $\eta^2(12z) \in M_1\left(\Gamma_0(144), \left(\frac{-1}{\bullet}\right)\right)$. Since $\eta^2(12z)$ is a Hecke eigenform (see, for example [84]), hence (1.22) and (1.23) yield

$$\eta^2(12z) \mid T_p = \sum_{n=0}^{\infty} \left(a(pn) + \left(\frac{-1}{p}\right) a\left(\frac{n}{p}\right)\right) q^n = \lambda(p) \sum_{n=0}^{\infty} a(n)q^n, \tag{6.32}$$

which gives

$$b(pn)+\left(\frac{-1}{p}\right)b\left(\frac{n}{p}\right)=\lambda(p)b(n). \tag{6.33}$$

Putting $n=1$ and noting that $b(1)=1$, we obtain $b(p)=\lambda(p)$. Since $b(p)=0$ for all $p\not\equiv 1 \pmod{12}$, we have $\lambda(p)=0$. From (6.33), we obtain

$$b(pn)=(-1)\left(\frac{-1}{p}\right)b\left(\frac{n}{p}\right). \tag{6.34}$$

Let $i\in\{5,7,11\}$. Suppose r is a non-negative integer such that the prime $p\equiv i \pmod{12}$ divides $12r+i$. Then substituting n by $12p^kn+12r+i$ in (6.34), we have

$$b\left(12\left(p^{k+1}n+pr+\frac{pi-1}{12}\right)+1\right)=(-1)\left(\frac{-1}{p}\right)b\left(12\left(p^{k-1}n+\frac{12r+i-p}{12p}\right)+1\right). \tag{6.35}$$

We note that $\frac{pi-1}{12}$ and $\frac{12r+i-p}{12p}$ are integer. Now, using (6.31) and (6.35), we arrive at

$$\mathrm{pod}_9\left(72p^{k+1}n+72pr+6pi-1\right)\equiv(-1)\left(\frac{-1}{p}\right)\mathrm{pod}_9\left(72p^{k-1}n+\frac{72r+6i-p}{p}\right) \pmod{16}. \tag{6.36}$$

For a prime $p\not\equiv 1 \pmod{12}$, we have

$$\left(\frac{-1}{p}\right)=\begin{cases}1, & \text{if } p\equiv 5 \pmod{12},\\ -1, & \text{if } p\equiv 7,11 \pmod{12}.\end{cases}$$

Hence, we conclude the proof of the theorem from (6.36). □

Proof of Theorem 6.8. From (3.4) of [112], we have

$$\sum_{n=0}^{\infty}\mathrm{ped}_5(2n+1)q^n=\frac{f_2^4f_5f_{20}}{f_1^3f_4f_{10}^2},$$

which under modulo 2 gives

$$\sum_{n=0}^{\infty}\mathrm{ped}_5(2n+1)q^{4n+1}\equiv\eta(4z)\eta(20z) \pmod{2}.$$

Proceeding similarly as the proof of Theorem 6.6, we complete the proof. □

Proof of Theorem 6.9. First we recall the following identity from [52, (8.4)]:

$$\sum_{n=0}^{\infty}\mathrm{ped}_9(6n+2)q^n\equiv 2f_2^2 \pmod{8}.$$

Extracting the terms involving even powers of q on both side of the anbove equation, we find that

$$\sum_{n=0}^{\infty} \mathrm{ped}_9(12n+2)q^n \equiv 2f_1^2 \pmod{8},$$

which gives

$$\sum_{n=0}^{\infty} \mathrm{ped}_9(12n+2)q^{12n+1} \equiv 2\eta^2(12z) \pmod{8}.$$

Again, proceeding in a similar manner as in the proof of Theorem 6.7, we complete the proof. □

Proof of Theorem 6.10. From (3.20) of [112], we have

$$\sum_{n=0}^{\infty} \mathrm{ped}_9(4n+3)q^n \equiv 3f_6^2 \pmod{12}.$$

Extracting the the terms involving even powers of q on both sides of the equation, we get

$$\sum_{n=0}^{\infty} \mathrm{ped}_9(8n+3)q^n \equiv 3f_3^2 \pmod{12}.$$

Again, extracting the the terms involving q^{3n} on both sides of the equation, we get

$$\sum_{n=0}^{\infty} \mathrm{ped}_9(24n+3)q^n \equiv 3f_1^2 \pmod{12},$$

which gives

$$\sum_{n=0}^{\infty} \mathrm{ped}_9(24n+3)q^{12n+1} \equiv 3\eta^2(12z) \pmod{12}.$$

Proceeding as similar to the proof of Theorem 6.7, we arrive at the desired result. □

Proof of Theorem 6.11. From (10.3) of [52], we have

$$\sum_{n=0}^{\infty} \mathrm{ped}_9(12n+5)q^n \equiv 6f_1^8 \pmod{18},$$

which gives

$$\sum_{n=0}^{\infty} \mathrm{ped}_9(12n+5)q^{3n+1} \equiv 6\eta^8(3z) \pmod{18}.$$

Let $\eta^8(3z) := \sum_{n=0}^{\infty} e(n)q^n$. Then $c(n) = 0$ if $n \not\equiv 1 \pmod 3$ and for all $n \geq 0$,

$$\mathrm{ped}_9(12n+5) = e(3n+1). \tag{6.37}$$

By Theorem 4.15, we have $\eta^8(3z) \in M_4(\Gamma_0(9))$. Since $\eta^8(3z)$ is a Hecke eigenform (see, for example [84]), hence (1.22) and (1.23) yield

$$\eta^8(3z) \mid T_p = \sum_{n=0}^{\infty} \left(e(pn) + p\ e\left(\frac{n}{p}\right)\right) q^n = \lambda(p) \sum_{n=0}^{\infty} e(n)q^n, \tag{6.38}$$

which gives

$$e(pn) + p\ e\left(\frac{n}{p}\right) = \lambda(p)e(n). \tag{6.39}$$

Putting $n = 1$ and noting that $e(1) = 1$, we obtain $e(p) = \lambda(p)$. Since $e(p) = 0$ for all $p \not\equiv 1 \pmod 3$, we have $\lambda(p) = 0$. From (6.39), we obtain

$$e(pn) = (-p)\ e\left(\frac{n}{p}\right). \tag{6.40}$$

Next, substituting n by $3p^kn + 3r + 2$ such that p divides $3r + 2$ in (6.40), we have

$$e\left(3\left(p^{k+1}n + pr + \frac{2p-1}{3}\right) + 1\right) = (-p)\ e\left(3\left(p^{k-1}n + \frac{3r+2-p}{3p}\right) + 1\right). \tag{6.41}$$

We note that $\frac{2p-1}{3}$ and $\frac{3r+2-p}{3p}$ are integer. Now, using (6.37) and (6.41), we obtain

$$\mathrm{ped}_9\left(12p^{k+1}n + 12pr + 8p + 1\right) \equiv (-p)\ \mathrm{ped}_9\left(12p^{k-1}n + \frac{12r+8+p}{p}\right) \pmod{18},$$

which concludes the proof of the theorem. $\square$